> called it, 'the attachment of the area to a given line segment.'." (Kolman 1961, p. 115)

Solving equations with the help of geometrical constructions avoids incommensurability. However, such methods are appropriate only for linear and quadratic equations. Cubic equations represented real technical complications, for they deal with volumes. Equations of higher degree were beyond the expressive boundaries of the language of synthetic geometry.

1.1.3. Algebra

Algebra is a creation of the Arabs. The ancient Greeks did not develop the algebraic way of thinking. This does not mean that they could not solve mathematical problems which we today call algebraic. Such problems were solved already in ancient Egypt and Babylonia. Nevertheless, the Egyptian and Babylonian mathematicians were enchanted by their calculative recipes and did not see the need to develop any more general methods. On the other hand the ancient Greeks excluded calculative recipes from mathematics and reduced almost the whole of mathematics to geometry. Therefore they lost contact with symbolic thinking and calculative manipulations. The reduction of algebraic problems to geometry has, moreover, one fundamental disadvantage. The second power of the unknown is in geometry represented by a square, the third power by a cube; but for higher powers of the unknown there is no geometric representation. Thus the language of synthetic geometry made it possible to grasp only a small fragment of the realm of algebraic problems, a fragment that was perhaps too narrow to stimulate the creation of an independent mathematical discipline.

From the cognitive point of view Arabic algebra is not more difficult than Euclid's *Elements*. If we compared the logical structure of the Arabic algebraic texts with the complex patterns of argumentation used in Euclid, we would find that it is often much simpler. The reason why the Greeks did not discover algebra cannot be the insufficient subtlety of their thought. It seems that there must have been some obstacle that prevented the Greeks from entering the sphere of algebraic thought. The first who entered this new land were the Arabs. There is no doubt that they learned from the Greeks what is a proof, what is a definition, what is an axiom. But the Arabic culture was very different from the Greek one. Its center was Islam, a religion which denied that transcendence could be grounded in the metaphor of sight. Therefore the close connection between knowledge and sight which formed the core of the Greek *epistéme*, was lost. The Greek word *theoria* is derived from *theoros*, which was a delegate of the polis who had to oversee a religious ceremony without taking part in it. A *theoria* was what the *theoros* saw. This means that for the Greeks a theory is what we see when we observe the course of events without participating in them. Thus the metaphor behind the Greek notion of theoretical knowledge was *view from a distance*. According to this metaphor, in order to get insight into a (mathematical) problem one has to separate oneself from everything that bounds one to the problem and could disturb the impartiality of his view.

This shows that algebra with its symbolic manipulations was alien to Greek thought. Algebraic manipulations are based not on insight but rather on feel. The goal is not to envision the solution (as it was in geometry) but rather to get a feel for how to obtain it, to get a feel for all the different modifications, transformations and tricks, to get a feel for the possibilities offered by the symbolic language. But these possibilities are not actualized; they are not opened to the gaze. In an algebraic derivation at each step we have only one expression at our disposal. We remember different calculations that we performed in the past and we perceive the actual expression against their background. We feel the analogies and similarities; we perceive the different hints that lead us through the jungle of transformations, through the inexhaustible amounts of possible substitutions straight to the result. Algebraic thought is always in flux; its transformations are incessantly proceeding. When Arabic algebra reached Europe, a dialogue started. It was a dialogue between the western spirit and a fundamentally different but equally deep spirit of algebra. This dialogue is led by the effort to visualize, to see; the effort to bring the tricks and manipulations before ones eyes and so to attain insight. As a tool of this visualization a new language has born: the symbolic language of algebra.

The language of algebra was developed gradually by Italian and German mathematicians during the fifteenth and sixteenth centuries (see Boyer and Merzbach 1989, pp. 312–316). The main invention of these mathematicians was the idea of a variable, which they called *cosa*, from the Italian word meaning “thing”. They called algebra *regula della cosa*, i.e., the rule of the thing, and understood it as a symbolic language, in which they manipulated letters just as we manipulate things. For instance, if we add to a thing an equally great thing, we obtain two things, what they wrote as $2r$ (to indicate the thing they usually used the first letter of the Latin word *res*). This new language is a return from geometrical construction to symbolic manipulations, from the iconic to the symbolic language.

1.1.3.1. Logical Power – Ability to Prove Modal Predicates

In comparison with the language of elementary arithmetic, the language of algebra has a fundamental innovation – it contains a symbol for the variable. Thus we can say that the algebraists succeeded in transferring the basic advantage of the language of geometry, its ability to *prove general propositions*, into the symbolic language. For instance, it is possible to prove that the sum of two even numbers is even, by a simple formal manipulation with symbols: $2l + 2k = 2(l + k)$. Thus the symbolic language reached the generality of the iconic language of synthetic geometry. This generality was achieved in geometry by using line segments of indefinite lengths, in algebra with the help of variables. Nevertheless, the new symbolic language of algebra surpassed in logical power the geometrical language. For example, let us take the formula for the solution of the quadratic equation

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.1}$$

The parameters a, b, c confer to this formula a generality analogous to that which the line segment of indefinite length bestows to geometrical proofs. But the x on the left-hand side means that the formula as a whole expresses an individual and the components of the formula represent the different steps of its calculation. In this way *the procedure of calculation gets explicitly expressed in language*.

The superiority of the language of synthetic geometry over that of elementary arithmetic is connected with the fact that the particular steps of a geometrical construction are not lost (as the steps of a calculation are lost). Each line or point used in the process of construction remains a constituent of the resulting picture. Nevertheless, what gets lost in the process of construction is the order of its particular steps. This is the reason why geometrical construction is usually supplemented by a commentary written in ordinary language, which indicates the precise order of its steps. It is important to notice that these commentaries do not belong to the iconic language of geometry.

The language of algebra, on the other hand, is able to *express the order of the steps of a calculation within the language*. Thus we need no further commentary on the above formula similar to the one we need on a geometrical construction. The formula represents the process of calculation.[7] It tells us that first we have to take the square of b, subtract from it four times the product of a and c, etc. So the process of solution becomes expressed in the language. The structure of an algebraic formula indicates the relative order of all steps necessary for the calculation. Thanks to this feature of the language of algebra, *modal predicates*, for instance, insolubility can be expressed within the language. The insolubility of the general equation of the fifth degree was proven at the beginning of the nineteenth century by Paolo Ruffini, Niels Henrik Abel and Evariste Galois. It is also possible to prove the insolubility of the problems of trisecting an angle, duplicating a cube, or constructing a regular heptagon. Thus the language of algebra is superior to the language of synthetic geometry. The proofs of insolubility are an illustration of the logical power of the language of algebra.

The language of geometry has no means to express or to prove that some problem is insoluble. The process of solution is something that the iconic language of geometry cannot express. As we have shown in the discussion of the explanatory power of the language of geometry, geometry is able to express the fact that a problem has no solution. Nevertheless, the problem of insolubility is a more delicate one. There is no doubt that for each angle there is an angle that is just one third of it, or that each equation of the fifth degree has five roots. The insolubility does not mean the non-existence of the objects solving the particular problem. It means that these objects, even if they exist, cannot be obtained using some standard methods. The language of algebra is the first language that is able to prove insolubility of a particular problem.

1.1.3.2. Expressive Power – Ability to Form Powers of any Degree

In geometry the unknown quantity is expressed as a line segment of indefinite length, the second power of unknown quantity is expressed as a square constructed over this line segment, and the third power of the unknown is a cube. Three-dimensional space does not let us go further

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[7] It is important to realize that algebra is able to express a procedure due to the circumstance that it contains an implicit notion of a function (or as Frege puts it in the passage that we quote on page 13, the mathematicians "*had got to the point of dealing with individual functions; but were not yet using the word, in its mathematical sense, and had not yet formed the conception of what it now stands for*"). It is the distinction between the function and the argument that makes it possible to determine from a formula the exact order in which the operations follow each other.

in this construction to form the fourth or fifth power of the unknown. We characterized this feature of the language of geometry as its expressive boundaries. The language of algebra is able to transcend these boundaries and form the fourth or fifth power of the unknown. We can find traces of geometrical analogies in the algebraic terminology of the fifteenth and sixteenth centuries; for instance, the third power of the unknown is called *cubus*. But nothing hindered algebraists from going beyond this third degree, beyond which Euclid was not allowed by the geometrical space. They called the second degree of the unknown *zensus* and denoted it z. That is why they wrote the fourth degree as *zz (zensus de zensu)*, the fifth as rzz, the sixth as zzz and so on. In this way the symbolic language of algebra transcended the boundaries placed on the language of geometry by the nature of space. Of course, we are not able to say what the fifth power of the unknown means, but this is not important. The language of algebra offers us rules for manipulation of such expressions independently of any interpretation.

The turn from geometrical construction to symbolic manipulation made it possible to discover the method of solving cubic equations.[8] This was the first achievement of western mathematics that surpassed the ancient heritage. It was published in 1545 by the Italian mathematician Girolamo Cardano in his *Ars Magna Sive de Regulis Algebraicis*. The history of this discovery is rather dramatic (see van der Waerden 1985, pp. 52–59). Details about this rather deep result will be presented in Chapter 2.2.2 of the present book. The language of algebra makes it possible to solve problems which in the language of synthetic geometry it is difficult even to formulate.

1.1.3.3. Explanatory Power – Ability to Explain the Insolubility of the Trisection of an Angle

The language of algebra makes it possible to understand why some geometrical problems, such as the trisection of an angle, the doubling of a cube and the construction of a regular heptagon, are insoluble with

[8] We present here the formula for the solution of the equation $x^3 = bx + c$ in modern notation:

$$x = \sqrt[3]{\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 - \left(\frac{b}{3}\right)^3}} + \sqrt[3]{\frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 - \left(\frac{b}{3}\right)^3}}$$

Of course Cardano never wrote such a formula. He formulated his rule verbally. Nevertheless, this result illustrates the expressive power of the language of algebra.

ruler and compasses. All problems solvable by means of ruler and compasses can be characterized as problems in which only line segments of lengths belonging to some finite succession of quadratic extensions of the field of rational numbers occur. Thus in order to show the insolubility of the three mentioned problems it is sufficient to show that their solution requires line segments whose length does not belong to any finite sequence of quadratic extensions of the field of rational numbers. This can be easily done (see Courant and Robbins 1941, pp. 134–139). The language of geometry does not make it possible to understand why the three mentioned problems are insoluble. From the algebraic point of view it is clear. The ruler-and-compasses constructions take place in fields that are too simple. This explanation illustrates the explanatory power of the language of algebra.

1.1.3.4. Integrative Power – Ability to Create Universal Analytic Methods

Euclidean geometry is a collection of disconnected construction tricks. Each problem is solved in a particular way. Thus Greek geometry is also based on memorizing. Instead of memorizing the complete recipes as the Egyptians did, only the fundamental ideas and tricks are to be remembered. However, there are still many of them. Algebra replaces these tricks by universal methods. This innovation was introduced by François Viète in his *In Artem Analyticam Isagoge* (Viète 1591). Before him, mathematicians used different letters for different powers of the unknown $(r, z, c, zz, rzz, \ldots)$, and so they could write equations having only one unknown, whose different powers were indicated by all these letters. Viètes idea was not to indicate the different powers of the same quantity with different letters, but to use the same letter and to indicate its power by a word. He used *A latus*, *A quadratum*, and *A cubus* for the first three powers of the unknown quantity A. Similarly he used *B longitudo*, *B planum* and *B solidum* for the powers of the parameter. In this way the letters expressed the identity of the quantity, while the words indicated the particular power. Thus Viète introduced the distinction between a parameter and an unknown.

Algebraists before Viète worked only with equations having numerical coefficients. This was a consequence of the use of different letters for powers of the unknown. The algebraists were fully aware that their methods were universal, fully independent of the particular values of the coefficients. Nevertheless, they were not able to express this universality in the language itself. Viète liberated algebra from the necessity to calculate with numerical coefficients only. His idea was to express the coefficients of an equation with letters as well. In order not to confuse coefficients with unknowns, he used vowels (A, E, I, O, U) to express unknowns and consonants ($B, C, D, \ldots$) to express coefficients. So the equation which we would write as $ax^3 - by^2 = c$ he would write as

> *B latus in A solidum – C quadratum in E planum equatur D quadrato-quadratum.*

It is important to note that for Viète the coefficients had dimensions (indicated by the words *latus*, *quadratum*, or *cubus*), just like the unknowns (indicated by the words *longitudo*, *planum*, or *solidum*), so that all terms of an equation had to be of the same (in our case of the fourth) dimension. So his symbolism was a cumbersome one, and many simplifications were needed until it reached its form used at present. But the basic gain, the existence of universal analytic methods, is already present.

Viètes *analytic art*, as he called his method, was based on expressing the unknown quantities and the parameters of a problem by letters. In this way the relations among these quantities could be expressed in the form of an equation containing letters for unknown quantities as well as for parameters. Solving such an equation we obtain a general result, expressing the solution of all problems of the same form. In this way generality becomes a constituent of the language. The existence of universal methods for the solution of whole classes of problems is the fundamental advantage of the language of algebra. The language of synthetic geometry does not know any universal methods. Geometry can express universal facts (facts which are true for a whole class of objects), but it operates with these facts using very particular methods. Algebra developed universal analytic methods, which played a decisive role in the further development of the whole mathematics. From algebra the analytic methods passed to *analytic geometry* (Descartes 1637) and *mathematical analysis of infinitesimals* (i.e., calculus, Euler 1748), then to physics in the form of *analytic mechanics* (Lagrange 1788) and *analytic theory of heat* (Fourier 1822) till they reached logic in *mathematical analysis of logic* (Boole 1847). Thus the integrative power of the language of algebra permeates vast regions of western thought.

1.1.3.5. Logical Boundaries – Casus Irreducibilis

Studying equations of the third degree, Cardano discovered a strange thing. If we take the equation $x^3 = 7x + 6$, and use the standard recipe for its solution, we obtain a negative number under the sign of the square root.[9] Cardano called this case *casus irreducibilis*, the insoluble case. In many respects it resembles the discovery of incommensurability. In both cases we are confronted with a situation in which the language fails. The attempts to express this situation in the language led to paradoxes. And in both cases the therapy consists in extending the realm of objects with the help of which the language operates. In the case of incommensurability it was necessary to introduce the irrational numbers, in the *casus irreducibilis* the complex numbers. After the introduction of irrational numbers, incommensurability is no longer paradoxical, it just indicates the fact that the diagonal of the unit square has an irrational length. Similarly, after the introduction of complex numbers the *casus irreducibilis* is no longer paradoxical, it just indicates that the formula expresses the roots of the equation in the form of a sum of two conjugate complex numbers.

Nevertheless, for Cardano the *casus irreducibilis* was a mysterious phenomenon. It took almost two centuries until the square roots of negative numbers were sufficiently understood. It is important to realize that Cardano did not look for the complex numbers. The complex numbers rather imposed themselves on him. Cardano would have been much happier if no *casus irreducibilis* had appeared. The discovery of complex numbers is thus an illustration of the "law" formulated by Michael Crowe:

> "New mathematical concepts frequently come forth not at the bidding, but against the efforts, at times strenuous efforts, of the mathematicians who create them." (Crowe 1975, p. 16)

[9] The equation $x^3 = 7x + 6$ has the solution $x = 3$ (the further solutions $x = -1$ and $x = -2$ were not considered, as the *cosa* cannot be less than nothing). Nevertheless, if we substitute $b = 7$ and $c = 6$ into the formula presented in note 8 on p. 33, we obtain a negative number under the sign of the square root:

$$x = \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{27}}}.$$

1.1.3.6. Expressive Boundaries – Transcendent Numbers

Even if algebraists were able to explain why nobody succeeded in solving the problem of the trisection of an angle, the problem of quadrature of a circle resisted algebraic methods. Gradually a suspicion arose that this problem is insoluble as well. Nevertheless, its insolubility is not for algebraic reasons. This suspicion found an exact expression in the distinction between algebraic and transcendental numbers. Transcendental numbers are numbers that cannot be characterized using the language of algebra. The first example of a transcendental number was given by Joseph Liouville in 1851. It is the number:

$$l = \sum_{n=1}^{\infty} 10^{-n!} = 10^{-1!} + 10^{-2!} + 10^{-3!} + 10^{-4!} + 10^{-5!} + 10^{-6!} + 10^{-7!} + \ldots$$
$$= 10^{-1} + 10^{-2} + 10^{-6} + 10^{-24} + 10^{-120} + 10^{-720} + \ldots$$
$$= 0,1100010\ldots(17 zeros)\ldots010\ldots(96 zeros)\ldots010\ldots(600 zeros)\ldots010\ldots$$

In the decimal expansion of this number the digit 1 is in the $n!^{th}$ places.[10] All other digits are zeros. This means that the digit one is on the first, second, sixth, twenty-fourth, ... decimal places. Even though this number is relatively easy to define, it does not satisfy any algebraic equation. This means that it is a transcendental number – it transcends the expressive power of the language of algebra (see Courant and Robbins 1941, p. 104–107). Liouville's number l illustrates the expressive boundaries of the language of algebra. In 1873 Charles Hermite proved the transcendental nature of e (the basis of natural logarithms) and in 1882 Ferdinand Lindemann proved the transcendental nature of π. Thus l, e, and π are quantities about which we can say nothing using the language of algebra.

1.1.4. Analytic Geometry

Analytic geometry originated from the union of several ideas or even traditions of thought that existed independently of each other for many centuries. The first of them was the idea of *co-ordinates*. In geography co-ordinates have been used since antiquity. One of the highlights of ancient geography was the *Introduction to geography* (*Geógrafiké*

[10] The symbol $n!$ represents the product of the natural numbers from 1 to n (thus $4! = 1.2.3.4 = 24$, while $6! = 1.2.3.4.5.6 = 720$).