representation has its limits. These limits were discovered in 1931 by Gödel, who after proving the completeness of the predicate calculus turned to arithmetic and attempted to prove the completeness also of this theory. These attempts resulted in the perhaps most surprising discovery of mathematics in the twentieth century – the discovery of the incompleteness of arithmetic and the improvability of its consistency. Nevertheless, the tools by means of which Gödel achieved his results transcend the language of predicate calculus. Gödel used a new kind of symbolic language, the theory of recursive functions. Therefore the proofs of incompleteness and improvability of consistency do not belong to the framework of predicate calculus. These results can be seen as an illustration of the expressive boundaries of the language of predicate calculus. The situation here is similar to the previous cases. The language of the next stage (in this case the language of the theory of recursive functions, computability, and algorithms) makes it possible to draw the boundaries of the given language. Similarly as the language of algebra made it possible to prove the non-constructability of the regular heptagon, and thus delineated the boundaries of the language of synthetic geometry; or as the language of the differential and integral calculus made it possible to prove the transcendence of π, and so to draw the boundaries of the language of algebra; also Gödel had to use a stronger language than the one, the boundaries of which he succeeded in drawing. In the language itself its boundaries are inexpressible. They only display themselves in the fact that all the attempts to prove, for instance, the consistency of arithmetic undertaken by Hilbert's school were unsuccessful. The language of the predicate calculus, however, did not make it possible to understand the reason for this systematic failure. Only when Gödel developed his remarkable method of coding and laid the foundations of the theory of recursive functions, did he create the linguistic tools necessary for demarcation of the expressive boundaries of the language of predicate calculus.

1.1.8. Set Theory

Infinity fascinated mankind from the earliest times. The distance of the horizon or the depths of the sea filled the human soul with a feeling of awe. When mathematics created a paradigm of exact, precise, and unambiguous knowledge, infinity because of its incomprehensibility and ambiguousness found itself beyond the boundaries of mathematics. The ancient Greeks could not imagine that the infinite (called

$\alpha\pi\epsilon\iota\rho o\nu$ by them) could become a subject of mathematical inquiry. The Pythagoreans, Plato, as well as Aristotle denied the possibility of a mathematical description of the $\alpha\pi\epsilon\iota\rho o\nu$. The situation started to change during the middle ages when the attribute of infinity was ascribed to God. This weakened or even eradicated the ambiguity and imperfection traditionally associated with the notion of infinity (see Kvasz 2004). God is perfect and so also must be his attributes, among them infinity. In this way infinity was divested of the negativity that was associated with this notion since antiquity. "The study of infinity acquired a noble purpose; it became a part of theology and not of science" (Vopenka 2000, p. 328). In the Renaissance this new, perfect, and unambiguous notion of infinity started to find its way from theology into mathematics. As an illustration of this process we can mention the *De Docta Ignorantia*, in which Nicholas of Cusa attempted to prove the Trinity using an infinitely large triangle:

> "It is already evident that there can be only one maximum and infinite thing. Moreover, since any two sides of any triangle cannot, if conjoined, be shorter than the third: it is evident that in the case of a triangle whose one side is infinite, the other two sides are not shorter. And because each part of what is infinite is infinite: for any triangle whose one side is infinite, the other sides must also be infinite. And since there cannot be more than one infinite thing, you understand transcendently that an infinite triangle cannot be composed of a plurality of lines, even though it is the greatest and truest triangle, incomposite and most simple.." (Nicholas of Cusa 1440, p. 22)

I quote this text not for analyzing the correctness or the persuasiveness of its arguments. Rather I would like to use it as an illustration of the distance that western thought has travelled since Antiquity. The freedom with which Nicholas of Cusa uses the notion of infinity is amazing. Such a text could not have been written by any philosopher of ancient Greece. After theology had broken the barrier that separated mathematics from the notion of infinity, a gradual transformation of all mathematics started. Euclid's $\epsilon\iota\vartheta\epsilon\iota\alpha$ (straight line), which had only a finite length, was replaced by our *straight line*, i.e., by an object of infinite extension. The Greek geometer needed his second postulate ("*To produce a straight line continuously in a straight line*") in order to secure the possibility of extending the $\epsilon\iota\vartheta\epsilon\iota\alpha$ as far as he wished;

on the other hand in the Renaissance the straight line "reached the infinity". Similarly the atoms of Democritus witnessed a revival in the form of indivisibles by Kepler and Cavalieri and were finally replaced by infinitesimals in the seventeenth century.

The penetration of the notion of infinity into mathematics was made possible by the change of attitude to this notion that appeared first in theology. Nevertheless, the notion of infinity was from the beginning accompanied by criticism. Many mathematicians and philosophers of the seventeenth and eighteenth century considered manipulations with infinitesimals to be doubtful or even wrong (we can mention Descartes and Berkeley). Therefore relatively early a countermovement started, the aim of which was to eliminate infinitesimals from mathematics. After the first attempts (Carnot 1797, Lagrange 1797) a successful way of eliminating infinitesimals was found by Bolzano in his *Rein analytischer Beweis* (Bolzano 1817) and fully developed by Cauchy in his *Cours d'Analyse de l'École Polytechnique* (Cauchy 1821).[16] But as we already mentioned, Cauchy and Bolzano based their method of elimination of infinitesimals on the intuitive notion of the continuum. Half a century later Dedekind, Cantor, and Weierstrass presented three constructions of the continuum and so brought Cauchy's project to a consummation. These constructions of the continuum, even if independent, all assumed the existence of some infinite system of objects. This indicated that the notion of infinity, which came into mathematics from theology, cannot be so easily eliminated. Even if Dedekind, Cantor, and Weierstrass succeeded in eliminating the infinitesimals, they succeeded at the price of introducing infinite systems of objects as actually existing.

Dedekind analyzed the notion of an infinite system in *Was sind und was sollen die Zahlen* (Dedekind 1888), where he introduced a definition of the infinite set, which we use until now (a set is infinite if it can be mapped onto its proper subset by a one-to-one mapping). Cantor was led to the study of infinite sets by his investigations of Fourier series. His main contribution to set theory was the *Grundlagen einer allgemeiner Mannigfaltigkeitslehre*. He gave here a definition of the notion of a set:

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[16] When we look into the *Course* we find out that Cauchy preserved the notion of an infinitesimal. Nevertheless, he changed its meaning. An infinitesimal for Cauchy is not an infinitely small number, as it was for his predecessors, but a variable that converges to zero. Thus the foundations of Cauchy's approach were built on the notion of a limit.

> "In general, by a manifold or a set I understand every multiplicity which can be thought of as one, i.e., every aggregate of determinate elements which can be united into a whole by some law. I believe that I am defining something akin to the Platonic $\varepsilon\iota\delta o\varsigma$ or $\iota\delta\varepsilon\alpha$ as well as to that which Plato called $\mu\iota\chi\tau o\nu$ in his dialogue Philebus or the Supreme Good." (Cantor 1883, p. 204; Ewald 1996, p. 916)

Cantor's work on the theory of infinite sets received at the beginning only minimal support and many influential mathematicians, such as Kummer or Kronecker, opposed it (Dauben 1979, pp. 133–140). The reason for this opposition lay at least partially in Cantor's turning against the general trend of mathematics of his times, which consisted in the elimination of infinity from mathematics. This trend strengthened after the discovery of Russell's paradox in 1902. Many mathematicians believed that the notion of infinity (introduced into mathematics from theology) is alien to the nature of mathematics and should be eliminated. As an illustration of this attitude we can mention Poincaré's words:

> "There is no actual infinity. The Cantorians forgot this, and so fell into contradiction. It is true that Cantorianism has been useful, but that was when it was applied to a real problem, whose terms were clearly defined, and then it was possible to advance without danger.." (Poincaré 1908, p. 499)

Russell's paradox is usually presented as the paradox of the set of all sets. Therefore an impression could emerge that it is the paradox of Cantorian set theory. Nevertheless, Cantor was fully aware of the problematic nature of the system of all sets, which he called the *Absolute*. Thanks to a theological interpretation of the Absolute, which he identified with God (Dauben 1979, pp. 120–148), Cantor avoided the formulation of any mathematical propositions about this notion. Thus the theological interpretation of his theory saved Cantor from paradoxes. The other two foundationalist approaches lay open to the full brunt of the logical paradoxes. Thus Poincaré's attack against the "Cantorians" was unjustified; the same paradoxes appeared also in the theories of Frege and Peano.

Despite criticism, set theory slowly started to gain popularity among mathematicians working in the field of the theory of functions of a real variable, measure theory, and general topology. These theories witnessed a rapid growth at the end of the nineteenth and the beginning of the twentieth centuries and so more and more mathematicians started to use the notions and methods of set theory. In 1908 Zermelo in his *Untersuchungen über die Grundlagen der Mengenlehre* found a way to avoid the paradoxes. Analyzing the works of Dedekind and Cantor, Zermelo formulated as axioms the rules that are necessary to form new sets. But he formulated these axioms so that they did not allow the construction of any paradoxical object, analogous to the set of all sets. A discussion of the axioms of set theory can be found in (Fraenkel and Bar-Hillel 1958). Thanks to the work of Zermelo, set theory was consolidated during the short period of six years after the discovery of the paradoxes[17] and became an important mathematical discipline with remarkable results and methods.

The next important shift in set theory appeared in 1914 when Hausdorff's *Grundzüge der Mengenlehre* appeared. This book summarized the results that had been achieved in set theory up to the date of its publication. But Hausdorff's book contains also a fundamental innovation – the notion of a function was there for the first time defined as a *set* of ordered pairs. Dedekind, Cantor, and Zermelo considered functions and sets as two fundamentally different kinds of things. For Hausdorff they were analogous. Hausdorff's definition of an ordered pair was a bit cumbersome. It was based on the assumption of the existence of two special objects, which Hausdorff indicated by the symbols 1 and 2. Using them he defined an ordered pair as $\{\{a, 1\}, \{b, 2\}\}$. The modern definition of an ordered pair as $\{\{a\}, \{a, b\}\}$, i.e., using no special objects, was introduced by Kuratowsky in 1921. Hausdorff's idea of interpreting functions as special sets presented a fundamental step towards a unification of mathematics on the basis of set theory. Set theory became the language in which almost the whole of mathematics is developed.

1.1.8.1. Logical Power – Proof of the Consistency of the Infinitesimal Calculus

Even though the roots of set theory go back to the program of arithmetization of mathematical analysis, which had as its aim to eliminate infinitesimals from mathematical analysis, it is interesting to notice

[17] In the framework of the logicist approach Russell proposed a solution of the logical paradoxes based on the theory of types (Russell 1908). Nevertheless, the theory of types was not accepted by the majority of the mathematical community.

that set theory led to a discovery, which made it possible to put the infinitesimals on solid foundations. The first construction of the system of non-standard real numbers was given by Robinson in his paper *Non-standard analysis* (Robinson 1961). Robinson's model was based on the notion of an ultrafilter, the existence of which follows from the axiom of choice. Therefore, construction of the hyperreal numbers as well as proof of the consistency of the theory of infinitesimals, which follows from this construction, can be seen as an illustration of the logical power of the language of set theory. By means of the language of set theory (and of model theory, which is based on this language) it became possible to lay logical foundations beneath many of Leibniz's and Euler's "proofs". These "proofs" were considered as unsound until Robinson showed that they were correct and so we can spare the quotation marks. Nonstandard analysis that grew out from Robinson's construction became in the meanwhile a theory with many important results and applications (see Albeverio et al. 1986, and Arkeryd et al. 1997).

1.1.8.2. Expressive Power – Transfinite Arithmetic

One of Cantor's surprising discoveries was his realization that in the successive construction of the so-called *derived sets* of a given set P of real numbers it is possible to continue also after we have made this construction an infinite number of times. At this point it is not important what precisely this operation means (its explanation can be found in Dauben 1979, p. 41). It is important rather that after Cantor created the *first derived set* P', and by the same construction formed from the first derived set the *second derived set* P'', the third derived set P''', etc. he came to the idea of prolonging the steps of construction of the derived sets beyond the limit of any number of steps that can be counted by natural numbers. On the technical level it is not so difficult, because among the derived sets $P', P'', P''', \dots$ there is an interesting relation. Each set $P^{(n)}$ is a subset of the previous one. Therefore if we have a finite series of derived sets $P', P'', \dots, P^{(n)}$, the last member of this series (which in this case is $P^{(n)}$) can be expressed as the intersection of all of the members of the series:

$$P^{(n)} = \bigcap_{k=1}^{n} P^{(k)} . \tag{1.8}$$

When instead of a finite sequence of derived sets $P', P'', \dots, P^{(n)}$ we take an *infinite* one $P', P'', P''', \dots, P^{(n)}, \dots$, Cantor's idea was to build an analogous intersection of all members of the sequence. Of course, an infinite sequence does not have a last element and therefore the intersection of all of its members will not be equal to some derived set $P^{(n)}$as it was in the case of the finite sequence. But despite the fact that we do not know from the very beginning what the result of this intersection will be (in contrast to the finite case, where it was sufficient to look at the last member of the series and we knew what the intersection was), the intersection is a well-defined set also in the case of the infinite series. A point x belongs to this intersection if and only if it is a member of each of the sets $P^{(k)}$. Cantor used the symbol $P^{(\infty)}$ to represent the intersection of an infinite sequence of derived sets:

$$P^{(\infty)} = \bigcap_{k=1}^{\infty} P^{(k)} . \tag{1.9}$$

After he had introduced the derived set of an infinite degree, he could start to create the derived sets of this set and thus to create the sets $P^{(\infty+1)}$, $P^{(\infty+2)}$, $P^{(\infty+3)}$, ..., until he reached the next infinite case. Then he could turn to the operation (1.9) and create $P^{(2\infty)}$. In this way the paper *Über unendliche lineare Punktmannigfaltigkeiten 2* (Cantor 1880) brought a decisive shift in the history of mathematics. Cantor made here the first step towards *transfinite arithmetic*. In the paper he used the symbols $\infty, \infty + 1$ or 2∞ to represent the steps of the process of creating derived sets, thus they were *indices*. His attention was directed towards the set P and he wanted to understand what happens with it by the successive derivations.[18] In his later papers Cantor replaced the symbol ∞ by the last letter of the Greek alphabet ω and interpreted this symbol not as an index but as a transfinite *number*. He introduced the distinction between ordinal and cardinal numbers, introduced for both of them the operations of addition and multiplication and created thus the transfinite arithmetic.

[18] The fact that Cantor discovered transfinite arithmetic in the study of iterations of a particular operation confirms the connection of set theory to iterative geometry. The iterative processes, by means of which mathematicians of the nineteenth century constructed their strange objects, had steps that could be numbered by natural numbers. Cantor prolonged the iterative process into the transfinite realm. If we examine what enabled Cantor to make this radically new step, we will find out that it was the operation of the intersection of an infinite system of sets and the understanding that an element belongs to this intersection if it belongs into *each one* of the intersected sets. I do not want to indicate that Cantor read Frege's *Begriffsschrift* (it appeared just one year before Cantor's paper). But I would like to stress that the increase of the logical precision in the foundations of the calculus and the parallel development of formal logic during the nineteenth century, which culminated in Frege's work, played a fundamental role also in Cantor's breakthrough into the transfinite realm.

Transfinite arithmetic is doubtlessly one of the most remarkable achievements of set theory. Mathematicians before Cantor were unaware that it is possible to discriminate different degrees of infinity or that it is possible to associate with them (cardinal or ordinal) numbers which can be added and multiplied like ordinary numbers. Therefore transfinite arithmetic is a good illustration of the expressive powers of the language of set theory.

1.1.8.3. Explanatory Power – Unveiling the Typicality of the Transcendent Numbers

As we already mentioned, the first transcendental number was discovered in 1851 by Liouville. In 1873 Hermite showed that also the number e is transcendental, and in 1882 Lindeman proved the transcendence of the number π. Thus the transcendental numbers slowly accumulated. Nevertheless, it still seemed as if the transcendental numbers were some rare exceptions and the overwhelming majority of real numbers were algebraic. Even though Lindeman proved a stronger result than just the transcendence of π, and created an infinite set of transcendental numbers (see Dörrie 1958, or Gelfond 1952), but he constructed the transcendental numbers by means of the algebraic ones, and so still nobody could suspect that there are fundamentally more transcendental numbers than there are algebraic ones.

Therefore when Cantor in 1873 proved that transcendental numbers form an uncountable set, while the algebraic numbers are only countably many, it turned out that exceptional numbers are rather the algebraic ones, and that a typical real number is transcendental. Even though Cantor's proof was a non-constructive one, and so it could not be used to find a new transcendental number, it was a remarkable discovery. After a short time it was followed by similar results, as for instance that a typical continuous function of a real variable has a derivative almost nowhere. Thus under the influence of set theory our view of the real numbers as well as of the functions of a real variable changed in a radical way. The objects which mathematicians of the nineteenth century considered exceptional turned out to be typical; and as exceptional (at least from the point of view of cardinality and measure) must be considered the objects of classical mathematics, such as algebraic numbers, differentiable functions or rectifiable curves. Set theory thus opened a new perspective on the universe of classical mathematics; it changed our view of which mathematical objects are typical and which are exceptional. This change is an illustration of the explanatory power of the language of set theory.

1.1.8.4. Integrative Power – The Ontological Unity of Modern Mathematics

Even though the first foundationalist program in mathematics was Frege's logicism, while set theory did not have at the beginning such broad ambitions, the truth is that the overwhelming majority of contemporary mathematics is done in the framework of set theory. Therefore while the axiomatic method unifies mathematics on the methodological level, set theory unifies it on the ontological one. If we take some mathematical object – be it a number, a space, a function, or a group – contemporary mathematics studies this object by means of its set-theoretical model. It considers natural numbers as cardinalities of sets, spaces as sets of points, functions as sets of ordered pairs, and groups as sets with a binary operation. Thanks to this viewpoint, mathematics acquired an unprecedented unity. We are used to it and so we consider it as a matter of course, but a look into history reveals the radical novelty of this unity of the whole of mathematics. We are justified in seeing the ontological unity of modern mathematics as an illustration of the integrative power of the language of set theory.

1.1.8.5. Logical and Expressive Boundaries

Set theory is one of the last re-codings which were created in the history of mathematics and so today a substantial part of all mathematical work is done in its framework. Therefore it is difficult to determine the logical and expressive boundaries of its language. The logical and expressive boundaries of a particular language can be most easily determined by means of a stronger language, which transcends these boundaries and so makes it possible to draw them. This is so, because the stronger language makes it possible to express things that were in the original language inexpressible. Nevertheless, it seems that mathematics has not surpassed the boundaries posed by the language of set theory. Therefore to characterize these boundaries remains an open problem for the future. What we can say today is that from the contemporary point of view the expressive, logical, explanatory, and integrative force of the language of set theory is total. The boundaries of the language of set theory are the boundaries of the world of contemporary mathematics and as such they are inexpressible.