LECTURE NOTES: SELECTED TOPICS FROM SET THEORY (2022/23)

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Contents

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1. Introduction

We will discuss areas of set theory which are relevant for other fields in mathematics:

- Strengthenings of the Continuum Hypothesis (CH) which are often more useful than the plain CH. For instance, $\mathsf{ZFC} + \diamondsuit_{\omega_1}(S)$ for every stationary $S \subseteq \omega_1$ implies that every Whitehead group on ω_1 is free (a result of Shelah, see [4] for more details; also see Section 8.2.1

for a brief outline of the problem); or, $\mathsf{ZFC} + \diamondsuit_{\omega_1}(\omega_1)$[1] implies that there exists a dense linear order which is complete in the ordering and satisfies ccc, and yet is not isomorphic to the real line (Suslin line). Kaplansky's conjecture, see Section 8.3.1, is an example of a problem which is decided in one way by CH alone.

- Strengthenings of the negation of the Continuum Hypothesis which are often more useful than the plain $\neg\mathsf{CH}$ or $2^\omega = \omega_2$: most importantly, the so called *forcing axioms*, such as Martin's Axiom MA_{ω_1} or even stronger Proper forcing axiom (PFA). In the above mentioned results, see [4], it shown that MA_{ω_1} implies there exists a Whitehead group of size ω_1 is which is not free. MA_{ω_1} also implies there are no Suslin lines. Kaplansky's conjecture is decided by PFA.
- Many apparently simple questions lead to answers which have a strictly stronger consistency strength then ZFC. The strength is usually measured in terms of *large cardinals*: we shall meet the more known ones, such as inaccessible, weakly compact, measurable and strongly compact cardinals. For instance the consistency strength of the theory $\mathsf{ZF}+$ "all subsets of $\mathbb{R}$ are Lebesgue-measurable" is exactly one inaccessible cardinal, while the consistency of $\mathsf{ZFC}+$ "the Lebesgue measure can be extended to all subsets of $\mathbb{R}$ (while losing translation-invariance, of course)" is exactly one measurable cardinal. There are also direct implications, the existence of a measurable cardinal implies that all $\mathbf{\Sigma}^1_2$ sets are Lebesgue-measurable.
- Many combinatorial principles are best analysed by means of infinite trees: we will discuss Aronszajn and Suslin trees and their constructions.

2. Preliminaries

2.1. Basic concepts

Notation: unless specified otherwise, $\kappa, \lambda, \mu, \dots$ will range over infinite cardinals. $\alpha, \beta, \gamma, \dots$ will range over ordinals. n, m, k will range over natural numbers. ξ, i may range over anything. Ord denotes the proper class of ordinal numbers, Card denotes the proper class of cardinal numbers, ω denotes the set of natural numbers. $\mathscr{P}(x)$ is the powerset of x, the set of all its subsets. $\{x \mid \varphi(x)\}$ denotes the class of all x satisfying φ; $\langle x_\xi \mid \xi < \kappa\rangle$ denotes a function f with domain κ and range $\{x_\xi \mid \xi < \kappa\}$ with $f(\xi) = x_\xi$. If $x = \{x_\xi \mid \xi < \kappa\}$ is a collection of sets, we may write either $\bigcup_{\xi<\kappa} x_\xi$ or $\bigcup\{x_\xi \mid \xi < \kappa\}$ for $\bigcup x$.

The following concepts should be familiar to the reader:

[1] $\diamondsuit_{\omega_1}(\omega_1)$ is often denoted just by $\diamondsuit_{\omega_1}$ or even just $\diamondsuit$ (see Definition 2.14 for more details). Note that by a result of Shelah $\diamondsuit_{\omega_1}(\omega_1)$ does not imply $\diamondsuit_{\omega_1}(S)$ for every stationary S ([8]).

- Axioms of $\mathsf{ZFC} = \mathsf{ZF} + \mathsf{AC}$, where AC denotes the Axiom of Choice (for every collection x of non-empty sets there is a function f with domain x such that $f(x) \in x$ for every x). AC is equivalent to many statements, in particular to the principle which says that every set can be well-ordered.[2]
- Well-orderings, ordinal and cardinal numbers. Ordinal numbers are canonical representatives for the equivalence classes with respect to the isomorphism relation on the class of all well-orderings. Historically, there are more definitions of cardinal numbers, all equal under AC. For concreteness: an ordinal numbers α is called a cardinal number if there is no $\beta < \alpha$ and a bijection between β and α. Ordinal numbers measure "length" (of an enumeration), cardinal numbers measure "size". It is important to remember that while these two concepts are the same for finite sets, they are different for infinite sets.
- The notion of cofinality: if α is a limit ordinal, then $\mathrm{cf}(\alpha)$ is the least cardinal κ for which there exists a strictly increasing sequence $\langle \alpha_\xi \,|\, \xi < \kappa \rangle$ whose supremum is α. It can be shown that $\omega \le \mathrm{cf}(\alpha) = \mathrm{cf}(\mathrm{cf}(\alpha)) \le \alpha$, and the sequence $\langle \alpha_\xi \,|\, \xi < \kappa \rangle$ can be assumed to be continuous at its limit points.
- κ is a regular cardinal if $\mathrm{cf}(\kappa) = \kappa$, it is singular otherwise. $\mathrm{cf}(\alpha)$ is always a regular cardinal.
- Cardinal addition and multiplication: for all κ, λ, $\kappa + \lambda = \lambda + \kappa = \kappa \cdot \lambda = \lambda \cdot \kappa = \max\{\kappa, \lambda\}$, where $\kappa + \lambda$ is the size of the disjoint union of κ, λ, and $\kappa \cdot \lambda$ the size of the product $\kappa \times \lambda$.
- Cardinal exponentiation: 2^κ denotes the size of $\mathscr{P}(\kappa)$. It can be shown that $\kappa < \lambda$ implies $2^\kappa \le 2^\lambda$ (but not in general $2^\kappa < 2^\lambda$).
- Cantor's theorem says $\kappa < 2^\kappa$.
- König's lemma makes a stronger claim that $\kappa < \mathrm{cf}(2^\kappa) \le 2^\kappa$.
- In particular, the size of the real numbers $\mathbb{R}$ is 2^ω and the cofinality of 2^ω must be uncountable.
- Ordinal addition and multiplication: $\alpha + \beta$ denotes the unique ordinals isomorphic to a well-ordered set which starts as α and continues as β, more formally, $\alpha + \beta$ denotes the order-type of the set $(\{0\} \times \alpha) \cup (\{1\} \times \beta)$ ordered lexicographically. This operation is not in general commutative, for instance $\omega = n + \omega < \omega + n$, for any $n < \omega$.
- Let $\aleph : \mathrm{Ord} \to \mathrm{Card} \setminus \omega$ be the unique isomorphism. Then $\aleph(\alpha) = \aleph_\alpha$ denotes the α-th infinite cardinal number.
- AC implies that for every x there is a (necessarily unique) κ with a bijection between x and κ. We write $|x| = \kappa$ to indicate that x has size κ.

[2] $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ can be shown to be well-orderable just in ZF; this is not the case for $\mathbb{R}$, a matter of confusion at the beginnings of set theory.

– AC implies that for every α, $\aleph_{\alpha+1}$ is a regular cardinal. The least singular cardinal is $\aleph_\omega$ with cofinality ω, $\aleph_{\omega+\omega}$ is the next one; $\aleph_{\omega_1}$ is a singular cardinal with cofinality ω_1.
– We will write ω_α instead of $\aleph_\alpha$ (double notation).
– We write κ^+ to denote the cardinal successor of κ; more generally $\kappa^{+\xi}$ is the ξ-th successor of κ.
– CH, the continuum hypothesis, says that the size of $\mathbb{R}$ is the least possible: ω_1. This implies that one can enumerate reals as a sequence $\langle r_\xi \,|\, \xi < \omega_1\rangle$; i.e. it is possible to carry out a recursive construction over the reals whose initial parts are at most countable.
– GCH, the general continuum hypothesis, says that for every κ, $2^\kappa = \kappa^+$. GCH is consistent with the axioms of ZFC.[3]

2.2. Cardinal arithmetics

Notation: Suppose $\langle \kappa_\xi \,|\, \xi < \mu\rangle$ is a sequence of cardinal numbers. Then $\sum_{\xi<\mu} \kappa_\xi$ denotes the size of the disjoint union of sets $\bigcup\{\{\xi\} \times \kappa_\xi \,|\, \xi < \mu\}$. If x, y are sets, then ${}^x y$ denotes the set of all functions with domain x and range in y. In particular, $|{}^x 2| = 2^{|x|}$. If μ is a cardinal number, then ${}^{<\mu}x$ denotes thet set $\bigcup_{\xi<\mu} {}^\xi x$. If $\kappa = |x|$, we write $|{}^{<\mu}x| = \kappa^{<\mu}$.

Lemma 2.1. *(i)* $\sum_{\xi<\mu} \kappa_\xi = \sup\{\kappa_\xi \,|\, \xi < \mu\} \cdot \mu$. *In particular if* $1 \leq \kappa$ *and* $\omega \leq \mu$, *then* $\kappa^{<\mu} = \sup\{\kappa^{|\alpha|} \,|\, \alpha < \mu\}$.
(ii) For every set x, $|{}^{<\omega}x| = |x| + \omega$. *In particular, if* $\mathcal{L}$ *is an at most countable first-order signature, then there are just countably many formulas in the language* $\mathcal{L}$.
(iii) The size of ${}^\omega x$ *cannot in general be computed just from the size of* x, *but at least we have* $(\kappa^\mu)^\nu = \kappa^{\mu\cdot\nu}$, *so in particular* $|{}^\omega\mathbb{R}| = |\mathbb{R}| = 2^\omega$.

Proof. (i) Let ν denote $\sup\{\kappa_\xi \,|\, \xi < \mu\}$. We show there exists an injection (a) from $\bigcup\{\{\xi\} \times \kappa_\xi \,|\, \xi < \mu\}$ into $\mu \times \nu$, and (b) conversely. For (a), the identity function works. For (b), it suffices to notice that there is an injection from μ, and also one from μ; since $\mu \cdot \nu = \max\{\mu, \nu\}$, this suffices.
(ii) Let $\kappa = |x|$. Clearly, $|{}^{<\omega}x| = \kappa^{<\omega} = \sum_{n<\omega} \kappa^n = \kappa + \omega$.
(iii) Obvious. $\square$

Regarding (iii), what can we say about the special case of ${}^\omega\aleph_\omega$? This interesting both from the point of set theory, and also from the point of history because this question was incorrectly answered by Bernstein, and König used Bernsteins's (incorrect) claim[4] to argue that 2^ω is not a cardinal; but König's

[3] First shown by Gödel in 1930's using the constructible universe L; can be also shown by forcing (a method developed by Cohen in 1960's).
[4] Bernstein proved in his dissertation an incorrect claim that $\aleph_\alpha^\omega = \aleph_\alpha \cdot 2^\omega$ for all α.

argument was otherwise solid and without the mistaken claim, it correctly showed that $\mathrm{cf}(2^\omega) \neq \omega$.

Example. Suppose $2^\omega = \omega_1$, then for all $1 \le n < \omega$, $\aleph_n^\omega = \aleph_n$ (proof easy). By a difficult theorem of Shelah, this breaks down at $\aleph_\omega$: even if GCH holds below $\aleph_\omega$, then $\aleph_\omega^\omega$ (in this case $= 2^{\aleph_\omega}$) is provably (in ZFC) bounded by $\aleph_{\omega_2}$, yet it is consistent that $2^{\aleph_\omega} = \aleph_{\omega+\alpha+1}$ for any $\alpha < \omega_1$. This "breaking-down" can take place in general at all singular cardinals with countable cofinality.

Example. The case of singular cardinals with uncountable cofinalities is completely different: Silver showed in 1970's that (to take a specific example) if GCH holds below $\aleph_{\omega_1}$, then it continues to hold at $\aleph_{\omega_1}$. In fact, it suffices if it holds at a *stationary* set below $\aleph_{\omega_1}$, we shall discuss stationarity below.

One of the reasons for this difference is the structure of *closed unbounded subsets* of limit cardinals which behaves pathologically at countable cofinalities (we will discuss this later).[5]

We conclude this section by showing how GCH determines the exponentiation function.

Lemma 2.2. *Assume* GCH *holds. Let* κ, μ *be infinite cardinals. Then if (i)* $\kappa \le \mu$, *then* $\kappa^\mu = 2^\mu = \mu^+$, *(ii)* $\mathrm{cf}(\kappa) \le \mu \le \kappa$, *then* $\kappa^\mu = \kappa^+$, *(iii)* $\mu < \mathrm{cf}(\kappa) \le \kappa$, *then* $\kappa^\mu = \kappa$.

Proof. (i) $\kappa^\mu \le 2^{\kappa\cdot\mu} = 2^\mu = \mu^+$.

(ii) $\kappa < \kappa^{\mathrm{cf}(\kappa)} \le \kappa^\mu \le \kappa^\kappa = 2^\kappa = \kappa^+$, and so $\kappa^{\mathrm{cf}(\kappa)} = \kappa^\mu = \kappa^+$.

(iii) $\kappa^\mu = \sum_{\alpha<\kappa} |\alpha|^\mu = \sup\{|\alpha|^\mu \mid \alpha < \kappa\} \cdot \mu$. Let $\nu = (|\alpha| \cdot \mu)^{|\alpha|\cdot\mu}$; note that $\nu < \kappa$. Then $|\alpha|^\mu \le \nu^\nu = \nu^+ \le \kappa$, and so $\kappa^\mu = \sup\{|\alpha|^\mu \mid \alpha < \kappa\} \cdot \mu = \kappa$. □

2.3. Filter of closed unbounded sets

Before we move to trees at uncountable cardinals, we need to review the notion of a closed unbounded sets. Let κ be a regular uncountable cardinals.

To motivate the notional of closed unbounded set, consider the following example. Let $f : \kappa \to \kappa$ be a function. Let us say that $\alpha < \kappa$ is a *closure point* of f if for all $\beta < \alpha$, $f(\beta) < \alpha$. Let us denote $\mathrm{CL}(f)$ the set of closure points.

Claim 2.3. *(i) The set* $\mathrm{CL}(f)$ *is* unbounded *in* κ, *that is for every* $\alpha < \kappa$ *there is some* β *such that* $\beta \in \mathrm{CL}(f)$ *and* $\alpha \le \beta$.

(ii) The set $\mathrm{CL}(f)$ *is* closed *in* κ, *that is if* $\alpha < \kappa$ *is a limit ordinal, and* $\mathrm{CL}(f) \cap \alpha$ *is unbounded, then* $\alpha \in \mathrm{CL}(f)$.

Proof. Ad (i). The proof is a special case of the Skolem hull argument for the construction of a substructure of κ which is closed under f and contains as a

[5]To anticipate a little: if α has cofinality ω, then there are disjoint closed unbounded subsets of α: the reason is that being closed is trivial for sequences of length ω; this cannot happen at uncountable cofinalities.

subset a given $\alpha \in \kappa$ (note that by transitivity of κ, $\alpha \subseteq \kappa$). Let $\alpha \in \kappa$ be given. By induction of length ω construct $\beta \supseteq \alpha$, $\beta \in \kappa$, which is closed under f. Set $\alpha = \alpha_0$, and if n is already constructed, $\alpha_{n+1} = \max(\alpha_n, \sup\{f(\gamma) + 1 \,|\, \gamma \in \alpha_n\})$. Set $\beta = \sup\{\alpha_n \,|\, n \in \omega\}$; by regularity of κ, $\beta \in \kappa$. It follows that $\beta \geq \alpha$ is a closure point of f.

Ad (ii). Trivial. □

The two properties of $\mathrm{CL}(f)$ identified above lead to the concept of an closed unbounded set. We say that $X \subseteq \kappa$ is *club* if it is unbounded and closed in κ.

Lemma 2.4. *If C and D are clubs in κ, then $C \cap D$ is a club in κ*

Proof. We first show that $C \cap D$ is closed. This is clear: if α is a limit ordinal and $C \cap \alpha$ and $D \cap \alpha$ are both unbounded in α, then by closedness of C, D, $\alpha \in C \cap D$.

The key of the proof is to show the unboundedness. Let $\alpha < \omega_1$ be given, we wish to find some $\beta \geq \alpha$ such that $\beta \in C \cap D$. Let us construct by recursion a sequence $\langle c_i \,|\, i < \omega\rangle$ of elements of C and $\langle d_i \,|\, i < \omega\rangle$ of elements of D as follows. Choose $c_0 \in C$ and $d_0 \in D$ so that $\alpha < c_0 < d_0$. In general, in the step $n+1$, choose $c_{n+1} \in C$ and $d_{n+1} \in D$ so that $\ldots c_n < d_n < c_{n+1} < d_{n+1}$. Let us denote $c = \sup\{c_i \,|\, i < \omega\}$ and $d = \sup\{d_i \,|\, i < \omega\}$. First note that $c = d$ and that c (and d) is a limit ordinal of countable cofinality. By closedness of C and D, $c \in C \cap D$. □

Corollary 2.5. *If $\{C_i \,|\, i < \mu\}$ is a set of clubs in κ for some $\mu < \kappa$, then $\bigcap_{i<\mu} C_i$ is a club in κ.*

Proof. This is a simple generalization of Lemma 2.4, using the regularity of κ. See proof of Lemma 2.8 for more details. □

Exercise. Let C be a club. Let us denote as D the set of all limit ordinals in C. Show that D is a club.

Exercise. Let C be a club and let $\mathrm{Lim}(C)$ be the set of limit points of C, where $\alpha \in C$ is a limit point of C if $C \cap \alpha$ is unbounded in α. Show that $\mathrm{Lim}(C)$ is a club (which is strictly smaller than C).

Lemma 2.4 allows us to define the *closed unbounded filter* generated by the club sets. Let us denote this filter as $\mathrm{Club}(\kappa)$:

$$\mathrm{Club}(\kappa) = \{X \subseteq \kappa \,|\, \text{there is a club } C \text{ such that } C \subseteq X\}.$$

We say that a filter F is *κ-complete* for a regular cardinal κ if for every family $\{X_i \,|\, i < \lambda\}$ of elements of F, where $\lambda < \kappa$, the intersection $\bigcap\{X_i \,|\, i < \lambda\}$ is in F.

Corollary 2.6. *The filter* $\mathrm{Club}(\kappa)$ *is κ-complete.*

Proof. Follows from Corollary 2.5. □

Note. Under AC, Club(κ) is never an ultrafilter (see in Theorem 2.11). The existence of an ω_1-complete ultrafilter on any regular κ is a very strong assumption which postulates the existence of the so called *measurable* cardinal.

Let us denote as NS(κ) the dual ideal to Club(κ):

$$\mathrm{NS}(\kappa) = \{X \subseteq \kappa \,|\, \kappa \setminus X \in \mathrm{Club}(\kappa)\}.$$

We call the ideal NS(κ) the *non-stationary ideal* on κ. The Lemma 2.6, the non-stationary ideal NS(κ) is κ-complete.[6] We say that $X \subseteq \kappa$ is *stationary* if $X \notin \mathrm{NS}(\kappa)$.

Lemma 2.7. *$X \subseteq \kappa$ is stationary iff $X \cap C \neq \emptyset$ for every club C.*

Proof. If X is stationary, then $\kappa \setminus X$ is not in Club(κ). This means that there is no C so that $C \subseteq \kappa \setminus X$, or equivalently for any club C, $C \nsubseteq \kappa \setminus X$, which is the same as $C \cap X \neq \emptyset$.

For the converse, just run the argument in the opposite direction. □

The club filter Club(κ) satisfies another important property, that of *normality*. Let X_i for $i < \kappa$ be subsets of κ. Let us define the diagonal intersection

$$\triangle_{i<\kappa} X_i = \{\xi < \kappa \,|\, \xi \in \bigcap_{\zeta<\xi} X_\zeta\}.$$

Lemma 2.8. *The filter* Club(κ) *is* normal*, that is it is closed under the diagonal intersections of length κ: If for every $i < \kappa$, X_i is an element of* Club(κ)*, then*

$$\triangle_{i<\kappa} X_i \in \mathrm{Club}(\kappa).$$

Proof. Let $\{C_i \,|\, i < \kappa\}$ be clubs such that $C_i \subseteq X_i$. It suffices to show that $D = \triangle_{i<\kappa} C_i$ is closed unbounded.

We first show that D is closed. Let α be a limit ordinal and $D \cap \alpha$ unbounded, we wish to show $\alpha \in D$. This is equivalent to demanding that for all $\beta < \alpha$, $\alpha \in C_\beta$. Fix such $\beta < \alpha$. Then for all γ, $\beta < \gamma < \alpha$, $\gamma \in D$ implies $\gamma \in C_\beta$; it follows $D \cap \alpha$ is unbounded in C_β, and hence $\alpha \in C_\beta$ as desired.

We now show that D is unbounded. Let $\alpha < \kappa$ be given, we wish to show there exists $\beta \geq \alpha$, $\beta \in D$. Set $\alpha_0 = \alpha$ and $A_0 = \emptyset$. Assume α_n and A_n are already constructed, we show how to construct α_{n+1} and A_{n+1}. Choose an increasing sequence $\langle a_\beta \,|\, \beta < \alpha_n \rangle$ such that $a_\beta \in C_\beta$ and $a_\beta > \alpha_n$ for each $\beta < \alpha_n$. Set $A_{n+1} = \{a_\beta \,|\, \beta < \alpha_n\}$ and $\alpha_{n+1} = \sup A_{n+1}$. Finally set $\beta = \sup \{\alpha_n \,|\, n < \omega\}$. In order to verify $\beta \in D$, we need to check that $\beta \in C_\gamma$ for each $\gamma < \beta$. Notice that for every $\gamma < \beta$ there exists $n < \omega$ such that

[6] Often, we say σ-complete instead of ω_1-complete.

$\gamma < \alpha_n$; it follows that for each $m \geq n$, there is some $a \in A_m \cap C_\gamma$. Hence $\beta \cap C_\gamma$ is unbounded and so $\beta \in C_\gamma$ as required. $\square$

Note that in general, we cannot hope that any proper filter F on κ is κ^+-complete – for every such F there is a family X_α, $\alpha < \kappa$, of elements in F such that $\bigcap_{\alpha<\kappa} X_\alpha = \emptyset$. It follows that the diagonal intersection is in some sense the best we can get.

Exercise.* Any normal filter F on κ is also κ-complete.

Intuitively, if set X is stationary, it means that it is not small in the sense of the club filter. "Stationarity" is therefore a measure of "largeness" for subsets of regular cardinals of uncountable cofinality. It has no analogue in case of ω, because ω has no limit points.

Remark 2.9. The club filter $\mathrm{Club}(\kappa)$ properly extends the Frechet filter $F(\kappa)$ on κ, where $X \in F(\kappa) \leftrightarrow X \setminus \kappa$ is bounded in κ. A typical subset of κ on which $F(\kappa)$ makes no decision, but $\mathrm{Club}(\kappa)$ does, is the set A of all limit ordinals in κ – A nor its complement $\kappa \setminus A$ is in $F(\kappa)$, but $A \in \mathrm{Club}(\kappa)$.

We said above that the club filter $\mathrm{Club}(\kappa)$ is not an ultrafilter:

Lemma 2.10. *Suppose κ is a regular cardinal. Then κ is a disjoint union of two stationary sets. In particular* $\mathrm{Club}(\kappa)$ *is not an ultrafilter.*

Proof. First note that if κ is a regular cardinal and $\mu < \kappa$ is also a regular cardinal, then $E^\kappa_\mu = \{\alpha < \kappa \,|\, \mathrm{cf}(\alpha) = \mu\}$ is a stationary set. It follows that if κ is at least ω_2, then E^κ_ω and $E^\kappa_{\omega_1}$ are two disjoint stationary subsets of κ.

Interestingly, there is no easy way how to imitate this simple proof for $\kappa = \omega_1$. It is necessary to use the Axiom Choice, and the so called *Ulam matrices* (see any text book on set theory).[7] $\square$

In fact, a stronger result hold:

Theorem 2.11 (Solovay)**.** *If κ is regular uncountable, then every stationary subset of κ is a disjoint union of κ-many stationary sets.*

Proof. (Sketch) For a successor cardinal κ, Ulam matrices mentioned in the proof of Lemma 2.10 give the stronger result that every stationary set is a disjoint union of κ-many stationary sets. For a limit cardinal κ, the proof is a bit more involved (see for instance [1, Theorem 2.27]). $\square$

We end the discussion of stationary sets by stating a very useful Fodor's lemma.

We say that a function $f : \kappa \to \kappa$ is *regressive* if $f(\alpha) < \alpha$ for every $\alpha > 0$.

[7] Uder some large cardinal hypotheses, it is consistent with ZF that $\mathrm{Club}(\omega_1)$ is an ultrafilter (for instance the Axiom of Determinacy implies this).

Theorem 2.12 (Fodor's lemma)**.** *If $f : \kappa \to \kappa$ is regressive, then there is a stationary set $S \subseteq \kappa$ on which f is constant.*

More generally, if $f : T \to \kappa$ is regressive, where $T \subseteq \kappa$ is stationary, then there a stationary set $S \subseteq T$ on which f is constant.

Proof. We will just show the case for $f : \kappa \to \kappa$, although the generalization to the second part featuring T is easy.

Assume for contradiction that for each $\alpha < \kappa$, the set set $f^{-1\prime\prime}\{\alpha\}$ is non-stationary, and fix for each α a club C_α such that

$$f^{-1\prime\prime}\{\alpha\} \cap C_\alpha = \emptyset. \tag{2.1}$$

By diagonal intersection, the set $\triangle_\alpha C_\alpha$ is a club. However, any $\xi \in \triangle_\alpha C_\alpha$ contradicts the fact that f is regressive: $\xi \in \bigcap_{\zeta<\xi} C_\zeta$ implies by (2.1) that $\xi \notin f^{-1\prime\prime}\{\zeta\} \leftrightarrow f(\xi) \neq \zeta$ for every $\zeta < \xi$. Thus $f(\xi) \geq \xi$, which contradicts the fact that f is regressive. □

2.4. Diamonds and squares

Jensen discovered several combinatorial principles which are true in L, but can hold also in more general universes and which can be used to construct mathematically interesting structures. Later on we will discuss Suslin trees and non-reflecting stationary sets, and show how these principles can be used to construct.

Definition 2.13. Let E be a subset of κ^+, then $\square_\kappa(E)$ holds if there is a sequence $\langle C_\alpha \,|\, \alpha < \kappa^+, \alpha \text{ limit}\rangle$ such that:

(i) C_α is a club in α;
(ii) $\mathrm{cf}(\alpha) < \kappa \to \mathrm{ot}(C_\alpha) < \kappa$, where $\mathrm{ot}(C_\alpha)$ denotes the order-type of C_α;
(iii) if $\bar\alpha < \alpha$ is a limit point of C_α, then $\bar\alpha \notin E$ and $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.

We write $\square_\kappa$ instead of $\square_\kappa(\emptyset)$. Note that by (iii), we can extend (ii) to $\mathrm{cf}(\alpha) = \kappa \to \mathrm{ot}(C_\alpha) = \kappa$. Exercise*. Note that $\square_\omega$ is trivially true (provable in ZFC).

Under the assumption $V = L$, $\square_\kappa$ is true for every infinite cardinal κ.

Definition 2.14. Let E be a stationary subset of κ^+. We say that $\diamondsuit_{\kappa^+}(E)$ holds if there is sequence $\langle S_\alpha \,|\, \alpha \in E\rangle$ such that $S_\alpha \subseteq \alpha$ for every α and for every $A \subseteq \kappa^+$,

$$\{\alpha \in E \,|\, S_\alpha = A \cap \alpha\} \text{ is stationary.}$$

Under $V = L$, $\diamondsuit_{\kappa^+}(E)$ is true for every stationary E. It is also known that for every infinite $\kappa \geq \omega_1$, $2^\kappa = \kappa^+$ is equivalent to $\diamondsuit_{\kappa^+}$. However, there are limitions for this result which connects GCH with diamond:

- This correspondence fails for $\kappa = \omega$. By a result of Jensen, CH plus $\neg\diamondsuit_{\omega_1}$ is consistent.

- Even for regular $\kappa \geq \omega_1$, $2^\kappa = \kappa^+$ does not imply diamond on the *critical cofinality* κ, i.e. $2^\kappa = \kappa^+$ is consistent with $\neg\Diamond_{\kappa^+}(E^{\kappa^+}_\kappa)$ (by a result of Shelah in Theorem 2.16, this is the only restriction).

Let us state the connection with GCH with more detail:

Theorem 2.15 (Gregory)**.** *Assume* GCH*. Let* κ *be an infinite cardinal. Denote* $E = E^{\kappa^+}_\omega$*. Then:*
(i) If $\operatorname{cf}(\kappa) > \omega$*, then* $\Diamond_{\kappa^+}(E)$ *holds.*
(ii) If moreover $\Box_\kappa$ *holds, then even in case* $\operatorname{cf}(\kappa) = \omega$*,* $\Diamond_{\kappa^+}(E)$ *holds.*

Shelah finally proved:

Theorem 2.16 (Shelah)**.** *Suppose* λ *is an uncountable cardinal satisfying* $2^\lambda = \lambda^+$*. Then* $\Diamond_{\lambda^+}(E)$ *holds for every stationary* $E \subseteq \{\alpha < \lambda^+ \mid \operatorname{cf}(\alpha) \neq \operatorname{cf}(\lambda)\}$*.*

Remark 2.17. Diamond as a canonical way to construct many disjoint stationary sets. Let κ be a regular cardinal. We say that two subsets A, B of κ are *almost-disjoint* if $|A \cap B| < \kappa$.

Claim 2.18. *Assume* $\langle S_\alpha \mid \alpha < \kappa\rangle$ *is a diamond sequence for* κ*. Then one can define* 2^κ*-many almost disjoint stationary subsets of* κ *"canonically from this diamond sequence".*

Proof. Let $\langle X_i \mid i < 2^\kappa\rangle$ be some enumeration of subsets of κ. For each i, define

$$D_i = \{\alpha < \kappa \mid X_i \cap \alpha = S_\alpha\}.$$

We claim that $\{D_i \mid i < 2^\kappa\}$ is an almost disjoint family of stationary subsets of κ. Since $\langle S_\alpha \mid \alpha < \kappa\rangle$ is a diamond sequence, each D_i is stationary. If $X_i \neq X_j$, then without loss of generality there is some $\xi \in X_i$ & $\xi \notin X_j$. It follows that D_i and D_j are disjoint in the interval (ξ, κ). $\Box$

Remark 2.19. Forcing diamond at ω_1. Forcing diamond is quite easy, in fact the Cohen forcing $\operatorname{Add}(\omega_1, 1)$ adds a diamond (see [7], Theorem 8.3, p.227).[8] Note that this immediately implies that $\Diamond(\omega_1)$ is consistent with 2^{ω_1} being arbitrarily large: just carry out Theorem 8.3 over a ground model satisfying CH where 2^{ω_1} is arbitrarily large (for instance by first forcing with $\operatorname{Add}(\omega_1, \kappa)$ for some κ over a model with CH).

3. Linear orders and trees

3.1. Kurepa's theorem

First we not some useful facts about countable linear orders.

[8] $\operatorname{Add}(\omega_1, 1)$ always preserves ω_1, but collapses 2^ω to ω_1 (that is, if CH holds in the ground model, then it does not collapse cardinals).

Definition 3.1. Let $(P,<)$ be a partially ordered set. We say that $A \subseteq P$ is an *antichain* if no two elements in A are comparable in $<$. We say that A is a *chain* if it is linearly ordered under $<$.

Definition 3.2. Let $(A,<)$ and $(B,<)$ be partial orders. We say that A *embeds* in B (or A is B-*embeddable*) if there is a function $f : A \to B$ which is a homomorphism with respect to the *strict* ordering $<$.

If $f : A \to B$ is as above, then f is 1-1 on chains: if $C \subseteq A$ is a chain and $c_1 \neq c_2$ are in C, then $f(c_1) \neq f(c_2)$ (if $c_1 < c_2$, then $f(c_1) < f(c_2)$). However f may not be 1-1 on all of its domain – see Kurepa's theorem below which shows that there is no bound on the size of $(E,<)$ which embeds in $\mathbb{Q}$.

When we want to say that an embedding $f : A \to B$ is 1-1 on all elements of A, we say it is a *1-1 embedding.*

Fact 3.3. *(i)* $(\mathbb{Q},<)$ *is an universal order for countable linear orders: if* $(A,<)$ *is an at most countable linear order, then there is an 1-1 embedding* $i : A \to \mathbb{Q}$.

(ii) In particular, if $(A,<)$ *is dense linear order without end-points, then* A *and* $\mathbb{Q}$ *is isomorphic.*

Proof. Ad (i). Let $\{a_0, a_1, \ldots\}$ be an infinite enumeration of A (if A is finite, the Fact is trivial). By induction construct $i = \bigcup_n i_n$, where i_{n+1} is defined on $\{a_0, \ldots, a_{n+1}\}$ and extends i_n by defining $i(a_{n+1})$ to be an element in $\mathbb{Q}$ such that i_{n+1} is an isomorphism between $\mathrm{dom}(i_{n+1})$ and $\mathrm{rng}(i_{n+1})$.

Ad (ii). Extend the above argument to a back-and-forth argument. □

Remark 3.4. A useful representation of $(\mathbb{Q},<)$, one which is easily generalizable to higher cardinals (see higher Aronszajn trees below), is

$$A = (\{f \in {}^{\omega}\omega \mid \{n < \omega \mid f(n) \neq 0\} \text{ is finite}\}, <_{\mathrm{lex}}),$$

or

$$B = (\{f \in {}^{\omega}2 \mid \{n < \omega \mid f(n) \neq 0\} \text{ is finite}\}, <_{\mathrm{lex}}),$$

where $<_{\mathrm{lex}}$ is the lexicographical ordering. (Remove the sequence of constant 0's if you do not wish to have the least element.)

Claim 3.5. *Let* $\vec{0}$ *denote the sequence in* ${}^{\omega}2$ *such that* $\vec{0}(n) = 0$ *for each* $n < \omega$. *Then* $A \setminus \{\vec{0}\}$ *and* $B \setminus \{\vec{0}\}$ *are both isomorphic with* $\mathbb{Q}$ *(with respect to the lexicographical orders defined above).*

Proof. We will focus on B, the case of A is identical. We show that the ordering $(A,<_{\mathrm{lex}})$ is countable, linear, dense and without endpoints. All of these follow immediately, except perhaps the density: let $x < y$ in B be given. Let n be the least such that $0 = x(n) < y(n) = 1$. Since there can only be finitely many 1's in x, let $k > n$ be the least such that $x(k) = 0$; then $z = x \restriction k \cup \{\langle k, 1\rangle\} \cup x \restriction (\omega \setminus (k+1))$ satisfies $x < z < y$. $\square$

The following is less known (see [11], p.284).

Theorem 3.6 (Kurepa). *Let $(E, <)$ be a partially ordered set. Then the following are equivalent:*

(i) E is embeddable in $\mathbb{Q}$.

(ii) E is the union of at most countably many antichains.

Proof. (i)→(ii). Define $A_n = f^{-1}(q_n)$ for each q_n which is in the range of f, where $\{q_0, q_1, \ldots\}$ is some enumeration of rationals. Each A_n must be an antichain if f is an embedding.

(ii)→(i). Consider $({}^{\omega}2, <)$, where $<$ is the lexicographical, and hence linear ordering. We find an embedding $g : E \to {}^{\omega}2$ such that $\mathrm{rng}(g)$ is countable. By Fact 3.3(i), $\mathrm{rng}(g)$ is then embeddable in $\mathbb{Q}$, and by composition, so is E. To construct g, proceed as follows.

Let first $f : E \to \omega$ be so defined so that $A_n = f^{-1}(n)$ is an antichain of E, and $E = \bigcup_{n \in \omega} A_n$. We assume here that f is onto ω (that is E is the union of countably many antichains); if not, modify the argument below accordingly. For $x \in E$ define $g(x)$ so that $g(x)(n) = 1$ iff $n \le f(x)$ and $\{y \in E \mid y \le x\} \cap A_n \ne \emptyset$.

We check that g is as required. First notice that each $g(x)$ has only finitely many 1's because $g(x)(n) = 1$ implies that $n \le f(x)$, where $f(x) \in \omega$. Thus $\mathrm{rng}(g)$ is only countable. Notice further that the following holds for all elements in E:

$$(3.2) \quad (\forall x, y \in E)\,(m, n \in \omega)\,\big(x \in A_m \to [g(x)(m) = 1\ \&\ (x < y\ \&\ y \in A_n\ \&\ m \le n) \to g(y)(m) = 1)]\big).$$

Assume now $x < y$ are in E, we want to show that $g(x) < g(y)$. Let $x \in A_m$ and $y \in A_n$ for some $n \ne m$, and let $l = \min(m, n)$. Clearly, for all $k \le l$, if $g(x)(k) = 1$, then $g(y)(k) = 1$. We now distinguish two cases.

Case 1: $m < n$, and so $l = m$. In this case, by (3.2), $g(x)(m) = g(y)(m) = 1$. For all $k > m$, it must be $g(x)(k) = 0$. So to ensure $g(x) < g(y)$, it suffices to find $k > m$ such that $g(y)(k) = 1$. However, $g(y)(n) = 1$ by (3.2), so we are done.

Case 2: $n < m$, and so $l = n$. In this case, we show $g(x)(n) = 0$ and $g(y)(n) = 1$, and so $g(x) < g(y)$. If $g(x)(n) = 1$, then by definition of the function g, we know there exists $z \in A_n$ and $z \le x$; hence $z \le x \le y$ implies $z \le y$ are two comparable elements in A_n and this is a contradiction. $g(y)(n) = 1$ follows by (3.2). $\square$

Corollary 3.7. *There is no upper bound on the size of a partially ordered set which can be embedded into $\mathbb{Q}$.*

Proof. In Kurepa's proof, we put no size restriction on the size of the antichains. □

We will see later that Kurepa's theorem provides a useful characterization of special Aronszajn trees.

3.2. Basic definitions for trees

The following definition is more or less standard (although it differs slightly from Kunen's use of T_α in [7]).

Definition 3.8. We say that $(T, <)$ is a *tree* if $(T, <)$ is a partial order such that for each $t \in T$, the set $\{s \in T \mid s < t\}$ is well-ordered by $<$. Let

$$\mathrm{ht}(t, T) = \mathrm{ot}(\{s \in T \mid s < t\}),$$

where "ot" denotes the order-type of a given well-ordered set. We define $T_\alpha = \{t \in T \mid \mathrm{ht}(t, T) = \alpha\}$. We set height$(T)$ to be the least α such that $T_\alpha = \emptyset$. We further set $T \restriction \alpha = \bigcup_{\beta<\alpha} T_\beta$ (which makes $T \restriction \alpha$ a subtree of T of height α).

We say that $S \subseteq T$ is a *subtree* of $(T, <)$ in the induced ordering $<$ if

$$\forall x \in S \; \forall y \in T \; (y < x \to y \in S).$$

Notice that if S is a subtree of T, then for all $x \in S$, $\mathrm{ht}(x, S) = \mathrm{ht}(x, T)$. If $x \in T$, then the set $\{y \in T \mid x < y \vee y < x\}$ of all nodes in the tree T above or below x is a subtree (T restricted to x).

Definition 3.9. For a regular cardinal $\kappa \geq \omega$, T is called a *κ-tree* if T has height κ, and $|T_\alpha| < \kappa$ for each $\alpha < \kappa$.

Very often, a κ-tree T is isomorphic to a subtree of the full κ-ary tree $(\kappa^{<\kappa}, \subseteq)$. More precisely, whenever T is *normal* (indeed, normal here means representable as a subtree of $(\kappa^{<\kappa}, \subseteq)$). See Definition 3.10.

Definition 3.10. A *normal κ-tree* is a tree T with the following properties:

(i) height$(T) = \kappa$;
(ii) $|T_0| = 1$;
(iii) $|T_\alpha| < \kappa$, for every $\alpha < \kappa$;
(iv) each node has ρ-many successors (exact number varies; $\rho < \kappa$);
(v) each $x \in T$ has some $y > x$ at each higher level of T;
(vi) if $\beta < \kappa$ is a limit ordinal, and $\mathrm{ht}(x, T) = \mathrm{ht}(y, T) = \beta$ and x, y have the same predecessors, then $x = y$.

See [5], p.122, Ex 9.6.

Lemma 3.11. *Every normal tree T is isomorphic to a subtree $\bar{T}$ of the full κ-ary tree $({}^{<\kappa}\kappa, \subseteq)$, where $\bar{T}_\beta$ consists of sequences with domain β. In fact, only the items (i),(ii),(iii),(vi) of normality are required.*

Proof. We define by induction isomorphisms $i_\alpha : T \restriction \alpha \to \bar{T} \restriction \alpha$ and $i = \bigcup i_\alpha : T \to \bar{T}$. Set $\bar{T}_0 = \{\emptyset\}$; by (ii), $i_1(r) = \emptyset$ is an isomorphism between T_0 and $\bar{T}_0$, where r is the unique root of T. Suppose we have constructed $i_\beta : T \restriction \beta \to \bar{T} \restriction \beta$ for each $\beta < \alpha$ and we wish to construct i_α.

Assume first that α is limit. Set $i_\alpha = \bigcup_{\beta<\alpha} i_\beta$.

Suppose α is a successor of a limit cardinal: $\alpha = \alpha' + 1$ where α' is limit. Then define i_α by extending $i_{\alpha'}$ setting for each $x \in T_\alpha$

$$i_\alpha(x) = \{\langle \beta, i_{\alpha'}(y) \rangle \mid \beta < \alpha' \ \&\ y < x \ \&\ \mathrm{ht}(y, T) = \beta\}.$$

By (vi), i_α is 1-1. It is obviously also an isomorphism.

Assume now that α is a successor of a successor ordinal β. Since $|T_\beta| < \kappa$ by (iii), one can naturally extend i_β to i_α by including the level T_β using some 1-1 function from T_β into κ.

Set $\bar{T} = \bigcup\{\mathrm{rng}(i_\alpha) \mid \alpha < \kappa\}$. □

If T is a tree and $B \subseteq T$, we say that B is a *branch* if it is a maximal (under inclusion) chain in T.

Definition 3.12. Let κ be a regular cardinal. We say that a κ-tree $(T, <)$ is an *Aronszajn tree* if it has no branch of size κ.

Remark 3.13. An Aronszajn κ-tree T is a typical example of an "incompact object": by definition, for each $\alpha < \kappa$, there is a branch B_α of height α in T – if T were to be "compact" (in the analogous sense as first-order logic is compact), then from the assumption that for each $\alpha < \kappa$, there exists a branch of height α, we should be able to conclude that there is a branch of height κ.

Note that it is important that the levels in the tree have size $< \kappa$. For instance, it is easy to construct a tree of height ω which has no infinite branch (Exercise). However, such a tree would not be an ω-tree; compare with König's theorem below.

Definition 3.14. We say that a regular cardinal $\kappa \geq \omega$ has the *tree property* if there are no κ-Aronszajn trees.

In view of the previous Remark, if κ has the tree property, then it is "compact" as far as branches in trees are concerned.

Theorem 3.15 (König)**.** *Every ω-tree T has an infinite branch. Hence ω has the tree property.*

Proof. We construct a branch B by induction on levels. Since T is an ω-tree, $|T_0| < \omega$. It follows there is some $t_0 \in T_0$ such that $S(t_0) = \{s \in T \,|\, t_0 < s\}$ is infinite (this is true because T_0 is finite, T is infinite, and every element in T is above an element of T_0). The set $S(t_0) \cap T_1$ is finite – pick $t_1 \in S(t_0) \cap T_1$ such that $S(t_1) = \{s \in T \,|\, t_1 < s\}$ is infinite. Proceed in the same fashion and pick t_n for each $n < \omega$. Then $B = \{t_n \,|\, n < \omega\}$ is a branch in T. □

Often one considers κ-trees with extra properties which simplify the presentation of various results.

Definition 3.16. A κ-tree T is called *well-pruned* if
(i) T has a single root: $|T_0| = 1$;
(ii) $\forall t \in T\, \forall \alpha \big(\mathrm{ht}(t, T) < \alpha < \kappa \to \exists y \in T_\alpha (x < y)\big)$.

Notice that every normal tree is well-pruned (conditions (ii) and (v)) in Definition 3.10.

Lemma 3.17. *Every κ-tree T has a subtree which is a well-pruned κ-tree.*

Proof. Hint. Consider the nodes $\{t \in T \,|\, |\{s \in T \,|\, s > t\}| = \kappa\}$; the subtree is any thinning of this to a single root. □

Well-pruned Aronszajn trees are bushy, in the terminology of Exercise 38, p.90, in [7]:

Lemma 3.18. *Assume κ is regular, T is a κ-Aronszajn tree, $\lambda < \kappa$, $x \in T$ and $|\{y \in T \,|\, y > x\}| = \kappa$. Then*

$$\exists \alpha > \mathrm{ht}(x, T)\big(|\{y \in T_\alpha \,|\, y > x\}| \geq \lambda\big).$$

Proof. Hint. By contradiction. Define a regressive function q on α's with $\mathrm{cf}(\alpha) = \mathrm{cf}(\lambda) = \lambda$, with $q(\alpha) < \alpha$ where T does not branch between levels $q(\alpha)$ and α. □

However, note that this does not imply that the trees must split into many nodes. The trees can be subtrees of the binary tree (see the construction of the ω_1-tree Suslin tree below).
stationary. If $X_i \neq X_j$, then without loss of generality there is some $\xi \in X_i \,\&\, \xi \notin X_j$. It follows that D_i and D_j are disjoint in the interval (ξ, κ).

4. ω_1-trees

4.1. Aronszajn trees

We now study the trees at ω_1. When we say an Aronszajn tree in this section, we mean an ω_1-Aronszajn tree unless stated otherwise.

Definition 4.1. We say that an Aronszajn tree is *special* if T is the union of countably many antichains.

Lemma 4.2. *The following are equivalent for an Aronszajn tree T,*
(i) T is special;
(ii) There is $f : T \to \omega$ such that if x, y are comparable in T, then $f(x) \neq f(y)$;
(iii) There is $f : T \to \omega$ which is 1-1 on chains;
(iv) T embeds in $\mathbb{Q}$.

Proof. (iv)→(iii). Let f be the embedding, we show that f' witnesses (iii), where $f' = i \circ f$, where i is any bijection between $\mathbb{Q}$ and ω. Let $C \subseteq T$ be a chain, and $t < s$ elements in C; then $f(t) < f(s)$, and so $f'(t) \neq f'(s)$.

(iii)→(ii). f in (iii) witnesses (ii).

(ii)→(i). Let f be as in (ii). Set $A_n = f^{-1}(n)$. Then each A_n is an antichain.

(i)→(iv). This is the Kurepa's theorem 3.6. $\square$

We will now construct an Aronszajn tree in ZFC.

Theorem 4.3. *There is an Aronszajn tree T^*. We construct T^* as a subtree of $T = \{s \in {}^{<\omega_1}\omega \mid s$ is 1-1$\}$ with $\subseteq$ as the ordering. In particular, our tree will be normal according to Definition 3.10.*

Proof. Consider the subtree $T = \{s \in {}^{<\omega_1}\omega \mid s \text{ is 1-1}\}$ of the tree ${}^{<\omega_1}\omega$. T cannot have an ω_1-branch, because it would yield a 1-1 function from ω_1 to ω. However, T is not required tree because it has uncountable levels, and so is not an ω_1-tree.

Let us define for s and t in ${}^{<\omega_1}\omega$ the following equivalence relation

$$s \approx t \leftrightarrow \mathrm{dom}(s) = \mathrm{dom}(t) \ \&\ \{\beta \in \mathrm{dom}(s) \mid s(\beta) \neq t(\beta)\} \text{ is finite.}$$

We call a sequence $\langle s_\alpha \mid \alpha < \omega_1 \ \&\ \mathrm{dom}(s_\alpha) = \alpha\rangle$ a *semi-branch* whenever

(i) $s_\alpha \restriction \beta \approx s_\beta$, for every $\beta \leq \alpha$.

(ii) $\omega \setminus \mathrm{rng}(s_\alpha)$ is infinite for each $\alpha < \omega_1$.

A semi-branch satisfying (i) and (ii) makes it easy to define an Aronszajn tree:

$$T^* = \{s \in T \mid \exists \alpha\ s \approx s_\alpha\}.$$

It is immediate to verify that T^* is an Aronszajn tree, and a subtree of T.

To finish the prove of the theorem, it suffices to construct a semi-branch $\langle s_\alpha \mid \alpha < \omega_1\rangle$ satisfying (i) and (ii) above. The construction is by induction on $\alpha < \omega_1$. For $\alpha + 1$, define $s_{\alpha+1} = s_\alpha \cup \{\langle \alpha, n\rangle\}$, where n is any natural number in $\omega \setminus \mathrm{rng}(s_\alpha)$ (this is possible by (ii)).

At a limit stage γ, first fix an increasing sequence $\langle \alpha_n \mid n < \omega\rangle$ with limit γ. Define $t \in T_\gamma$ as the union $t = \bigcup_n t_n$, where each t_n is in T_{α_n} and $t_n \approx s_{\alpha_n}$ (which implies $t_n \restriction \beta \approx s_\beta$ for each $\beta \leq \alpha_n$). The sequence $\langle t_n \mid n < \omega\rangle$ is defined by induction. First set $t_0 = s_{\alpha_0}$. To construct t_{n+1} when we have already constructed t_n, consider first t^*_{n+1} defined as $t_n \cup (s_{\alpha_{n+1}} \setminus s_{\alpha_n})$. The domain of t^*_{n+1} is equal to α_{n+1} and by the induction assumption on t_n and the properties of $\langle s_\alpha \,|\, \alpha < \gamma\rangle$,

$$t^*_{n+1} \approx s_{\alpha_{n+1}}. \tag{4.3}$$

However, while $s_{\alpha_{n+1}}$ is 1-1, t^*_{n+1} may not be 1-1 because of the finite disagreement (4.3). Define t_{n+1} by making finitely many changes to t^*_{n+1} to ensure:

(i) $t_{n+1} \approx t^*_{n+1}$.
(ii) t_{n+1} is 1-1.

This can be done because $\omega \setminus \mathrm{rng}(s_{\alpha_{n+1}})$ is infinite, and so there is plenty of room to make t_{n+1} 1-1. It follows $t_{n+1} \in T_{\alpha_{n+1}}$.

Finally, the range of t may have used up all of ω, so we define s_γ by setting $s_\gamma(\alpha_n) = t(\alpha_{2n})$, thus leaving $t(\alpha_{2n+1})$'s outside the range of s_γ. Note that still $s_\gamma \restriction \beta \approx s_\beta$ for every $\beta < \gamma$, because α_n's are bounded below each $\beta < \gamma$, and so $s_\gamma \restriction \beta \approx t_n \restriction \beta \approx s_\beta$, for any n such that $\beta < \alpha_n$. □

Definition 4.4. An Aronszajn tree T is called a *Suslin tree* if all antichains in T are at most countable.

Note that T^* is never a Suslin tree (see [7],p.90, ex 39):

Lemma 4.5. *Suppose that T is an Aronszajn tree and a subtree of the tree $\{s \in {}^{<\omega_1}\omega \,|\, s$ is 1-1$\}$. Then T is not Suslin.*

Proof. Notice that for each $n \in \omega$, $A_n = \{s \in T \,|\, \exists\alpha\; \mathrm{dom}(s) = \alpha + 1 \;\&\; s(\alpha) = n\}$ is an antichain. To see this, let $s \neq t$ be in A_n, and assume for contradiction that $s \subsetneq t$. Then $s(\alpha) = n$ and $t(\alpha) = t(\alpha') = n$, where $\mathrm{dom}(s) = \alpha + 1$ and $\mathrm{dom}(t) = \alpha' + 1$. This contradicts that t is 1-1. Now, many A_n's can be just countable, but by the pigeon hole principle, there must be some n such that A_n is uncountable (the set $A = \{s \in T \,|\, \exists\alpha\; \mathrm{dom}(s) = \alpha + 1\}$ is uncountable and $A = \bigcup_n A_n$). □

Note that the above Lemma does not claim that T must be special: $A \neq T$ because it consists of successor levels only, and so T may not be a union of countably many antichains.

Note however (as is stated in [1]) that if T is any Aronszajn tree as in Lemma 4.5, then $S = \bigcup_{\alpha<\omega_1} T_{\alpha+1}$ together with the induced ordering is a special Aronszajn tree. [Every sequence $s \in S$ has the maximal element; compare with the construction of the special Aronszajn tree below.]

Remark 4.6. Let T be the tree constructed in Theorem 4.3. Then ZFC does not prove that T is non-special because there is a generic extension (see Baumgartner's construction below) where T is special. Even more strongly, Baumgartner's result shows that $\mathsf{ZFC} + \mathsf{MA}_{\omega_1}$ implies that all Aronszajn trees are special: this implies

$$\text{ZFC} \nvdash \exists \text{ a non-special } \omega_1\text{-Aronszajn tree},$$

while we do have (see below)

$$\text{ZFC} \vdash \exists \text{ a special } \omega_1\text{-Aronszajn tree}.$$

We will now construct a special Aronszajn tree (following [11], p.257, Theorem 5.2).

We first introduce a useful definition ([11], p.245):

Definition 4.7. Let $(P, <)$ be a partially ordered set. Then σP denotes the set of all bounded well-ordered subsets of P ordered as follows: $s \leq t$ iff s is an initial segment of t. Define also $\sigma' P = \{t \in \sigma P \,|\, t \text{ has the greatest} = \max \text{ element}\}$.

Note that σP is always a tree: for $t \in \sigma P$, the set $\{s \,|\, s < t\}$ is isomorphic to the ordinal $(\mathrm{ot}(t), \in)$, and is thus well-ordered. As for $\sigma' P$, note that $\sigma' P = \{t \in \sigma P \,|\, \mathrm{ot}(t) \text{ is a successor ordinal}\}$. $\sigma' P$ is thus in general not the subtree of σP (limit levels are omitted), but $\sigma' P$ is a tree in its own right.

Denote $\max : \sigma' P \to P$ the function which takes the max. Then max is a strictly increasing mapping.

Lemma 4.8. *$\sigma'\mathbb{Q}$ is a tree of height ω_1 which is the union of countably many antichains.*

Proof. For some illustration, there are some elements of $\sigma'\mathbb{Q}$: $\{0, 1, 3, 17\}$, $\{1, 2, 5/2\}$, $t = \{1, 2-1/2, 2-1/3, 2-1/4\ldots, 2\}$. Note that $\mathrm{ot}(t) = \omega+1$, but t is on the ω-th level of the tree $\sigma'\mathbb{Q}$ because the set $\{s \,|\, s < t\}$ has oder-type ω. In general, if $\mathrm{ot}(t) = \alpha + 1$ for a limit ordinal $\alpha < \omega_1$, then $t \in (\sigma'\mathbb{Q})_\alpha$.[9] Since successor order-types of well-ordered subsets of $\mathbb{Q}$ are unbounded in ω_1, the height of $\sigma'\mathbb{Q}$ is ω_1. Since $\mathbb{Q}$ is countable, $\sigma'\mathbb{Q}$ has no uncountable branches. Also, $\sigma'\mathbb{Q}$ is the union of countably many antichains because for each $q \in \mathbb{Q}$, the set $A_q = \{s \in \sigma'\mathbb{Q} \,|\, \max s = q\}$ is an antichain. However, $\sigma'\mathbb{Q}$ has uncountable levels, so is not an ω_1-tree, and hence not a (special) ω_1-Aronszajn tree. $\square$

We will find now a subtree T of $\sigma'\mathbb{Q}$ which will be a special ω_1-Aronszajn tree.

Theorem 4.9. *There is a special ω_1-Aronszajn tree.*

Proof. As in Theorem 4.3, $\sigma'\mathbb{Q}$ is almost the right tree, except that it is too wide. We will rectify that.

[9] Show by induction that for every limit α, $\{\beta + 1 \,|\, \beta < \alpha\}$ has the same order-type as α.

We will construct a subtree T of $\sigma'\mathbb{Q}$ by induction on levels $\alpha < \omega_1$. The following the the induction hypothesis for level α:

$(*)_\alpha$: For each $\gamma < \beta < \alpha$, each $t \in T_\gamma$, and each $x \in \mathbb{Q}$ such that $x > \max t$, there is an $s \in T_\beta$ such that $t < s$ and $x > \max s$. Moreover each level T_β for $\beta < \alpha$ is at most countable.

Case $\alpha = \beta + 1$. We assume that $(*)_\alpha$ holds, and wish to define T_α so that $(*)_{\alpha+1}$ holds. Define

$$T_\alpha = \{t \cup \{x\} \mid t \in T_\beta \ \&\ x \in \mathbb{Q} \ \&\ x > \max t\}.$$

Since $|T_\beta| \leq \omega$, $|T_\alpha| \leq \omega$; also since all relevant $x \in \mathbb{Q}$ are added, $(*)_{\alpha+1}$ holds.

Case α is limit. It will be here that we need to exercise some care to have T_α at most countable. Let $\langle \alpha_n \mid n < \omega\rangle$ be an increasing sequence cofinal in α.

Fix $t \in \bigcup_{\beta<\alpha} T_\beta$, and $x \in \mathbb{Q}$ such that $x > \max t$. Let $h(t)$ be the ordinal $\beta < \alpha$ such that $t \in T_\beta$. Let $m = \min\{n \mid \alpha_n \geq h(t)\}$. Using $(*)_\beta$ for $\beta < \alpha$, construct inductively an increasing sequence $\langle t_k \mid k < \omega\rangle$ of elements in $\bigcup_{\beta<\alpha} T_\beta$ such that $t_0 = t$, $t_k \in T_{\alpha_{m+k}}$ and $\max t_k < x$. We can also ensure that $\sup\{\max t_k \mid k < \omega\} = x$.[10] Let

$$s_{t,x} = \bigcup\nolimits_k t_k \cup \{x\}.$$

Notice that $s_{t,x} \in \sigma'\mathbb{Q}$ and $t < s_{t,x} \in T_\alpha$. Finally, set

$$T_\alpha = \{s_{t,x} \mid t \in \bigcup\nolimits_{\beta<\alpha} T_\beta \ \&\ x \in \mathbb{Q} \ \&\ x > \max t\}.$$

Clearly, $|T_\alpha| \leq \omega$ and $(*)_{\alpha+1}$ holds (if $\gamma < \beta < \alpha$, $t \in T_\gamma$ and $x > \max t$, then $s_{t,x}$ was constructed to ensure that $\max s_{t,x} \restriction \beta < x$).

The function max which maps t to $\max t$ embeds T in $\mathbb{Q}$ and witnesses that T is special. $\square$

4.2. Suslin trees

Definition 4.10. Let κ be a regular cardinal. We say that a κ-tree T is Suslin if every chain and antichain in T has size $< \kappa$.

In the context of Suslin trees, it is more convenient to demand something extra:

Definition 4.11. A κ-tree T is called *ever-branching* (or *ever-splitting*) if for all $x \in T$, the set $\{y \in T \mid y > x\}$ is not a chain.

There are some simplifications:

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[10] This is possible by the induction hypothesis which ensures that the max's of t_n's can be chosen to be cofinal in x.

Lemma 4.12. *(i) Assume κ is an infinite cardinal, T is a tree such that $|T| = \kappa$ and every chain and antichain has size $< \kappa$. Then κ is regular, and T is a κ-tree.*

(ii) If T is an ever-splitting κ-tree such that every antichain has size $< \kappa$, then every chain has size $< \kappa$. In other words, T is Suslin.

Proof. Ad (i). [7], Lemma 5.6.

Ad (ii). If B were a chain of size κ, then by induction pick an increasing sequence $b_\alpha \in B$, $\alpha < \kappa$, and $f(b_\alpha)$ such that $f(b_\alpha)$ is incomparable with any element of B above b_α. Then $\{f(b_\alpha) \,|\, \alpha < \kappa\}$ is a antichain of size κ. Note that we have essentially used that T is ever-splitting. □

The following is according to [7], p.82.

Theorem 4.13. *Assume $\Diamond_{\omega_1}$. Then there exists an ever-splitting ω_1-tree which is Suslin. This tree can be taken to be the subtree of the binary tree ${}^{<\omega_1}2$.*

We first state some lemmas first. The key idea of the construction is that we should "seal off" all possible antichains in building T. However, there are more potential antichains – 2^{ω_1} – than levels of construction – ω_1. This is where $\Diamond_{\omega_1}$ comes in. Recall the definition of the $\Diamond_{\omega_1}$-sequence:

Definition 4.14. $\langle S_\alpha \,|\, \alpha < \omega_1 \rangle$ is called a $\Diamond_{\omega_1}$-sequence if $S_\alpha \subseteq \alpha$, and if X is any subset of ω_1, then the set

$$\{\alpha < \omega_1 \,|\, X \cap \alpha = S_\alpha\}$$

is stationary. Note that one can consider the sequence is defined only at limit ordinals $\alpha < \omega_1$.

The magic of an $\Diamond_{\omega_1}$-sequence is that can be used to guess any subset ω_1 whose defining properties reflect down on a club in ω_1. This is the meaning behind the following lemma:

Lemma 4.15. *Let $T = (\omega_1, \prec)$ be an ω_1-tree, then:*

(i) $\{\alpha < \omega_1 \,|\, T \restriction \alpha = \alpha\}$ is a club.

(ii) If $A \subseteq \omega_1$ is a maximal antichain in T, then

$$C_A = \{\alpha < \omega_1 \,|\, T \restriction \alpha = \alpha \ \&\ A \cap T \restriction \alpha \text{ is a maximal antichain in } T \restriction \alpha\}$$

is club in ω_1.

Proof. Note that this has a Löwenheim-Skolem flavour, but not for the structure $(\omega_1, \prec)$, but rather for a structure of the type $(\mathscr{P}(\omega_1), \omega_1, \prec, \in, \dots)$ because we talk about antichains etc. It is easier to describe the L-S argument directly, rather than to invoke some abstract statements.

Ad (i). Closure is easy. So let α_0 be given. We need to find $\alpha \geq \alpha_0$ such that $T \restriction \alpha = \alpha$. First analyze the meaning of $T \restriction \alpha = \alpha$:

$$T \restriction \alpha = \alpha \leftrightarrow \forall \xi \big(\mathrm{ht}(\xi, T) < \alpha \leftrightarrow \xi < \alpha\big).$$

This suggests that we define some closure functions f, g to make this true: define $f(\xi) = \mathrm{ht}(\xi, T)$ (to ensure the direction from right to left), and $g(\xi) = \bigcup T_\xi$ (from left to right).

Let α be an ordinal $\geq \alpha_0$ closed under f, g.

Ad (ii). Similarly. Note that $A \cap T \restriction \alpha$ will always be an antichain, so we worry about maximality. $A \cap T \restriction \alpha$ is maximal if

$$(\forall \xi \in T \restriction \alpha)(\exists \alpha \in A \cap T \restriction \alpha)\ \xi \mid\mid \alpha.$$

This suggest we define a function $h(\xi) =$ some $\alpha \in A$ such that $\xi \mid\mid \alpha$ in T. Note that h is correctly defined because A is maximal in T, so such a witness always exists. Obtain α by closing under f, g, h (or equivalently, take the intersection of the three clubs determined by f, g, h). □

The following lemma shows how to "seal the antichains":

Lemma 4.16 (Sealing-off lemma).
Let $T = (\omega_1, \prec)$ be an ever-splitting ω_1-tree, and $\langle S_\alpha \,|\, \alpha < \omega_1\rangle$ a $\Diamond_{\omega_1}$-sequence. Suppose for all limit $\alpha < \omega_1$:

$$\big(T \restriction \alpha = \alpha \ \&\ S_\alpha \textit{ max antichain in } \alpha\big) \to (\forall x \in T_\alpha)(\exists y \in S_\alpha)\ y \prec x.$$

Then T is an ω_1-Suslin tree.

Proof. Let $A \subseteq \omega_1$ be a maximal antichain. By Lemma 4.15(ii), the set C_A is a club, and hence the intersection of C_A and the stationary set $\{\alpha < \omega_1 \,|\, S_\alpha = A \cap \alpha\}$ is non-empty. Fix some α such that $\alpha = T \restriction \alpha$ and $S_\alpha = T \restriction \alpha \cap A$ is a maximal antichain in $T \restriction \alpha$. Now, by our assumption any $x \in T$ such that $\mathrm{ht}(x, T) \geq \alpha$ must be above some element of S_α, and so A cannot have any elements on levels $\geq \alpha$. It follows that $A = S_\alpha$, and so A is countable. □

We now review the construction of the tree.

Proof. (of Theorem 4.13). Fix a diamond sequence $\langle S_\alpha \,|\, \alpha < \omega_1\rangle$. The tree will have levels $T_n = \{k \,|\, 2^n - 1 \leq k < 2^{n+1} - 1\}$ for $n < \omega$, and $T_{\omega+\alpha} = \{\omega(1+\alpha) + n \,|\, n < \omega\}$, so that $T_{\omega+\alpha} = \omega(1+\alpha) + \omega$, for $0 \leq \alpha < \omega_1$.

At successor level, split each node into two (compatibly with what the levels should be).

At a limit level α, first consider the situation when S_α is not a maximal antichain or $\alpha \neq T \restriction \alpha$. The point is that we need to extend only ω-many branches through $T \restriction \alpha$. By induction hypothesis $T \restriction \alpha$ is countable, so let $\{x_m \,|\, m < \omega\}$ be some enumeration of $T \restriction \alpha$. For each x_m, pick an increasing sequence of nodes $x_m(n) > x_m, n < \omega$, the heights of which converge to α. Put

$\omega\alpha + m$ above all $\{x_m(n) \,|\, n < \omega\}$. If S_α is a maximal antichain in $T \restriction \alpha = \alpha$, then for each x_m first pick some $x_m \prec y_m$ such that for some $a \in S_\alpha$, $a \prec y_m$ (this is possible because by assumption $A \cap \alpha$ is maximal) and start building $x_m(n)$'s, for $m < \omega$, above y_m. □

Remark 4.17. Some facts about ω_1-Suslin trees. Let $(T, <)$ be an ω_1-Suslin tree. Then $(T, <)$ can be naturally viewed as a forcing notion where for $t, s \in T$ we set that t is stronger than s if $t > s$. Since T does not contain uncountable antichains, T as a forcing notion is ccc. A more complicated argument can be used to show that T is ω_1-distributive, i.e. does not add new countable sequences of ordinals (see [5], Lemma 15.28). It is easy to see that forcing with T adds a new branch through T, and therefore T is no longer Suslin (nor Aronszajn) in the generic extension V^T. Thus to destroy a Suslin tree, it is enough to force with it.

The completion of the forcing $(T, <)$ is a complete Boolean algebra which is ccc and ω_1-distributive (sometimes called the *Suslin algebra*). Such an algebra may not exist (consistently).

The product forcing $T \times T$ is not ccc even even if T is Suslin. Exercise*. This shows that ccc is *consistently* not *productive*, i.e. the product of two ccc forcings may not be ccc. However, it is also consistently true that ccc is productive (e.g. under MA_{ω_1}).

5. Higher trees

5.1. κ^+-Aronszajn trees

This section roughly follows [Todorcevic, p.273].

Let κ be a regular uncountable cardinal. Consider first the following generalization of $(\mathbb{Q}, <)$:

$$\mathbb{Q}_\kappa = (\{f \in {}^\omega\kappa \,|\, \{n < \omega \,|\, f(n) \neq 0\} \text{ is finite}\}, <_{\text{lex}}).$$

Clearly, $|\mathbb{Q}_\kappa| = \kappa$, and the ordering is linear and dense.

We will use $\mathbb{Q}_\kappa$ in place of $\mathbb{Q}$ to construct, under some cardinal arithmetic assumptions, a special κ^+-Aronszajn tree. The following lemma captures the key property of $\mathbb{Q}_\kappa$.

Lemma 5.1. *Let κ be a regular uncountable cardinal. Then for all $a <_{\text{lex}} b$ in $\mathbb{Q}_\kappa$ there exists a strictly increasing sequence $\langle a_i \,|\, i < \kappa\rangle$ with $a_0 = a$ and $\lim(a_i) = b$.*

Proof. Exercise*. □

Theorem 5.2 (Specker)**.** *Assume $\kappa^{<\kappa} = \kappa \geq \aleph_0$ (in particular, κ is regular). Then there is a κ-special κ^+-Aronszajn tree.*

Proof. Follow proof of Theorem 4.9; in particular use the induction assumption $(*)_\alpha$ for each $\alpha < \kappa^+$. Make the following modifications:

(i) At every $\alpha < \kappa$ of cofinality $< \kappa$, enumerate all increasing sequences of order-type α of points in $T \restriction \alpha$ and put a rational number larger than all of the points in the sequence on top. By the assumption $\kappa^{<\kappa} = \kappa$, T_α will have size $\leq \kappa$.

(ii) At every $\alpha < \kappa$ of cofinality κ, proceed as in Theorem 4.9, making crucial use of Lemma 5.1. Note that in this step it is essential that we have added all possible branches in cofinalities $< \kappa$ (we need to make sure that at limit stages we can take limits).

□

5.2. κ^+-Suslin trees for a regular κ

Theorem 5.3. *If $\kappa^{<\kappa} = \kappa$ and $\diamondsuit_{\kappa^+}(E^{\kappa^+}_\kappa)$ holds, then there exists a κ^+-Suslin tree. The tree can be taken to be a subtree of the binary tree, and to be κ-complete.*

Proof. The levels are similar as in Theorem 4.13, except that $\alpha < \kappa^+$, and T_α is a subset of $\kappa\alpha + \kappa$ (it could be calculated what it is exactly, but this is not really important).

Fix $\langle S_\alpha \,|\, \alpha \in E\rangle$ witnessing the diamond, where $E = E^{\kappa^+}_\kappa$.

The construction is by induction. At each successor level, each node splits into two.

At each $\alpha < \kappa^+$, with $\mathrm{cf}(\alpha) < \kappa$, the number of all branches in $T \restriction \alpha$ is at most $|T \restriction \alpha|^{|\mathrm{cf}(\alpha)|}$ since very branch $b \in [T \restriction \alpha]$ is determined by a cofinal chain in $T \restriction \alpha$. By the induction assumption, $|T \restriction \alpha| \leq \kappa$, and by $\kappa^{<\kappa} = \kappa$, we get that the number of all branches is $\leq \kappa$. So extend all branches.

Note that this permits the possibility that even if $\alpha = T \restriction \alpha$ and $A \subseteq T \restriction \alpha$ is a maximal antichain, then it is not necessarily the case that each branch in $[T \restriction \alpha]$ has an element above some member of A (if $b \in [T \restriction \alpha]$, then for each $\beta < \alpha$ there is some $a_\beta || \beta$, $a_\beta \in A$; however $\beta < a_\beta$ is possible for every $\beta < \alpha$). So it may happen that potential antichains are not sealed off at $\alpha < \kappa^+$ with cofinality $< \kappa$.

This is the reason to use E, where the antichains are sealed off as usual. The κ-completeness of the tree is also essential: if $\alpha \in E$, then fix $\{\gamma_i \,|\, i < \kappa\}$ converging to α. Extend each $x \in T \restriction \alpha$ (possibly taking into account S_α where appropriate) into a branch – at each $i < \kappa$, the ordinal γ_i has cofinality $< \kappa$, so at limit stages there are nodes which "connect the dots" to make a branch. □

5.3. κ^+-Suslin trees for a singular κ

Assume κ is a singular cardinal. Since $\kappa^{\mathrm{cf}(\kappa)} > \kappa$, at stage $\mathrm{cf}(\kappa) < \kappa$ we cannot extend all the branches. The following is taken from [3], p.141–148.

Theorem 5.4. *Assume that κ is an uncountable cardinal (regular or singular). If there is a stationary set $E \subseteq \kappa^+$ such that both $\square_\kappa(E)$ and $\diamondsuit_{\kappa^+}(E)$ hold, then there is a κ^+-Suslin tree.*

In particular, if GCH *holds and $\square_\kappa$ holds, then there is a κ^+-Suslin tree (by Theorem 2.16 and Lemma 7.3).*

Proof. Hints.

At ordinals in E, the construction of the tree is designed to seal-off the antichains.

The construction of the tree is by induction. The successor step is trivial (just split every node into two nodes). Let α be a limit ordinal.

If $\alpha \notin E$, we define the level T_α so that every node $x \in T \restriction \alpha$ lies on some cofinal branch in $T \restriction \alpha$ (and we put some node on level T_α above all the nodes in that branch). To arrange this, we define canonically from the square sequence a cofinal branch b^α_x which contains x. Let $\langle c^\alpha_i \mid i < \gamma_\alpha \rangle$ be some enumeration of C_α. We find the least node (in the canonical ordering of $\kappa^+ \times \kappa$) on levels $T_{c^\alpha_i}$, starting with i such that $\mathrm{ht}(x, T \restriction \alpha) \le i$. At limit stages c^α_j, j limit, if there is one, pick the unique node on level $T_{c^\alpha_j}$ which extends the branch $b^\alpha_x \restriction c^\alpha_j$ built so far.

If $\alpha \in E$, and S_α is a maximal antichain in $T \restriction \alpha$, we proceed as in the previous paragraph, but we only extend those $x \in T \restriction \alpha$ which lie above some element of S_α. If S_α is not a maximal antichain, we extend every node x.

It is easy to see, using the properties of the diamond sequence, that if the construction does not break down, then T is a κ^+-Suslin tree.

We verify that the construction does not break. Let α be the least ordinal such that for some $x \in T \restriction \alpha$, b^α_x cannot be built. This can only happen if for some limit j, with $\mathrm{ht}(x, T) < c^\alpha_j$, there is no node on level $T_{c^\alpha_j}$ above $b^\alpha_x \restriction c^\alpha_j$. However c^α_j is a limit point of C_α, and so (*) $C_{c^\alpha_j} = C_\alpha \cap c^\alpha_j$, and also (**) $c^\alpha_j \notin E$. By (**), x was considered at stage in $T \restriction c^\alpha_j$ for the construction of $b^{c^\alpha_j}_x$, and by (*) the branch b^α_x restricted to c^α_j is precisely the branch $b^{c^\alpha_j}_x$. However, by our assumption that α is the least counterexample, there must be a node on the level $T_{c^\alpha_j}$ above $b^{c^\alpha_j}_x$, and so the branch b^α_x can be built, which contradicts our assumption that it cannot.

Note that it seems that the condition that the order type of each C_α is $\le \kappa$ is not used explicitly. However this is essentially used to ensure $\square_\kappa(E)$ from $\square_\kappa$. $\square$

6. Large cardinals

6.1. Measurable cardinals

Recall that $U \subseteq \mathscr{P}(\kappa)$ is a non-principal κ-complete ultrafiter on κ if it is an ultrafilter which does not contain singletons, and is closed under intersections of fewer than κ-many sets.

Definition 6.1. We say that an uncountable cardinal κ is *measurable* if there exists a κ-complete non-principal ultrafilter U on κ.

It is easy to see that if κ is measurable, then it contains the complements of all bounded sets, and κ is a regular cardinal. It is in fact much more than that: measurable cardinals are an example of a *large cardinal*: they are inaccessible and more. For instance every weakly compact cardinal, see Definition 6.20, is measurable (but not conversely).

We will prove some consequences of measurability. We will use the concept of an ultrapower of the universe V, to give all these proofs a unified (and elegant) setting.

6.1.1. The ultrapower construction

For more details regarding the ultrapower construction, read Section 5 "Elementary embeddings" in Kanamori's book [6]. For more context about the measure problem, read Section 2 "Measurability" also in Kanamori's book.

We sketch the construction: Suppose U is a normal (and hence κ-complete) ultrafilter on κ. For $f, g : \kappa \to V$ let us define

$$f =_U g \leftrightarrow \{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U.$$

$=_U$ is an equivalence on the class of all functions $f : \kappa \to V$. The equivalence classes are denoted by $[f]_U = [f]$. The universe of the ultrapower $\mathrm{Ult}(V, U) = M$ is the class of all equivalence classes $[f]$.[11] Let us further define

$$[f] \in [g] \leftrightarrow \{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\} \in U.$$

Los theorem states that this definition extends to all formulas: for all formulas φ,

$$M \models \varphi([f_1], \ldots, [f_n]) \leftrightarrow \{\alpha < \kappa \mid \varphi(f_1(\alpha), \ldots, f_n(\alpha))\} \in U.$$

For every $x \in V$, let c_x be the function with domain κ such that for every $\alpha < \kappa$, $c_x(\alpha) = x$. It is easy to check that if we define $j_U = j : V \to M$ by $j(x) = [c_x]$, then j is an elementary embedding.

Lemma 6.2. *If U is σ-complete (i.e. closed under countable intersections), then M is well-founded.*

[11] Each $[f]$ is technically speaking a proper class, but it is possible to make the definition formally correct. We will omit it here.

Proof. Suppose for contradiction there is an finite $\in$-decreasing sequence of elements in M $\langle [f_n] \,|\, n < \omega \rangle$ with $[f_0] \ni [f_1] \ni \dots$. For each n let $X_n \in U$ be the set of all $\alpha < \kappa$ such that $f_{n+1}(\alpha) \in f_n(\alpha)$. By σ-completeness there is some $\alpha \in \bigcap_{n<\omega} X_n$, and so $f_0(\alpha) \ni f_1(\alpha) \ni \dots$, contradicting that $\in$ is well-founded in V. □

By Mostowski theorem, there is a "transitive collapse isomorphism" i from M onto a transitive proper class $\bar{M}$.[12] It is usual to identify M with $\bar{M}$ and "forget" about the isomorphism. For instance if f is a function from κ to ordinals, then by Los theorem

$$([f] \text{ is an ordinal})^M$$

though technically speaking it is not an ordinal in V; since i is an isomorphism, we have

$$(i([f]) \text{ is an ordinal})^{\bar{M}}$$

and $i([f])$ is really an ordinal in V because $\bar{M}$ is a transitive model of ZFC and ordinals are absolute for such models. But as we said above, it is customary to treat $[f]$ as an ordinal in V.

Lemma 6.3. *Suppose U a normal ultrafilter and let M denote the ultrapower via U.*

(i) For all α, $\alpha \le j(\alpha)$.
(ii) $j(\alpha) = \alpha$ for every $\alpha < \kappa$, but $\kappa \le [id] < j(\kappa)$. We say that κ is the critical point *of j.*
(iii) Let A be a subset of κ. Define $f : \kappa \to [\kappa]^{<\kappa}$ by $f(\alpha) = A \cap \alpha$. Then $A = [f]$.
(iv) M contains all subsets of κ, and hence all cardinals up to and including κ^+ are computed correctly by M.
(v) $[id] = \kappa$, and $\kappa^+ < j(\kappa) < (2^\kappa)^+$.

Proof. (i) The proof is by induction. Suppose the claim holds for every $\beta < \alpha$. Since α is an upper bound of $\{\beta \,|\, \beta < \alpha\}$, by elementarity $j(\alpha)$ is an upper bound of $\{j(\beta) \,|\, \beta < \alpha\}$; since by induction $\beta \le j(\beta)$, we have $j(\alpha) \ge \alpha$. Note that $j(\alpha)$ need not be equal to the supremum of $\{j(\beta) \,|\, \beta < \alpha\}$; see for instance claim (ii).

(ii). To show $j(\alpha) = \alpha$, it suffices by (i) to argue that $[f] < j(\alpha)$ implies $[f] < \alpha$. $[f] < j(\alpha) \leftrightarrow \{\xi < \kappa \,|\, f(\xi) < \alpha\} \in U$; by κ-completeness of U there must be some $\beta < \alpha$ with $\{\xi < \kappa \,|\, f(\xi) = \beta\} \in U$. For the second part, it suffices to notice that $\kappa \le [id] < j(\kappa)$ (κ is the supremum of all of $j(\alpha) = \alpha$ for $\alpha < \kappa$).

[12] The function i is defined by well-founded recursion setting $i([f]) = \{i([g]) \,|\, [g] \in [f]\}$.

(iii). By Los theorem, $[f]$ is a subset of $j(\kappa)$ of size $< j(\kappa)$. Recall that for every $\xi < \kappa$, $j(\xi) = \xi$. Suppose $\xi \in A$; then $\xi \in f(\alpha)$ for every $\alpha > \xi$, and so $\{\alpha < \kappa \mid \xi \in f(\alpha)\} \in U$, and hence $j(\xi) = \xi \in [f]$. Conversely, if $\xi \in [f]$, then there is some $\alpha < \kappa$ with $\xi \in f(\alpha) = \alpha \cap A$ and so $\xi \in A$.

(iv). By (iii), if $A \subseteq \kappa$, then $[f] = A$ for some f. Every well-order of size at most κ can be coded by a subset of κ, so M has the right information about ordinals of size at most κ, and so $\kappa^{+M} = \kappa^+$.

(v). We show that $[id]$ is the least upper bound of ordinals in κ; in other words if $[f] < [id]$, then $[f] = \alpha$ for some $\alpha < \kappa$. $[f] < [id] \leftrightarrow \{\xi < \kappa \mid f(\xi) < \xi\} \in U$. It follows as in Fodor's lemma (because U is a normal filter) that the regressive function f is constant on a set in U, and hence $[f] = \alpha$ for some $\alpha < \kappa$. □

Remark 6.4. The normality of U is essential for proving $[id] = \kappa$; if U is non-normal and κ-complete, then the ultrapower M is still well-founded, but $[id] > \kappa$ is possible.

6.1.2. Some consequences of measurability

A sort of converse to Lemma 6.3(iii) is of a separate interest, related to the notion of a *weakly compact* cardinal and an *ineffable* cardinal discussed in Sectin 6.2.

Lemma 6.5. *Suppose U is a normal ultrafilter on κ, and let $f = \langle S_\alpha \mid \alpha < \kappa\rangle$ is such that for all α, $S_\alpha \subseteq \alpha$ (i.e. f has the form of a diamond sequence). Then there is $S \subseteq \kappa$ such that*

$$X_S = \{\alpha < \kappa \mid S \cap \alpha = S_\alpha\}$$

is in U (and so in particular is stationary). We say that S threads f *on a* U-large set*, or that f* coheres on a U-large set.[13]

Proof. By Los theorem, $[f]$ is a subset of κ. Let us write $[f] = S$. By Lemma 6.3(ii), $j(S) \cap \kappa = S$, and so by Los theorem, using $[id] = \kappa$, this is equivalent to

$$X_S = \{\alpha < \kappa \mid S \cap \alpha = S_\alpha\} \in U.$$

□

We will see that if we relax the requirement and ask that for each $\langle S_\alpha \mid \alpha < \kappa\rangle$ coheres on a stationary set, and then further only on an unbounded set, we will get the notion of an ineffable, and a weakly compact cardinal, respectively (see Section 6.2).

[13]Sequences of the form like f are called *lists*, a generalization of a tree. With this terminology, if S threads f, we can also say that it is a cofinal branch in f.

Exercise. Convince yourself that if we strengthen the requirement and ask for coherence on a closed unbounded set, then such a principle is contradictory.

Remark 6.6. Note that if $\langle S_\alpha \,|\, \alpha < \kappa \rangle$ is a diamond sequence, then it coheres on a stationary set for all subsets of κ. The converse asks that *every* sequence of the form $\langle S_\alpha \,|\, \alpha < \kappa \rangle$ coheres on a large set: obviously, some sequences always cohere: for instance if $S \subseteq \kappa$, then $\langle S \cap \alpha \,|\, \alpha < \kappa \rangle$ coheres everywhere. It is interesting to learn (as we will prove later on), that the existence of an "ugly" sequence $\langle S_\alpha \,|\, \alpha < \kappa \rangle$ on an inaccessible κ which does not cohere on any unbounded set is equivalent to the fact that there is an "ugly" graph on κ without a clique or indenpendent set of size κ, or an "ugly" κ-tree without a cofinal branch.

In Theorem 6.7 we show that measurability of κ implies $\diamondsuit_\kappa$. In Theorem 6.25 we will show, by a more difficult argument, the same for an ineffable cardinal. It is a major open question, whether this can be further extended to a weakly compact cardinal.

Theorem 6.7. *If κ is measurable, then $\diamondsuit_\kappa$ holds.*

Proof. We will define by recursion a sequence $\langle S_\alpha \,|\, \alpha < \kappa \rangle$ and show that it is a diamond sequence. Let $<$ be some fixed well-order of $H(\kappa)$. Suppose the sequence was constructed for every $\beta < \alpha$. If there is some $S \subseteq \alpha$ such that $\langle \beta < \alpha \,|\, S \cap \beta = S_\beta \rangle$ is non-stationary, i.e. $\langle S_\beta \,|\, \beta < \alpha \rangle$ does not guess S stationarily often, let S_α be the $<$-least such S. If there is no such S, set $S_\alpha = \alpha$.

Let $j_U : V \to M$ be an ultrapower embedding via some normal ultrafilter U on κ. Assume for contradiction there is some $E \subseteq \kappa$ which is not guessed by $\langle S_\alpha \,|\, \alpha < \kappa \rangle$ stationarily often and let E be the $j(<)$-least such. By Lemma 6.3(iii), $E = [\langle E_\alpha \,|\, \alpha < \kappa \rangle]$ with $E_\alpha = E \cap \alpha$ for every α.

By elementarity $j(\langle S_\alpha \,|\, \alpha < \kappa \rangle)$ is a sequence of length $j(\kappa)$ which maps $\kappa = [id]$ to E. By Los theorem, this means

$$\{\alpha < \kappa \,|\, \langle S_\alpha \,|\, \alpha < \kappa \rangle(\alpha) = E_\alpha\} \in U \text{ iff } \{\alpha < \kappa \,|\, S_\alpha = E \cap \alpha\} \in U. \tag{6.4}$$

So E is guessed on a set in U, and since U extends the filter of closed unbounded subsets of κ, this set must be at least stationary. This a contradiction because we assumed that E is not guessed stationarily often. $\square$

Remark 6.8. Since κ is a regular cardinal in every ultrapower (equivalently, the set Reg of regular cardinals $< \kappa$ is in every ultrapower), we have actually showed $\diamondsuit_\kappa(\text{Reg})$. More generally, the same argument shows that $\diamondsuit_\kappa(S)$ holds for every stationary S such that there is a normal ultrafilter U with $S \in U$. It might be tempting to say that there may be a normal ultrafilter for every S, but this is not the case: the stationary set Sing of singular cardinals below κ is the complement of Reg and hence cannot be in any normal ultrafilter.

Note that $\diamondsuit_\kappa$ does not characterize any large cardinal notion: in L, $\diamondsuit_\kappa$ holds for every regular κ (even limit). But in L, $\diamondsuit_\kappa$ is a consequence of the global structure of L; in Theorem 6.7 we derived $\diamondsuit_\kappa$ from a local "large cardinal" property of κ.

Large cardinals tend to imply a lot of reflection for structures at κ. Let us prove some of the more known.

Theorem 6.9. *If* GCH *holds below* κ *and* κ *is measurable,* $2^\kappa = \kappa^+$ *(conversely: failure of* GCH *at* κ *reflects on a large set below* κ*).*

Proof. Let M be an ultrapower over some normal measure U. By our assumption, the set $X = \{\alpha < \kappa \,|\, 2^\alpha = \alpha^+\}$ is equal to whole of κ, and is therefore in U. By Los theorem and by $[id] = \kappa$, this implies $(2^\kappa = \kappa^+)^M$. We need to argue that this holds in V: but this follows from the fact that $\mathscr{P}(\kappa) \subseteq M$ and $(\kappa^+)^M = \kappa^+$ (Lemma 6.3), and so a bijection between $\mathscr{P}(\kappa)$ and κ^+ in M also witnesses $2^\kappa = \kappa^+$ in V. Note that it is enough to assume that there is some normal ultrafilter which contain X. □

Another form of reflection is the following:

Theorem 6.10. *If* κ *is a measurable cardinal, then every stationary* $S \subseteq \kappa$ *reflects,* $\mathsf{SR}(\kappa)$*.*

Proof. Let M be an ultrapower over some normal ultrafilter U. Let $S \subseteq \kappa$ be stationary. We wish to show there some $\alpha < \kappa$ of uncountable cofinality such that $S \cap \alpha$ is stationary in α. By elementarity $j(S)$ is a stationary subset of $j(\kappa)$, and $j(S) \cap \kappa = S$ is stationary in κ in V, and hence a fortiori in M. So we have (There is some α of uncountable cofinality such that $j(S)$ reflects at $\alpha)^M$. By elementarity of j this implies in V there is some α of uncountable cofinality such that S reflects at α. □

Theorem 6.11. *If* κ *is a measurable cardinal, then every* κ*-tree has a cofinal branch,* $\mathsf{TP}(\kappa)$*.*

Proof. Let M be an ultrapower over some normal ultrafilter U. Let T be a κ-tree, which we can assume to be a subset of V_κ (if not, take an isomorphic copy). Since all levels of T has size $< \kappa$, there are not moved by j, and so $j(T)$ restricted to nodes in V_κ is equal to T. By elementarity $j(T)$ has height $j(\kappa)$, so in particular there must be some node $t \in j(T)$ on level $\kappa < j(\kappa)$. Since $t \in M \subseteq V$, t is a cofinal branch in T. □

Remark 6.12. Notice that Theorem 6.9 requires that M contains all subsets of κ, while Theorem 6.10 and 6.11 only require that κ is the critical point of

j. This will lead us to weaker principles such as ineffable, weakly compact and Mahlo cardinal.

Measurability implies that the universe V is very "wide"; more precisely, it is inconsistent with the $V = L$. We will need the following fact related to L. Lest state a definition first:

Definition 6.13. A transitive proper-class model M of ZFC is called an *inner model.*

Notice that by definition an inner model is a subclass of V: $M \subseteq V$. This makes it different from an "outer model" such as a forcing extension $V[G]$ of V for which we have $V \subseteq V[G]$. By the following fact, it is consistent that L is the only inner model.

Fact 6.14. *L is the least inner model under $\subseteq$: if M is an inner model, then $L \subseteq M$. In particular, if $V = L$, then L is the only inner model.*

Proof. The main idea is to show that if M is an inner model, then L constructed inside of M, L^M, is again L. It follows $L = L^M \subseteq M$. □

Theorem 6.15. *If there is a measurable cardinal, then $V \neq L$.*

Proof. Suppose $V = L$ and let M be an ultrapower via some normal measure at κ, where κ is the least measurable cardinal. By the fact above, $M = L$ because $L \subseteq M$. By elementarity $j(\kappa)$ is the least cardinal in $M = L$. But also $\kappa < j(\kappa)$, a contradiction. □

Finally, let us show that the existence of any embedding from V to M gives us a measurable cardinal; in other words, the existence of elementary embeddings between transitive models of ZFC implies the existence of certain large cardinals.

Lemma 6.16. *Suppose $j : V \to M$ is an elementary embedding into some inner model M with critical point κ, then κ is a measurable cardinal.*

Proof. Define

$$U = \{X \subseteq \kappa \,|\, \kappa \in j(X)\}.$$

It is easy to check that U is a normal κ-complete ultrafilter on κ.

Note that is not important for the definition of U that the domain of j is the whole universe V: it suffices if the domain of j contains all subsets of κ. □

6.2. Weakly compact and ineffable cardinals

6.2.1. Partition relations

For many of the results proved above from the assumption of a measurable cardinal, weaker combinatorial concepts are sufficient. They have many equivalent definitions but let us start with the most widely known.

Recall the "arrow notation": we write

$$\kappa \to (\mu)^2_2$$

if every undirected graph on κ has either an independent set of size κ or a clique of size μ. More generally, we write

$$\kappa \to (\mu)^n_\gamma$$

for every function[14] $f : [\kappa]^n \to \gamma$ there exists a homogenous set H of size μ, i.e. a set $H \subseteq \kappa$ such that there is a single color $i < \gamma$ such that f gives to every n-element subset of H the same color i.

Remark 6.17. We can consider even more general partitions: for us, only the following are relevant: (i) $f : [\kappa]^\omega \to \gamma$, where $[\kappa]^\omega$ is the set of all countable subsets of κ; in this case f is homogenous on H if all elements of $[H]^\omega$ have the same color. (ii) $f : [\kappa]^{<\omega} \to \gamma$, where $[\kappa]^{<\omega}$ is the set of all finite subsets of κ; here H is homogenous if for every $n < \omega$, all n-element subsets of H have the same color (but the color for each n can be different).

Remark 6.18. It is customary to identify an n-element subset $\{x_1, \dots, x_n\}$ of κ with the n-tuple $(x_1, \dots, x_n)$ with the assumption $x_1 < x_2 < \cdots x_n$.

Recall that Ramsey's theorem states

$$\omega \to (\omega)^m_n, \text{ where } m, n \geq 1. \tag{6.5}$$

Let us first discuss the limits of possible generalizations of Ramsey's theorem (6.5).

Theorem 6.19. *The following hold:*

(i) (Erdös-Rado) $\omega \not\to (\omega)^{<\omega}_2$.
(ii) (Erdös-Rado) For every regular $\kappa \geq \omega$, $\kappa \not\to (\omega)^\omega_2$.
(iii) (Gödel, Erdös-Kukutani) For every regular $\kappa \geq \omega$, $2^\kappa \not\to (3)^2_\kappa$.
(iv) (Sierpinski, Kurepa) For every regular $\kappa \geq \omega$, $2^\kappa \not\to (\kappa^+)^2_2$.

The item (iii) in the previous theorem is the best possible: a theorem of Erdös and Rado shows that if increase 2^κ by one cardinal, we get a positive result:[15] for instance for every regular $\omega \leq \kappa$,

$$(2^\kappa)^+ \to (\kappa^+)^2_2.$$

Let us now sketch a proof of Theorem 6.19.[16]

[14]View f as a coloring which to every n-element subset of κ assigns a colour $\delta < \gamma$.

[15]For more details, see Kanamori's book [6], Theorem 7.3.

[16]For more details, see Kanamori's book [6], Proposition 7.1, Exercise 7.4 and Proposition 7.5.

Proof. (i). Let $f : [\omega]^{<\omega}$ be defined by setting $f(k_1, \dots, k_n) = 0$ iff $k_1 \le n$, and 1 otherwise. Suppose H is an infinite homogeneous set and let k be its least element. Then for every $n \ge k$, f restricted to $[H]^n$ must have value 0 because $f(k, k_2, \dots, k_n) = 0$, for any increasing $k < k_2 < \dots < k_n$; however, since H is infinite, there must be some $l > n$ in H such that $(l, l_2, \dots, l_n)$ is in $[H]^n$, and f assigns value 1 to this sequence. A contradiction.

(ii). Let $\prec$ be some well-order of $[\kappa]^\omega$. Set $f(x) = 0$ iff x is the $\prec$-least element in $[x]^\omega$, and 0 otherwise. Suppose H an infinite homogeneous set, and let $x \in [H]^\omega$ be the $\prec$-least element of $[H]^\omega$. Then $f(x) = 0$ by the definition of f: x is the $\prec$-least element of $[H]^\omega$, and hence also of $[x]^\omega$. By homogeneity, it follows that f gives to all elements of $[x]^\omega$ value 0. Let $x_0 \subsetneq x_1 \cdots \subsetneq x$ be a strictly increasing infinite chain of countable subsets of x. It is easy to check that this gives an infinite $\prec$-decreasing chain $\cdots x_2 \prec x_1 \prec x_0$, which is a contradiction because $\prec$ is a well-order.

(iii). We identify 2^κ with ${}^\kappa 2$. Let $\langle f_\alpha \mid \alpha < 2^\kappa \rangle$ some enumeration of ${}^\kappa 2$. Set $F(\alpha, \beta)$ to be the least $i < \kappa$ such that f_α and f_β are identical below i but $f_\alpha(i) \neq f_\beta(i)$. It is easy to check that F cannot have a homogeneous set of size more than 2. *Exercise.* Show that there is a function from two-elements subsets of real numbers in the interval $(0, 1)$ into ω such that every homogeneous set has size at most 10.

(iv). The proof proceeds by showing that ${}^\kappa 2$ ordered lexico-graphically by $<_{\mathrm{lex}}$ does not have a decreasing or increasing chain of order-type κ^+.[17] Let us give a hint to this argument: Suppose for contradiction that $\{f_\alpha \mid \alpha < \kappa^+\}$ is a strictly increasing chain (the argument is analogous for the decreasing chain), and let $\gamma \le \kappa$ be least such that $X = \{f_\alpha \restriction \gamma \mid \alpha < \kappa^+\}$ has size κ^+; by thinning out and reenumerating if necessary, we can assume that X has the property that if $\alpha \neq \beta < \kappa^+$, then $f_\alpha \restriction \gamma \neq f_\beta \restriction \gamma$. For every $\alpha < \kappa^+$, let $\xi_\alpha < \gamma$ be such that $f_\alpha \restriction \xi_\alpha = f_{\alpha+1} \restriction \xi_\alpha$ and $f_\alpha(\xi_\alpha) = 0$ and $f_{\alpha+1}(\xi_\alpha) = 1$. By the pigeon hole principle, there is some fixed $\xi < \gamma$ such that $\xi = \xi_\alpha$ for κ^+-many α (let us denote this set Y). Since γ was chosen as the least one and $\xi < \gamma$ there must be $\alpha_1 < \alpha_2$ in Y such that $f_{\alpha_1} \restriction \xi = f_{\alpha_2} \restriction \xi$: we have $f_{\alpha_1} <_{\mathrm{lex}} f_{\alpha_1+1} \le_{\mathrm{lex}} f_{\alpha_2}$, but also $f_{\alpha_1}(\xi) = 0$, $f_{\alpha_1+1}(\xi) = 1$, $f_{\alpha_2}(\xi) = 0$, which is a contradiction.

Then one concludes the argument by pointing out that if F is defined by setting $F(\alpha, \beta) = 0$ iff $f_\alpha <_{\mathrm{lex}} f_\beta$, then a homogeneous set of size κ^+ gives a decreasing or increasing chain of length κ^+. For more details see [6], Proposition 7.5. Note that this limiting result is strongest possible: Erdös proved in 1940s that $(2^\kappa)^+ \to (\kappa^+)^2_2$ always holds. □

[17] Recall that the usual linear order on $\mathbb{R}$ (which has size 2^ω) does not have a decreasing or increasing chain of order-type ω_1, while it has such a chain for every countable $\alpha < \omega_1$. The argument uses the separability of $\mathbb{R}$ (the existence of a countable dense set); the present result is a bit more general.

The above theorem does not a priori refute $\kappa \to (\kappa)^2_2$, set let us take it as the definition of a new concept which gives the Ramsey property – which holds on ω – to an uncountable cardinal κ.

Definition 6.20. We say that a cardinal $\kappa > \omega$ is *weakly compact* if

$$\kappa \to (\kappa)^2_2.$$

Lemma 6.21. *If κ is weakly compact, it must be inaccessible.*

Proof. First notice that κ must be regular: If not let, us write κ as a disjoint union $\kappa = \bigcup_{i<\gamma} X_i$ where $|X_i| < \kappa$ for each $i < \gamma$, where $\gamma < \kappa$. Define $F : [\kappa]^2 \to 2$ by setting $F(\alpha, \beta) = 0$ iff for some $i < \gamma$, $\alpha, \beta \in X_i$. It is easy to see that F cannot have a homogeneous set of size κ.

To show that κ is strong limit, we use Theorem 6.19(iv). Suppose for contradiction there is a cardinal $\mu < \kappa$ and $2^\mu \geq \kappa$. Then in $\kappa \to (\kappa)^2_2$ holds, so must $2^\mu \to (\kappa)^2_2$. Since $\kappa \geq \mu^+$, we get $2^\mu \to (\mu^+)^2_2$, a contradiction. $\square$

There is an obvious strengthening of the concept of weak compactness which postulates that there should be not only an unbounded homogeneous set, but stationary.

Definition 6.22. We say that a cardinal $\kappa > \omega$ is *ineffable* if for every function

$$F : [\kappa]^2 \to 2$$

there exists a homogeneous set H which is stationary in κ.

It is not obvious, but ineffable cardinals are strictly stronger than weakly compact (there are many weakly compact cardinals below the least ineffable). We will show below in Theorem 6.25 that they imply $\diamondsuit_\kappa$ like measurable cardinals. For weakly compact cardinals this is open.

6.2.2. Some consequences of weak compactness

We fill prove a part of the following theorem:

Theorem 6.23. *Suppose κ is a infinite cardinal. Then the following are equivalent.*

(i) $\kappa \to (\kappa)^2_2$.
(ii) *κ is inaccessible and there are no κ-Aronszajn tree (i.e. the tree property holds, $\mathsf{TP}(\kappa)$).*

Proof. ($(i) \to (ii)$). Inaccessibility of κ follows by Lemma 6.21. Let us now argue that the existence of a large homogeneous set for graphs on κ implies the existence of a large homogeneous set – i.e. a cofinal branch – for certain direct graphs on κ, namely κ-trees.

Let $(T, <_T)$ be a κ-tree; we can assume that $T = \kappa$ by taking an isomorphic copy in necessary. We extend $<_T$ into a linear order $\prec$ and apply the partition property to it. $\prec$ is a generalization of the Kleene-Brouwer ordering on finite sequences of natural numbers.

In preparation for the definition let us define the "projection" of a node ξ on level $\geq \alpha$ on the level α of the tree:

$$\pi_\alpha(\xi) \text{ is the unique node } \zeta \in T_\alpha \text{ with } \zeta \leq_T \xi,$$

where T_α is the α-th level of T. If $\alpha, \beta < \kappa$ are comparable, we set

$$\alpha <_T \beta \leftrightarrow \alpha \prec \beta.$$

If $\alpha, \beta < \kappa$ are incomparable in $<_T$, we set

$$\alpha \prec \beta \leftrightarrow \pi_\delta(\alpha) < \pi_\delta(\beta),$$

where δ is the least ordinal such that $\pi_\delta(\alpha) \neq \pi_\delta(\beta)$. It is easy to check that $\prec$ is a linear order which extends $<_T$; note for instance that the following hold

$$\alpha \prec \beta <_T \gamma \to \alpha \prec \gamma.$$

Define $F : [\kappa]^2 \to 2$ by setting $F(\alpha, \beta) = 0$ iff $\alpha \prec \beta$, and $F(\alpha, \beta) = 1$ iff $\beta \prec \alpha$. Let H be an F-homogeneous set of size κ.

Let $\alpha < \kappa$ be fixed. Since T is a κ-tree, there is some ρ_α such that $\xi > \rho_\alpha$ implies ξ is on level greater or equal to α. By definition of $\prec$, if $\xi, \zeta \geq \rho_\alpha$, and $\xi \prec \zeta$, then either $\pi_\alpha(\xi) \prec \pi_\alpha(\zeta)$ or $\pi_\alpha(\xi) = \pi_\alpha(\zeta)$. It follows that if $f''[H] = 0$, then $\{\pi_\alpha(\xi) \,|\, \rho_\alpha < \xi, \xi \in H\}$ is a non-$\prec$-decreasing set of nodes in T_α, and if $f''[H] = 1$, it is a non-$\prec$-increasing set of nodes in T_α. In either case, since $|T_\alpha| < \kappa$, there must be some σ_α and some node $b_\alpha \in T_\alpha$ such that

$$\text{for all } \xi > \sigma_\alpha, \text{ if } \xi \in H, \text{ then } \pi_\alpha(\xi) = b_\alpha.$$

Do the above construction for every α, obtaining a sequence of nodes $\langle b_\alpha \,|\, \alpha < \kappa\rangle$. We show that two nodes b_α, b_β have a common larger node in the $<_T$-ordering which makes them comparable, and hence $\langle b_\alpha \,|\, \alpha < \kappa\rangle$ is a cofinal branch. Fix b_α, b_β, with $\alpha < \beta$, and let $\sigma = \max(\sigma_\alpha, \sigma_\beta)$; then for every $\xi \in H$, $\xi > \sigma$, $b_\alpha = \pi_\alpha(\xi) <_T \xi$ and $b_\beta = \pi_\beta(\xi) <_T \xi$.

$((ii) \to (i))$. We will not prove this. If you are interested, see [6, Theorem 7.8]. □

6.2.3. Some consequences of ineffability

Recall the notion of "threading" in Lemma 6.5. The following lemma is a key to proving that ineffability implies $\diamondsuit_\kappa$.

Lemma 6.24. *Let $\kappa > \omega$ be regular. Then κ is ineffable iff whenever $\langle S_\alpha \,|\, \alpha < \kappa\rangle$ is such that $S_\alpha \subseteq \alpha$ for all $\alpha < \kappa$, there is a $S \subseteq \kappa$ such that $\{\alpha < \kappa \,|\, S \cap \alpha = S_\alpha\}$ is stationary.*

Proof. $(\leftarrow)$. Let $F : [\kappa]^2 \to 2$ be given. For each α, let us define $S_\alpha : \alpha \to 2$ by setting $S_\alpha(\nu) = F(\nu, \alpha)$, for $\nu < \alpha$. By our assumption there is a stationary set S such that for each $\alpha \in S$, $S \restriction \alpha = S_\alpha$ (identifying subsets of α with their characteristic functions). S is regressive for arguments ≥ 2, so by Fodor's lemma there is a stationary set $S^* \subseteq S$ such that for some fixed $i < 2$, $S(\nu) = i$ for every $\nu \in S^*$. It follows that S^* is homogeneous for F: for $\nu < \alpha$ in S^*, $F(\nu, \alpha) = S_\alpha(\nu) = S \restriction \alpha(\nu) = S(\nu) = i$.

$(\rightarrow)$. (This is optional). See Devlin's book [3, Theorem 2.1, p. 313]. □

Theorem 6.25. *Suppose κ is ineffable. Then $\Diamond_\kappa$ holds.*

We will not prove this. If you are interested, see [3, Theorem 2.4, p. 315].

7. Large cardinals – in disguise

7.1. Stationary reflection

Suppose $\kappa \geq \omega_2$ is a regular cardinal.

Definition 7.1. We say that $S \subseteq \kappa$ *reflects* if there is some $\alpha < \kappa$ of uncountable cofinality such that $S \cap \alpha$ is stationary in α. We call α a *reflection point* (of S).

Let us fix some notation. Suppose S is stationary, then

$$r(S) = \{\alpha \mid \mathrm{cf}(\alpha) > \omega, S \cap \alpha \text{ is stationary}\},$$

and

$$nr(S) = S \setminus r(S).$$

Notice that we do not necessarily have $r(S) \subseteq S$, while $nr(S) \subseteq S$ is true by definition.

Let us state some easy observations related to this concept:

Lemma 7.2. *Suppose $\kappa \geq \omega_2$ is a regular cardinal.*

(i) If $\kappa = \mu^+$, then E^κ_μ does not reflect.
(ii) If S is stationary, then $nr(S)$ is stationary.
(iii) If every stationary subset of κ reflects, then for every stationary S, $r(S)$ is stationary.

Proof. (i) Let us denote $E = E^\kappa_\mu$. If $\alpha < \kappa$ has uncountable cofinality δ, then $\delta < \kappa$, and there exists a closed unbounded set C in α of order-type δ such that for every α in C we have

$$\mathrm{cf}(\alpha) < \delta \leq \mu < \kappa.$$

It follows that $E \cap C \cap \alpha$ is empty. Since α was arbitrary, it follows E does not reflect.

(ii) Let C be a closed unbounded set. We wish to show that $nr(S) \cap C$ is non-empty. Let α be the least element of $\mathrm{Lim}(C) \cap S$, where $\mathrm{Lim}(C)$ is the set of limit points of C ($\mathrm{Lim}(C) \cap S$ is non-empty because S is stationary). If $\mathrm{cf}(\alpha) = \omega$, then $\alpha \in nr(S)$ because $r(S)$ contains only ordinal with uncountable cofinality. So suppose $\mathrm{cf}(\alpha) > \omega$. Then $\mathrm{Lim}(C) \cap \alpha$ is a closed unbounded set disjoint from $S \cap \alpha$ because α is the least element of $\mathrm{Lim}(C) \cap S$. It follows $S \cap \alpha$ is not stationary, and hence $\alpha \in nr(S)$.

(iii). Let $C \subseteq \kappa$ be a closed unbounded set. We wish to show that $r(S) \cap C$ is non-empty. Since $S \cap C$ is stationary, the reflection applied to $S \cap C$ gives the desired claim. Note that without the blanket assumption that every stationary set reflects, it is possible that $r(S)$ contains just one element: if S is a non-reflecting stationary set and $S^* = S \cup \omega_1$, then $r(S^*) = \{\omega_1\}$. □

Is it possible that every stationary subset of a regular limit cardinal κ reflects? Or, is it possible that every stationary subset of $E^{\omega_2}_{\omega}$ reflects?[18]

The following lemmas says that under certain assumptions, there are non-reflecting stationary subsets of $E^{\omega_2}_{\omega}$; see Definition 2.13 for the square sequence.

Lemma 7.3. *Assume $\square_{\omega_1}$ holds. Then there is a non-reflecting stationary subset $E \subseteq E^{\omega_2}_{\omega}$.*

Proof. Let $\langle D_\alpha \,|\, \alpha \in \mathrm{Lim}(\omega_2)\rangle$ witnesses $\square_{\omega_1}$. We define a new sequence $\langle C_\alpha \,|\, \alpha \in \mathrm{Lim}(\omega_2)\rangle$ which witnesses $\square_{\omega_1}(E)$ for some $E \subseteq E^{\omega_2}_{\omega}$. This implies that E does not reflect: if $\alpha < \omega_2$ has uncountable cofinality, then $\mathrm{Lim}(C_\alpha) \cap E$ is empty by (iii) from Definition 2.13.

First note that by the condition $\mathrm{cf}(\alpha) < \omega_1 \to \mathrm{ot}(D_\alpha) < \omega_1$, we have

$$E^{\omega_2}_{\omega} = \bigcup_{\delta < \omega_1} E_\delta,$$

where for each $\delta < \omega_1$:

$$E_\delta = \{\alpha \in E^{\omega_2}_{\omega} \,|\, \mathrm{ot}(D_\alpha) = \delta\}.$$

By ω_2-completeness of the non-stationary ideal and the fact that $E^{\omega_2}_{\omega}$ is stationary, there is some $\delta < \omega_1$ such that $E_\delta = E$ is stationary.

The following holds:

(*) For every $\alpha < \omega_2$ with uncountable cofinality, $|\mathrm{Lim}(D_\alpha) \cap E| \leq 1$.

Assume for contradiction there are two (limit) ordinals $\xi < \xi' < \alpha$ in $\mathrm{Lim}(D_\alpha) \cap E$. Then $D_\xi = D_\alpha \cap \xi$ and $D_{\xi'} = D_\alpha \cap \xi'$, so the order-type of D_ξ must be smaller than the order-type of $D_{\xi'}$, but this contradicts the assumption that they both should have order-type δ.

[18] We know from the previous lemma that $E^{\omega_2}_{\omega_1}$ cannot reflect so we need to limit ourselves to $E^{\omega_2}_{\omega}$.

Let us now define a new sequence $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\omega_2)\rangle$ as follows: if $\mathrm{ot}(D_\alpha) \leq \delta$, set $C_\alpha = D_\alpha$, and if $\mathrm{ot}(D_\alpha) > \delta$, set $C_\alpha = \{\gamma \in D_\alpha \mid \mathrm{ot}(D_\alpha \cap \gamma) > \delta\}$. Check that the sequence $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\omega_2)\rangle$ witnesses $\square_{\omega_1}(E)$. $\square$

We can ask whether the assumption of $\square_{\omega_1}$ was essential for the proof of the existence of a non-reflecting stationary subset of $E^{\omega_2}_\omega$. The answer is more complicated than it seems: the fact that all subset of $E^{\omega_2}_\omega$ reflect is consistent with ZFC if the existence of certain large cardinals (more precisely of a Mahlo cardinal) is consistent. And conversely, the consistency of a Mahlo cardinal implies that it is consistent that all stationary subsets of $E^{\omega_2}_\omega$ reflect. Let φ denote the statement "All stationary subsets of $E^{\omega_2}_\omega$ reflect", then we have:

$$\mathrm{Con}(\mathsf{ZFC} + \varphi) \leftrightarrow \mathrm{Con}(\mathsf{ZFC} + \text{"There is a Mahlo cardinal"}).$$

We say that φ and "There is a Mahlo cardinal" are *equiconsistent*.

We will sketch a proof of one direction using a measurable cardinal.

Theorem 7.4. *If κ is a measurable cardinal, then there is generic extension $V[G]$ in which $\kappa = (\omega_2)^{V[G]}$ and all stationary subsets of $E^{\omega_2}_\omega$ in $V[G]$ reflect.*

Proof. (Sketch). Let $\mathbb{P}$ denote the Levy collapse $\mathrm{Col}(\omega_1, <\kappa)$[19] and let G be $\mathbb{P}$-generic filter over V. Then by standard forcing arguments $(\omega_1)^{V[G]} = \omega_1^V$ and $(\omega_2)^{V[G]} = \kappa$.

Let $j : V \to M$ be an ultrapower elementary embedding with critical point κ. $\mathbb{P}$ is an element of M and $j(\mathbb{P})$ is by elementarity equal to $\mathrm{Col}(\omega_1, < j(\kappa))$ and the restriction of $\mathrm{Col}(\omega_1, < j(\kappa))$ to κ is equal to $\mathbb{P}$.

By standard arguments it follows that j lifts[20] in $V[G][H]$ to an elementary embedding j^*:

$$j^* : V[G] \to M[G][H],$$

where $G * H$ is $j(\mathbb{P})$-generic over V.

The rest of the argument is a more complicated version of the proof for Theorem 6.10 (compare). If S is a stationary subset of $E^{\omega_2}_\omega$, then $j(S)$ is a stationary subset of $E^{j(\kappa)}_\omega$ in $M[G][H]$. By the construction $j(S) \cap \kappa = S$ is a stationary subset of E^κ_ω in $V[G]$ and hence in $M[G]$. But we need to prove more:

$$(*)\ S \text{ is stationary in } M[G][H].$$

This is done by arguing that an ω_1-closed forcing (like our $\mathbb{P}$) preserves stationarity of S: stationarity of S is preserved from the model $M[G]$ to the larger model $M[G][H]$, and this finishes the proof. $\square$

[19] Conditions are countable functions f from $\omega_1 \times \kappa$ to κ with $f(\alpha, \beta) < \beta$.

[20] This means that $j^* \restriction V = j$.

7.2. The tree property

Recall that we proved in Theorem 6.23 that $\kappa \to (\kappa)^2_2$ implies that κ is inaccessible and there are no κ-Aronszajn trees (κ satisfies the "tree property", $\mathsf{TP}(\kappa)$). As a fact we mentioned that the converse is true as well. This leaves the intriguing possibility that $\mathsf{TP}(\kappa)$ can hold even for a successor cardinal. In this section, in Theorem 7.6 we sketch a proof that this is indeed the case: If there is a weakly compact cardinal, then there is a generic extension where $\mathsf{TP}(\omega_2)$ holds.

Let us first start with an informative observation related to the existence of Aronszajn trees. It implies that being a κ-tree with levels of size $< \kappa$ is the only interesting case as far as existence or non-existence of Aronszajn trees on κ is concerned: if T of height κ is allowed to have levels of size κ, then it is easy to construct in ZFC such a tree without a cofinal branch; conversely, as Theorem 7.5 says, if the levels of T have size $< \lambda$ for some cardinal $\lambda < \kappa$, then T always has a cofinal branch.

Theorem 7.5. *Suppose κ is regular and $\lambda < \kappa$ is an infinite cardinal. Suppose T is a κ-tree such that every level of T has size $< \lambda$. Then T has a cofinal branch.*

Proof. Let $(T, <_T)$ be a κ-tree; we can assume that $T = \kappa$ by taking an isomorphic copy in necessary. We assume that T is normal in the sense of Definition 3.10 (only item (vi) is really important). Recall that T_α denotes the collection of nodes on level α of the tree.

Assume first that λ is a regular cardinal. Let us choose arbitrarily $\xi_\alpha \in T_\alpha$ for each $\alpha < \kappa$ with cofinality λ. Suppose $\xi \in T_\alpha$ and $\xi \neq \xi_\alpha$. Let $\sigma(\xi)$ be the least ordinal $\zeta < \alpha$ such that $\pi_\zeta(\xi) \neq \pi_\zeta(\xi_\alpha)$.[21] Let $\rho_\alpha < \alpha$ be above all the $\sigma(\xi)$, $\xi \neq x_\alpha$, $\xi \in T_\alpha$ (such a ρ_α exists because the cofinality of α is λ, and $|T_\alpha| < \lambda$). Then for every two nodes $\xi \neq \xi_\alpha$ in T_α, $\pi_{\rho_\alpha}(\xi) \neq \pi_{\rho_\alpha}(\xi_\alpha)$. The function ρ is regressive, and so by Fodor's lemma there is a $\gamma < \kappa$ and a stationary set S composed of ordinals of cofinality λ such that $\rho_\alpha = \gamma$ for every $\alpha \in S$. For every $\alpha \in S$, let $\gamma_\alpha = \pi_\gamma(\xi_\alpha)$. Since $|T_\alpha| < \lambda < \kappa$, there is a stationary $S^* \subseteq S$ such that for all $\alpha \in S^*$, $\gamma_\alpha = x$ for some fixed $x \in T_\gamma$. It follows that $\langle \xi_\alpha \,|\, \gamma < \alpha, \alpha \in S^* \rangle$ determines a cofinal branch in T: for $\alpha < \beta$ in S^*, if ξ_α and ξ_β were incomparable, then $\pi_\alpha(x_\beta) \neq x_\alpha$, and by the construction (because $\gamma < \alpha < \beta$), $\pi_\gamma(\pi_\alpha(x_\beta)) \neq \pi_\gamma(x_\alpha)$; but this contradicts the fact that they both should be equal to x.

As an exercise, argue that the regular case can be used to prove the theorem also for a singular λ. $\square$

[21] Recall that $\pi_\zeta(\xi)$ the unique node on level $\zeta <_T$ below ξ.

We will not prove the following theorem, but keep in mind that it says that some "trace" of weak compactness can consistently hold even at such small cardinals as ω_2.[22]

Theorem 7.6 (Mitchell)**.** *If there is a weakly compact cardinal, then there is a generic extension where* $\mathsf{TP}(\omega_2)$ *holds.*

8. Forcing and forcing axioms

8.1. Preservation of stationary sets and Aronszajn trees

[This section requires basic understanding of forcing.]

We have discussed the compactness principles $\mathsf{SR}(\kappa)$ (stationary reflection) and $\mathsf{TP}(\kappa)$ (the tree property); see Section 7.

Let us now state some lemmas which are used in proofs for results like Theorems 7.4 and 7.6.

Recall that if $\mathbb{P} = (\mathbb{P}, \leq)$ is a partially ordered set with the greatest element 1; then we say that $p, q \in \mathbb{P}$ are *compatible*, and write $p \,||\, q$, if there is $r \in \mathbb{P}$ with $r \leq p, q$. We say that p, q are *incompatible* if there are not compatible. We say that $A \subseteq \mathbb{P}$ is an *antichain* if all $p \neq q \in A$ are incompatible. For future use, let us also define that $D \subseteq P$ is *dense* if for every p there is some $q \leq p$ in D; D is *dense below* p if for every $p' \leq p$ there is some $q \leq p'$ in D.

Definition 8.1. We say that $\mathbb{P}$ is ccc (countable chain condition) if every antichain in $\mathbb{P}$ is at most countable. We say that $\mathbb{P}$ is σ-closed (or ω_1-closed) if whenever $\langle p_i \,|\, i < \alpha \rangle$, $\alpha < \omega_1$ is a decreasing sequence in $\leq$ in $\mathbb{P}$, then there is some p such that $p \leq p_i$ for all $i < \alpha$.

Typical examples are these:

Definition 8.2. $\mathrm{Add}(\omega, \alpha)$, $0 < \alpha$, is a set of all functions p such that $\mathrm{dom}(p) \subseteq \alpha \times \omega$, $|\mathrm{dom}(p)| < \omega$, and $\mathrm{rng}(p) \subseteq \{0, 1\}$. $\mathrm{Add}(\omega, \alpha)$ is called the Cohen forcing (at ω). It adds α-many new subsets of ω. Δ-lemma implies that $\mathrm{Add}(\omega, \alpha)$ is ccc for every α.

Definition 8.3. $\mathrm{Coll}(\omega_1, \alpha)$, $\omega_1 \leq \alpha$, is a set of all functions p such that $\mathrm{dom}(p) \subseteq \omega_1$, $|\mathrm{dom}(p)| \leq \omega$, and $\mathrm{rng}(p) \subseteq \alpha$. $\mathrm{Coll}(\omega_1, \alpha)$ is called a collapsing forcing. It adds a surjection from ω onto α. It is easy to check it is σ-closed.

Definition 8.4. Suppose $\mathbb{P}$ is a partially ordered set. Then the product $\mathbb{P} \times \mathbb{P}$ is defined as the partially ordered set whose domain are pairs (p, q), $p, q \in \mathbb{P}$, with the ordering defined by coordinates.

[22]Also recall that ZFC proves $\neg\mathsf{TP}(\omega_1)$, so ω_1 is provably "incompact" in this sense.

Note that $\mathrm{Add}(\omega, \alpha) \times \mathrm{Add}(\omega, \alpha)$ is isomorphic to $\mathrm{Add}(\omega, \alpha)$ so the product is ccc. If $(T, \leq_T)$ is a Suslin tree, the the partially ordered set $(T, \geq_T)$ (notice the reversed ordering) is ccc (by the definition of Susliness), but $T \times T$ is not ccc (exercise).[23]

Exercise. Show that $\mathrm{Add}(\omega, \alpha)$ is not σ-closed, $\mathrm{Coll}(\omega_1, \alpha)$ is not ccc, and a Suslin tree $(T, \geq_T)$ is not σ-closed.

Lemma 8.5. *Let $\mathbb{P}$ be ccc. Then:*

(i) $\mathbb{P}$ preserves stationary subsets of ω_1.

(ii) If $\mathbb{P} \times \mathbb{P}$ is ccc, then $\mathbb{P}$ preserves all ω_1-Aronszajn trees. More strongly, it does not add new cofinal branches to ω_1-trees.

Proof. (i). Suppose S is a stationary set. We show that if p forces that $\dot{C}$ is a club in ω_1, then p forces that $S \cap \dot{C}$ is non-empty. The key observation is that there exists a club D such that

$$p \Vdash \check{D} \subseteq \dot{C}.$$

With this observation, the proof is finished easily: since S is stationary, $S \cap D$ is non-empty; if $\xi \in \check{D} \cap \check{S}$, then $p \Vdash \xi \in \dot{C} \cap \check{S}$.

To see that the observation holds, note that using ccc of $\mathbb{P}$ one can build by induction a countable sequence $\langle X_n \,|\, n < \omega \rangle$ such that (i) X_n is an at most countable set of ordinals below ω_1, (ii) for every $n < \omega$, $p \Vdash \dot{C} \cap \check{X}_n \neq \emptyset$,[24] and (iii) the supremum of each X_n is strictly smaller than the least element of X_{n+1}. Since $\dot{C}$ is forced by p to be a club, p must force that the supremum of $\bigcup X_n$ is in $\dot{C}$. By repeating the argument, we get in V a club D whose elements are forced by p into $\dot{C}$.

(ii). (Sketch) Suppose for contradiction that $\dot{b}$ is forced by p to be a new cofinal branch. This means that for every $\alpha < \omega_1$ there are incompatible extensions p_1, p_2 of p such that p_1 forces $\dot{b} \restriction \alpha = b_1$ and p_2 forces $\dot{b} \restriction \alpha = b_2$, and $b_1 \neq b_2$. Then using the ideas in Footnote 23, one can construct an uncountable antichain in $\mathbb{P} \times \mathbb{P}$. □

Note that (ii) of the previous lemma cannot be strengthened to just $\mathbb{P}$ being ccc: if $\mathbb{P}$ is a Suslin tree $(T, \geq_T)$, then forcing with T adds a new cofinal branch to T.

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[23] Hint. Let X be a collection of nodes in T such that (i) if $t \in X$, then t has two immediate successors t_1 and t_2 on the next level of T, (ii) on every level of T there is at most one $t \in X$ and if t, t' are in X, then the levels of t, t' are sufficiently far apart (it is enough if the difference is at least 2) and (iii) the heights of $t \in X$ are cofinal in ω_1. Argue that $A = \{(t_1, t_2) \,|\, t \in X\}$ is an uncountable antichain in $T \times T$: If (t_1, t_2) and (x_1, x_2) are in A, then $t_1 < x_1$ implies t_2 is incompatible with x_2.

[24] For instance as follows: for every $\xi < \omega_1$, $p \Vdash (\exists \alpha) f(\check{\xi}) = \alpha$. By ccc, there is an at most countable set X such that $p \Vdash f(\xi) \in \check{X}$.

Lemma 8.6. *Let $\mathbb{P}$ be σ-closed. Then:*

(i) $\mathbb{P}$ preserves stationary subsets of ω_1.

(ii) $\mathbb{P}$ preserves all ω_1-Aronszajn trees. More strongly, it does not add new cofinal branches to ω_1-trees.

Proof. (i). Suppose S is a stationary set and p forces that $\dot{C}$ is a club. For each $p' \le p$, we find $q \le p'$ which forces $\dot{C} \cap \check{S} \neq \emptyset$. This means that conditions forcing that $\dot{C} \cap \check{S} \neq \emptyset$ are dense below p and this suffices.

By induction on ω_1, construct a decreasing sequence of elements $\langle q_\alpha \,|\, \alpha < \omega_1 \rangle$ below p' and a strictly increasing continuous sequence $\langle \xi_\alpha \,|\, \alpha < \omega_1 \rangle$ of ordinals below ω_1 such that q_α forces that ξ_α is in $\dot{C}$. The set $D = \{\xi_\alpha \,|\, \alpha < \omega_1\}$ is a club, and hence there is some α such that $\xi_\alpha \in S \cap D$. It follows $p_\alpha \Vdash \xi_\alpha \in \dot{C} \cap \check{S}$.

(ii) (Sketch) Suppose p forces that $\dot{b}$ is a new cofinal branch. By induction on $2^{<\omega}$ (finite sequences of zeros and ones) construct a tree of conditions $\langle p_s \,|\, s \in 2^{<\omega} \rangle$ below p such that for each s, $p_{s^\frown\langle 0 \rangle} \le p_s$ and $p_{s^\frown\langle 1 \rangle} \le p_s$ are incompatible conditions which force $\dot{b} \restriction \alpha = \check{b}_1 \neq \check{b}_2 = \dot{b} \restriction \alpha$ for some well-chosen α. For $x \in 2^\omega$, let p_x be a lower bound of $\langle p_{x \restriction n} \,|\, n < \omega \rangle$. It can be arranged that there is a level $\gamma < \omega_1$ of the tree such that for every pair $x \neq y$ of elements of 2^ω, p_x and p_y decide $\dot{b} \restriction \gamma + 1$ differently, which implies that the γ-th level of the tree has at least $2^\omega > \omega$ many nodes, a contradiction. □

8.2. Martin's Axiom

One of the (few) drawbacks of the forcing method for a typical mathematician is that it requires deeper knowledge of set-theoretical methods (compare with the previous section). Forcing axioms are a way of applying forcing to classical mathematical problems without requiring too much of set theory: it is enough to define an appropriate partially ordered set $(\mathbb{P}, \le)$ and then show by a combinatorial argument that $\mathbb{P}$ is ccc to get the required result.

Recall the notions of antichains and dense sets defined at the beginning of Section 8.1. Let us further define that $G \subseteq P$ is a filter if G contains the greatest element of P (we also assume it has one), for every $p, q \in G$ there is some $r \in \mathbb{P}$ with $r \le p, q$, and if $p \in G$ and $p \le q$, then $q \in G$.

Let us define the most widely known forcing axiom:

Definition 8.7 (Martin's axiom, MA_{ω_1})**.** Whenever $\mathbb{P}$ is ccc and $\mathcal{D}$ is a collection of ω_1-many dense sets in $\mathbb{P}$, then for every p there is a filter G containing p which intersects every element of $\mathcal{D}$.

Recall that if $\mathcal{D}$ has size ω, then the respective principle is provable:

Lemma 8.8 (Rasiowa-Sikorski)**.** *Suppose $\mathbb{P}$ is a partially ordered set and $\mathcal{D}$ is a countable collection of dense sets. Then for every p there is a filter G such that $p \in G$ and G meets every element of $\mathcal{D}$.*

Proof. Construct by induction a decreasing sequence of elements in $\mathbb{P}$, $\langle p_n \mid n < \omega\rangle$ with $p_0 = p$ and $p_{n+1} \in D_n$. Then define

$$G = \{q \in \mathbb{P} \mid \exists n < \omega, p_n \leq q\}.$$

□

Remark 8.9. MA_{ω_1} is not provable in ZFC, but by using a forcing argument, it holds that if ZFC is consistent, then so is $\mathsf{ZFC} + \mathsf{MA}_{\omega_1}$.

Let us show some consequences of MA_{ω_1} to illustrate its use:

Theorem 8.10. $\mathsf{ZFC} + \mathsf{MA}_{\omega_1}$ *proves* $\neg\mathsf{CH}$.

Proof. Suppose for contradiction that $2^\omega = \omega_1$, and let $\langle x_\alpha \mid \alpha < \omega_1\rangle$ enumerate all subsets of ω. Recall that the Cohen forcing $\mathbb{C} = \mathrm{Add}(\omega, 1)$ is ccc.[25] Define dense sets D_α for $\alpha < \omega_1$ and D_m for $m < \omega$:

$$D_\alpha = \{p \in \mathbb{C} \mid \exists n < \omega, p(n) \neq x_\alpha(n)\}, D_m = \{p \in \mathbb{C} \mid m \subseteq \mathrm{dom}(p)\}.$$

Let G be a filter meeting every D_α and D_m. Let x be the union of conditions in G. It is a function (because G is a filter) from ω into 2 (because G meets every D_m). It further follows $x \neq x_\alpha$ for every $\alpha < \omega_1$ because for every α there is some n the domain of x with $x(n) \neq x_\alpha(n)$ (because G meets every D_α). This contradicts the fact that $\langle x_\alpha \mid \alpha < \omega_1\rangle$ enumerates all subsets of ω. □

Recall that SH denotes the *Suslin hypothesis* which states that there are no ω_1-Suslin trees.

Theorem 8.11. $\mathsf{ZFC} + \mathsf{MA}_{\omega_1}$ *proves* SH.

Proof. Suppose for contradiction that $(T, \leq_T)$ is a Suslin tree. Then $(T, \geq_T)$ is a ccc partial order. Define dense sets D_α for $\alpha < \omega_1$,

$$D_\alpha = \{t \in T \mid \mathrm{ht}(t, T) \geq \alpha\}.$$

Let G be a filter meeting every D_α. Then G determines a cofinal branch through T, a contradiction (recall that being Suslin implies being Aronszajn).

□

[25]For $\mathrm{Add}(\omega, 1)$ this follows without a reference to the Δ-system lemma because $|\mathrm{Add}(\omega, 1)| = \omega$.

8.2.1. Whitehead's problem

Recall the following theorem which we used to motivate some of the notions discussed in these lectures:

Theorem 8.12 (Shelah [8]). *(i) If $\diamondsuit_{\omega_1}(S)$ holds for all stationary $S \subseteq \omega_1$ (a consequence of $V = L$), then every Whitehead group of size ω_1 is free.*
(ii) $\mathsf{ZFC} + \mathsf{MA}_{\omega_1}$ implies there is a non-free Whitehead group of size ω_1.

With regard to (i), let us just say that with $\diamondsuit_{\omega_1}(S)$ (in fact a weaker principle suffices) one can imitate the proof of the known fact that every countable Whitehead group is free. We will not discuss (i) here in detail, the reader may read [4].

The theorem in (ii) shows that some additional combinatorial assumptions above ZFC are necessary to prove the result in (i).[26]

Let us briefly review the underlying group-theoretic assumptions (let us write W-group for "Whitehead group") to motivate the definition of the forcing notion to use with MA_{ω_1}.

- Recall that a free group can be characterized as follows: A is free if for every group B and every homomorphism $\pi : B \to A$ onto A there is an (injective) homomorphism $\varphi : A \to B$ such that $\pi(\varphi(a)) = a$ for all $a \in A$. (The existence of φ follows from the fact that A has a basis).
- We say that a group A is a W-group if for every group B and every homomorphism $\pi : B \to A$ onto A with kernel $\mathbb{Z}$, there exists a homomorphism $\varphi : A \to B$ such that $\pi(\varphi(a)) = a$ for all $a \in A$.
- It was known that every countable W-group is free.
- We say that a group A is ω_1*-free* if every countable subgroup of A is free.
- It was known that every W-group of size ω_1 is ω_1-free.
- Before Shelah's result it was also known that there are groups A such that:
  - (i) $|A| = \omega_1$,
  - (ii) A is not free,
  - (iii) A is ω_1-free,[27]
  - (iv) every countable subgroup of A is included in a countable subgroup B of A such the quotient A/B is ω_1-free.

[26] Shelah later proved in [9, 10] that CH is *not* sufficient for the result in (i): this is analogous to the existence of an ω_1-Suslin tree which is implied by $\diamondsuit_{\omega_1}(\omega_1)$ but not by CH alone.

[27] Hence ω_1-free does not imply free; which makes the property of being "free" non-compact in the sense of the compactness properties discussed in this lecture.

Suppose A is a group satisfying the properties (i)–(iv) from the last bullet. It is good to see A as a potential counterexample to the Whitehead's conjecture – i.e. while not free, it may still be Whitehead: we now define a partial order which will make it Whitehead with respect to a fixed B. Let B be an arbitrary group and $\pi : B \to A$ a homomorphism onto A. Let $\mathbb{P}(B, A)$ be a partial order such that if G is a filter ensured by MA_{ω_1} which meets a certain family of ω_1 many dense sets in $\mathbb{P}$, then $\bigcup G = \varphi$ is a homomorphism from A into B with $\pi(\varphi(a)) = a$.

Definition 8.13. $\mathbb{P}(B, A)$ is a collection of all φ such that there exists some finitely generated pure subgroup[28] S of A and φ is a homomorphism from S into B with $\pi(\varphi(a)) = a$ for all $a \in S$. The ordering on $\mathbb{P}$ is by reverse inclusion.

To apply MA_{ω_1} one needs to verify:

(1) For $a \in A$, let us write $D_a = \{\varphi \in \mathbb{P} \,|\, a \in \mathrm{dom}(\varphi)\}$. One needs to verify that each D_a is dense in $\mathbb{P}$.
(2) $\mathbb{P}$ is ccc.

The second condition is hard and captures the combinatorial core of the argument. First notice that the Δ-lemma argument would imply $\mathbb{P}$ is ccc if each φ were *finite*. But $\varphi \in \mathbb{P}$ are countable so the Δ-lemma alone will not help. We are saved by the fact that S in the definition of $\mathbb{P}$ is *finitely-generated* which together with (iv) can be used to argue for ccc.

Remark 8.14. Without MA_{ω_1}, the same argument would need go by defining a forcing iteration of length ω_2 with finite support such that at each stage $\alpha < \omega_2$ one would force with a partial order as in Definition 8.13 for some A, B (some bookkeeping function would make sure that every potential counterexample would eventually appear in our iteration). One can appreciate to what extent MA_{ω_1} makes the argument more accessible for a non-set-theoretician.

Remark 8.15. The reader may speculate which of the two additional axioms – $(\forall S)\Diamond_{\omega_1}(S)$ or MA_{ω_1} – are more "natural" or "in keeping with mathematical intuition", based on the implications they have for the Whitehead's problem – a problem which emerged in mainstream mathematics, and not in set theory:

$(\forall S)\Diamond_{\omega_1}(S)$ implies that the result which true for countable groups just from ZFC (that every countable W-group is free) extends to ω_1. This may sound convincing, until one realizes that ω is an "inaccessible" cardinal (regular and strong limit), and ω_1 is a successor cardinal, so some differences might be in fact natural (compare with the fact that every ω-tree has a cofinal branch, but there is in ZFC an ω_1-tree without a cofinal branch). Perhaps ZFC is too

[28]In our context, it means that A/S is torsion free.

weak to accentuate the difference between ω and ω_1 in all contexts: under this interpretation, MA_{ω_1} can be identified as a natural principle which lends set theory the power to make visible the distinctions which may be under the resolution ZFC.

8.3. Proper forcing axiom

The consequences of MA_{ω_1} are limited by the fact that it only applies to partial orders which are ccc. It took some time to find a good generalization of MA_{ω_1}; one of the reasons was that it was hard to find a good analogue of the following two properties of ccc forcing notions:

(A) Any forcing iteration with finite support of arbitrary length such that each iterand of the iteration is forced to be ccc is ccc.

(B) MA_{ω_1} is equivalent to its restriction to ccc partial orders *of size* $\le \omega_1$.

The combined consequence of (A) and (B) is that to get a model with MA_{ω_1}, it suffices to iterate with finite support ccc partial orders of size ω_1, and in the resulting extension, all cardinals are preserved (by ccc) and MA_{ω_1} holds (by (B) and some bookkeeping device to successive deal with all ccc partial orders of size ω_1).

Let sketch the prove of (B):

Theorem 8.16. MA_{ω_1} *is equivalent to its restriction to ccc partial orders* of size $\le \omega_1$.

Proof. Suppose $\mathbb{P}$ is a ccc forcing notion and $\mathcal{D} = \{D_\alpha \,|\, \alpha < \omega_1\}$ is a collection of ω_1-dense sets in $\mathbb{P}$. We wish to find a filter over $\mathbb{P}$ meeting every D_α using only the version of MA_{ω_1} which is applicable to forcing notions of size $\le \omega_1$: $\mathbb{P}$ may be much larger so we need to find some ccc forcing notion $\mathbb{P}^{**} \subseteq \mathbb{P}$ of size $\le \omega_1$ with a filter G which (when extended to all of $\mathbb{P}$) will meet every element of $\mathcal{D}$.

Let for each $\alpha < \omega_1$ be $A_\alpha \subseteq D_\alpha$ some antichain in $\mathbb{P}$ maximal for the following property:

(*) for every $d \in D_\alpha$ there is some $a \in A_\alpha$ with $a \,||\, d$.

Notice that because D_α is dense it actually holds:

(**) for every $p \in \mathbb{P}$ there is some $a \in A_\alpha$ with $a \,||\, p$, i.e.

$$\text{there is some } q \in \mathbb{P} \text{ with } q \le a, p. \tag{8.6}$$

(Naive approach). Since $\mathbb{P}$ is ccc, each A_α is at most countable, and hence $\mathbb{P}^* = \bigcup_{\alpha<\omega_1} A_\alpha$ is subset of $\mathbb{P}$ of size ω_1. Define $D^*_\alpha = \{p \in \mathbb{P}^* \,|\, (\exists a \in A_\alpha)\ p \le a\}$. If each D^*_α is dense in $\mathbb{P}^*$, we are done: by MA_{ω_1}, there is a filter G over $\mathbb{P}^*$ which meets every D^*_α, and by definition of D^*_α it meets some $a \in A_\alpha \subseteq D_\alpha$. But D^*_α may not be dense because (8.6) only ensures that

$\bar{D}_\alpha = \{p \in \mathbb{P} \mid (\exists a \in A_\alpha)\ p \leq a\}$ is dense. There is also an additional concern whether $\mathbb{P}^*$ is actually ccc (perhaps some conditions are compatible in $\mathbb{P}$ but not in $\mathbb{P}^*$, which may give rise to a large antichain).

(Corrected approach). Using a Löwenheim-Skolem type argument, let $\mathbb{P}^{**}$ be the minimal closure of $\mathbb{P}^*$ under the operation h which assigns to each pair (p, p') of compatible condition some witness for compatibility, i.e. some $q \in \mathbb{P}$ with $q \leq p, p'$. Then $\mathbb{P}^{**}$ has size ω_1 and it satisfies the following two properties:

$$\mathbb{P}^{**} \text{ is ccc,} \tag{8.7}$$

because by the closure of $\mathbb{P}^{**}$ under h, any antichain in $\mathbb{P}^{**}$ is an antichain in $\mathbb{P}$, and

$$(\forall p \in \mathbb{P}^{**})(\forall \alpha < \omega_1)(\exists a \in A_\alpha)(\exists q \in \mathbb{P}^{**})\ q \leq p, a. \tag{8.8}$$

Define $D^{**}_\alpha = \{p \in \mathbb{P}^{**} \mid (\exists a \in A_\alpha)\ p \leq a\}$. Then by (8.8) each D^{**}_α is dense in $\mathbb{P}^{**}$, and we are done by applying MA_{ω_1} to $\mathbb{P}^{**}$: first let $G^{**} \subseteq \mathbb{P}^{**}$ be a filter meeting every D^{**}_α, and then define

$$G = \{p \in \mathbb{P} \mid (\exists g \in G^{**})\ g \leq p\}.$$

It is easy to check that G is as required. □

It was Shelah who discovered a fruitful generalization of ccc which satisfies an analogue of (A), and with large cardinals, a version of (B). This is the notion of a *proper* forcing. The definition of this concept is beyond the scope of this lecture (see for instance [5] for details). Let us just mention that every ccc and σ-closed forcing is proper, as are variants of the "tree-based" forcings, such as the Sacks forcing.

Definition 8.17. PFA (proper forcing axiom) says that if $\mathbb{P}$ is proper and $\mathcal{D}$ is a collection of ω_1 many dense sets in $\mathbb{P}$, then there is a filter G meeting them all.

Note that since every ccc forcing is proper, we easily get that PFA implies MA_{ω_1}.

The following analogues of (A) and (B) discussed for ccc forcings are true for the proper forcings:

- (α) Any forcing iteration with *countable* support of arbitrary length such that each iterand of the iteration is forced to be proper is proper.
- (β) PFA is *not* equivalent to its restriction to proper partial orders *of size* $\leq \omega_1$. However, using a supercompact cardinal, PFA is consistent.[29]

[29] A supercompact cardinal κ is used to "guess" every potential proper forcing of an arbitrary size, and in the resulting proper iteration of length κ, PFA holds.

While MA_{ω_1} does not put any upper bound on the size of 2^ω (it only implies that 2^ω must be regular and greater than ω_1), PFA implies $2^\omega = \omega_2$.

The known consequences of PFA are often strengthenings of the consequences of MA_{ω_1}: for instance MA_{ω_1} implies that all ω_1-Aronszajn trees are special (and hence there are no ω_1-Suslin trees); PFA implies that any two ω_1-Aronszajn trees are very similar.[30] PFA also implies $\mathsf{TP}(\omega_2)$.[31]

Importantly, PFA also decides another "mainstream mathematics" problem, this time from functional analysis – the Kaplansky's conjecture. See the next section for a brief description.

Let us illustrate the use of PFA by showing that it implies the tree property at ω_2.

Theorem 8.18. PFA *implies* $\mathsf{TP}(\omega_2)$.

Proof. (Sketch) Suppose for contradiction that T is an ω_2-Aronszajn tree (we identify the domain of T with ω_2). Let M be an elementary submodel of size ω_1 with $\omega_1 \subseteq M$, in some large enough $H(\theta)$, for instance in $H(\omega_3)$, with $T \in M$.

Let $o_M = M \cap \omega_2$ (o_M is an ordinal of size ω_1). By elementarity $T \cap M \in M$ is a tree of height o_M without cofinal branches. Let $\mathbb{P}$ be a σ-closed forcing which collapses ω_2 to ω_1 and $\dot{\mathbb{Q}}$ the ccc forcing which specializes T by finite conditions. By elementarity $\mathbb{P}*\dot{\mathbb{Q}} \in M$ and it is easy to see that $\mathbb{P}*\dot{\mathbb{Q}}$ is a proper forcing. By PFA, let us consider all ω_1 dense sets in M in $\mathbb{P} * \dot{\mathbb{Q}}$ and a filter G which meets them all. G determines an initial part of the collapsing function f which would be added by $\mathbb{P}$: working over M, this collapsing function is a sujrection $f : o_M \to \omega_1$. Since f is a generic filter over M (because it meets all ω_1 dense sets in M), $M[f]$ is a generic extension of M. Working in $M[f]$, let $g : T \cap M \to \omega$ be the specializing function (corresponding to $\dot{\mathbb{Q}}$) which maps $T \cap M$ into ω such that

$$t <_T s \text{ implies } g(t) \neq g(s). \tag{8.9}$$

Again $M[f][g]$ is a model of set theory. Now we reach a contradiction because any node $w \in T$ on the level o_M determines a cofinal branch through $T \cap M$ of length ω_1 which is supposed to be mapped injectively into ω by (8.9). □

Remark 8.19. MA_{ω_1} is equiconsistent with ZFC. In contrast, PFA has a very large consistency strength: by current results in inner model theory, it implies consistency of many Woodin cardinals (much larger than measurable). The conjecture is that PFA is equiconsistent with the existence of a supercompact

[30] "club-isomorphic", which we will not define here.

[31] PFA does not imply $\mathsf{SR}(\omega_2)$, but a strengthening of PFA called Martin's maximum (MM) does imply $\mathsf{SR}(\omega_2)$.

cardinal. This should be viewed as a positive thing: PFA can thus decide statements which cannot be decided by MA_{ω_1} simply based on the consistency strength of the respective theories. An example is $\mathsf{TP}(\omega_2)$; the consistency strength of $\mathsf{TP}(\omega_2)$ is a weakly compact cardinal – since MA_{ω_1} is equiconsistent with ZFC, it cannot imply $\mathsf{TP}(\omega_2)$. In Theorem 8.18 we showed that PFA does imply $\mathsf{TP}(\omega_2)$.

8.3.1. Kaplansky's conjecture

Recall that a *Banach algebra* A is an associative algebra over the complex numbers that is at the same time a *Banach space*, i.e. a normed space which is complete with respect to the metric induced by the norm. The norm must satisfy the multiplicative inequality

$$||xy|| \leq ||x||\,||y||.$$

The multiplicative inequality makes the multiplication continuous on A (if $(x_n) \to x$ and $(y_n) \to y$, then $(x_n y_n) \to xy$).

The algebra is *unital* if it has a multiplicative inverse whose norm is 1.

A prototypical example of a unital Banach algebra is the algebra of continuous complex valued functions defined on some non-empty compact Hausdorff space X, denoted $C(X)$, such as the unit interval $[0, 1]$. The norm is the usual supremum norm $||f|| = \sup\{|f(x)| \,|\, x \in X\}$, and the multiplication is defined by $fg(x) = f(x)g(x)$. This makes $C(X)$ a commutative Banach algebra with the identity function being the unit for multiplication.

Kaplansky conjectured in 1948 that any Banach algebra homomorphism from $C(X)$, for a non-empty compact Hausdorff space X, into any other Banach algebra is necessarily continuous (and thus the notion of continuity – which depends on the norm – is reduced to purely *algebraic* properties of $C(X)$).

Theorem 8.20. *(i) (Dales, Esterle, 1976)* CH *implies that Kaplansky's conjecture fails.*

(ii) (Solovay, Woodin, 1976) It is consistent relative to ZFC *that* $2^\omega = \omega_2$*,* MA *holds, and Kaplansky's conjecture is true.*

(iii) (Todorcevic, 1989, see [12]) PFA *implies that Kaplansky's conjecture is true.*

We will not give further details (see for instance [2] for more details), but put the result in the set-theoretical context considered in this lecture.

Notice that we have discussed three independence of mainstream mathematical questions from the axiom of set theory: Suslin hypothesis, Whitehead's problem, and Kaplansky's conjecture. In one direction, they were all decided by a form of CH: $\diamondsuit_{\omega_1}(\omega_1)$, $(\forall S)\diamondsuit_{\omega_1}(S)$, and CH, respectively. In the other direction, they were all decided by a form of $\neg\mathsf{CH}$: MA_{ω_1}, MA_{ω_1}, PFA, respectively. It is instructive to notice that $V = L$ decides all three in one way, while PFA decides all three the other way. It is up to the reader to speculate which solution is the more intuitively "correct".

Remark 8.21. On a more technical note: Solovay's and Woodin's result from 1976 proceeds as follows: first we force MA_{ω_1} by the usual ccc finite-support iteration of length ω_2, denoted $\mathbb{P}$; then we define a ccc forcing $\dot{\mathbb{Q}}$ such that in $V[\mathbb{P} * \dot{\mathbb{Q}}]$ Kaplansky's conjecture holds. The whole forcing $\mathbb{P} * \dot{\mathbb{Q}}$ is ccc, but it is not same as saying that MA_{ω_1} implies Kaplansky's conjecture.[32] Todorcevic showed that PFA does imply Kaplansky's conjecture: the difference is that in PFA one can also consider collapsing forcing notions which can "morally speaking" (though not literally) turn ω_2 into ω_1 using only ω_1 many dense sets; over this (partial collapse) a ccc forcing notions is used (this is similar to the argument in Theorem 8.18).

[32] MA_{ω_1} says that there is a filter for any ccc forcing and any collection of ω_1 dense sets in that forcing; a true generic extension will meet *all*, in our context ω_2 many, dense sets. It may be open whether Kaplansky's conjecture can consistently fail with MA_{ω_1}.

9. For the exam 2023: Theorems and lemmas to learn

There will be a written exam. You will receive a question from the list below and you should prove the relavant result in a self-contained, clearly written way. The proof must contain all relevant definitions appearing in the theorem or lemma. You will submit the written proof and I will grade it.

Questions:

- 3.6,
- 4.2,
- 4.3,
- 4.13,
- 6.2, 6.3,
- 6.5, 6.7,
- 6.9, 6.10, 6.11,
- 6.19, 6.21,
- 6.23,
- 7.2, 7.3,
- 7.5,
- 8.8, 8.10, 8.11,
- 8.16