Domácí úkol na 22. 4.

$\cos\left(\frac{1}{z}\right)$: • izolovaná sing. v 0

• $\cos(y) = 1 - \frac{y^2}{2} + \frac{y^4}{4!} - \dots = \sum_{k=0}^{+\infty} (-1)^k \frac{y^{2k}}{(2k)!} \quad \forall y \in \mathbb{C}$

$$\cos\left(\frac{1}{z}\right) = \sum_{k=0}^{+\infty} (-1)^k \left(\frac{1}{z}\right)^{2k} \cdot \frac{1}{(2k)!} \quad \forall z \in \mathbb{C} \setminus \{0\}$$

$$= \sum_{k=-\infty}^{-1} (-1)^k z^{2k} \frac{1}{(-2k)!} + 1$$

$$= \sum_{\substack{k=-\infty \\ k \text{ sudé}}}^{-1} (-1)^{k/2} z^k \frac{1}{(-k)!} + 1$$

$$\cos(y) \stackrel{?}{=} \sum_{k=0}^{+\infty} \frac{\cos^{(k)}(0)}{k!} y^k$$

$u\ y^2$: $\frac{\cos^{(2)}(0)}{2!} = -\frac{\cos(0)}{2!}$

AM 22.4.

Aplikace residuové věty

1) $\int_0^{2\pi} \frac{P(\sin x, \cos x)}{Q(\sin x, \cos x)} dx$

$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right), \quad \cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$\gamma(t) := e^{it}$ ... $\sin(t) = S \circ \gamma(t)$, kde $S(z) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$t \in [0, 2\pi]$ $\quad \cos(t) = C \circ \gamma(t)$, kde $C(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\int_\gamma \frac{1}{z} \frac{P\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)}{Q\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)} dz \overset{\text{res. věta}}{=} 2\pi i \sum_{\substack{z_k \text{ res} \\ \in K(0,1)}} \operatorname{res}_{z_k} \left(\frac{1}{z} \frac{P(\dots)}{Q(\dots)}\right)$$

$$\| $$

$$\int_0^{2\pi} \frac{1}{\cancel{e^{it}}} \frac{P(\sin t, \cos t)}{Q(\sin t, \cos t)} \cdot i\cancel{e^{it}} dt$$

Př: $\int_0^{2\pi} \frac{\cos x}{5 + 4\cos x} dx = 2\pi\left(\frac{1}{4} - \frac{5}{12}\right) = 2\pi \frac{\cancel{8}^{-2}}{12} = -\frac{\cancel{4}\pi}{3}$

$= -\frac{\pi}{3}$

$$F(z) = \frac{\frac{1}{2}\left(z + \frac{1}{z}\right)}{5 + 4\frac{1}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{1}{z} = \frac{\frac{1}{2}\frac{z^2+1}{\cancel{z}}}{\frac{5z + 2z^2 + 2}{\cancel{z}}} \frac{1}{z} = \frac{1}{2} \frac{z^2 + 1}{z(2z^2 + 5z + 2)}$$

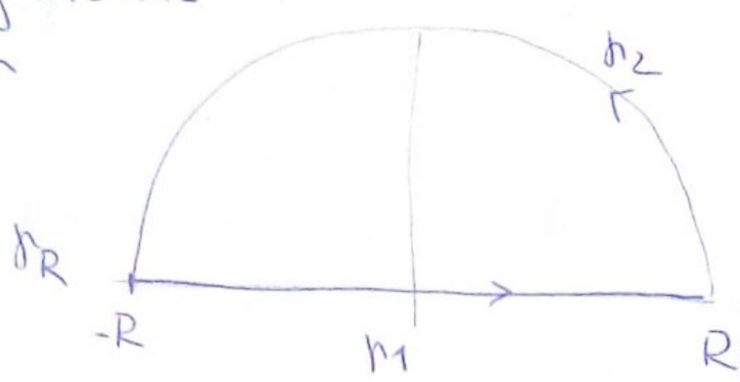
$$z_{1,2} = \frac{-5 \pm \sqrt{25 - 16}}{4} = -\frac{5}{4} \pm \frac{3}{4} = \begin{cases} -\frac{1}{2} \in U(0,1) \\ -2 \notin U(0,1) \end{cases}$$

$$\operatorname{res}_0 F(z) = \frac{1}{4}, \quad \operatorname{res}_{-\frac{1}{2}} F(z) = \operatorname{res}_{-\frac{1}{2}} \frac{1}{2}\frac{z^2+1}{z} \Big/ 2z^2 + 5z + 2 =$$

$$= +\frac{1}{2} \frac{5/4}{-1/2} \Big/ -\frac{4}{2} + 5 = -\frac{5}{4} \cdot \frac{1}{3} = \frac{-5}{12}$$

2) Integrály a rez. fcí

$$I = \int_{-\infty}^{+\infty} \frac{x^2}{(x^2+2x+2)^2}\,dx \longrightarrow \int_\gamma F(z)\,dz$$



$$F(z) := \frac{z^2}{(z^2+2z+2)^2}$$

$$\int_{\gamma_1} F(z)\,dz \xrightarrow{R\to+\infty} I$$

$$\int_{\gamma_2} F(z)\,dz \longrightarrow 0$$ podle Lemmatu o velké kružnici

$$\int_{\gamma_R} F(z)\,dz \longrightarrow I$$

$\parallel$ rez. věta

$2\pi i$ (součet reziduí v horní polorovině) $= \pi$

$$F(z) = \frac{z^2}{((z+1)^2+1)^2} = \frac{z^2}{[(z+1+i)(z+1-i)]^2}$$

$\Rightarrow$ F má pól nás. 2 v bodech $-1 \pm i$ nás zajímá $-1+i$

$$\operatorname{res}_{-1+i} F(z) = \lim_{z\to -1+i} \left(F(z)(z+1-i)^2\right)' =$$

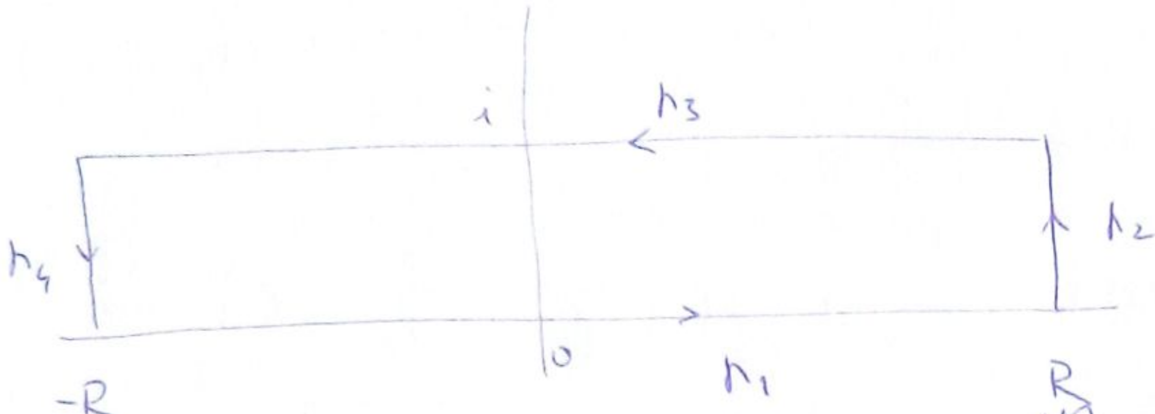
$$= \lim_{z\to-1+i} \left(\frac{z^2}{(z+1+i)^2}\right)' = \lim_{z\to-1+i} 2\frac{z}{z+1+i}\cdot\frac{z+1+i-z}{(z+1+i)^2}$$

$$= 2\frac{(-1+i)(1+i)}{(2i)^3} = \frac{1}{4}\,\frac{i^2-1}{-i} = \frac{1}{2}\,\frac{1}{i}$$

Ke 4) V $\mathbb{C}$ není věta o substituci

$$I=\int_{-\infty}^{+\infty} e^{-(x+i)^2}\,dx \neq \left|\begin{matrix} y=x+i\\ dy=dx\end{matrix}\right| \text{ atd. \dots NE!}$$

ale použijeme residuovou větu



$$F(z)=e^{-z^2} \qquad \int_{\gamma_3} F(z)\,dz \xrightarrow{R\to+\infty} -\int_{-\infty}^{+\infty} e^{-(t+i)^2}\,dt = -I$$

$$\gamma_3(t) = i+t,\ t\in[-R,R]$$

$$\int_{\gamma_1} F(z)\,dz \xrightarrow{R\to+\infty} \int_{-\infty}^{+\infty} e^{-t^2}\,dt = \sqrt{\pi}$$

Pokud $\int_{\gamma_2} e^{-z^2}dz$ a $\int_{\gamma_4} e^{-z^2}dz \xrightarrow{R\to+\infty} 0 \to I=\sqrt{\pi}$

$$\left|\int_{\gamma_2} e^{-z^2}dz\right| = \left|\int_0^1 e^{-(R+it)^2}\, i\,dt\right| \le \int_0^1 e^{-(R^2-t^2)}\,dt$$

$$\gamma_2(t)=R+it,\ t\in[0,1] \qquad \le \int_0^1 e^{-R^2+1}\,dt \xrightarrow[R\to+\infty]{} 0$$

podobně γ_2

pozn. stejně funguje pro $\int_{-\infty}^{+\infty} e^{-(x+\alpha i)^2}dx$, $\alpha\in\mathbb{R}$.

sin, cos, exp ... holomorfní
sinh, cosh

$$\sin(x+y) = \sin x \cos y + \sin y \cos x \quad \forall x \in \mathbb{R}$$

$$\sin(z+w) \overset{?}{=} \sin z \cos w + \sin w \cos z \quad \forall z, w \in \mathbb{C}$$

↑ ano, podle věty o jednoznačnosti

1) $F(z) := \sin(z+y) - (\sin z \cos y + \sin y \cos z)$ pro $y \in \mathbb{R}$ pevné

- F je holomorfní v $\mathbb{C}$
- $F(z) = 0 \quad \forall z \in \mathbb{R}$

$\Rightarrow$ V. o jedn. $F(z) = 0 \quad \forall z \in \mathbb{C}$

2) $G(w) := \sin(z+w) - (\sin z \cos w + \sin w \cos z)$, $z \in \mathbb{C}$ pevné!

- $G \in \mathcal{H}(\mathbb{C})$
- $G(w) = 0 \quad \forall w \in \mathbb{R}$ (z 1))

$\Rightarrow \quad G \equiv 0$ v $\mathbb{C}$

věta o jedn.

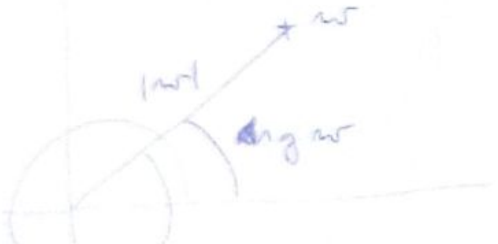
Př 2: $y = it$, $t \in \mathbb{R} \Rightarrow \cos(it) = \frac{1}{2}\left(e^{-t} + e^{+t}\right)$
$= \cosh(t)$

Př 3: Inverzní fce k exp: $e^z = w$; vyjádři z pomocí w

$z = \alpha + i\beta$; $e^{\alpha + i\beta} = w \Rightarrow |w| = e^{\alpha}$

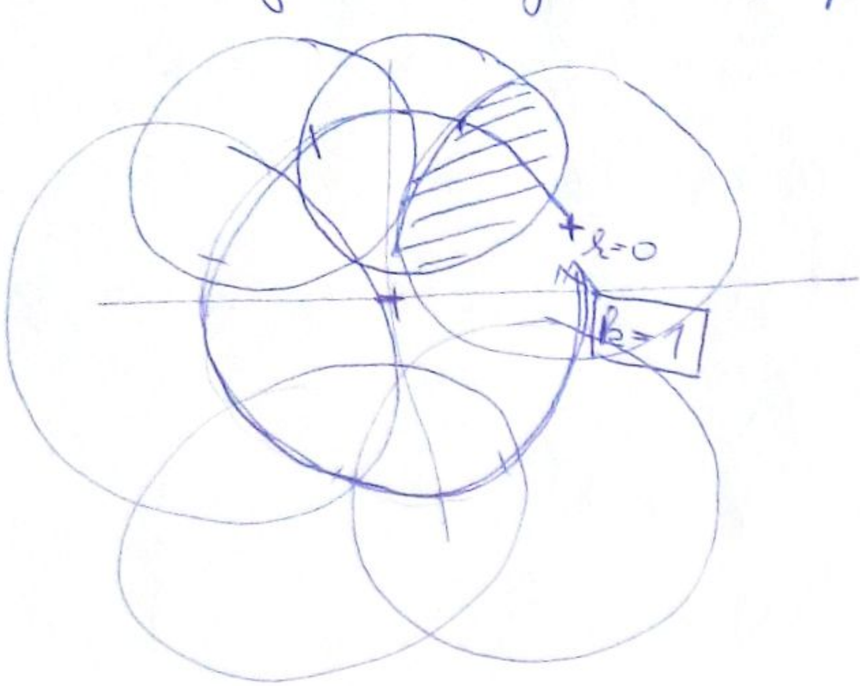
$\alpha = \lg|w|$, $\beta = \arg w$

$$\underbrace{z = \lg|w| + i(\arg w + 2\pi k), \quad k \in \mathbb{Z}}_{\text{komplexní logaritmus}}$$



Komplexní lg

$$\ln z := \lg|z| + i \arg z + 2\pi k i \quad \text{pro j. } k \in \mathbb{Z}$$



$$\sqrt{-1} = \exp\left(\tfrac{1}{2}\ln(-1)\right) = \exp\left(\tfrac{1}{2}\left(\underbrace{\lg 1}_{=0} + i\pi + 2k\pi i\right)\right)$$

$$-1 = 1 \cdot e^{i\pi}$$

$$= e^{\frac{i\pi}{2}} \cdot e^{k\pi i} = (-1)^k i, \quad k \in \mathbb{Z}$$

$\hookrightarrow = \cos k\pi + i \underbrace{\sin k\pi}_{=0}$, $\cos k\pi = (-1)^k$

$\hookrightarrow$ $\cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = i$

i^i: $\quad i = e^{\frac{\pi}{2} i} \qquad i^i = e^{-\frac{\pi}{2}} e^{-2k\pi}, \quad k \in \mathbb{Z}$

Fourierova transformace

$f \in L^1(\mathbb{R}^m)$ tj. $\int_{\mathbb{R}^m} |f| \in \mathbb{R}$

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^m} f(x)\, e^{-2\pi i (x,\xi)}\, dx \quad \text{je definován?}$$

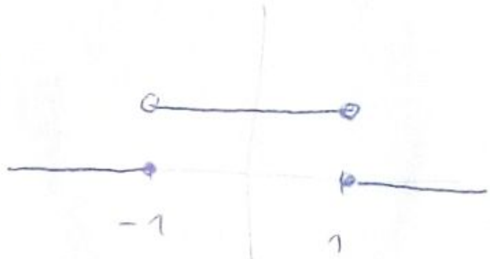
$$|f(x)\, e^{-2\pi i (x,\xi)}| \le |f(x)| \in L^1(\mathbb{R}^m) \to \text{ano!}$$

- $$\hat f(\xi) = \int_{\mathbb{R}^m} f(x)\, e^{-2\pi i (x,\xi)}\, dx = \int_{\mathbb{R}^m} f(x)\, e^{2\pi i (x,-\xi)}\, dx$$

$$= \check f(-\xi)$$

- $\check{\hat f} = f$ obecně neplatí!

Př: $f(t) := \chi_{(-1,1)}(t)$, $m = 1$



$$\hat f(\xi) = \int_{\mathbb{R}} \chi_{(-1,1)}(t)\, e^{-2\pi i (t,\xi)}\, dt =$$

$$= \int_{-1}^{1} e^{-2\pi i t\xi}\, dt = \int_{-1}^{1} \cos(2\pi t\xi) \mp i \cancel{\sin(2\pi t\xi)}\, dt$$

$$= \left[\frac{\sin 2\pi t\xi}{2\pi\xi} \right]_{t=-1}^{1} = \frac{\sin 2\pi\xi}{\pi\xi}$$

$$\hat{f}(\xi) = \frac{\sin 2\pi\xi}{\pi\xi}$$

$$\int_{\mathbb{R}} |\hat{f}(\xi)|\, d\xi = \int_{\mathbb{R}} \frac{|\sin 2\pi\xi|}{\pi\xi}\, d\xi = +\infty \; !$$

$$\Rightarrow \hat{f} \notin L^1(\mathbb{R})$$

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f sféricky symetrická? $\Leftrightarrow$ $\forall x \in \mathbb{R}^n$, A ON matice:

$$f(x) = f(Ax)$$

$$\hat{f}(A\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i (x, A\xi)}\, dx =$$

$$(x, A\xi) = (\underbrace{A^T x}_{y}, \xi)$$

$$= \int_{\mathbb{R}^n} \underbrace{f(Ay)}_{f(y)}\, e^{-2\pi i (y, \xi)}\, \underbrace{\det A^T}_{1}\, dy$$

$$= \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i (y, \xi)}\, dy = \hat{f}(\xi)$$