JEM004 Macroeconomics
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Lecture Notes
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3 The Ramsey-Cass-Koopmans Model

3.1 Preview

- Solow - problem = exogenous saving rate
- Ramsey - solution = households decide how much is optimal to eat and save, so they and their children have the best possible life (i.e. utility maximization)
- Result - saving rate as a function of capital k
  - no possibility of over-saving
  - dependance on interest rate
  - effect on speed on convergence, revisited

3.2 Assumptions

3.2.1 Firms

- are owned by households, produce goods, hire capital and labor, pay rent and wages, profits are transferred to the households
- each firm has access to production technology $Y = F(K, AL)$ which satisfies the same assumptions as in Solow model
- knowledge growing at rate g (exogenous), A(0) is normalized to 1
  i.e. $\dot{A}(t) = gA(t)$ and $A(t) = e^{gt}A(0) = e^{gt}$

3.2.2 Households

- work for real wage, which they use for consumption, or they save it in the form of accumulating assets, on which they later get interest income
- **representative household**
  - 1 economy = 1 infinitely lived household
  - $L(t)$ - size of population at time t, $L(0)$ is normalized to 1

- population growing at rate n (exogenous)
  i.e. $\dot{L}(t) = nL(t)$ and $L(t) = e^{nt}L(0) = e^{nt}$

- household members try to maximize their current happiness and happiness of ALL their future descendants - **overall utility**

$$U = \int_0^\infty u[c(t)]e^{nt}e^{-\rho t}dt \tag{1}$$

  - **utility** = satisfaction, happiness, positive value
    * contingent on situation, person, time, etc.
    * in theory: described by functional form, easy to measure
    * in practice: hard to measure, only revealed preferences (comparative)
  - in this model, utility is derived from **individual consumption** (per capita) $c(t) = \frac{C(t)}{L(t)}$ only[1]
    * in other literature: utility from leisure, cheating, betting; disutility from work, waiting
  - summing up the utility for all members of the household from the beginning ($t = 0$) to eternity $\left(\text{in discreet time } \sum_{t=0}^{\infty}, \text{ in continuous time } \int_0^\infty dt\right)$
  - weight e^{nt} accounts for **growing population**, as $u[c(t)]L(t) = u[c(t)]e^{nt}$
  - weight $e^{-\rho t}$ accounts for **time preference** (utility in future is less valuable than utility today) - i.e. impatience and selfishness across generations

- moreover, we assume functional form of utility[2]

$$u[c(t)] = \frac{c(t)^{1-\theta} - 1}{1-\theta}, \quad \text{where} \quad \theta > 0$$

  - $u'_c = c(t)^{-\theta} > 0$ and $u''(c) = -\theta c(t)^{-\theta-1} < 0$ -> u(c) is **increasing and concave**; together with assumption $\rho > n$ it ensures that **lifetime utility does not diverge** (e.g. household would not have infinite lifetime utility) which is needed for well defined solution
  - called **constant intertemporal elasticity of substitution (CIES)** utility function, where intertemporal elasticity of substitution σ, defined as $\sigma = -\frac{u'(c)}{cu''(c)}$, is equal to $1/\theta$. Thus, higher the θ, lower the willingness to move the consumption between today to tomorrow (i.e. intertemporally).
  - also called **constant relative risk aversion (CRRA)** utility function, where risk aversion coefficient equals $1/\sigma = \theta$

[1] Please, do notice the change in notation from the lecture on Solow model - now "small" versions of variables are per capita/worker values, e.g. $c(t)$ above, while intensive (per effective worker) values are denoted with hats, e.g. $\hat{k}(t)$.

[2] In the special case when $\theta \to 1$, the utility function simplifies into $\ln c$.

- household can accumulate **assets**, either in form of ownership claims to capital rented to firms, or loans to other households (negative loan = debt)
  - both forms have to pay the same rate of return $r(t)$
  - representative household framework $\Rightarrow$ no loans
  - $a(t)$ - asset holdings per person, i.e. capital income per capita is $r(t)a(t)$ and household's **capital income** at time t is $L(t)r(t)a(t)$
- each household member inelastically supplies 1 unit of time for wage $w(t)$, therefore household's **labor income** at time t is $L(t)w(t)$
- income of household = labor income + capital income; from this income it finances consumption of all its members $c(t)L(t)$ and purchase of additional assets
- The overall **budget constraint** of household at time t is therefore

$$\frac{d(Assets)}{dt} = L(t)w(t) + L(t)r(t)a(t) - c(t)L(t)$$

  We are interested in the change of holdings of assets per person, i.e. $\dot{a}$. As $a(t) = Assets(t)/L(t)$ then $\dot{a}(t) = \frac{d(Assets)}{dt}\frac{1}{L(t)} - \frac{Assets}{L(t)}\frac{\dot{L}(t)}{L(t)} = \frac{d(Assets)}{dt}\frac{1}{L(t)} - na$.

  Therefore, if we divide overall budget constraint by $L(t)$, we can write budget constraint in per capita terms:

$$\dot{a}(t) = w(t) + [r(t) - n]a(t) - c(t) \tag{2}$$

- **No Ponzi game restriction:** instantaneous budget constraint can be integrated into the lifetime budget constraint, which would imply that household's present value of lifetime consumption cannot exceed present value of income. However, if the household can borrow an unlimited amount at the rate $r(t)$, it would have an incentive to borrow amount higher than present value of income for current consumption, and then in future always borrow enough to cover interest (but not principal). To prevent this, we restrict the present value of assets to be positive, i.e.

$$\lim_{t\to\infty} a(t)e^{-\int_0^t (r(v)-n)dv} \geq 0 \tag{3}$$

  where $e^{-\int_0^t (r(v)-n)dv}$ is discount factor based on continuous discounting with changing interest rate $r(t)$. With homogenous infinitely living households, however, we cannot have Ponzi game in equilibrium.

3.3 Behavior of Firms

- again, we will use variables in intensive form, i.e. $\hat{k} = K/AL,\ \hat{y} = Y/AL = f(\hat{k})$
- firm rents capital at rental rate R - cost is RK
- capital depreciates at rate δ, therefore net rate of return to a household (owner of the capital) is $(R - \delta)$. As household can either hold assets in capital or as loans to other households at interest rate r, it must hold $R - \delta = r$
- firm pays its labor force L wage w
- no costs of adjustment of capital in time $\Rightarrow$ the problem of maximizing the present value of profit reduces to the problem of maximizing profit in each period

Firm's profit at any period can be expressed as

$$\begin{aligned}\pi &= F(K, AL) - (r+\delta)K - wL = AL\left[f(\hat{k}) - (r+\delta)\hat{k} - \frac{w}{A}\right] \\ &= AL\left[f(\hat{k}) - (r+\delta)\hat{k} - we^{-gt}\right]\end{aligned}$$

As the size of effective workforceAL at any time t is given and eventually everybody is employed, firm can only choose the level of effective capital rented $\hat{k}$ such that

$$\frac{\partial \pi}{\partial \hat{k}} = AL\left[f'(\hat{k}) - (r+\delta)\right] = 0$$

Therefore, the optimal choice of level of capital per effective worker is such that $f'(\hat{k}) = (r+\delta)$. Indeed, we just confirm that in the competitive equilibrium with CRS production function, factors are paid their marginal products , i.e. [3]

$$\begin{aligned}r &= f'(\hat{k}) - \delta &\quad (4)\\ w &= \left[f(\hat{k}) - f'(\hat{k})\hat{k}\right]e^{gt} &\quad (5)\end{aligned}$$

3.4 Behavior of Households

The household's optimization problem is to maximize utility U in equation (1), subject to its budget constraint (2), limitation on debt (3), initial stock of assets $a(0)$ and inequality restriction $c(t) \geq 0$

$$\begin{aligned}\max_{c(t)} U &= \int_0^T u[c(t)]e^{-(\rho-n)t}dt \\ s.t. \quad & \dot{a}(t) = w(t) + [r(t) - n]a(t) - c(t) \\ & a(0) = a_0,\ c(t) \geq 0 \\ & \lim_{t\to\infty} a(t)e^{-\int_0^t (r(v)-n)dv} \geq 0\end{aligned}$$

[3]Labor is paid its marginal labor, i.e. w $= \partial Y/\partial L = ALf'(\hat{k})\frac{\partial \hat{k}}{\partial L} + Af(\hat{k}) = ALf'(\hat{k})\frac{K}{AL^2} + Af(\hat{k}) =$ $= A\left[-f'(\hat{k})\hat{k} + f(\hat{k})\right] = \left[f(\hat{k}) - f'(\hat{k})\hat{k}\right]e^{gt}$.

We use present-value Hamiltonian framework to solve this optimization problem

$$H = u[c(t)]e^{-(\rho-n)t} + \mu(t)[w(t) + [r(t) - n]a(t) - c(t)]$$

The first-order conditions for this problem are

$$\begin{aligned}
(i)\,\frac{\partial H}{\partial c} &= 0 \Rightarrow u'(c)e^{-(\rho-nt} = \mu \\
(ii)\,\frac{\partial H}{\partial a} &= -\dot{\mu} \Rightarrow [r(t) - n]\mu = -\dot{\mu} \\
\lim_{t\to\infty} & \quad \mu(t)a(t) = 0
\end{aligned}$$

We use equation (i) to plug in for μ in the equation (ii). To do so, we need to find value of $\dot{\mu}$, therefore we differentiate equation (i) with respect to time, obtaining

$$\dot{\mu} = u''(c)\dot{c}\,e^{-(\rho-n)t} - u'(c)(\rho - n)e^{-(\rho-n)t}.$$

After rearranging and plugging in the equation (ii) we get

$$\begin{aligned}
r(t) - n &= -\frac{\dot{\mu}}{\mu} = \frac{-u''(c)\dot{c}\,e^{-(\rho-n)t} + u'(c)(\rho - n)e^{-(\rho-n)t}}{u'(c)e^{-(\rho-n)t}} = -\frac{u''(c)\dot{c}}{u'(c)} + (\rho - n) \\
r(t) - \rho &= -\frac{u''(c)\,c}{u'(c)}\frac{\dot{c}}{c}
\end{aligned}$$

where last equation is known as **Euler equation**. It describes how household decides between consumption today (immediate utility) and tomorrow (trade-off between positive return on savings r and decrease in utility over time ρ).

- if $r = \rho \Rightarrow \dot{c}/c = 0 \Rightarrow$ flat consumption profile $c(t) = c, \ \forall t$
- to save (i.e. sacrifice consumption today for more consumption tomorrow) household would have to be compensated by higher r than ρ

Recall that we are using CIES form of utility function (i.e. $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$), where intertemporal elasticity of substitution is $\sigma = -\frac{u'(c)}{cu''(c)} = \frac{1}{\theta}$. Therefore, we can rewrite Euler equation in the form of differential equation for consumption

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) \tag{6}$$

- if $r > \rho \Rightarrow \dot{c}/c > 0$ (incentive to save for future consumption)
- if $r < \rho \Rightarrow \dot{c}/c < 0$ (incentive to consume today)
- higher $\theta \Rightarrow$ lower willingness to substitute intertemporally (save)

We are also able to rewrite transversality condition. From F.O.C (ii) we get

$$\frac{\dot{\mu}}{\mu} = -(r(t) - n) \quad \Rightarrow \mu(t) = \underbrace{\mu(0)}_{=1} e^{-\int_0^t (r(v)-n)dv}$$

$$\lim_{t\to\infty} a(t)e^{-\int_0^t (r(v)-n)dv} = 0$$

3.5 Competitive Market Equilibrium

- combine behavior of households and firms
- **a** = **k** - all of the capital stock must be owned by household members (loans = 0)
- again, it will be convenient to rewrite everything into intensive form. Recall

$$\begin{aligned}
\hat{k} &= \frac{k}{A} = ke^{-gt} \\
\dot{\hat{k}} &= \dot{k}e^{-gt} - gke^{-gt} = \dot{k}e^{-gt} - g\hat{k} \quad \Rightarrow \; \dot{k} = \dot{\hat{k}}e^{gt} + g\hat{k}e^{gt} \\
\hat{c} &= \frac{c}{A} = ce^{-gt} \\
\dot{\hat{c}} &= \dot{c}e^{-gt} - gce^{-gt} = \dot{c}e^{-gt} - g\hat{c} \quad \Rightarrow \; \dot{c} = \dot{\hat{c}}e^{gt} + g\hat{c}e^{gt}
\end{aligned}$$

- both firms and household face the same prices w and r, we therefore can use the results from firms' maximization in solving household's problem. Therefore, we plug in the expressions for rental rate(4) and wage(5) into budget constraint (2)

$$\begin{aligned}
\dot{k} &= w + rk - c - nk \\
\dot{\hat{k}}e^{gt} + g\hat{k}e^{gt} &= \left[f(\hat{k}) - f'(\hat{k})\hat{k}\right] e^{gt} + (f'(\hat{k}) - \delta)\hat{k}e^{gt} - \hat{c}e^{gt} - n\hat{k}e^{gt} \\
\dot{\hat{k}} + g\hat{k} &= f(\hat{k}) - f'(\hat{k})\hat{k} + f'(\hat{k})\hat{k} - \delta)\hat{k} - \hat{c} - n\hat{k} \\
\dot{\hat{k}} &= f(\hat{k}) - (g + n + \delta)\hat{k} - \hat{c} \qquad (7)
\end{aligned}$$

- the differential equation (7) determines the evolution of capital $\hat{k}$ and output $\hat{y}$ over time, however, we still need to determine the path of $\hat{c}$. For this, we naturally use the Euler equation and plug in for interest rate r

$$\begin{aligned}
\frac{\dot{c}}{c} &= \frac{1}{\theta}(r - \rho) \\
\frac{\dot{\hat{c}}\,e^{gt} + g\hat{c}\,e^{gt}}{\hat{c}\,e^{gt}} &= \frac{1}{\theta}(f'(\hat{k}) - \delta - \rho) \\
\frac{\dot{\hat{c}}}{\hat{c}} + g &= \frac{1}{\theta}(f'(\hat{k}) - \delta - \rho) \\
\frac{\dot{\hat{c}}}{\hat{c}} &= \frac{1}{\theta}(f'(\hat{k}) - \delta - \rho - \theta g) \qquad (8)
\end{aligned}$$

- we also have to rewrite the transversality condition

$$\lim_{t\to\infty} k(t)e^{-\int_0^t (r(v)-n)dv} = \lim_{t\to\infty} \hat{k}e^{-\int_0^t (f'(\hat{k})-\delta-n-g)dv} = 0 \qquad (9)$$

- equations (7) and (8) form a system of two differential equations in $\hat{k}$ and $\hat{c}$, which together with initial condition on $\hat{k}(0)$ and transversality condition (9) determine the time paths of $\hat{k}$ and $\hat{c}$ and thus the evolution of the economy

3.6 Dynamics of the Economy

Let us draw the **phase diagram** of this system of differential equations, taking into consideration transversality condition $\lim_{t\to\infty} \hat{k}e^{-\int_0^t (f'(\hat{k})-\delta-n-g)dv} = 0$

$$\begin{aligned}\dot{\hat{k}} &= f(\hat{k}) - (g+n+\delta)\hat{k} - \hat{c} \\ \frac{\dot{\hat{c}}}{\hat{c}} &= \frac{1}{\theta}(f'(\hat{k}) - \delta - \rho - \theta g)\end{aligned}$$

For locus $\dot{\hat{k}} = 0$ we get the expression for $\hat{c}$ as a function of $\hat{k}$ - $\hat{c} = f(\hat{k}) - (g+n+\delta)$. From the Figure 1 we see that consumption is an increasing function of capital up to the point where $f'(\hat{k}_{GOLD}) - \delta = g + n$, and then changes to decreasing function. Level of $\hat{k}_{GOLD}$ which maximizes consumption is called, similarly as in Sollow model, golden rule level of capital. For all points lying above the locus $\hat{c} > f(\hat{k}) - (g+n+\delta)$ and therefore $\dot{\hat{k}} < 0$, i.e. level of capital per effective worker is decreasing. Opposite is true for points lying under the locus.

Note that locus $\dot{\hat{c}} = 0$ is independent of the level of $\hat{c}$, thus it directly pinpoints the equilibrium level of $\hat{k}^*$ which will have to satisfy condition $f'(\hat{k}^*) - \delta = \rho + \theta g$. Therefore, locus $\dot{\hat{c}} = 0$ will be a vertical line through this level of capital. Moreover, as transversality condition implies that $f'(\hat{k}^*) - \delta > g + n$, we see that $\hat{k}^* < \hat{k}_{GOLD}$, i.e. the vertical line will be to the left of the golden rule level of capital $\hat{k}_{GOLD}$.[4] For all points lying left to the locus $\hat{k} < \hat{k}^* \Rightarrow f'(\hat{k}) > f'(\hat{k}^*) \Rightarrow \dot{\hat{c}} > 0$, i.e. level of consumption per effective worker is increasing. Opposite is true for points lying right to the locus.

The phase diagram of this system is depicted in the Figure 2. We see that this system of differential equations have **3 equilibria**: point 0 ($\hat{c} = 0, \hat{k} = 0$), point where $\hat{c} = 0$ and $\hat{k} = \hat{k}^{**}$ (i.e. where we spend all output on depreciation of capital) and point $(\hat{c}^*, \hat{k}^*)$. However, we are only interested in equilibria with positive consumption $\hat{c} > 0$. This equilibrium is unstable with saddle path. For further analysis of transitional dynamics, see Romer, p.60 (+ I will discuss it on the lecture).

Saddle path :

- **policy function:** for each level of capital per effective worker $\hat{k}$ there is a unique level of consumption $\hat{c}$ that is consistent with household's optimisation problem as well as law of motion for capital.
- shape depends on the parameters of the model: e.g higher θ (higher preference for today's consumption) implies that on the path to the steady state, household will have high levels of consumption but the convergence will be slower (the saddle

[4]Note that this fact has two implication for the steady state characteristics of the economy. First, there is no inefficient oversaving (like in Solow). However, optimizing households does not save enough to attain the maximum consumption.

path will be close to $\dot{\hat{k}} = 0$ locus). On the other hand, if θ is low, households will sacrifice current consumption for faster convergence to the steady state with high level of consumption in future.

3.7 Introduction of government - policy analysis

- new agent in the economy = government
- collects money = taxation
  - what to tax: labor income, consumption (VAT), capital income, profits of firms
  - how: lump sum, flat (proportional), progressive (brackets)
- spends money
  - own consumption ("overheads") + public goods (education, infrastructure) - enters households' utility = G
  - transfers (redistribution of income, e.g. retirement benefits) = V
- **Government's budget constraint** (generalized for flat rate case):

$$G + V = \tau_w wL + \tau_a r(Assets) + \tau_c C + \tau_f(firm's\ earnings)$$

- **Question:** How do government's policies (taxation / spending) affect the steady state of economy?

In all analyzed cases we assume zero technological growth and by g we denote government consumption per capita (instead of growth rate of technology). We compare the situations with the steady-state values without existence of government

3.7.1 Lump sum tax τ + nonproductive spending G

- firms' problem unchanged - determine $r = f'(k) - \delta; w = f(k) - f'(k)k$
- government's budget constraint: $G = \tau L; \quad \tau = G/L = g$
- household's budget constraint: $\dot{a} = w + ra - na - c - \tau$

Hamiltonian for household's problem:

$$H = u(c)e^{-(\rho-n)t} + \mu[w + ra - na - c - \tau]$$

- $\frac{\partial H}{\partial c}$ and $\partial H/\partial a$ do not change => Euler equation is unchanged

In the equilibrium we plug in for w, r (firm's problem) and g (gvt BC) and replace $k = a$

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\theta}[f'(k) - \delta - \rho] \\ \dot{k} &= f(k) - (n+\delta)k - c - g \end{aligned}$$

- k^* unchanged, c^* lower (exactly to offset government spending)
- Reason: lump sum tax = take part of income, decision making unchanged

3.7.2 Flat labor income tax τ_w + nonproductive spending G

- firms' problem unchanged - determine $r = f'(k) - \delta; w = f(k) - f'(k)k$
- government's budget constraint: $G = \tau_w wL; \quad \tau_w = g/w$
- household's budget constraint: $\dot{a} = (1 - \tau_w)w + ra - na - c$

Hamiltonian for household's problem:

$$H = u(c)e^{-(\rho-n)t} + \mu[(1-\tau_w)w + ra - na - c]$$

- $\frac{\partial H}{\partial c}$ and $\partial H/\partial a$ do not change => Euler equation is unchanged

In the equilibrium we plug in for w, r (firm's problem) and g (gvt BC) and replace $k = a$

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\theta}[f'(k) - \delta - \rho] \\ \dot{k} &= f(k) - (n+\delta)k - c - g \end{aligned}$$

- k^* unchanged, c^* lower (exactly to offset government spending)
- Reason: inelastic supply of labor - HH cannot adjust (like lump sum tax)

3.7.3 Flat capital income tax τ_a + nonproductive spending G

- firms' problem unchanged - determine $r = f'(k) - \delta; w = f(k) - f'(k)k$
- government's budget constraint: $G = \tau_a raL; \quad \tau_a = g/(ra)$
- household's budget constraint: $\dot{a} = w + (1 - \tau_a)ra - na - c$

Hamiltonian and F.O.C.'s for household's problem:

$$\begin{aligned} H &= u(c)e^{-(\rho-n)t} + \mu[w + (1-\tau_a)ra - na - c] \\ \frac{\partial H}{\partial c} &= 0 : \quad u'(c)e^{-(\rho-n)t} = \mu \\ \frac{\partial H}{\partial a} &= -\dot{\mu} \quad \mu[(1-\tau_a)r - n] = \dot{\mu} \end{aligned}$$

- new Euler equation therefore looks $\frac{\dot{c}}{c} = \frac{1}{\theta}[(1-\tau_a)r - \rho]$

In the equilibrium we plug in for w, r (firm's problem) and g (gvt BC) and replace $k = a$

$$\begin{aligned}\frac{\dot{c}}{c} &= \frac{1}{\theta}\Big[(1-\tau_a)(f'(k)-\delta) - \rho\Big] \\ \dot{k} &= f(k) - (n+\delta)k - c - g\end{aligned}$$

- k^* lower, c^* lower
- Reason: HH's adjust accumulation of assets to keep consumption -> lower capital investment -> lower total output -> even lower consumption
- if taxation affects decision making of HH = **distortionary taxation**

3.7.4 Flat capital income tax τ_a + transfers V

- firms' problem unchanged - determine $r = f'(k) - \delta; w = f(k) - f'(k)k$
- government's budget constraint: $V = \tau_a raL; \quad \tau_a = v/(ra)$
- household's budget constraint: $\dot{a} = w + (1-\tau_a)ra - na - c + v$

Hamiltonian and F.O.C.'s for household's problem:

$$\begin{aligned}H &= u(c)e^{-(\rho-n)t} + \mu[w + (1-\tau_a)ra - na - c + v] \\ \frac{\partial H}{\partial c} &= 0: \quad u'(c)e^{-(\rho-n)t} = \mu \\ \frac{\partial H}{\partial a} &= -\dot{\mu} \quad \mu[(1-\tau_a)r - n] = \dot{\mu}\end{aligned}$$

- new Euler equation therefore looks $\frac{\dot{c}}{c} = \frac{1}{\theta}[(1-\tau_a)r - \rho]$

In the equilibrium we plug in for w, r (firm's problem) and v (gvt BC) and replace $k = a$

$$\begin{aligned}\frac{\dot{c}}{c} &= \frac{1}{\theta}\Big[(1-\tau_a)(f'(k)-\delta) - \rho\Big] \\ \dot{k} &= f(k) - (n+\delta)k - c\end{aligned}$$

- k^* lower, c^* lower
- Reason: even though taxes come back in the form of transfers, HH's still adjust accumulation of assets due to lower rate of return -> lower capital investment -> lower total output -> lower consumption
- still **distortionary taxation**

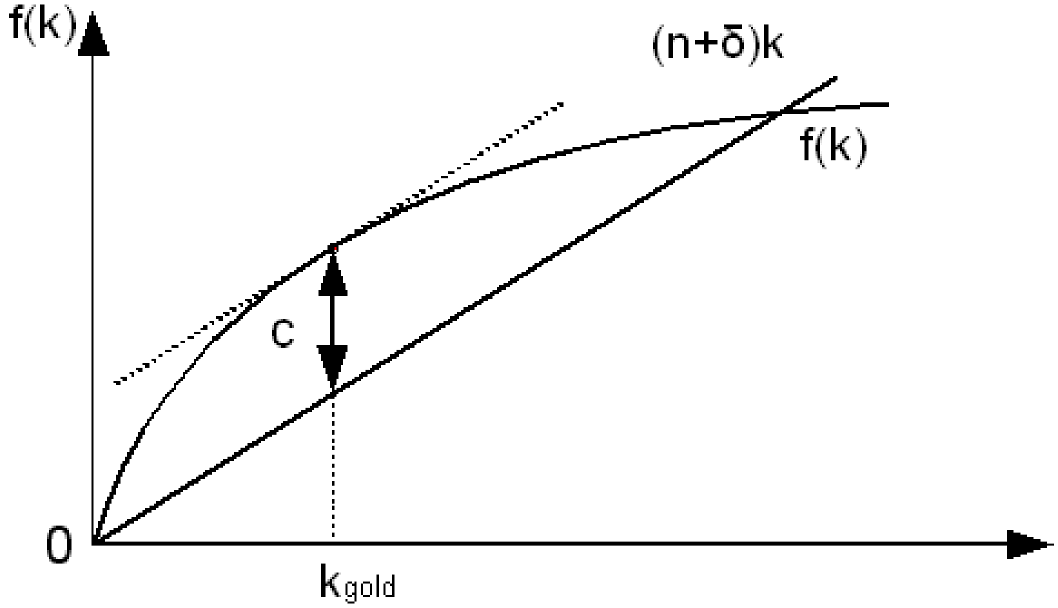

Figure 1: Consumption as a function of k - RHS of $\dot{k} = 0$ locus.

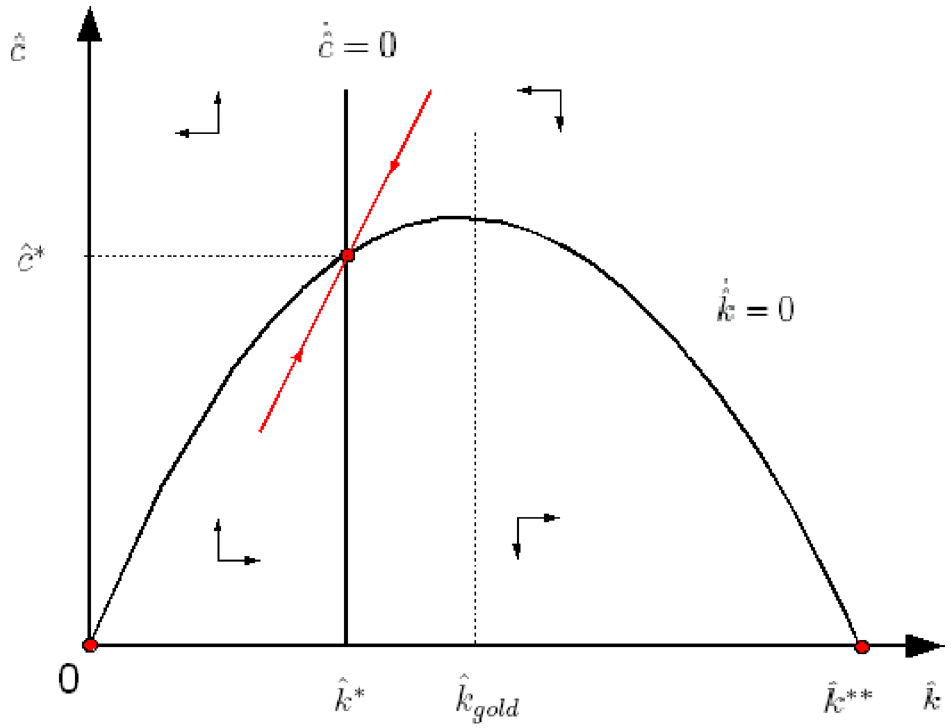

Figure 2: Phase diagram of the Ramsey model.