Algorithms and datastructures I
Lecture 11: Master theorem, Strassen algorithm, k-th smallest element

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April 28 2020

Divide & Conquer

“Divide and conquer is an algorithm design paradigm based on multi-branched recursion. A divide-and-conquer algorithm works by recursively breaking down a problem into two or more sub-problems of the same or related type, until these become simple enough to be solved directly. The solutions to the sub-problems are then combined to give a solution to the original problem.”

John von Neumann

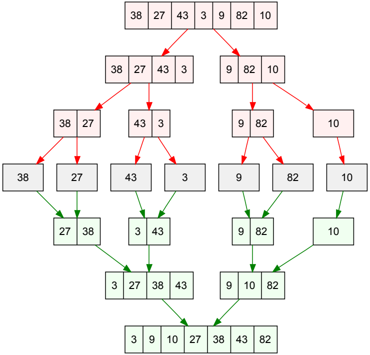
MergeSort $(x_1, \ldots, x_n)$, John von Neumann, 1945

1. if $n = 1$: Return (x_1).
2. $(y_1, \ldots, y_{\lfloor \frac{n}{2} \rfloor}) \leftarrow$MergeSort $(x_1, \ldots, x_{\lfloor \frac{n}{2} \rfloor})$
3. $(z_1, \ldots, z_{\lceil \frac{n}{2} \rceil}) \leftarrow$MergeSort $(x_{\lfloor \frac{n}{2} \rfloor + 1}, \ldots, x_n)$
4. Return Merge $((y_1, \ldots, y_{\lfloor \frac{n}{2} \rfloor}), (z_1, \ldots, z_{\lceil \frac{n}{2} \rceil}))$.

Time complexity (for $n = 2^k$)

$$T(n) = 2T(\frac{n}{2}) + \Theta(n)$$

$$T(1) = 1$$



Multiplication (Karatsuba 1960)

$$
\begin{aligned}
X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\
Y &= \boxed{C}\boxed{D} = C \cdot 10^{\frac{n}{2}} + D \\
X \cdot Y &= AC \cdot 10^{n} + (AD + BC) \cdot 10^{\frac{n}{2}} + BD \\
&= AC \cdot 10^{n} + ((A+B)(C+D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD
\end{aligned}
$$

Anatolii Alexeievitch Karatsuba

$$T(n) = 3T(\frac{n}{2}) + cn$$

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^{i} cn = cn \cdot \frac{\left(\frac{3}{2}\right)^{\log n+1} - 1}{\left(\frac{3}{2}\right) - 1} = \Theta\left(n \cdot \left(\frac{3}{2}\right)^{\log n}\right) = \Theta\left(n \cdot \left(\frac{3^{\log n}}{2^{\log n}}\right)\right) = \Theta(3^{\log n}) \\
&= \Theta\left(\left(2^{\log 3}\right)^{\log n}\right) = \Theta\left(2^{\log 3 \log n}\right) = \Theta\left(\left(2^{\log n}\right)^{\log 3}\right) = \Theta\left(n^{\log 3}\right) = \Theta\left(n^{1.59\cdots}\right).
\end{aligned}
$$

What about general case?

General recurrence

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\begin{aligned}
T(1) &= 1 \\
T(n) &= aT(\lfloor \frac{n}{b} \rfloor) + \Theta(n^c)
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# of subprob	size of subprob.	time per subprob	time per level

Lets do the math

$$T(N) = \Theta\left(\sum_{i=0}^{k} N^c \frac{a^i}{b^{ic}}\right)$$

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Question

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Question

What if $n \leq b^k$ for some integer k?

Easy: Put $b^k \leq n \leq b^{k+1}$ and then $T(b^k) \leq T(n) \leq T(b^{k+1})$

Master theorem

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has solution:

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Strassen's algorithm, 1969

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

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7 multiplications instead of 8 $\Rightarrow$ time complexity $T(n) = 7T\left(\frac{n}{2}\right) + \Theta(n^2) = \Theta(n^{\log_2 7}) = O(n^{2.808})$.

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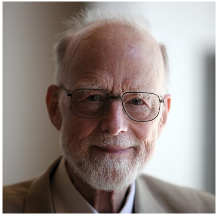
Current record: $(n^{2.373})$ with really big constant factors

Quickselect

Problem: Find k-th smallest element of a sequence $(x_1, \ldots, x_n)$.



Sir Tony Horae

QuickSelect($(x_1, \ldots, x_n)$, k), Sir Tony Horae 1961

1. Choose pivot p.
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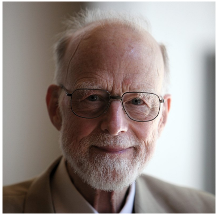

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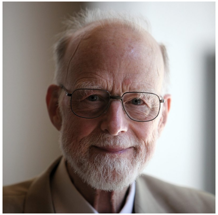

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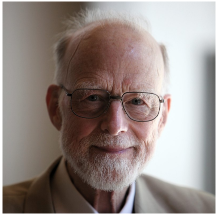

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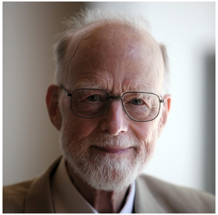

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$\mathbb{E}[\# \text{ of trials}] = \sum_{i \geq 1} i \cdot \Pr[\text{we do precisely } i \text{ trials}]$
$\Pr[\text{we do precisely } i \text{ trials}] = (1-p)^i p$
Method 2:
$\mathbb{E}[\# \text{ of trials}] = D = 1 + p0 + (1-p)D$
$D(1+1+p) = 1$

1. Assume that p is always maximum.
   Time complexity: $\Theta(N^2)$.
2. Assume that p is median.
   Time complexity: $T(N) = T\left(\frac{N}{2}\right) + \Theta(N) = \Theta(N)$.
3. Assume that p is "almost median" (at least $\frac{1}{4}$ of elements is smaller than p and $\frac{1}{4}$ is bigger than p).
   Time complexity: $T(N) = T\left(\frac{3N}{2}\right) + \Theta(N) = \Theta(N)$.
4. Randomized choice of almost median: $\Pr[\text{random element is almost median}] \geq \frac{1}{2}$

Lemma

Let V be an event that occurs in trial with probability p.
The expected number of trials to first occurrence of V is $\frac{1}{p}$.

Expected number of trials is 2; time complexity $\Theta(N)$.

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   Every stage reduces problem to $\frac{3}{4}$.
   Time complexity: $\Theta(N)$.