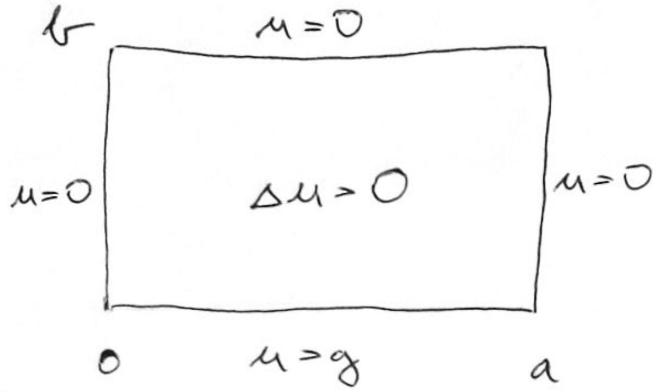
Fourierova metoda



$\boxed{\Delta u = 0}$

$$u(x,y) = X(x)Y(y) \longrightarrow \Delta u(x,y) = X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\xrightarrow{u \neq 0 \ /:XY} \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \longrightarrow \frac{X''}{X}(x) = -\frac{Y''}{Y}(y) = \lambda$$

$\lambda \in \mathbb{R}$

$$\longrightarrow X'' = \lambda X \quad \text{na } (0,a)$$

$$Y'' = -\lambda Y \quad \text{na } (0,b)$$

$\boxed{u = 0, \quad x = 0 \vee a} \longrightarrow X(0) = X(a) = 0$

Pomocný problém: $\begin{cases} X'' - \lambda X = 0 & \text{na } (0,a) \\ X(0) = X(a) = 0 \end{cases}$

Existuje pro nějaké λ netriviální řešení?

Char. rce: $r^2 - \lambda = 0$, $r^2 = \lambda$

1) $\lambda > 0$, $r = \pm\sqrt{\lambda}$: f.s. $\{\sinh(\sqrt{\lambda}x), \cosh(\sqrt{\lambda}x)\}$

$X(0) = 0 \Rightarrow$

$X(x) = \alpha \sinh(\sqrt{\lambda}x)$

nelze X

2) $\lambda = 0$, $r = 0$ f.s. $\{1, x\}$

$X(0) = 0 \Rightarrow X(x) = \alpha x \Rightarrow \alpha = 0$ X nelze

3) $\lambda < 0$: $r = \pm\sqrt{-\lambda}\, i$ f.s.: $\{\sin(\sqrt{-\lambda}\, x), \cos(\sqrt{-\lambda}\, x)\}$

$X(0) = 0 \Rightarrow X(x) = \alpha \cdot \sin(\sqrt{-\lambda}\, x)$

$X(a) = 0 \Rightarrow \sin(\sqrt{-\lambda}\, a) = 0 \Rightarrow \sqrt{-\lambda}\, a = k\pi$ pro $k \in \mathbb{N}$

$$\lambda = -\frac{k^2\pi^2}{a^2} \; ; \; \sqrt{-\lambda} = \frac{k\pi}{a}$$

Řešení pomocného problému:

$$\forall k \in \mathbb{N}; \; \lambda = -\frac{k^2\pi^2}{a^2}, \; \sqrt{-\lambda} = \frac{k\pi}{a}$$

$$X_k(x) = \sin\left(\frac{k\pi x}{a}\right)$$

Dopočti Y: $Y'' - \frac{k^2\pi^2}{a^2} Y = 0$ v $(0, b)$

$$Y(b) = 0$$

$r = \pm \frac{k\pi}{a}$; f.s.: $\{\sinh(\frac{\pi k}{a}(y-b)), \cosh(\frac{k\pi}{a}(y-b))\}$

$$Y_k(y) = \sinh(y-b)$$

→ Řešení problému $u=0$ $\boxed{\Delta u = 0}$ $u=0$: (nahoře $u=0$, dole ?)

$\forall k \in \mathbb{N}$: $u_k(x,y) = X_k(x) Y_k(y) = \sinh\left[\frac{k\pi}{a}(y-b)\right] \sin\left(\frac{k\pi}{a} x\right)$

$= \sinh\left(\frac{k\pi}{a}(y-b)\right) \sin\frac{k\pi x}{a}$

Jak splníme podm. v $y = 0$?

Hledáme u jako:

$$u(x,y) = \sum_{k=1}^{+\infty} \alpha_k \sinh\left(\frac{k\pi}{a}(y-b)\right) \sin\frac{k\pi x}{a}$$

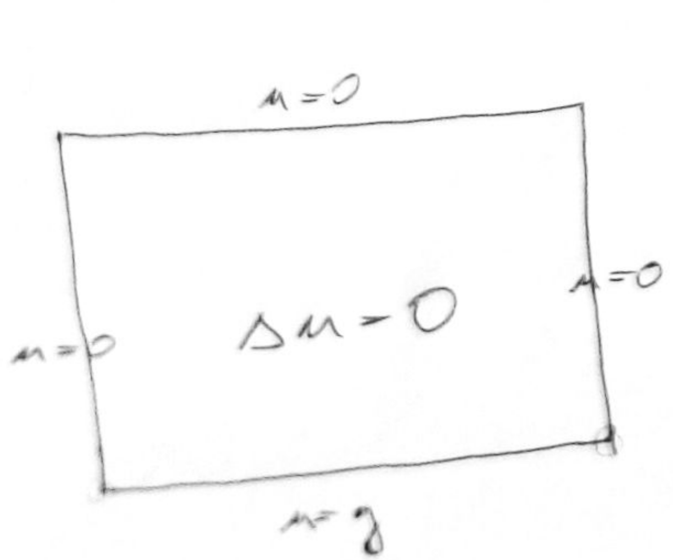
$$g(x) = u(x,0) = \sum_{k=1}^{+\infty} \underbrace{\alpha_k \sinh\frac{k\pi(-b)}{a}}_{\beta_k} \sin\left(\frac{k\pi x}{a}\right)$$

Sinová Fourierova řada g prodloužené liše na $(-a,a)$.

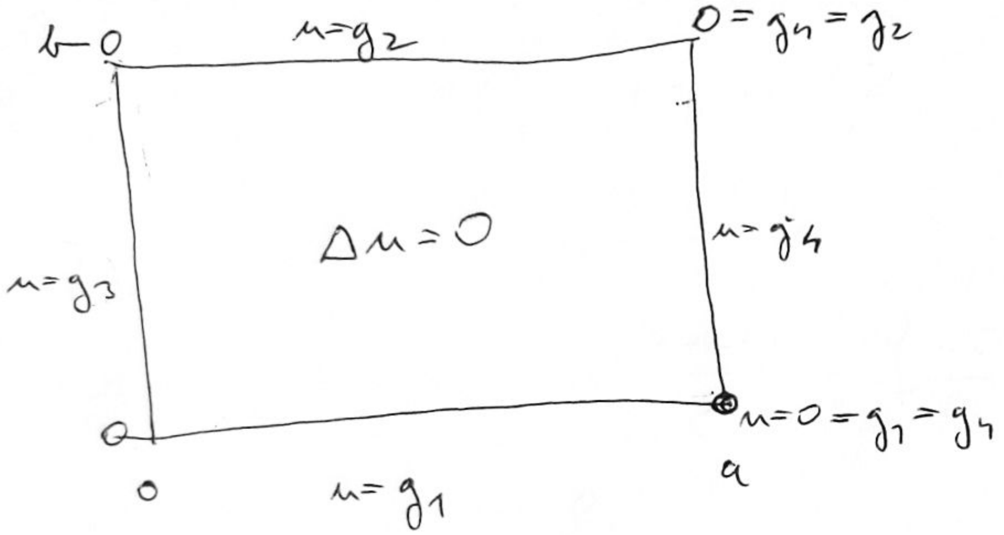
$$\beta_k = \frac{2}{a}\int_0^a g(t) \sin \frac{k\pi t}{a}\, dt$$

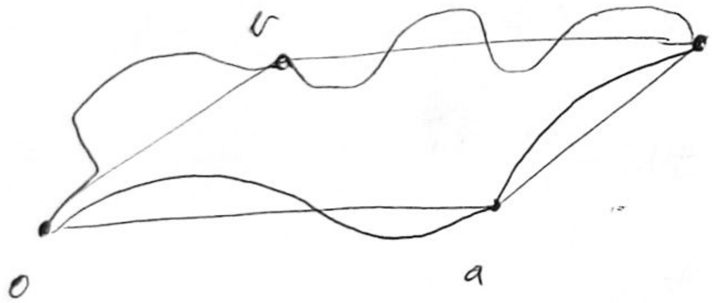
$$\Rightarrow \alpha_k = \frac{2}{a}\int_0^a g(t) \sin\frac{k\pi t}{a}\, dt \Big/ \sinh\left(\frac{k\pi(-b)}{a}\right)$$

$$u(x,y) = \sum_{k=1}^{+\infty} \frac{2}{a}\int_0^a g(t) \sin\frac{k\pi t}{a}\, dt \; \frac{\sinh\left(\frac{k\pi}{a}(y-b)\right)}{\sinh\left[\frac{k\pi}{a}(-b)\right]} \sin\frac{k\pi x}{a}$$



Př 6





$u_1 \sim g_1$; $u_2 \sim g_2$; $u_3 - g_3$, $u_4 \sim g_4$

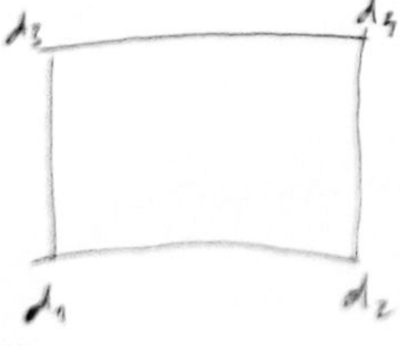
$\longrightarrow$ $u = u_1 + u_2 + u_3 + u_4$

Př 7 bez podmínky souhlasnosti v rozích

$$u(x,y) = v(x,y) + c(x,y)$$

kde c řeší $\Delta c = 0$ a v v rozích má stejné hodnoty jako u

$c(x,y) = c_0 + c_1 x + c_2 y + c_3 xy$ řeší $\Delta c = 0$



$$c_0 = d_1$$
$$c_0 + \quad + c_2 b \quad = d_3$$
$$c_0 + c_1 a + \quad = d_2$$
$$c_0 + c_1 a + c_2 b + c_3 ab = d_4$$

matice soustavy $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & b & 0 \\ 1 & a & 0 & 0 \\ 1 & a & b & ab \end{pmatrix}$

$\det A \neq 0$ pokud $a \cdot b \neq 0$.

$\rightarrow$ c ex. a řešíme příklad 6 pro v.

$\Delta v = 0$ v obdélníku

v splňuje na hranici obdélníku $u - c$

$\rightarrow$ nabývá v rozích hodnoty 0.

Jiný přístup k Fourierově metodě

Máme PDR $\Delta u = 0$; $\partial_x^2 u + \partial_y^2 u = 0$

Rozvineme u pro pevné y do báze z. Hilbertova prostoru.

$\{w_k\}$ (např. $L^2(0,a)$)

$$u(x,y) = \sum_{k=1}^{+\infty} \frac{(u, w_k)(y)\, w_k(x)}{\|w_k\|^2} = \sum_{k=1}^{+\infty} p_k(y)\, w_k(x)$$

Dosadíme do rce: $$\sum_{k=1}^{+\infty} p_k(y)\, \underline{\underline{w_k''(x)}} + p_k''(y)\, w_k(x) = 0$$

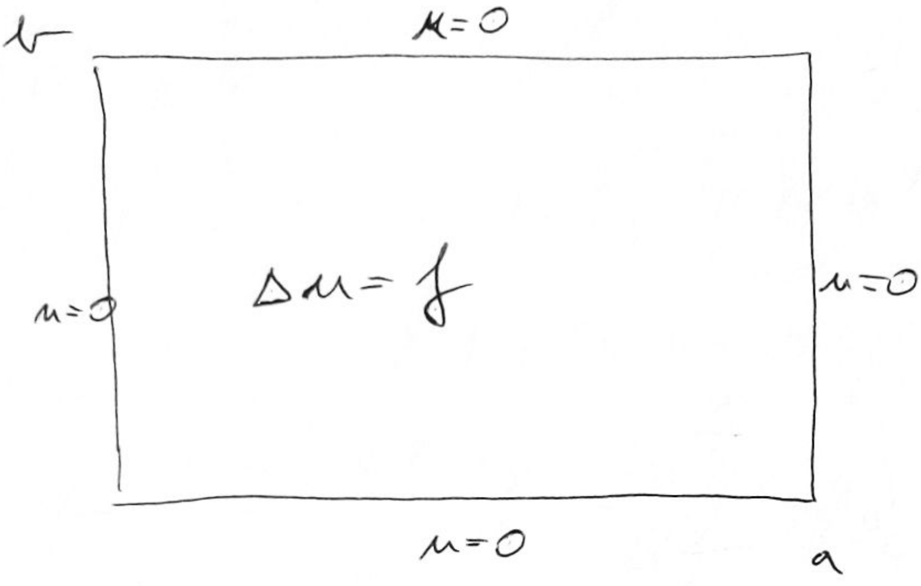
Vhodná volba w_k : $w_k''(x) = w_k(x) \cdot \lambda_k$ -> úloha na vlastní čísla

$w_k(0) = w_k(a) = 0$

$$\Rightarrow \sum_{k=1}^{+\infty} \left(p_k(y)\lambda_k + p_k''(y) \right) w_k(x) = 0$$

$$\Rightarrow \underline{\underline{p_k \lambda_k + p_k'' = 0}}$$

Příklad 8



$\cdot\ \Delta u = f$

bázi $\left\{ \sin \frac{l\pi x}{a} \right\}_{l=1}^{+\infty}$

projekce rce do báze: $u_l(y) := \int_0^a u(t,y) \sin \frac{l\pi t}{a}\, dt$

$$u(x,y) = \frac{2}{a} \sum_{l=1}^{+\infty} u_l(y) \sin\left(\frac{l\pi x}{a}\right)$$

dosadíme do $\Delta u = f$:

$$\frac{2}{a} \sum_{l=1}^{+\infty} u_l''(y) \cdot \sin\left(\frac{l\pi x}{a}\right) + u_l(y)\left(\sin \frac{l\pi x}{a}\right)'' = \sum_{l=1}^{+\infty} f_l(y) \sin \frac{l\pi x}{a}$$

$$\frac{2}{a} \sum_{l=1}^{+\infty} \left[u_l''(y) - \frac{l^2\pi^2}{a^2} u_l(y) \right] \sin \frac{l\pi x}{a} = \sum_{l=1}^{+\infty} f_l(y) \sin \frac{l\pi x}{a}$$

$$\Rightarrow \quad u_l''(y) - \frac{l^2\pi^2}{a^2} u_l(y) = \frac{a}{2} f_l(y)$$

$$+ \quad u_l(0) = u_l(b) = 0$$