a number, or we can choose a unit commensurable with the diagonal, but then we will be unable to express the length of the side. So the incommensurability of the side and diagonal of the square reveals the boundaries of the expressive power of the language of elementary arithmetic.[6]

1.1.2. Synthetic Geometry

Since the earliest stages of their development the great agricultural civilizations of antiquity were confronted with several problems that are closely linked to geometry. By measuring fields, planning irrigation channels, constructing roads, building granaries and fortification walls, as well as many other practical activities, they collected knowledge about different geometrical objects. Even the term "*geometry*", which has its origin in the Greek words for earth (*geos*) and measure (*metros*), reveals the practical roots of this discipline. However, despite the quantity of practical knowledge collected in these ancient civilizations, geometry did not become an independent discipline. It was embedded into the framework of arithmetical calculative recipes just like calculation of the volume of the cylindrical granary, quoted above from the *Rhind papyrus*. Even though the problem is a geometrical one, it is treated in a purely arithmetical way. Geometry as an independent mathematical discipline was created in ancient Greece. Thales and Pythagoras, the founders of geometry, made according to ancient tradition long trips to Egypt and to Mesopotamia where they learned the practical geometrical knowledge of these civilizations. They brought this knowledge back to their homeland and presumably enriched it with many new discoveries that bear their names. But what is even more important, they created a new mathematical language suitable for representing geometrical objects.

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[6] This limitation was later removed when mathematicians introduced real numbers as their basic number system. But we are interested here in the language of elementary arithmetic in its original form. And for the kind of elementary arithmetic that takes number to be a collection of units, the incommensurability of the side and diagonal of the square is a paradox that defies comprehension. Later this paradox was resolved by the introduction of real numbers and it was replaced by the positive fact that $\sqrt{2}$ is an irrational number. Here we see one of the fundamental strategies of dealing with paradoxes. It consists in the extension of the language and in the reinterpretation of the paradox so that it becomes a positive fact. Nevertheless, it took many centuries until the incommensurability was resolved inside of the symbolic language itself.

Geometrical language, in contrast to the language of arithmetic, is an iconic and not a symbolic language. Its expressions are pictures formed of points, line segments and circles, rather than formulas formed of linear sequences of arithmetic symbols. From this point of view the Greeks moved from the symbolic language of arithmetic to the iconic language of geometry. They had good reasons for this, because, as we will see, geometrical language surpasses the language of elementary arithmetic in logical as well as in expressive power. It makes it possible to prove general theorems and to represent incommensurable line segments such as the side and the diagonal of a square.

1.1.2.1. Logical Power – Ability to Prove Universal Theorems

The new language of geometry was originally developed in close connection with arithmetic. We have in mind the famous Pythagorean theory of figurate numbers (see Boyer and Merzbach 1989, p. 62). Certainly, its content is arithmetical – it deals with numbers. But its form is quite new. Using small dots in sand or pebbles (*psefos*) it represents numbers geometrically – as square numbers (i.e., numbers the *psefoi* of which can be arranged into a square, like $4, 9, 16, \dots$), triangular numbers (like $3, 6, 10, \dots$), and so on. With the help of this geometrical form, arithmetical predicates can be visualized. For instance an even number is a number the *psefoi* of which can be ordered into a double row. So the *arithmetical* property of being odd or even becomes a property expressible in the new *geometrical* language. This very fact, that arithmetic properties became expressible in the language makes it possible to prove universal theorems (and not only particular statements, as was the case until then). For instance the theorem that the sum of two even numbers is even can be easily proved using this Pythagorean language. It follows from the fact that if we connect two double rows, one to the end of the other, we will again get a double row. Therefore the sum of *any two* even numbers must be even. We see that the Pythagoreans introduced an important linguistic innovation into mathematics. With the help of their figurate numbers they were able to prove general mathematical theorems.

The discovery of incommensurability led the Greeks to abandon the Pythagorean arithmetic basis of their new geometrical language and to separate geometrical forms from the arithmetic content. In this process the new language, the iconic language of geometry, was created. From the logical point of view this new geometrical language is much stronger than the language of elementary arithmetic. It makes it possible to prove universal statements. The language of geometry is able to do this, thanks to an expression of a new kind – a *segment of indefinite length*. (In fact this was the essence of the Pythagorean innovation – they were able to prove that the sum of any even numbers is even, because the double row which represented an even number could be of any length. The geometrical form is independent of the particular arithmetical value to which it is applied.) If we prove some statement for such a segment, in fact we have proved the statement for a segment of any length, which means that we have proved a general proposition.

The segment of indefinite length is not a variable, because it is an expression of the iconic and not of the symbolic language. That means it does not refer to, but rather represents the particular objects (side of a triangle, radius of a circle, etc.). Of course, any concrete segment drawn in the picture has a precise length, but this length is not used in the proof, which means that the particular length is irrelevant. This substantiates the interpretation of geometrical pictures as a language. In a proof we ascribe to a segment that has a precise length a universal meaning, thus we treat it as a linguistic expression.

1.1.2.2. Expressive Power – Ability to Overcome Incommensurability

The language of geometry is superior to that of elementary arithmetic also regarding its expressive power. For the language of arithmetic the incommensurability of the side and the diagonal of a square means that there are lengths which cannot be expressed by numbers. From the geometrical point of view, the side and diagonal of the square are ordinary segments. Thus the iconic language of synthetic geometry has expressive superiority over the language of elementary arithmetic.

The ratio of the lengths of two incommensurable line segments cannot be expressed by numbers. Nevertheless, the language of geometry makes it possible to compare such lengths. Eudoxus created for this purpose his theory of proportions that can be found in Book V. of Euclid's *Elements*. This theory is based on the following definition:

> "Magnitudes are said to be *in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and forth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the later equimultiples respectively taken in corresponding order." (Euclid V., Def. 5)

This definition makes it possible to compare two pairs of incommensurable magnitudes and show that for instance the areas of two "circles are to one another as the squares on the diameters" (Euclid XII, Prop. 2). The areas of a circle and of the square on its diameter are incommensurable magnitudes (as we know today) but despite this incommensurability the language of geometry makes it possible to prove that their ratio is constant for all circles. This illustrates the superiority of the expressive power of the language of synthetic geometry over the language of elementary arithmetic, which was unable to say anything about two incommensurable magnitudes.

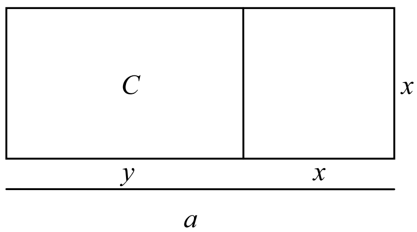
1.1.2.3. Explanatory Power – Ability to Explain the Non-Existence of a Solution of a Problem

As an illustration of the logical boundaries of the language of elementary arithmetic we showed that from the arithmetical point of view it is incomprehensible why some problems, as for instance

$$x + y = 10, \qquad x \cdot y = 40$$

have no solution. The language of synthetic geometry can explain this strange fact:

> "One advantage of the appeal to geometry can now be mentioned to illustrate the gain in *explanatory power*. Evidently there are no numbers x and y whose sum is 10 and product 40, and the Babylonian scribes seem to have avoided discussing such questions. However, we can now see why there are no such numbers. In the language of application of areas we have to put a rectangle of area 40 on a segment of the length 10 leaving a square behind.



> The area xy of the large rectangle C varies with x (and therefore with y) but is greatest when the rectangle is a square. In that case $x = y = a/2$, and the area is $a^2/4$.

> Therefore we can solve the problem provided $a^2/4$ exceeds the specified area C. In our example $100/4 = 25$ is not greater than 40, so no numbers can be found. The discussion of the feasibility of finding a solution is indeed to be found in Euclid (Book VI, Prop. 27) immediately preceding the solution of the quadratics themselves." (Gray 1979, p. 24)

Thus the language of synthetic geometry enables one to understand the conditions under which problems of elementary arithmetic have solutions. In this way the language of geometry makes it possible to express explicitly the boundaries of the language of arithmetic; the boundaries that in the language of elementary arithmetic itself are inexpressible.

1.1.2.4. Integrative Power – The Unity of Euclid's Elements

So far we have shown that the language of synthetic geometry has logical, expressive and explanatory superiority over the language of elementary arithmetic. A further advantage of this language is that it makes it possible to create a unifying approach to mathematics. Perhaps the best illustration of this unifying approach is Euclid's *Elements*. Euclid united number theory, plain geometry, theory of proportions, and geometry in three dimensions into one organic whole. Euclid's *Elements* are only partially original. Substantial parts of it were taken from older sources that are now lost. The *theory of proportions* that comprises Book V of the *Elements* was taken probably from Eudoxus. From him stems also the *method of exhaustion* that can be found in Book XII. The *classification of irrational magnitudes* contained in Book X was taken from Theaetetus and the number theory contained in Books VII.–IX. from Archytas. Euclid's original contribution was probably the theory of the five Platonic solids contained in Book XIII. Thus the *Elements* are a compilation that contains ideas of several mathematicians. Despite their heterogeneous content they form a unity which captures the reader by its stringent logical structure. And this structural unity of the Elements can be seen as the best illustration of the integrative power of the language of synthetic geometry.

1.1.2.5. Logical Boundaries – Existence of Insoluble Problems

There are three problems, formulated during the early development of Greek geometry, which turned out to be insoluble using just the elementary methods of ruler-and-compasses construction. These problems are: *to trisect an angle*, *to duplicate a cube*, and to *construct a square with the same area as a circle*. The insolubility of these problems was proved with modern algebra and complex analysis, that is, in languages of higher expressive and explanatory power than that of the language of synthetic geometry. From the point of view of synthetic geometry the fact that nobody managed to solve these problems, despite the efforts of the best mathematicians, must have been a paradox. It indicated the logical boundaries of the language of synthetic geometry.

Many solutions of these problems were presented using curves or methods which have been characterized by some historians of mathematics as*ad hoc* (Gray 1979, p. 16). As an example we can take the *trisectrix*, a curve invented by Hippias, or the method of *neusis*, invented by Archimedes (see Boyer and Merzbach, 1989, pp. 79 and 151). Nevertheless, these new curves did not belong to the language of synthetic geometry, as characterized in the introduction. They contained points which could not be constructed using only ruler and compass but were given with the help of other mechanical devices. So even if these new methods are important from the historical perspective and have attracted the attention of historians (see Knorr 1986), they fall outside the scope of our analysis.

1.1.2.6. Expressive Boundaries – Equations of Higher Degrees

The new geometrical language prevailed in Greek mathematics to such a degree that Euclid, when confronted with the problem of solving a quadratic equation, presented the solution as a geometrical construction. It was natural, because in the geometrical setting it was not necessary to care whether the magnitudes involved in the problem were commensurable or not.

> "The discovery of incommensurability and the impossibility of expressing the proportion of any two segments as a proportion of two natural numbers led the Greeks to start to use proportions between geometrical magnitudes instead of arithmetical proportions and with their help to express general proportions between magnitudes.... In order to solve the equation $cx = b^2$, the Greeks regarded the b^2 as a given area, c as a given line segment and x the unknown segment. They transformed thus the problem into the construction of an oblong whose area and one side are known. Or as they

> called it, 'the attachment of the area to a given line segment.'." (Kolman 1961, p. 115)

Solving equations with the help of geometrical constructions avoids incommensurability. However, such methods are appropriate only for linear and quadratic equations. Cubic equations represented real technical complications, for they deal with volumes. Equations of higher degree were beyond the expressive boundaries of the language of synthetic geometry.

1.1.3. Algebra

Algebra is a creation of the Arabs. The ancient Greeks did not develop the algebraic way of thinking. This does not mean that they could not solve mathematical problems which we today call algebraic. Such problems were solved already in ancient Egypt and Babylonia. Nevertheless, the Egyptian and Babylonian mathematicians were enchanted by their calculative recipes and did not see the need to develop any more general methods. On the other hand the ancient Greeks excluded calculative recipes from mathematics and reduced almost the whole of mathematics to geometry. Therefore they lost contact with symbolic thinking and calculative manipulations. The reduction of algebraic problems to geometry has, moreover, one fundamental disadvantage. The second power of the unknown is in geometry represented by a square, the third power by a cube; but for higher powers of the unknown there is no geometric representation. Thus the language of synthetic geometry made it possible to grasp only a small fragment of the realm of algebraic problems, a fragment that was perhaps too narrow to stimulate the creation of an independent mathematical discipline.

From the cognitive point of view Arabic algebra is not more difficult than Euclid's *Elements*. If we compared the logical structure of the Arabic algebraic texts with the complex patterns of argumentation used in Euclid, we would find that it is often much simpler. The reason why the Greeks did not discover algebra cannot be the insufficient subtlety of their thought. It seems that there must have been some obstacle that prevented the Greeks from entering the sphere of algebraic thought. The first who entered this new land were the Arabs. There is no doubt that they learned from the Greeks what is a proof, what is a definition, what is an axiom. But the Arabic culture was very different from the Greek one. Its center was Islam, a religion which denied that transcendence could be grounded in the metaphor of sight. There-