unit square. These functions gave rise to a considerable refinement of the basic concepts of differential and integral calculus and gave rise to a whole new branch of mathematics – the theory of functions of a real variable. In the course of their study it turned out that the methods of the calculus can be applied only to a rather narrow class of "decent" functions. The rest of the functions lie beyond the expressive boundaries of the language of the differential and integral calculus.

1.1.6. Iterative Geometry

Differential and integral calculus were born in very close connection to analytical geometry. Perhaps this was one of the reasons why mathematicians for a long time considered Descartes' way of generating curves (i.e., point by point, according to a formula) to be the correct way of visualizing the universe of mathematical analysis. They thought that it would be enough to widen the realm of formulas used in the process of generation, and to accept also infinite series, integrals or perhaps other kinds of analytical expressions instead of polynomials. They believed that the symbolic realm of functions and the iconic realm of curves were in coherence. Leibniz expressed this conviction with the following words:

> "Also if a continuous line be traced, which is now straight, now circular, and now of any other description, it is possible to find a mental equivalent, a formula or an equation common to all the points of this line by virtue of which formula the changes in the direction of the line must occur. There is no instance of a face whose contour does not form part of a geometric line and which can not be traced entire by a certain mathematical motion. But when the formula is very complex, that which conforms to it passes for irregular." (Leibniz 1686, p. 3)

First doubts about the possibility of expressing of an arbitrary curve by an analytical expression occurred in the discussion between Euler and d'Alembert on the vibrating string. The vibrations of a string are described by a differential equation that was derived in 1715 by Taylor. In 1747 d'Alembert found a solution of this equation in the form of travelling waves. Nevertheless, as the differential equation describing the string is an analytical formula, d'Alembert assumed that the initial shape of the string must be given in an explicit form of an analytical expression so that it can be substituted into the equation. Then by solving the equation we obtain the shape of the string in later moments of time. Euler raised against this technical assumption the objection that nature need not care about our analytical expressions. Thus if we gave the string a particular shape using our fingers, the string would start to vibrate independently of whether this initial shape is given by an analytical expression or not. Neither Euler nor d'Alembert was able to bring arguments in favor of his position and thus the problem of the relation between physical curves and analytical formulas stopped soon after it started.

The question of the relation between geometrical curves and analytical formulas got a new stimulus at the beginning of the nineteenth century when Fourier derived the differential equation of heat conduction and developed methods for its solution. Fourier was one of the first scientists who started to use discontinuous functions (in the contemporary sense)[13]. The use of discontinuous functions in mechanics would be absurd; if a function describes the motion of a particle, then its discontinuity would mean that the particle disappeared in one place and appeared in another. Similarly unnatural is the use of a discontinuous function in the theory of the vibrating string. There the discontinuity would mean that the string is broken and thus the vibration would stop. But if a function describes the distribution of heat in a body, then a discontinuous function is something rather natural. It corresponds to a situation of a contact between bodies with different temperatures. So the transition from mechanics to thermodynamics broadened the realm of suitable functions.

Besides a radical extension of the realm of suitable functions, Fourier's *Théorie analytique de la chaleur* (Fourier 1822) contains a method that makes it possible for an almost arbitrary function to find an analytical expression that represents it. Suppose that we have a function $f(x)$ given by means of a chart of its values or of a graph. Fourier's method consists in the calculation of particular numbers (today called *Fourier's coefficients*) by means of (a numerical or graphical) integra-

[13] Already Euler used the term "*discontinuous function*" but he used it in a different sense than we do today. For instance, he called the absolute value function, i.e., the function $f(x) = |x|$, discontinuous because on the interval $(-\infty, 0)$ it is given as $f(x) = -x$, while on the interval $(0, +\infty)$ as $f(x) = x$. Euler called a function discontinuous if on different parts of its domain it was defined by means of different analytic expressions.

tion of the function:

$$A_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos(kx)\mathrm{d}x\,,$$

$$B_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\sin(kx)\mathrm{d}x\,,$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x\,.$$

Using these coefficients it is possible to express the function $f(x)$ that was formerly given only numerically or graphically, in the form of an analytical expression, today called *Fourier's series:*

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k\cos(kx) + \sum_{k=1}^{\infty} B_k\sin(kx)\,.$$

In this way Fourier entered the discussion between Euler and d'Alembert on the relation between curves and expressions. Fourier showed that "almost" any function can be represented in the form of an analytical expression. Therefore d'Alembert could answer to Euler, that even if nature does not care about analytical expressions, we can take care of them ourselves. The only problem was the word "almost". A series of prominent mathematicians such as Lejeune–Dirichlet, Riemann, Weierstrass, Lebesgues, and Kolmogorov tried to determine more precisely this "almost". In their works a series of strange functions appeared: Dirichlet's function in 1829, Riemann's function in 1854, Weierstrass' function in 1861, and Kolmogoroff's function in 1923 (Manheim 1964, or Hardy and Rogosinski 1944). The theory of Fourier series enforced the refinement of several notions of mathematical analysis, first of all the notions of function and of integral. In the framework of this theory both Riemann's and Lebesgues' integrals were introduced and finally in the works of mathematicians such as Jordan, Darboux, Peano, Borel, Baire, and Lebesgues a new mathematical discipline – *the theory of functions of a real variable* was born (see Kline 1972, pp. 1040–1051).

As already mentioned, in the study of the functions of a real variable a series of strange objects was discovered. In the nineteenth century these new functions were considered as "pathological" cases (Imre Lakatos will call them "monsters"). Mathematicians still considered the generation of curves point by point in accordance with an analytical formula to be a method which, perhaps with the exception of a few "pathological" cases, gave an adequate geometrical representation of the universe of functions. Towards the end of the nineteenth century these "pathological" functions accumulated in a sufficient number to become the subject of independent study. Mathematicians found their several common attributes, for instance, that a typical "pathological" function has almost nowhere a derivative and has no length. One of the most surprising discoveries was that what originally appeared as pathological exceptional cases were typical for functions of a real variable. This discovery led to a gradual emancipation of the notion of a function from its dependence on analytical expressions. It turned out that the close connection between the notion of a function and its analytical expression led to several distorted views. Nevertheless, as Picard noticed, these distorted views were useful:

> "If Newton and Leibniz had thought that continuous functions do not necessarily have a derivative – and this is the general case – the differential calculus would never have been created." (Picard as quoted in Kline 1972, p. 1040)

Many of the "pathological" functions discovered during the nineteenth century have in common that they are constructed not by Descartes' method of point by point construction in accordance with an analytical formula. They are generated as limits of an infinite iterative process. When in Descartes' method we plot a point on the graph of a function, it never changes. In contrast to this in the new method of generation of geometrical shapes, in each step of iteration the whole curve is drawn anew. The shape is obtained as the limit to which the curves, drawn in the particular steps of the iterative process, converge. Therefore the study of these new forms can be called *iterative geometry*. And just as Descartes' method can be seen as a visualization of the language of algebra (where the central notion is the notion of a polynomial and Descartes ascribed geometrical form to these algebraic objects), iterative geometry can be seen as the *visualization of the language of differential and integral calculus*. The central notion of the calculus is the notion of a limit transition and the new method of generation of geometric forms by means of an iterative process unveils the incredible richness of forms contained in the notion of the limit transition. Thanks to the methods of computer graphics, which make it possible to see the new forms created in iterative geometry on the screen of a computer, the new objects of iterative geometry, often called fractals, became known to a wider public and found their way even into art (see Peitgen and Richter 1986).

1.1.6.1. Logical Power – the Ability to Prove the Existence of Solutions of Differential Equations

Just as the language of analytic geometry made it possible to prove the fundamental theorem of algebra, according to which every polynomial has at least one root, the language of iterative geometry offers tools for the proof of theorems of existence and uniqueness of solutions for wide classes of differential equations. Differential equations are equations in which unknown functions together with their derivatives occur. It seems that the first differential equation was Newton's second law, which can be written in the form (suggested by Euler) as:

$$F = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\,.$$

This equation relates the acting force F to the acceleration of the body (i.e., the second derivative of the position x of the body) caused by this force. In the early period of the development of the theory of differential equations, mathematicians wanted to determine the solution of a differential equation in the explicit form of an analytical formula. They developed methods by means of which it was possible to solve wide classes of such equations. In most of these methods they started from the assumption that the solution will be a combination of functions of some special form (polynomial, exponential, etc.) depending on several parameters. Then they substituted these combinations into the *differential* equation and so obtained a system of *algebraic* equations for the parameters. After solving them they could determine the sought solution of the original differential equation.

Relatively early however they discovered that some differential equations also have so-called *singular solutions*. These were solutions which remained undetected by standard methods. A more systematic analysis of the singular solutions was offered in 1776 by Lagrange. Apart from the singular solutions, there were broad classes of (nonlinear) differential equations for which the standard methods simply did not work. At the beginning of the nineteenth century these technical problems were complemented by a general skepticism towards symbolic methods. Therefore in his lectures between 1820 and 1830

Cauchy put the emphasis on proving the existence and uniqueness of solutions of differential equations, before he started discussing their properties. Cauchy's methods used in these proofs were improved in 1876 by Lipschitz and in 1893 by Picard and Peano.

These proofs of existence and uniqueness of solutions of differential equations are technically too demanding to be expounded here. For our purposes it is sufficient to draw attention to one aspect of them. In their proofs, Cauchy, Lipschitz, Picard, and Peano used a new technique in constructing the function, which represented the sought solution of the differential equation. Instead of determining the function using some analytical expressions, i.e., the symbolic language of differential and integral calculus (as was common until then), they determined the sought function as a geometrical object generated by means of an iterative process. The *method of successive approximations*, as this new method is called, is akin to the iterative methods by means of which the "pathological" functions were generated. Of course, here the intention of mathematicians was the opposite to that present in the creation of "monsters". Now they restricted the iterative process by different conditions of uniformity of convergence so that the functions obtained as limits of the iterative process were "decent". But this difference is not so important. Whether we use the iterative process to create a "monster" that serves as a counterexample to some theorem, or a "decent" function that is a solution of a differential equation; in both cases we create an object by a fundamentally new method. Instead of plotting points following an analytical expression we employ an iterative process.

When Cauchy presented his proofs, nobody (with the probable exception of Bolzano) realized the radical novelty of his method of constructing functions. The new method served the goal of building a theoretical foundation for the theory of differential equations and so it did not attract suspicion. The revolutionary new technique was used to reach conservative goals – to prove that every sufficiently "decent" differential equation has a unique solution. Iterative geometry attracted attention only later, when it led to new unexpected results. Nevertheless, from the epistemological point of view it does not matter how we use a technical innovation. Its novelty is measured not by the surprise which the new results generate (that is a by-product belonging to the sociology of knowledge) but rather by the epistemological properties of these results. So even if the proof of existence and uniqueness of the solutions of differential equations did not raise much stir, it illustrates the logical power of the language of iterative geometry.

1.1.6.2. Expressive Power – the Ability to Describe Fractals

One of the first mathematicians who realized the problems to which the method of iterative generation of curves could lead was Bolzano. In 1834 he discovered a function that was not differentiable at any point (Sebestik 1992, pp. 417 – 431 and Rusnock 2000, p. 174). This example contradicts our intuition which we developed in the study of curves of analytic geometry. The derivative of a function at a particular point determines the direction in which the curve representing the function sets on when it leaves that point. A function that has no derivative at any point corresponds to a curve that cannot be drawn. If we attach our pencil to a particular point of the curve, we do not know in which direction to pull it. Or more precisely, we cannot pull it in any direction, because the curve changes its course at each of its point and so it does not have even the shortest segment in any fixed direction. The optimism of Leibniz expressed in the quotation at the beginning of this chapter is thus demolished.

The first fractals appeared as isolated counterexamples to some theorems of mathematical analysis. In 1918 Hausdorff found a property which these strange objects have in common. If we calculate their so-called *Hausdorff dimension* we will obtain not an integer, as in case of ordinary geometrical objects, but a fraction or even a real number. The Hausdorff dimension is for a point 0, for an ordinary curve it is 1, for a surface 2, and for a solid 3. The dimensions of fractals are somewhere in-between. So *Cantor's set* has Hausdorff dimension approximately 0.6309; *Koch's curve* 1.2619; and *Sierpinski's triangle* 1.5850 (see Peitgen, Jürgens and Saupe 1992). This is also the origin of the name *fractal* – the name indicates the fractional (i.e., non-integer) value of the Hausdorff dimension of these objects. Later another interesting property was discovered – their selfsimiliarity. The selfsimiliarity of fractals means that when we take a small part of a fractal and magnify it sufficiently, we obtain an object identical with the original one.

Later it turned out that fractals play an important role in chaotic dynamics and in the description of turbulence. So scientists slowly stopped viewing iterative geometry as a vagary of mathematicians and started to see it as an independent language for the description of geometric forms. Thus besides synthetic geometry that generates its objects by ruler and compasses, and analytic geometry that generates its objects point by point according an analytical formula, iterative geometry represents a third kind of language that can be used in the description of geometrical forms. The terms of this language are fractals. To try to describe a fractal by a formula, i.e., to generate it by Descartes' method, is hopeless. It is not difficult to see that the universe of iterative geometry is fundamentally richer than the universe of Cartesian curves. Nevertheless, the universe of analytic geometry can be delineated inside of the universe of iterative geometry as the region that we obtain when we restrict the iterative process by sufficiently strong conditions of uniformity of convergence. The Cartesian world of analytic geometry is the "smooth part" of the universe of iterative geometry – just as the universe of Euclidean curves is the "quadratic part" of the world of analytic geometry.

1.1.6.3. Explanatory Power – The Ability to Explain the Insolubility of the Three-Body Problem

In classical mechanics we call the problem of determining the trajectories of the motion of two bodies with masses m_1 and m_2 which interact only by gravitational attraction the *two-body problem*. This problem is also called *Kepler's problem* because Kepler, analyzing the data of the positions of the planet Mars, found (empirically) the basic properties of the solutions of the two-body problem: the elliptical form and the acceleration in the perihelion. Of course in Kepler's days the two-body problem could not even be formulated, because the law of universal gravitation was not known. This law was discovered by Newton who also solved the two-body problem under some simplifying conditions. The complete solution of this problem was found by Johannes Bernoulli in 1710.

The *three-body problem* is analogous to the previous one; the only difference is that instead of the trajectories of two bodies we have to determine the trajectories of three bodies. Nevertheless, all regularities that were present in Kepler's problem (the elliptical form of the trajectories; the simple law describing the acceleration in the perihelion) are totally lost after the transition to three bodies. Despite the efforts of the best mathematicians of the eighteenth and nineteenth centuries, such as Euler, Lagrange, Laplace, and Hamilton, the three-body problem remained unsolved. For all the mentioned mathematicians to solve this problem meant to solve it analytically, i.e., to find explicit formulas that would determine the position of each of the three interacting bodies at every temporal moment. But in a similar way to the quintic equations in algebra, the three-body problem in mechanics defied all efforts of solution. Today we know that this problem is insoluble. But the insolubility of the three-body problem is not caused by the poverty of language, as was the case in algebra. In the case of quintic equations it turned out that the universe of algebraic formulas is simply too poor and does not contain the roots of quintic equations. The insolubility of the three-body problem has a totally different reason – *deterministic chaos*. The discovery of deterministic chaos was one of the most important achievements of the mathematics of the nineteenth century. It was made by Poincaré in November 1889, when he found a mistake in his paper *On the three-body problem and the equations of dynamics*, (Poincaré 1890) for which he won in January 1889 the prestigious Prize of King Oscar II of Sweden. The dramatic circumstances of the discovery, which led to the withdrawal of the whole edition of the issue of *Acta Mathematica* which contained the original version of Poincaré's paper and to a new printing of the whole issue with the corrected version of the paper, at Poincaré's expense, are presented in the literature (see Diacu and Holmes 1996, or Barrow–Green 1997).

For our purposes it is important to realize that the discovery of the so-called *homoclinic trajectory*, which leads to the chaotic behavior of a system of three bodies, was made possible thanks to the language of iterative geometry. Poincaré first introduced a special transformation, the consecutive *iterations* of which represent the global dynamics of the system. He then discovered the chaotic behavior of the system of three bodies thanks to a careful analysis of these iterations. A further area, in which chaotic behavior was discovered, was meteorology. In 1961 Lorenz found chaos in a dynamic system by means of which he modeled the evolution of weather. When the chaotic behavior represented by this model was studied, a remarkable new object, the so-called *Lorenz attractor* was discovered. Lorenz's discovery was followed by several others and so gradually it turned out that many dynamic systems show chaotic behavior – from weather and turbulent flow to the retina of the human eye. In the study of chaotic systems, iterative geometry is used as the basic framework. Therefore we can say that the understanding of chaotic behavior is an illustration of the explanatory power of the language of iterative geometry. More detailed exposition of the theory of chaos can be found for instance in (Peitgen, Jürgens and Saupe 1992).

1.1.6.4. Integrative Power – The Description of Natural Forms

Fractals were originally discovered as counterexamples of some theorems of mathematical analysis. Therefore their purpose was destructive rather than integrative. Even at the beginning of the twentieth century when fractals had accumulated in a sufficient number so that some of their common properties could be found, they were still more an illustration of the ingenuity and the imaginative force of mathematicians than something useful. When in 1967 in the journal *Science* Mandelbrot's paper *How long is the coast of Britain?* appeared, it turned out that fractals are not a mere creation of the imagination of mathematicians, but can be used in describing natural phenomena. Ten years later in his book *Fractal Geometry of Nature* (Mandelbrot 1977) Mandelbrot drew attention to the fact that many natural objects, such as clouds or trees, in many respects resemble fractals. Thus a series of forms that were formerly ignored by science became subject to mathematical study:

> "Why is geometry often described as 'cold' and 'dry'? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line... The existence of these patterns challenges us to study these forms that Euclid leaves aside as being formless, to investigate the morphology of the amorphous. Mathematicians have disdained this challenge." (Mandelbrot 1977, p. 13)

And indeed, when we consider the techniques for generating geometrical forms offered by iterative geometry, we will find that they are able to generate forms that are surprisingly similar to the form of a tree, a leaf of a fern, the line of the seashore, or the relief of a mountain. Thus iterative geometry makes it possible to create faithful representations of natural forms. By means of the language of analytic geometry it would be impossible to achieve anything similar. The close relation of iterative geometry to natural forms is not so surprising, if we realize that every multicellular organism is the result of the iterative process of cell division. Therefore it seems to be natural that the language that generates its objects by iterations of a particular transformation is suitable for description of the forms of living things. The discovery of this (iterative) unity of all living forms can be therefore seen as an illustration of the integrative force of the language of iterative geometry. Where the previous languages saw only unrelated, haphazard, amorphous forms, iterative geometry finds order and unity.

1.1.6.5. Logical Boundaries – Convergence of Fourier Series

The theory of Fourier series played an important role in the creation of iterative geometry. It supplied a sufficiently rich realm of functions which were constructed as limits of an iterative process. Thus the theory of Fourier series was the birthplace of many examples, notions, and methods of iterative geometry. Nevertheless, it is rather interesting that the question of the convergence of Fourier series enforced the creation of a new language, the language of set theory, because the language of iterative geometry was not sufficiently strong to answer that question. Our experience hitherto with the development of the language of mathematics makes it possible to understand this fact and see it more as a rule than an exception in the evolution of mathematics. The situation with *analytic geometry* was similar. Analytic geometry was created thanks to the Cartesian visualization of the polynomials, but later it turned out that the language of polynomials was too narrow for an adequate description of the phenomena which we encounter in analytical geometry. It was necessary to create the differential and integral calculus which is much more adequate for characterization of different properties of analytic curves. Or we can take the example of the *differential and integral calculus*. This was created in close connection to analytic geometry, but in the end it turned out that if we wish to get an undistorted picture of the fundamental notions of the calculus, it is much better to base it on iterative than on analytic geometry. So it seems that the fragment with the help of which we enter the universe of a particular new symbolic or iconic language (in the case of iterative geometry this fragment was the theory of the Fourier series) is in most cases not appropriate to answer the deep new questions posed by this new universe. The lengths of many simple curves of analytic geometry can be calculated only by means of the differential and integral calculus; similarly the question of existence of a solution of a differential equation can be understood by means of iterations of some (contractive) mapping.

Therefore we should not be surprised that the question of the convergence of Fourier series can only be answered in the framework of set theory. We need first of all the notion of the Lebesgues integral, on which the whole modern theory of Fourier series is based, and this notion presupposes measure theory. Therefore we can say that the question of the convergence of Fourier series transcends the logical boundaries of the language of iterative geometry. The logical power of this language is insufficient to answer that question. Furthermore, between the theory of Fourier series and set theory there is also an interesting historical connection. It was the study of convergence of Fourier series that led Cantor to the discovery of set theory. As was noticed by Zermelo, the editor of Cantor's collected works, in his commentary to the paper *Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen*:

> "We see in the theory of Fourier series the birthplace of Cantor's set theory." (Cantor 1872)

1.1.6.6. Expressive Boundaries – Non-Measurable Sets

Even if the richness of the universe of fractals may seduce one to believing that the language of iterative geometry is strong enough to express any subset on the real line, nevertheless, there are sets of real numbers that cannot be expressed by means of this language. These are, for example, the non-measurable sets, the existence of which is based on the axiom of choice. It is precisely the non-constructive character of the axiom of choice that is the reason why the different sets, which can be defined by means of this axiom, cannot be constructed using an iterative process. Thus the existence of non-measurable sets illustrates the expressive boundaries of the language of iterative geometry.

1.1.7. Predicate Calculus

The history of logic started in ancient Greece, where there were two independent logical traditions. One of them has its roots in Plato's Academy and was codified by Aristotle in his *Organon*. From the contemporary point of view the Aristotelian theory can be characterized as a *theory of inclusion of classes* (containing also elements of quantification theory). Thanks to Aristotle's influence during the late Middle Ages, this logical tradition had a dominant influence on the development of logic in early modern Europe. The second tradition was connected with the Stoic school and from the contemporary point of view it can be characterized as *basic propositional calculus* (first of all a theory of logical connectives). In antiquity, probably as a result of the antagonism between the Peripatetic and the Stoic schools, these