NMAI057 – Linear algebra 1

Tutorial 1 – with solutions

Date: September 29, 2020 TA: Pavel Hubáček

Problem 1. Over $\mathbb{R}$, find all solutions for the system of linear equations

$$\begin{aligned} x + 2y &= 5 \\ 2x - y &= 0 \end{aligned}$$

Solution. We use the matrix representation for the system and reduce it as follows

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

After the reduction, we get $x = 1$ a $y = 2$.

Problem 2. Over $\mathbb{R}$, find all solutions for the system of linear equations

$$\begin{aligned} x - 3z &= 1 \\ -2x + 6z &= -2 \end{aligned}$$

Solution. We use the matrix representation for the system and reduce it as follows

$$\left(\begin{array}{cc|c} 1 & -3 & 1 \\ -2 & 6 & -2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

We can put a real parameter $t \in \mathbb{R}$ for the z variable and express x in terms of the parameter:

$$\begin{aligned} x - 3t &= 1 \\ x &= 1 + 3t \end{aligned}$$

The solution set of the system over $\mathbb{R}$ is

$$\{(x, z)^T \in \mathbb{R}^2 \mid x - 3z = 1\} = \{(1 + 3t, t)^T \mid t \in \mathbb{R}\} = \{(1, 0)^T + t(3, 1)^T \mid t \in \mathbb{R}\} .$$

Problem 3. Over $\mathbb{R}$, find all solutions for the system of linear equations

$$\begin{aligned} x + \ \ y - z &= 1 \\ 2x + 2y + z &= 5 \\ x - \ \ y - z &= -1 \end{aligned}$$

Solution. We use the matrix representation for the system and reduce it as follows

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 2 & 1 & 5 \\ 1 & -1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 0 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

From the final matrix, we get that $z = 1$, $y = 1$, and $x = 1$. Thus, the vector $(1, 1, 1)^T$ is the unique solution over $\mathbb{R}$.

Problem 4. Over $\mathbb{R}$, find all solutions for the system of linear equations

$$\begin{aligned} x + \ \ y - z &= 1 \\ 2x + 2y + z &= 5 \end{aligned}$$

Solution. We use the matrix representation for the system and reduce it as follows (which we already did in Problem 3)

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 2 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

From the second row of the last matrix, we get that $z = 1$. From the first row of the last matrix, we see that y can take an arbitrary real value, and we express x in terms of the parameter $y \in \mathbb{R}$ and the value of z:

$$\begin{aligned} x + y - z &= 1 \\ x + y - 1 &= 1 \\ x &= 2 - y \end{aligned}$$

This gives the solution space

$$\{(2 - y, y, 1)^T \mid y \in \mathbb{R}\} = \{(2, 0, 1)^T + y(-1, 1, 0)^T \mid y \in \mathbb{R}\} \ .$$

Note that we can easily verify that $(2-y, y, 1)^T$ solves the linear system for all $y \in \mathbb{R}$ by checking that any such vector satisfies both original equations:

$$\begin{aligned} 2 - y + y - 1 = 2 - 1 + y(-1 + 1) &= 1 \\ 4 - 2y + 2y + 1 &= 5 \end{aligned}$$

We have verified that $(2 - y, y, 1)^T$ is a solution for all $y \in \mathbb{R}$.

Problem 5. Over $\mathbb{R}$, find all solutions for the system of linear equations

$$\begin{aligned} 2x + 2y + z &= 5 \\ x - \quad y - z &= -1 \end{aligned}$$

Solution.

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 2 & 2 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 4 & 3 & 7 \end{array}\right)$$

From the second row of the last matrix, we see that z can take an arbitrary real value, and we express y in terms of the parameter $z \in \mathbb{R}$:

$$\begin{aligned} 4y + 3z &= 7 \\ 4y &= 7 - 3z \\ y &= \tfrac{1}{4}(7 - 3z) \end{aligned}$$

From the first row of the last matrix, we can express x in terms of the parameter $z \in \mathbb{R}$:

$$\begin{aligned} x - y - z &= -1 \\ x - \tfrac{1}{4}(7 - 3z) - z &= -1 \\ x &= -1 + \tfrac{1}{4}(7 - 3z) + z = \tfrac{1}{4}(3 + z) \end{aligned}$$

This gives the solution space

$$\left\{ \left(\tfrac{1}{4}(3+z), \tfrac{1}{4}(7-3z), z\right)^T \mid z \in \mathbb{R} \right\} = \left\{ \left(\tfrac{3}{4}, \tfrac{7}{4}, 0\right)^T + z\left(\tfrac{1}{4}, -\tfrac{3}{4}, 1\right)^T \mid z \in \mathbb{R} \right\} .$$

Problem 6. Over $\mathbb{R}$, find all solutions for the system of linear equations

$$\begin{aligned} x_1 + \quad x_2 + x_3 + x_4 &= 3 \\ x_1 - 2x_2 - x_3 - x_4 &= 1 \end{aligned}$$

Solution. We use the matrix representation for the system and reduce it as follows

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & -2 & -1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & -3 & -2 & -2 & -2 \end{array}\right)$$

From the second row of the last matrix, we see that x_3 and x_4 can take arbitrary real values, and we express x_2 in terms of the parameters $x_3, x_4 \in \mathbb{R}$:

$$\begin{aligned} -3x_2 - 2x_3 - 2x_4 &= -2 \\ -3x_2 &= -2 + 2x_3 + 2x_4 \\ x_2 &= \tfrac{2}{3}(1 - x_3 - x_4) \end{aligned}$$

From the first row of the last matrix, we can now express x_1 in terms of the parameters $x_3, x_4 \in \mathbb{R}$:

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 3 \\
x_1 + \tfrac{2}{3}(1 - x_3 - x_4) + x_3 + x_4 &= 3 \\
x + 1 &= 3 - \tfrac{2}{3}(1 - x_3 - x_4) - x_3 - x_4 = \tfrac{7}{3} - \tfrac{1}{3}(x_3 + x_4)
\end{aligned}
$$

This gives the solution space

$$
\begin{aligned}
&\left\{ \left(\tfrac{7}{3} - \tfrac{1}{3}(x_3 + x_4), \tfrac{2}{3}(1 - x_3 - x_4), x_3, x_4\right)^T \mid x_3, x_4 \in \mathbb{R} \right\} \\
&= \left\{ \left(\tfrac{7}{3}, \tfrac{2}{3}, 0, 0\right)^T + x_3 \left(-\tfrac{1}{3}, -\tfrac{2}{3}, 1, 0\right)^T + x_4 \left(-\tfrac{1}{3}, -\tfrac{2}{3}, 0, 1\right)^T \mid x_3, x_4 \in \mathbb{R} \right\} .
\end{aligned}
$$

The solution set is a plane containing the point $(\frac{7}{3}, \frac{2}{3}, 0, 0)^T$.

Problem 7. Over $\mathbb{R}$, find all solutions for the system of linear equations

$$
\begin{aligned}
2y - 3z &= -1 \\
x - 5y + 4z &= 1 \\
-3x + \ y + 2z &= -3
\end{aligned}
$$

Solution. We use the matrix representation for the system and reduce it as follows

$$
\left(\begin{array}{ccc|c} 0 & 2 & -3 & -1 \\ 1 & -5 & 4 & 1 \\ -3 & 1 & 2 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -5 & 4 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & -14 & 14 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -5 & 4 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & -7 & -7 \end{array}\right)
$$

The unique solution is the vector $(2, 1, 1)^T$.