expressive boundaries, more and more sophisticated and subtle techniques are developed. We will illustrate the growth of the logical, expressive, explanatory, and integrative power, as well as the shifts of the logical and expressive boundaries of the language of mathematics by some suitable examples.[3]

Mathematics has a tendency to improve its languages by addition. So, for instance, we are used to introducing the concept of variable into the language of arithmetic (enabling us to write equations in this language) and often we choose the field of real numbers as a base (so that the language is closed with respect to limits). This is very convenient from the pragmatic point of view, because it offers us a strong language in which we can move freely without any constraints. But, on the other hand, it makes us insensitive to historically existing languages. The old languages do not appear to us as independent systems with their own logical and expressive powers. They appear only as fragments of our powerful language. Since our aim is the epistemological analysis of the language of mathematics, we try to characterize every language as closely as possible to the level on which it was created. We ignore later emendations which consist of incorporation of achievements of the later development into the former languages (for instance of the concept of variable into arithmetic). In this book the language of arithmetic will be a language without variables. We think that such a stratification of the language of mathematics into different historical layers will be interesting also for logical investigations, showing the order in which different logical tools appeared.

1.1.1. Elementary Arithmetic

Counting is as old as mankind. In every known language there are special words expressing at least the first few numbers. For instance the Australian tribes around Cooper bay call 1 – *guna*, 2 – *barkula*, 3 – *barkula guna*, and 4 – *barkula barkula* (Kolman 1961, p. 15). With the development of society it became necessary to count greater quantities of goods and so different aids were introduced in counting: fingers, pebbles, or strings with knots. A remarkable tool was found in Moravia

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[3] The logical and expressive boundaries of a particular language can be expressed only by means of a later language; a language that is strong enough to enable us to construct a situation by the description of which the original language fails. Therefore for characterization of the logical and expressive boundaries of a particular language we will use later languages.

(Ifrah 1981, p. 111). It was used by our ancestors in the Paleolithic era (19 000 – 12 000 BC) probably for counting bigger quantities. It is a bone of a young wolf on which 55 cuts are visible. The first 25 cuts are rendered in groups by five and the 25th cut is twice as long as the others. This rendering shows traces of quinary notation. Further development of civilization and the related necessity to count bigger and bigger quantities led to a fundamental discovery, which changed mankind – the invention of numerals as special symbols designed for counting. Practically every civilization created its own numeral system, a system that made it possible to reduce counting to manipulation with symbols. We know many different ways in which this reduction can be achieved. Some civilizations took as the basis of their numeral systems 60, others took 10; some civilizations introduced a positional notation, others used a non-positional one. An overview of the principles of different numeral systems can be found in the book of George Ifrah (Ifrah 1981). What all numeral systems have in common is the creation of the first *symbolic language* in history, the language of elementary arithmetic.

The language of elementary arithmetic is the simplest symbolic language. It is based on manipulations with numerical symbols. There are many variants of this language; the most common contains ten symbols for numerals $0, 1, \dots, 9$ and the symbols $+$, $-$, x, :, and $=$. A basic feature of this language is that it has no symbol for a variable. For this reason it is impossible in this language to express any general statement or write a general formula. The rules for division or for multiplication, as general statements, are inexpressible in this language. They cannot be *expressed* in the language, but only *shown*. For instance the rule that multiplication by 10 consists in writing a 0 at the end of the multiplied number cannot be expressed in the language. It can only be shown on specific examples such as $17 \times 10 = 170$, or $327 \times 10 = 3270$. From such examples one understands that the particular numbers are unimportant and one grasps the universal rule.

1.1.1.1. Logical Power – Verification of Singular Statements

Typical statements of elementary arithmetic[4] are singular statements such as:

$$135 + 37 = 172 \qquad \text{or} \qquad 24 \times 8 = 192.$$

The language of elementary arithmetic contains implicit rules, which make possible verification of such statements with the help of manipulation with symbols, i.e., on a purely syntactical basis. One consequence of the fact that the language of elementary arithmetic did not contain variables was the necessity of formulating all problems with concrete numbers. Let us take for example a problem from the *Rhind papyrus:*

> "Find the volume of a cylindrical granary of diameter 10 and height 10.
>
> Take away 1/9 of 10, namely 1 1/9; the remainder is 8 2/3 1/6 1/18. Multiply 8 2/3 1/6 1/18 times 8 2/3 1/6 1/18; it makes 79 1/108 1/324 times 10; it makes 790 1/18 1/27 1/54 1/81 cubed cubits. Add 1/2 of it to it; it makes 1185 1/6 1/54, its contents in khar. 1/20 of this is 59 1/4 1/108. 59 1/4 1/108 times 100 hekat of grain will go into it." (Fauvel and Gray 1987, p. 18)

Instead of rewriting the problem as an equation and solving it in a general way, as we would do today, the scribe has to take the numbers from the formulation and to perform with them particular arithmetic operations, until he gets what he needs. The general method, which is inexpressible in the language, is shown with the help of concrete calculations. The language of elementary arithmetic was the first formal language in history, which made it possible to solve problems by manipulation with symbols. Its logical power is restricted to verification of particular statements.

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[4] In his paper *Funktion und Begriff* Frege mentioned as a typical proposition of elementary arithmetic the identity $2 + 3 = 5$. In his *Grundlagen der Arithmetik* Frege criticized Kant and as an example on which it is obvious that the propositions of arithmetic cannot be founded on intuition he mentions $135664 + 37863 = 173527$ (Frege 1884, p. 17). This example clearly shows that the language of elementary arithmetic is based on rules for formal manipulation with symbols.

1.1.1.2. Expressive Power – Ability to Express Arbitrarily Large Numbers

The language of elementary arithmetic is able to express arbitrarily large natural numbers. This may seem nothing special, as we are used to negative, irrational and complex numbers and so the natural numbers seem to us as a rather poor and limited system, where it is not possible to subtract or divide without constraints. Nevertheless, if we leave out of consideration these results of later developments, maybe we will be able to feel the fascination which must have seized the ancient Egyptian (Babylonian, Indian, Chinese) scribe when he realized, that *with the help of numbers it is possible to count everything*. This is the basis of bureaucratic planning, which was one of the most important discoveries of the ancient cultures. The universality of bureaucracy is based on expressive power of the language of arithmetic. Traces of fascination with the expressive power of the language of arithmetic can be found in an affinity for large or special numbers in mythology, in the Kabbala, in the Pythagoreans and even in Archimedes. His book *The Sand-reckoner* is devoted to a demonstration of the expressive power of the language of elementary arithmetic. Archimedes shows that numbers are able to count even the grains of sand on the earth:

> "There are some, King Gelon, who think that the number of the sand is infinite in multitude; and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regarding it as infinite, yet think that no number has been named which is great enough to exceed its multitude. And it is clear that they who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the earth, including in it all the seas and the hollows of the earth filled up to a height equal to that of the highest of the mountains, would be many times further still from recognizing that any number could be expressed which exceeded the multitude of the sand so taken. But I will try to show you *by means of geometrical proofs*, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described, but also that

> of a mass equal in magnitude to the universe. ... I say then that, even if a sphere were made up of the sand, as great as Aristarchus supposes the sphere of the fixed stars to be, I shall still prove that, of the numbers named in the *Principles* [lost work of Archimedes], some exceed in multitude the number of the sand which is equal in magnitude to the sphere referred to ....” (Archimedes 1952, pp. 520–521 and Heath 1921, pp. 81–85)

Of course, the geometrical proofs by means of which Archimedes shows that it is possible to “name” a number that is greater than the quantity of sand in the whole universe, do not belong to arithmetic. But the very fact that such numbers can be named illustrates the expressive power of the language of elementary arithmetic. The Egyptian or Babylonian scribes were unable to prove such a proposition, but they felt that everything can be counted and thus taken into records.

1.1.1.3. Explanatory Power – The Language of Elementary Arithmetic is Nonexplanatory

The language of arithmetic is *nonexplanatory*. This is obvious from preserved mathematical texts, which have the form of collections of recipes, comprising a sequence of instructions without any explanation. This characteristic of the language has been noticed also by historians. For instance, Gray speaks about “contradictory and nonexplanatory results” (Gray 1979, p. 3).

1.1.1.4. Integrative Power – The Language of Elementary Arithmetic is Nonintegrative

It is also obvious that the language of elementary arithmetic is *nonintegrative*. It does not make it possible to conceive any kind of unity or order. Mathematical texts are mere collections of unrelated particular cases. Ordering of problems is based on their content (problems on calculation of areas of fields, volumes of granaries, etc.) instead of on their mathematical form. This means that the ordering principle comes from outside the language.

1.1.1.5. Logical Boundaries – Existence of Insoluble Problems

The old Babylonians could encounter the fact that some problems have no solution (Gray 1979, p. 24). On Babylonian tables we can find a problem that leads (in modern notation) to the following system of equations

$$x + y = 10, \qquad x \cdot y = 16,$$

which has two solutions: (2, 8) and (8, 2). The Babylonians were able to solve it in the framework of elementary arithmetic. Nevertheless, if we take a problem slightly different from the previous one:

$$x + y = 10, \qquad x \cdot y = 40,$$

the Babylonian procedure collapses. In the framework of elementary arithmetic it is impossible to see what has happened. A procedure that works perfectly in some circumstances turns out to be useless if we only slightly change the conditions.

1.1.1.6. Expressive Boundaries – Incommensurability of the Side and the Diagonal of a Square

The Pythagoreans developed a qualitatively new kind of formal language. It was the iconic language of synthetic geometry. Nevertheless, at the beginning they connected this new geometrical language with an interesting kind of "arithmetical atomism". The Pythagoreans supposed that every quantity, among others also the side and the diagonal of a square, comprise a finite number of units. So proportion of the lengths of the side and the diagonal of the square equals the proportion between the numbers of units from which they are composed. The discovery of the incommensurability of the side and the diagonal of the square refuted the Pythagorean atomism.[5] It shows, however, that the language of geometry is more general than that of arithmetic. In arithmetic the side and diagonal of a square cannot be included in one calculation. We can either choose a unit commensurable with the side, but then it will be impossible to express the length of the diagonal by

[5] The Pythagoreans believed that every quantity can be expressed by means of natural numbers. In the case of length measurements they measured the length of a particular segment using an appropriately chosen unit. Then the length of the segment could be expressed in the form of a proportion of the segment and the unit. The Pythagoreans believed that each such proportion has the form of the ratio of two natural numbers. In this case they called the segment and the unit commensurable. One of the most fundamental discoveries of ancient Greek mathematics was the discovery that in the case of the square there is no unit that would be commensurable with both its side and its diagonal. There is an extensive literature dealing with this rather famous discovery (see Heath 1921, pp. 90–91; Gray 1979, pp. 11–13; Dauben 1984, pp. 52–57; or Boyer and Merzbach 1989, pp. 79–81).

a number, or we can choose a unit commensurable with the diagonal, but then we will be unable to express the length of the side. So the incommensurability of the side and diagonal of the square reveals the boundaries of the expressive power of the language of elementary arithmetic.[6]

1.1.2. Synthetic Geometry

Since the earliest stages of their development the great agricultural civilizations of antiquity were confronted with several problems that are closely linked to geometry. By measuring fields, planning irrigation channels, constructing roads, building granaries and fortification walls, as well as many other practical activities, they collected knowledge about different geometrical objects. Even the term "*geometry*", which has its origin in the Greek words for earth (*geos*) and measure (*metros*), reveals the practical roots of this discipline. However, despite the quantity of practical knowledge collected in these ancient civilizations, geometry did not become an independent discipline. It was embedded into the framework of arithmetical calculative recipes just like calculation of the volume of the cylindrical granary, quoted above from the *Rhind papyrus*. Even though the problem is a geometrical one, it is treated in a purely arithmetical way. Geometry as an independent mathematical discipline was created in ancient Greece. Thales and Pythagoras, the founders of geometry, made according to ancient tradition long trips to Egypt and to Mesopotamia where they learned the practical geometrical knowledge of these civilizations. They brought this knowledge back to their homeland and presumably enriched it with many new discoveries that bear their names. But what is even more important, they created a new mathematical language suitable for representing geometrical objects.

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[6] This limitation was later removed when mathematicians introduced real numbers as their basic number system. But we are interested here in the language of elementary arithmetic in its original form. And for the kind of elementary arithmetic that takes number to be a collection of units, the incommensurability of the side and diagonal of the square is a paradox that defies comprehension. Later this paradox was resolved by the introduction of real numbers and it was replaced by the positive fact that $\sqrt{2}$ is an irrational number. Here we see one of the fundamental strategies of dealing with paradoxes. It consists in the extension of the language and in the reinterpretation of the paradox so that it becomes a positive fact. Nevertheless, it took many centuries until the incommensurability was resolved inside of the symbolic language itself.