5 . Sequences and series of functions

We can define not only the sequences (and series) of real numbers but also sequences on general topology space or both sequences and series on linear topology spaces (X,τ) or even normed vector space $(X,\|\cdot\|)$.

Then we define a **sequence** as a mapping of $\mathbb{N}$to (X,τ) or $(X,\|\cdot\|)$ and we denote it

$$\begin{array}{llll} \{x_n\}_{n=1}^{\infty}: & \mathbb{N} \longrightarrow (X,\tau) & : n \mapsto x_n & \text{or} \\ \{x_n\}_{n=1}^{\infty}: & \mathbb{N} \longrightarrow (X,\|\cdot\|) & : n \mapsto x_n & . \end{array}$$

There is possible introduce also limit of sequence and sum of sequence or conception of convergence (or divergence) amd summability by following way.

$$\begin{array}{lll} x_n \stackrel{\tau}{\to} x & \stackrel{\text{def.}}{\Longleftrightarrow} & (\forall U \ \tau\text{-neighbourhood of } 0)\,(\exists n_0)\,(\forall n \geq n_0)\ x_n - x \in U \\ \sum\limits_{n=1}^{\infty} x_n \stackrel{\tau}{=} s & \stackrel{\text{def.}}{\Longleftrightarrow} & (\forall U \ \tau\text{-neighbourhood of } 0)\,(\exists n_0)\,(\forall n \geq n_0)\ x_1 + x_2 + \cdots + x_n - s \in U \\ x_n \stackrel{\|\cdot\|}{\to} x & \stackrel{\text{def.}}{\Longleftrightarrow} & ||x_n - x|| \to 0 \\ \sum\limits_{n=1}^{\infty} x_n \stackrel{\|\cdot\|}{=} s & \stackrel{\text{def.}}{\Longleftrightarrow} & ||\sum\limits_{k=1}^{n} x_k - s|| \to 0 \\ x_n \ \tau\text{-convergent in } X & \stackrel{\text{def.}}{\Longleftrightarrow} & (\exists x \in X)\ x_n \stackrel{\tau}{\to} x \\ x_n \ \tau\text{- summable in } X & \stackrel{\text{def.}}{\Longleftrightarrow} & (\exists s \in X)\ \sum\limits_{n=1}^{\infty} x_n \stackrel{\tau}{=} s \\ x_n \ \|\cdot\|\text{-convergent in } X & \stackrel{\text{def.}}{\Longleftrightarrow} & (\exists x \in X)\ x_n \stackrel{\|\cdot\|}{\to} x \\ x_n \ \|\cdot\|\text{-summable in } X & \stackrel{\text{def.}}{\Longleftrightarrow} & (\exists s \in X)\ \sum\limits_{n=1}^{\infty} x_n \stackrel{\|\cdot\|}{=} s \end{array}$$

Given $S \subset \mathbb{R}$, we can define

$$\mathcal{F}(S) := \{f : S \longrightarrow \mathbb{R}\}$$

space of all real functions on the set S. As for any $f, g \in \mathcal{F}(S)$, $\alpha \in \mathbb{R}$ also $f + g \in \mathcal{F}(S)$ and $\alpha f \in \mathcal{F}(S)$ and operations of addition and multiples fulfil corresponding axioms $\mathcal{F}(S)$ create vector space.

We can introduce also topologies on $\mathcal{F}(S)$.

1. Weak topology can be defined by base of neighbourhoods of 0

$$U_{n,K} := \{f \in \mathcal{F}(S); \sup_{x\in K} |f(x)| \leq \frac{1}{n}\},\ \text{where } n \in \mathbb{N} \text{ and } K \subset S,\ K \text{ finite} .$$

2. Strong topology can be defined by norm

$$||f|| := \sup_{x\in S} |f(x)|, \quad f \in \mathcal{F}(S)\,,$$

then base of neighbourhoods consists of $U_n = \{f \in \mathcal{F}(S); ||f|| \leq \frac{1}{n}\}$, where $n \in \mathbb{N}$. Now we can use the presented definitions of limits, sums, pointwise convergence or summability (in the case of weak topology) and uniform convergence or summability (in the case of strong topology). But this conception suppose some basic knowledges of topology and functional analysis.

Therefore we shall define limits, sums, convergence and summability by another way. We can imagine sequences of real functions on S as a map

$$\begin{array}{llll} \{f_n\}_{n=1}^{\infty}: & \mathbb{N} \longrightarrow \mathcal{F}(S) & : n \mapsto f_n & \text{or} \\ & \mathbb{N} \times \mathbb{R} \longrightarrow \mathbb{R} & : (n;x) \mapsto f_n(x)\,. & \end{array}$$

pointwise and uniform limit

def. 23

$$S \subset \mathbb{R},\ f, f_n : S \longrightarrow \mathbb{R} :$$
$$f_n \to f \text{ pointwise on } S \quad \overset{\text{def.}}{\Longleftrightarrow} \quad (\forall x \in S)\ f_n(x) \to f(x)$$
$$f_n \to f \text{ uniformly on } S \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \sup_{x \in S} |f_n(x) - f(x)| \to 0$$

We also denote $f = \lim_{n\to\infty} f_n$ pointwise or uniformly on S and we say f is pointwise or uniform limit of sequence f_n on S or f_n tends to f pointwise or uniformly on S. (There is sometimes used notation $f_n \rightrightarrows f$ on S for uniform limit.)

pointwise and uniform sum

def. 24

$$S \subset \mathbb{R},\ f, f_n : S \longrightarrow \mathbb{R} :$$
$$\sum_{n=1}^{\infty} f_n = f \text{ pointwise on } S \quad \overset{\text{def.}}{\Longleftrightarrow} \quad (f_1 + f_2 + \cdots + f_n) \to f \text{ pointwise on } S$$
$$\left(\text{or } (\forall x \in S)\ \sum_{n=1}^{\infty} f_n(x) = f(x) \right)$$
$$\sum_{n=1}^{\infty} f_n = f \text{ uniformly on } S \quad \overset{\text{def.}}{\Longleftrightarrow} \quad (f_1 + f_2 + \cdots + f_n) \to f \text{ uniformly on } S$$
$$\left(\text{or } \sup_{x \in S} \Big| \sum_{k=1}^{n} f_k(x) - f(x) \Big| \to 0 \right)$$

It is said f is pointwise or uniform sum of sequence f_n on S or series of sequence f_n tends to f pointwise or uniformly on S.

pointwise and uniform convergence and summability

def. 25

$$S \subset \mathbb{R},\ f_n : S \longrightarrow \mathbb{R} :$$
$$f_n \text{ pointwise convergent on } S \quad \overset{\text{def.}}{\Longleftrightarrow} \quad (\exists f \text{ real function on } S)\ f_n \to f \text{ pointwise on } S$$
$$f_n \text{ uniformly convergent on } S \quad \overset{\text{def.}}{\Longleftrightarrow} \quad (\exists f \text{ real function on } S)\ f_n \to f \text{ uniformly on } S$$
$$f_n \text{ pointwise summable on } S \quad \overset{\text{def.}}{\Longleftrightarrow} \quad (\exists f \text{ real function on } S)\ \sum_{n=1}^{\infty} f_n = f \text{ pointwise on } S$$
$$f_n \text{ uniformly summable on } S \quad \overset{\text{def.}}{\Longleftrightarrow} \quad (\exists f \text{ real function on } S)\ \sum_{n=1}^{\infty} f_n = f \text{ uniformly on } S$$

Bolzano - Cauchy

st. 108

$$S \subset \mathbb{R},\ f_n : S \longrightarrow \mathbb{R} :$$
$$f_n \text{ pointwise convergent on } S \iff (\forall x \in S)\,(\forall \epsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall m, n \geq n_0)\ \ |f_n(x) - f_m(x)| < \epsilon$$
$$f_n \text{ uniformly convergent on } S \iff (\forall \epsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall m, n \geq n_0)\,(\forall x \in S)\ \ |f_n(x) - f_m(x)| < \epsilon$$
$$\left(\text{or } \sup_{x \in S} |f_n(x) - f_m(x)| < \epsilon \right)$$

proof. It is consequence of next definitions and statement 52. ■

Similar statement holds also for pointwise or uniformly summable sequences.

It is obvious that sequence f_n uniformly convergent on both S and T is also uniformly convergent on union $S \cup T$. It holds naturally for pointwise convergence, too.

st. 109

$$\boxed{\begin{array}{l} f_n, f:]a; a+\Delta[\longrightarrow \mathbb{R},\ \Delta > 0: \\ \lim\limits_{x\to a^+} f_n(x) = a_n \in \mathbb{R},\ \text{and}\ f_n \to f\ \text{uniformly on}\ ]a; a+\Delta[\quad \Longrightarrow a_n \to A \in \mathbb{R}\ \text{and}\ \lim\limits_{x\to a^+} f(x) = A \end{array}}$$

proof. I. $a_n \to A \in \mathbb{R}$: Given arbitrary $\epsilon > 0$. As f_n uniformly convergent on $]a; a+\Delta[$ we have n_1 such that for any $m, n \geq n_1$

$$\sup_{x\in]a;a+\Delta[} |f_n(x) - f_m(x)| < \frac{\epsilon}{3}\,.$$

Let $m, n \geq n_1$ given also arbitrary. The existence of finite limit $\lim\limits_{x\to a^+} f_n(x) = a_n \in \mathbb{R}$ ensures existence of $\delta_n > 0$, $\Delta > \delta$ such that for any $x \in]a; a+\delta_n[$

$$|f_n(x) - a_n| < \frac{\epsilon}{3}$$

and similarly existence of $\lim\limits_{x\to a^+} f_m(x) = a_m \in \mathbb{R}$ provides $\delta_m > 0$ such that for any $x \in]a; a+\delta_m[$

$$|f_m(x) - a_m| < \frac{\epsilon}{3}\,.$$

Let $x_0 := a + \frac{1}{2}\min\{\delta_n, \delta_m\}$, then

$$|a_n - a_m| \leq \underbrace{|a_n - f_n(x_0)|}_{<\frac{\epsilon}{3}} + \underbrace{|f_n(x_0) - f_m(x_0)|}_{<\frac{\epsilon}{3}} + \underbrace{|f_m(x_0) - a_m|}_{<\frac{\epsilon}{3}} < \epsilon\,.$$

This means $\{a_n\}_{n=1}^{\infty}$ is Cauchy sequence and according Bolzano-Cauchy statement 52 it is convergent, so $a_n \to A \in \mathbb{R}$.
II. $\lim\limits_{x\to a^+} f(x) = A$: Given $\epsilon > 0$ arbitrary. As $f_n \to f$ uniformly on $]a; a+\Delta[$ we have n_2 such that for any $n \geq n_2$

$$\sup_{x\in]a;a+\Delta[} |f_n(x) - f(x)| < \frac{\epsilon}{3}\,.$$

As $a_n \to A$ we have also n_3 such that for any $n \geq n_3$

$$|a_n - A| < \frac{\epsilon}{3}\,.$$

Let $n_0 := \max\{n_2, n_3\}$. The existence of limit $\lim\limits_{x\to a^+} f_{n_0}(x) = a_{n_0} \in \mathbb{R}$ ensure existence of $\delta > 0$ such that for any $x \in]a; a+\delta[$

$$|f_{n_0}(x) - a_{n_0}| < \frac{\epsilon}{3}\,.$$

Then for any $x \in]a; a+\delta[$

$$|f_n(x) - A| \leq \underbrace{|f(x) - f_{n_0}(x)|}_{<\frac{\epsilon}{3}} + \underbrace{|f_{n_0}(x) - a_{n_0}|}_{<\frac{\epsilon}{3}} + \underbrace{|a_{n_0} - A|}_{<\frac{\epsilon}{3}} < \epsilon\,. \blacksquare$$

This statement is not true for pointwise convergence.

ex. 12 **W**e can take $f_n(x) = (1-x)^n$ for $x \in]0;1[$. This sequence tends pointwise to $f(x) = 0$ on $]0;1[$. But $a_n = \lim_{x\to 0^+} f_n(x) = 1$ for any $n \in \mathbb{N}$ and $\lim_{x\to 0^+} f(x) = 0$.

We can formulate another two similar statements for limits and one consequence about continuity.

st. 110
$$\begin{array}{|l|}\hline f_n, f :]a-\Delta; a[\longrightarrow \mathbb{R},\ \Delta > 0 : \\ \lim_{x\to a^-} f_n(x) = a_n \in \mathbb{R},\ \text{and } f_n \to f \text{ uniformly on }]a-\Delta; a[\implies a_n \to A \in \mathbb{R} \text{ and } \lim_{x\to a^-} f(x) = A \\ \hline\end{array}$$

proof. It is similar. ■

st. 111
$$\begin{array}{|l|}\hline f_n, f :]a-\Delta; a[\cup]a; a+\Delta[\longrightarrow \mathbb{R},\ \Delta > 0 : \\ \lim_{x\to a} f_n(x) = a_n \in \mathbb{R},\ \text{and } f_n \to f \text{ uniformly on }]a-\Delta; a[\cup]a; a+\Delta[\implies \\ \hfill \implies a_n \to A \in \mathbb{R} \text{ and } \lim_{x\to a} f(x) = A \\ \hline\end{array}$$

proof. It is consequence of last two statements. ■

st. 112
$$\begin{array}{|l|}\hline f_n, f : I \longrightarrow \mathbb{R},\ I \subset \mathbb{R} \text{ interval } : \\ f_n \text{ continuous on } I \text{ and } f_n \to f \text{ uniformly on } I \implies f \text{ continuous on } I \\ \hline\end{array}$$

proof. We have to realize that a function f is continuous at a iff $\lim_{x\to a} f(x) = f(a)$ (similarly for continuity from left or right). The rest is a consequence of the last three statements. ■

Now we shall consider integrals and derivatives of limit of sequence of function.

st. 113
$$\begin{array}{|l|}\hline f_n, f : [\alpha;\beta] \longrightarrow \mathbb{R},\ \alpha,\beta \in \mathbb{R} : \\ (R)\int_\alpha^\beta f_n \text{ exists and } f_n \to f \text{ uniformly on } [\alpha;\beta] \implies f (R)\int_\alpha^\beta f \text{ exists and } (R)\int_\alpha^\beta f_n \to (R)\int_\alpha^\beta f \\ \hline\end{array}$$

proof. We shall denote $a_n := \sup_{x\in[\alpha;\beta]} |f_n(x) - f(x)|$. As $f_n \to f$ uniformly on $[\alpha;\beta]$ then $a_n \to 0$. We have for any $x \in [\alpha;\beta]$

$$f_n(x) - a_n \le f(x) \le f_n(x) + a_n \qquad \text{and}$$

$$\int_\alpha^\beta f_n - (\beta-\alpha)\, a_n \le \int_\alpha^\beta (f_n - a_n) \le dolni \int_\alpha^\beta f \le horni \int_\alpha^\beta f \le \int_\alpha^\beta (f_n + a_n) \le \int_\alpha^\beta f_n + (\beta-\alpha)\, a_n\,.$$

After limiting we obtain

$$\lim_{n\to\infty} \int_\alpha^\beta f_n \le dolni \int_\alpha^\beta f \le horni \int_\alpha^\beta f \le \lim_{n\to\infty} \int_\alpha^\beta f_n$$

and $(R)\int_\alpha^\beta f = \lim_{n\to\infty} \int_\alpha^\beta f_n$ exists. ■

Similar statement is not true for pointwise convergence.

ex. 13

We can take $f_n(x) = nx\left(1-x^2\right)^n$ for $x \in [0;1]$. This sequence tends pointwise to $f(x) = 0$ on $[0;1]$. But for any $n \in \mathbb{N}$ we have $\int_0^1 f_n\,dx = \frac{n}{2n+2} \to \frac{1}{2}$ and $\int_0^1 f\,dx = 0$.

st. 114	$f_n, f :]\alpha;\beta[\longrightarrow \mathbb{R},\ \alpha,\beta,\in\mathbb{R}:$ $f_n \to f$ pointwise on $]\alpha;\beta[$ and $(\forall x \in]\alpha;\beta[)\ f'_n(x)\in\mathbb{R}$ exists and f'_n uniformly convergent on $]\alpha;\beta[\implies$ $\implies f_n \to f$ uniformly on $]\alpha;\beta[$ and $(\forall x\in]\alpha;\beta[)\ f'(x)\in\mathbb{R}$ exists and $f'_n \to f'$ uniformly on $]\alpha;\beta[$

proof. I. f_n is uniformly convergent on $]\alpha;\beta[$:
We choose one $x_0 \in]\alpha;\beta[$ and we have $f_n(x_0)\to f(x_0)$. Given $\epsilon>0$ arbitrary. There is some n_1 such that $|f_n(x_0)-f_m(x_0)|<\frac{\epsilon}{2}$ for any $m,n\ge n_1$. As f'_n is uniformly convergent on $]\alpha;\beta[$ there is also n_2 such that $\sup\limits_{x\in]\alpha;\beta[} |f'_n(x)-f_m(x)| < \frac{\epsilon}{2(\beta-\alpha)}$ for any $m,n\ge n_2$. So for any $m,n\ge n_0:=\max\{n_1,n_2\}$ and any $x\in]\alpha;\beta[$ (for instance $x\ge x_0$) we can estimate

$$|f_n(x)-f_m(x)| \le \underbrace{|f_n(x)-f_m(x)-f_n(x_0)+f_m(x_0)|}_{\le|(f'_n(c_1)-f'_m(c_1))(x-x_0)|\le\frac{\epsilon}{2(\beta-\alpha)}|x-x_0|\le\frac{\epsilon}{2}} + \underbrace{|f_n(x_0)-f_m(x_0)|}_{<\frac{\epsilon}{2}} < \epsilon\,,$$

having used the mean value theorem for the function $h:=f_n-f_m$ on interval $[x_0;x]$ and so we know there is some $c_1\in[x_0;x]\subset]\alpha;\beta[$ (c_1 depends on n, m, x and x_0) such that $h(x)-h(x_0)=h'(c_1)\,(x-x_0)$.
II. $f'_n \to f'$:
Given $a\in]\alpha;\beta[$ arbitrary. We shall define

$$g_n(x) := \frac{f_n(x)-f_n(a)}{x-a} \text{ and } g(x) := \frac{f(x)-f(a)}{x-a} \text{ for } x\in]\alpha;a[\cup]a;\beta[\,.$$

Then $\lim\limits_{x\to a} g_n(x) = f'_n(a)$ and $g_n\to g$ pointwise on $]\alpha;a[\cup]a;\beta[$. We shall prove g_n converge also uniformly on this interval $]\alpha;a[\cup]a;\beta[$. Indeed for arbitrary $\epsilon>0$ there is some n_3 such that for all $m,n\ge n_3$ $\sup\limits_{x\in]\alpha;\beta[}|f'_n(x)-f'_m(x)|<\epsilon$ and so we can again estimate for any $x\in]\alpha;a[\cup]a;\beta[$ (for instance $x\ge a$)

$$|g_n(x)-g_m(x)| \le \left|\frac{f_n(x)-f_n(a)-f_m(x)+f_m(a)}{x-a}\right| \le \left|\frac{(f'_n(c_2)-f'_m(c_2))\,(x-a)}{x-a}\right| < \epsilon\,.$$

using the mean value theorem for $h=f_n-f_m$ on the interval $]a;x[$. Now we shall use the statement 111 and we obtain existence of limit $\lim\limits_{x\to a} g(x) = \lim\limits_{n\to\infty} f'_n(a)$ and hence with regard to definition of g the existence of derivative $f'(a)=\lim\limits_{n\to\infty} f'_n(a)$. ■

We see from proof we can suppose $f_n(x_0)$ is convergent only for one point x_0.

Similar statement is not true for pointwise convergence of f'_n even if f_n itself is uniformly convergent.

ex. 14

We can take $f_n(x)=\frac{1}{n}\arctan nx$ for $x\in]-1;1[$. This sequence tends uniformly to $f(x)=0$ on $]-1;1[$. But for any $n\in\mathbb{N}$ we have $f'_n=\frac{1}{1+n^2x^2}\to g$, where $g=\begin{cases}1 & \text{for } x=0\\ 0 & \text{otherwise}\end{cases}$.

The same statements hold for series, too.

st. 115	$f_n, f :]a;a+\Delta[\longrightarrow\mathbb{R},\ \Delta>0:$ $\lim\limits_{x\to a^+} f_n(x)=a_n\in\mathbb{R}$, and $\sum\limits_{n=1}^{\infty} f_n = f$ uniformly on $]a;a+\Delta[\implies \sum\limits_{n=1}^{\infty} a_n = A\in\mathbb{R}$ and $\lim\limits_{x\to a^+} f(x)=A$

proof. It is consequence of statement 109. ■

We shall not formulate similar statements for limit from left and limit.

st. 116

$$f_n, f : I \longrightarrow \mathbb{R},\ I \subset \mathbb{R} \text{ interval } :$$
$$f_n \text{ continuous on } I \text{ and } \sum_{n=1}^{\infty} f_n = f \text{ uniformly on } I \quad \Longrightarrow f \text{ continuous on } I$$

proof. It is consequence of statement 112. ■

st. 117

$$f_n, f : [\alpha;\beta] \longrightarrow \mathbb{R},\ \alpha,\beta \in \mathbb{R} :$$
$$(R)\int_\alpha^\beta f_n \text{ exists and } \sum_{n=1}^{\infty} f_n = f \text{ uniformly on } [\alpha;\beta] \quad \Longrightarrow (R)\int_\alpha^\beta f \text{ exists and } \sum_{n=1}^{\infty} (R)\int_\alpha^\beta f_n = (R)\int_\alpha^\beta f$$

proof. It is consequence of statement 113. ■

st. 118

$$f_n, f :]\alpha;\beta[\longrightarrow \mathbb{R},\ \alpha,\beta, \in \mathbb{R} :$$
$$\sum_{n=1}^{\infty} f_n = f \text{ pointwise on }]\alpha;\beta[\text{ and}$$
$$(\forall x \in]\alpha;\beta[)\ f_n'(x) \in \mathbb{R} \text{ exists and } f_n' \text{ uniformly sumable on }]\alpha;\beta[\quad \Longrightarrow$$
$$\Longrightarrow \sum_{n=1}^{\infty} f_n = f \text{ uniformly on }]\alpha;\beta[\text{ and } (\forall x \in]\alpha;\beta[)\ f'(x) \in \mathbb{R} \text{ exists and } \sum_{n=1}^{\infty} f_n' = f' \text{ uniformly on }]\alpha;\beta[$$

proof. It is consequence of statement 114. ■

Now we shall present three tests about uniform summability of series of functions. They are similar to ones for series of numbers.

Weierstrass test of uniform summability

st. 119

$$K \subset \mathbb{R},\ f_n : K \longrightarrow \mathbb{R} :$$
$$(\forall n \in \mathbb{N})\,(\exists a_n \in \mathbb{R})\ \sup_{x\in K} |f_n(x)| \le a_n \text{ and } \sum_{n=1}^{\infty} a_n \in \mathbb{R} \Longrightarrow \sum_{n=1}^{\infty} f_n \text{ uniformly summable on } K$$

proof. We use Bolzano-Cauchy statemet for summability. For arbitrary $\epsilon > 0$ there is n_0 such that for any $m, n \ge n_0$ $m < n$ we have $\sum_{k=m+1}^{n} a_k < \epsilon$ and also

$$\sup_{x\in K} |\sum_{k=m+1}^{n} f_k(x)| \le \sum_{k=m+1}^{n} \sup_{x\in K} |f_k(x)| \le \sum_{k=m+1}^{n} a_k < \epsilon\,. \blacksquare$$

Sequences of functions have similar properties as that of numbers.

def. 26

$$K \subset \mathbb{R}, f_n : K \longrightarrow \mathbb{R} :$$
$$f_n \text{ decreasing on } K \overset{\text{def.}}{\Longleftrightarrow} (\forall n \in \mathbb{N})\,(\forall x \in K)\ f_{n+1}(x) \le f_n(x)$$

The definition of increasing sequence of function is similar.

Abel test of uniform summability

st.

$$K \subset \mathbb{R}, f_n, g_n : K \longrightarrow \mathbb{R} :$$
$$f_n \text{ uniformly summable on } K,$$
$$(\forall n \in \mathbb{N})\,(\forall x \in K)\ g_n(x) \ge 0\,,\ g_1 \text{ bounded on } K \text{ and } g_n \text{ decreasing on } K \Longrightarrow$$
$$\Longrightarrow f_n g_n \text{ uniformly summable on } K$$

proof. The function g_1 is bounded by some constant $M \in \mathbb{R}$. As g_n is decreasing and positive on K we have $M \geq g_1(x) \geq g_k(x) \geq 0$ and $g_k(x) - g_{k+1}(x) \geq 0$ for any k. Given $\epsilon > 0$ arbitrary. As f_n is uniformly summable on K according to the Bolzano-Cauchy theorem about summability there is some n_0 such that for any $m, k \geq n_0$ $k > m$ we have $\sup_{x \in K} |f_{m+1}(x) + \cdots + f_k(x)| < \frac{\epsilon}{2M}$. Then for arbitrary $m, n \geq n_0$ and arbitrary $x \in K$ we shall use Abel partial summation

$$\sum_{k=m+1}^{n} f_k(x) g_k(x) = \sum_{k=m+1}^{n-1} \left(f_{m+1}(x) + \cdots + f_k(x)\right)\left(g_k(x) - g_{k+1}(x)\right) + \left(f_{m+1}(x) + \cdots + f_n(x)\right) g_n(x)$$

and estimate

$$|\sum_{k=m+1}^{n} f_k(x) g_k(x)| \leq \sum_{k=m+1}^{n-1} \underbrace{|f_{m+1}(x) + \cdots + f_k(x)|}_{\leq \frac{\epsilon}{2M}} \underbrace{(g_k(x) - g_{k+1}(x))}_{\geq 0} + \\ + \underbrace{|f_{m+1}(x) + \cdots + f_n(x)|}_{\leq \frac{\epsilon}{2M}} \underbrace{g_n(x)}_{\geq 0} \leq \frac{\epsilon}{2M} \underbrace{g_{m+1}(x)}_{\leq M} \leq \frac{\epsilon}{2} . \quad (1)$$

As $x \in K$ was arbitrary also

$$\sup_{x \in K} |\sum_{k=m+1}^{n} f_k(x) g_k(x)| \leq \frac{\epsilon}{2} < \epsilon$$

and again according to Bolzano-Cauchy teorem $f_n g_n$ is uniformly summable on K. ■

Dirichlet test of uniform summability

st. 121

$K \subset \mathbb{R}, f_n, g_n : K \longrightarrow \mathbb{R}$: $(\exists M \in \mathbb{R})\,(\forall n \in \mathbb{N})\ \sup_{x \in K} |f_1(x) + f_2(x) + \cdots + f_n(x)| \leq M$, $(\forall n \in \mathbb{N})\,(\forall x \in K)\ g_n(x) \geq 0$, g_n decreasing on K and $g_n \to 0$ uniformly on $K \Longrightarrow$ $\Longrightarrow f_n g_n$ uniformly summable on K

proof. Given $\epsilon > 0$ arbitrary. As g_n is uniformly approaching to 0 on K according to the definition there is some n_0 such that for any $n \geq n_0$ we have $\sup_{x \in K} |g_n(x)| < \frac{\epsilon}{3M}$. Then for arbitrary $m, n \geq n_0$ and arbitrary $x \in K$ we shall again use Abel partial summation and estimate

$$|\sum_{k=m+1}^{n} f_k(x) g_k(x)| \leq \sum_{k=m+1}^{n-1} \underbrace{|f_{m+1}(x) + \cdots + f_k(x)|}_{\leq 2M} \underbrace{(g_k(x) - g_{k+1}(x))}_{\geq 0} + \\ + \underbrace{|f_{m+1}(x) + \cdots + f_n(x)|}_{\leq 2M} \underbrace{g_n(x)}_{\geq 0} \leq 2M \underbrace{g_{m+1}(x)}_{\leq \frac{\epsilon}{3M}} \leq \frac{2\epsilon}{3} . \quad (2)$$

As $x \in K$ was arbitrary also

$$\sup_{x \in K} |\sum_{k=m+1}^{n} f_k(x) g_k(x)| \leq \frac{2\epsilon}{3} < \epsilon$$

and $f_n g_n$ is uniformly summable on K. ■

Dini

st.121 $f_n, f : [\alpha, \beta] \longrightarrow \mathbb{R}$:
f_n, f continuous on $[\alpha, \beta], f_n \to f$ on $[\alpha, \beta], (\forall x \in [\alpha, \beta])\ f_n(x) \le f_{n+1}(x) \Longrightarrow f_n \rightrightarrows f$ on $[\alpha, \beta]$

proof. Suppose f is increasing.

Let $\epsilon > 0$ arbitrary. For any $t \in [\alpha, \beta]$ there is some $n(t)$ such that for all $k \ge n(t)$

$$f(t) - f_k(t) < \epsilon \; .$$

Then there exists some $\delta(t) > 0$ such that for all $x \in]t - \delta(t), t + \delta(t)[$

$$|f(x) - f(t)| < \epsilon \; .$$

Interval

$$[\alpha, \beta] = \Cup_{t \in [\alpha, \beta]}]t - \delta(t), t + \delta(t)[$$

is compact. So there is a finite number $t_1, \ldots, t_m \in [\alpha, \beta]$ such that

$$[\alpha, \beta] = \Cup_{k=1}^{m}]t - \delta(t), t + \delta(t)[\; .$$

For any $x \in [\alpha, \beta]$ there is p such that $x \in]t_p - \delta(t_p), t_p + \delta(t_p)[$. For any $n \ge n_0 = \max\{n(t_1), \ldots, n(t_m)\}$

$$f(x) - f_n(x) \le f(x) - f_{n_0}(x) \le f(x) - f_{n(t_p)}(x) < \epsilon \; ,$$

the last inequalities hold due to monotony.

So for any $\epsilon > 0$ and any $n \ge n_0$

$$\sup_{x \in [\alpha, \beta]} |f(x) - f_n(x)| < \epsilon$$

and $f_n \rightrightarrows f$ on $[\alpha, \beta]$. ■

5 . Power series

Power series are series of sequences of type

$$\{a_n \, (x - c)^n\}_{n=0}^{\infty} \text{ or } \{a_n x^n\}_{n=0}^{\infty} \, .$$

st.122 $a_n x_0^n$ summable $\Longrightarrow |a_n x^n|$ poinwise summable on $] - |x_0|; |x_0|[$

proof. Given $x \in \mathbb{R}$ such that $|x| < |x_0|$. Let us denote $q := \frac{|x|}{|x_0|}$. As $\sum_{n=0}^{\infty} a_n x_0^n$ the sequence $a_n x_0^n$ tends to 0 and so it is bounded by some $M \in \mathbb{R}$. For any $n \in \mathbb{N}$ we have $|a_n x_0^n| \le M$ and also

$$|a_n x^n| \le |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \le |a_n x_0^n| q^n \le M q^n \, .$$

As $\sum\limits_{n=0}^{\infty} Mq^n = \frac{M}{1-q}$ is finite also $\sum\limits_{n=0}^{\infty} a_n x^n$ is finite according to the comparison test. ■

Similar statement holds also for $\sum\limits_{n=0}^{\infty} a_n (x-c)^n$ and $x \in \mathbb{R}$ such that $|x-c| < |x_0 - c|$.

st.123 $\boxed{a_n x_0^n \text{ summable } \Longrightarrow |n a_n x^n| \text{ poinwise summable on }]-|x_0|;|x_0|[}$

proof. It is similar to that of last statement. Given $x \in \mathbb{R}$ such that $|x| < |x_0|$. Let us denote $q := \frac{|x|}{|x_0|}$. Again the sequence $a_n x_0^n$ tends to 0 and so it is bounded by some $M \in \mathbb{R}$. Therefore

$$|n a_n x^n| \leq |a_n x_0^n| n \left|\frac{x}{x_0}\right|^n \leq M n q^n .$$

By for instance ratio test $\sum\limits_{n=0}^{\infty} M n q^n = \frac{M}{1-q}$ is finite. Hence $\sum\limits_{n=0}^{\infty} a_n x^n$ is finite according to the comparison test. ■

radius of summability

def.27 $\boxed{\begin{array}{l} a_n \text{ sequence :} \\ R \overset{\text{def.}}{=} \sup\{r \geq 0; \sum\limits_{n=0}^{\infty} |a_n| r^n \text{ is finite } \} \end{array}}$

st.124 $\boxed{\begin{array}{l} a_n \text{ sequence :} \\ |x| < R \Longrightarrow a_n x^n \text{ summable} \\ |x| > R \Longrightarrow a_n x^n \text{ is not summable} \end{array}}$

proof. I.: Given $x \in \mathbb{R}$, $|x| < R$. There is $|x| < r_1 < R$ such that $\sum\limits_{n=0}^{\infty} |a_n| r_1^n$ is finite. According to the last statement also $\sum\limits_{n=0}^{\infty} |a_n x^n|$ is finite and $\sum\limits_{n=0}^{\infty} a_n x^n$ as well.
II.: We shall carry it out by contradiction. Suppose $\sum\limits_{n=0}^{\infty} a_n x_0^n$ is finite for some $x_0 \in \mathbb{R}$, $|x_0| > R$. There is some $r_2 \in \mathbb{R}$, $R < r_2 < |x_0|$. According to the last statement $\sum\limits_{n=0}^{\infty} |a_n| r_2^n = \sum\limits_{n=0}^{\infty} |a_n r_2^n|$ is finite. But this contradicts the definition of R. ■

Similar statement holds also for $\sum\limits_{n=0}^{\infty} a_n (x-c)^n$.

The set of $x \in \mathbb{R}$ for which the sequence $a_n (x-c)^n$ is summable (or series of this sequence is convergent) is called set of convergence of this power series. According the last statement this set creates an interval with boundary poins $c - R$ and $c + R$ and $c \in \mathbb{R}$ is called **centre of convergence** of power series and $R \in \mathbb{R}^*$, $R \geq 0$ **radius of convergence** of power series.

We can calculate this radius for instance by Cauchy root test $\frac{1}{R} = \limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$ (for $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} = 0$ it is ∞ and for $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ it is 0).

st.125 $\boxed{\begin{array}{r} a_n x^n \text{ pointwise summable on }]-r;+r[\text{ and } [\alpha;\beta] \subset]-r;+r[\quad \Longrightarrow \\ \Longrightarrow a_n x^n = f \text{ uniformly summable on } [\alpha;\beta] \end{array}}$

proof. Let $r_1 := \max(|\alpha|, |\beta|)$, There is some $r_2 \in \mathbb{R}$, $r_1 < r_2 < r$. As $\sum\limits_{n=0}^{\infty} a_n r_2^n$ is finite ($r_2 \in]-r; r[$) also $\sum\limits_{n=0}^{\infty} |a_n r_1^n|$ is finite. As $[\alpha; \beta] \subset [-r_1; r_1]$ for any $x \in [\alpha, beta]$ we have $|x| \leq r_1$ and $|a_n x^n| \leq |a_n r_1^n|$. Therefore $a_n x^n$ is uniformly summable on $[\alpha; \beta]$ according to the Weierstrass test. ■

st.126 $a_n x^n$ pointwise summable on $[0; r] \Longrightarrow a_n x^n$ uniformly summable on $[0; r]$

proof. We shall use Abel test for uniform summability. Let $f_n(x) := a_n r^n$, these functions are constants therefore uniformly summable on all $\mathbb{R}$ as $\sum\limits_{n=0}^{\infty} a_n r^n$ finite. Let $g_n(x) := \left(\frac{x}{r}\right)^n \geq 0$, then $\{g_n(x)\}_{n=0}^{\infty}$ create the decreasing sequence on $[0, r]$ and $g_0(x) = 1$ is bounded on $[0; r]$. According to the Abel test $f_n(x) g_n(x)$ uniformly summable on $[0; r]$ and $f_n(x) g_n(x) = a_n r^n \left(\frac{x}{r}\right)^n = a_n x^n$. ■

st.127

$$\sum_{n=0}^{\infty} a_n x^n = f \text{ pointwise on }]-r; r[\quad \Longrightarrow$$

$\Longrightarrow f$ continuous on]-r;r[

$$\Longrightarrow (\forall x \in]-r; r[)\, f'(x) \in \mathbb{R} \text{ exists and } f' = \sum_{n=0}^{\infty} n a_n x^{n-1} \text{ pointwise on }]-r; r[$$

$$\Longrightarrow (\forall x \in]-r; r[)\, F(x) = (R) \int_0^x f(\xi)\, d\xi \in \mathbb{R} \text{ exists and}$$

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \text{ pointwise on }]-r; r[$$

proof. I. Consequence of statements 116 and 125.
II. Consequence of statements 118 and 125.
III. Consequence of statements 117 and 125. ■

st.128 $\sum\limits_{n=0}^{\infty} a_n x^n = f$ pointwise on $[0; r] \quad \Longrightarrow \lim\limits_{x \to r^-} f(x) = f(r)$

proof. Consequence of statements 115 (for the right limit) and 126. ■

st.129 $(\exists \Delta > 0) \ \sum\limits_{n=0}^{\infty} a_n x^n = 0$ pointwise on $]-\Delta; \Delta[\quad \Longrightarrow (\forall n \in \mathbb{N})\, a_n = 0$

proof. As function $\sum\limits_{n=0}^{\infty} a_n x^n$ is continuous at 0 and pointwise summable on some $[-r; r]$, $0 < r < \Delta$, they and their derivatives are continuous at 0, too. Therefore $0 = \lim\limits_{x \to 0} \sum\limits_{n=0}^{\infty} a_n x^n = a_0$, $0 = \lim\limits_{x \to 1} \sum\limits_{n=0}^{\infty} n a_n x^{n-1} = 1 a_1$, $0 = \lim\limits_{x \to 2} \sum\limits_{n=0}^{\infty} n\,(n-1)\, a_n x^{n-2} = 2 a_2 \qquad \ldots$ ■

Taylor, McLaurint series

st.130

$f :]c-r; c+r[\longrightarrow \mathbb{R}$:
$(\forall n \in \mathbb{N})$ $f^{[n]}$ exists on $]c-r; c+r[$ and
$\frac{|x-c|^n}{n!} \sup\limits_{|\xi-c|\le|x-c|} |f^{[n]}(\xi)| \to 0$ pointwise on $]c-r; c+r[\Longrightarrow$
$\Longrightarrow \sum\limits_{n=0}^{\infty} \frac{f^{[n]}(c)}{n!}(x-c)^n = f$ poinwise on $]c-r; c+r[$

proof. We shall use Taylor theorem for f on $[c; x]$ (suppose for instance $x > c$)

$$f(x) = \sum_{k=0}^{n} \frac{f^{[k]}(c)}{k!}(x-c)^k + \frac{f^{[n+1]}(\xi)}{(n+1)!}(x-c)^{n+1} ,$$

where $\xi \in [c; x]$. We have for partial sums $s_n(x) := \sum\limits_{k=0}^{n} \frac{f^{[k]}(c)}{k!}(x-c)^k$

$$|s_n(x) - f(x)| \le \left| \frac{f^{[n+1]}(\xi)}{(n+1)!}(x-c)^{n+1} \right| \le \frac{|x-c|^{n+1}}{(n+1)!} \sup_{|\xi-c|\le|x-c|} |f^{[n+1]}(\xi)| \to 0 \,. \blacksquare$$

expansion of e^x

st.131 $(\forall x \in \mathbb{R}) \qquad e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$

proof. Let us put $f(x) := e^x$ and use for it and $c = 0$ statement 130. We have $f'(x) = f''(x) = \cdots = f^{[k]}(x) = e^x$ for any $k \in \mathbb{N}$, hence $f(0) = f'(0) = f''(0) = \cdots = f^{[k]}(0) = 1$. For any $x \in \mathbb{R}$ there is some $r > 0$ such that $x \in]-r; r[$ and

$$\frac{|x|^n}{n!} \sup_{|\xi|\le|x|} |f^{[n]}\xi| \le \frac{|x|^n}{n!} \sup_{|\xi|\le|x|} |e^{\xi}| \le \frac{r^n}{n!} e^r \to 0 \,. \blacksquare$$

expansion of $\cos x$

st.132 $(\forall x \in \mathbb{R}) \qquad \cos x = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

proof. We shall use the statement for $f(x) := \cos x$ and $c = 0$. Similarly we have $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$ etc. so $f^{[2k+1]}(x) = (-1)^k \sin x$ and $f^{[2k]}(x) = (-1)^k \cos x$ for any $k \in \mathbb{N}$. Hence $f^{[2k+1]}(0) = 0$ and $f^{[2k]}(0) = (-1)^k$. For any $x \in \mathbb{R}$

$$\frac{x^n}{n!} \sup_{|\xi|\le|x|} |f^{[n]}\xi| \le \frac{x^n}{n!} \to 0 \,. \blacksquare$$

expansion of $\sin x$

st.133 $(\forall x \in \mathbb{R}) \qquad \sin x = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

proof. It is similar. ■

expansion of $\ln x$

st.134

$(\forall x \in]-1;1])$	$\ln(1+x) = \sum\limits_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	or
$(\forall x \in]0;2])$	$\ln x = \sum\limits_{n=0}^{\infty} (-1)^{n-1} \frac{(x+1)^n}{n}$	

proof. We can use again the statement 130 for $\ln x$ and $c = 1$ but only on $]\frac{1}{2}; \frac{3}{2}[$. Therefore it is better to use sum of geometrical sequence. Let us denote by $f(x) := \sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ for any $x \in]-1;1]$. We can do derivative of power series step by step on $]-1;1[$ and we obtain $f'(x) = \sum\limits_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = \sum\limits_{k=0}^{\infty} (-x)^k = \frac{1}{1+x}$. Therefore by integrating $f(x) = \ln(1+x) + C$, where C is some constant. From $f(0) = 0$ we have $C = 0$. As $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = f$ pointwise on $[0;1]$ also limit $\lim\limits_{x \to 1^-} \ln(x+1) = \sum\limits_{n=1}^{\infty} (-1)^{n-1} \lim\limits_{x \to 1^-} \frac{x^n}{n}$ according to the statement 128. ■

expansion of $\arctan x$

st.135 $(\forall x \in]-1;1])$ $\qquad \arctan x = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

proof. It is similar, we use

$$\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\right)' = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n+1}}{2n+1}\right)' = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$$

and therefore $\sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x$. For $x = 1$ the same result holds like at the proof of statement 134.
■

st.136 $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n} = -\ln 2$ and $\sum\limits_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \frac{\pi}{4}$

proof. Consequences of 134 and 135. ■

Expansion of $\ln x$ enables us to prove several important formulas.

Wallis formula

st.000 $\frac{1}{n}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \longrightarrow \pi$ $\qquad$ or $\qquad \frac{2^{4n}}{n\binom{2n}{n}^2} \longrightarrow \pi$

proof. Lets introduce integrals

$$S_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx \; .$$

Integration by parts yields us $S_{n+2} = \frac{n+1}{n+2} S_n$ and so

$$S_0 = \frac{\pi}{2},\ S_1 = 1,\ S_2 = \frac{1}{2}\cdot\frac{\pi}{2},\ S_3 = \frac{2}{3}\cdot 1,\ S_4 = \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2},\ S_5 = \frac{4}{5}\cdot\frac{2}{3}\cdot 1,\ \dots$$
$$\dots,\ S_{2n} = \frac{2n-1}{2n}\cdot\dots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},\ S_{2n+1} = \frac{2n}{2n+1}\cdot\dots\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1\ .$$

Because $\sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x$ it holds $S_{2n+1} \le S_{2n} \le S_{2n-1}$ or

$$\frac{(2n)!!}{(2n+1)!!} \le \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2} \le \frac{(2n-2)!!}{(2n-1)!!}$$

and

$$\frac{2}{2n+1}\cdot\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 = \frac{((2n)!!)^2}{(2n+1)!!(2n-1)!!} \le \pi \le \frac{(2n-2)!!(2n)!!}{((2n-1)!!)^2} = \frac{1}{n}\cdot\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\ .$$

The result follows from inequality

$$\pi \le \frac{1}{n}\cdot\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \le \frac{2n+1}{2n}\pi \longrightarrow \pi\ .\ \blacksquare$$

Stirling formula

st.000 $\boxed{\frac{n!\epsilon n}{n^{n+\frac{1}{2}}} \longrightarrow \sqrt{2\pi}}$

proof. We shall use expansion of $\ln x$ for $x \in]-1;1[$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \quad (3)$$
$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \quad (4)$$

and their difference

$$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) = 2x\left(1 + \frac{1}{3}x^2 + \frac{1}{5}x^4 + \dots\right) \ge 2x\ .$$

We put $x := \frac{1}{2n+1}$ to the inequality and get

$$\ln\left(1+\frac{1}{n}\right) > \frac{1}{n+\frac{1}{2}} \qquad \text{and so} \qquad \frac{1}{\epsilon}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} > 1\ .$$

Sequence

$$c_n = \frac{n!\mathrm{e}^n}{n^{n+\frac{1}{2}}} \ge 0$$

is decreasing because $\frac{c_n}{c_{n+1}} = \frac{1}{\mathrm{e}}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} > 1$ and bounded above.

From () we know the sequence c_n convergs to some $c \in \mathbb{R}$.

The value of c can be calculated from the Wallis formula because

$$\frac{c_n^4}{c_{2n}^2}=\frac{\left(\left(\frac{\mathrm{e}}{n}\right)^n\cdot\frac{n!}{\sqrt{n}}\right)^4}{\left(\left(\frac{\mathrm{e}}{2n}\right)^{2n}\cdot\frac{(2n)!}{\sqrt{2n}}\right)^2}=\frac{2}{n}\cdot\frac{(2^n\cdot n!)^4}{((2n)!)^2}=\frac{2}{n}\cdot\frac{(2^n\cdot n(n-1)\cdot\dots\cdot 3\cdot 2\cdot 1)^4}{(2n(2n-1)(2n-2)\cdot\dots\cdot 3\cdot 2\cdot 1)^2}=$$

$$=\frac{2}{n}\cdot\frac{(2n(2n-2)\cdot\dots\cdot 6\cdot 4\cdot 2)^4}{(2n(2n-2)\cdot\dots\cdot 4\cdot 2)^2\,((2n-1)\cdot\dots\cdot 3\cdot 1)^2}=\frac{2}{n}\cdot\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\longrightarrow 2\pi\ .$$

As $\frac{c_n^4}{c_{2n}^2}\to c^2$ we have $c=\sqrt{2\pi}$. ■

Euler constant

st.000 $\boxed{\exists a\in\mathbb{R}\qquad \sum_{k=1}^{n}\frac{1}{k}-\ln n\longrightarrow a}$

proof. From expansion of $\ln x$ for $x\in]-1;1[$

$$\ln(1+x)=x-\frac{1}{2}x^2+\underbrace{\left(\frac{1}{3}x^3-\frac{1}{4}x^4\right)}_{\geq 0}+\underbrace{\left(\frac{1}{5}x^5-\frac{1}{6}x^6\right)}_{\geq 0}+\dots\geq x-\frac{x^2}{2}\quad\text{and}\tag{5}$$

$$\ln(1+x)=x-\underbrace{\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)}_{\geq 0}-\underbrace{\left(\frac{1}{4}x^4-\frac{1}{5}x^5\right)}_{\geq 0}+\dots\leq x\tag{6}$$

we have $x-\frac{x^2}{2}\leq\ln(1+x)\leq x$.

Lets denote

$$a_n:=\sum_{k=1}^{n}\frac{1}{k}-\ln n\quad\text{and}\quad b_n:=\sum_{k=1}^{n-1}\frac{1}{k}-\ln n\ .$$

We put $x:=\frac{1}{k}$ into last inequality, calculate

$$\frac{1}{k}-\frac{1}{2k^2}\leq\ln\left(1+\frac{1}{k}\right)\leq\frac{1}{k}\qquad\text{and}\qquad 0\leq\frac{1}{k}-\ln(k+1)+\ln k\leq\frac{1}{2k^2}\ .$$

From this inequality we have two properties of b_n. As

$$0\leq b_n=\sum_{k=1}^{n-1}\frac{1}{k}-\ln n\leq\frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k^2}\leq\frac{1}{2}\left(1+\underbrace{\sum_{k=2}n-1\frac{1}{k(k-1)}}_{1+\frac{1}{n-1}}\right)\leq\frac{3}{2}\ ,$$

so b_n is bounded above. As

$$b_{n+1}-b_n=\frac{1}{n}-\ln(n+1)+\ln n=\frac{1}{n}-\ln\left(1+\frac{1}{n}\right)\geq 0\ ,$$

so b_n is increasing.

From we know sequence b_n converges to some real number $a \in \mathbb{R}$ and

$$a_n = \frac{1}{n} + b_n \to 0 + a = a \ . \blacksquare$$

ex. 16 **(uniform summability)**

(1.9) $\frac{1}{x^2+n^2}$ uniformly summable on $\mathbb{R}$

We have $\frac{1}{x^2+n^2} \leq \frac{1}{n^2}$ for any $x \in \mathbb{R}$. As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is finite we can use Weierstrass test.

(1.) $\frac{x^2}{e^{nx}}$ uniformly summable on $[0;\infty[$

We shall put $f_n(x) := \frac{x^2}{e^{nx}}$ and inquire its maximum on $[0;\infty[$. Function f_n is non-negative, $f_n(0) = 0$ and limit $\lim_{x\to\infty} f_n(x) = 0$. As derivative $f_n'(x) = \frac{x(2-xn)}{e^{nx}}$ is 0 for $x_n = \frac{2}{n}$ we have maximum $f_n(\frac{2}{n}) = \frac{4}{e^2n^2}$. We denote $a_n := \frac{4}{e^2n^2} = f_n(x_n) \geq \sup_{x\in[0;\infty]} |f_n(x)|$. As $\sum_{n=1}^{\infty} \frac{1}{e^2n^2}$ is finite we can use again Weierstrass test.

(1.) $\frac{x}{e^{nx}}$ uniformly summable on $[1;\infty[$

We shall again put $f_n(x) := \frac{x}{e^{nx}}$ and inquire its maximum on $[1;\infty[$. Function f_n is non-negative, $f_n(1) = \frac{1}{e^n}$ and limit $\lim_{x\to\infty} f_n(x) = 0$. As derivative $f_n'(x) = \frac{1-nx}{e^{nx}}$ is negative on $[1;\infty]$ we have maximum $a_n := \frac{1}{e^n} = f_n(1) \geq \sup_{x\in[1;\infty]} |f_n(x)|$. As $\sum_{n=1}^{\infty} \frac{1}{e^n}$ is finite we can use again Weierstrass test.

(1.) $\frac{\sqrt{x}}{ne^{\frac{x}{n}}}$ is not uniformly summable on $[0;\infty[$

We can again put $f_n(x) := \frac{\sqrt{x}}{ne^{\frac{x}{n}}}$ and inquire its maximum on $[0;\infty[$. Function f_n is non-negative, $f_n(0) = 0$ and limit $\lim_{x\to\infty} f_n(x) = 0$. As derivative $f_n'(x) = \frac{n-2x}{2\sqrt{x}n^2e^{\frac{x}{n}}}$ is 0 for $x_n = \frac{n}{2}$ we have maximum $f_n(\frac{n}{2}) = \frac{1}{\sqrt{2ne}} \geq \sup_{x\in[0;\infty]} |f_n(x)|$. But we cannot use Weierstrass test as $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2ne}}$ is not finite. Fortunately we can conclude $f_n(x)$ is not uniformly summable on $[0;\infty[$ as it is not summable for $x := 1$, $\sum_{n=1}^{\infty} f_n(1) = \sum_{n=1}^{\infty} \frac{1}{ne^{\frac{1}{n}}} = \infty$.

(From the result $f_n(x_n)$ is not summable follows no conclusion for $f_n(x)$, we can consider following example.) $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly on $\mathbb{R}$, where

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{for } x = n \\ 0 & \text{otherwise} \end{cases} \quad , f(x) = \begin{cases} 1 & \text{for } x = 1 \\ \frac{1}{2} & \text{for } x = 2 \\ \frac{1}{3} & \text{for } x = 3 \\ \dots & \\ \frac{1}{k} & \text{for } x = k \\ \dots & \\ 0 & \text{otherwise} \end{cases}$$

Function f_n has its maximum in $x_n = n$, $a_n := \frac{1}{n} \geq \sup_{x\in\mathbb{R}} |f_n(x)|$ but $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ and we cannot use Weierstrass test. In spite of this, for partial sums $s_n := f_1 + \cdots + f_n$ we have $\sup_{x\in\mathbb{R}} |s_n(x) - f(x)| = \frac{1}{n+1}$ hence $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly on $\mathbb{R}$.

(1.) $(1-x)\,x^n$ is not uniformly summable on $[0;1]$
We can calculate for $f_n(x) = (1-x)\,x^n$ partial sums

$$s_n(x) = \sum_{k=0}^{n} (1-x)\,x^n = (1-x) \sum_{k=0}^{n} x^n = (1-x)\,\frac{1-x^{n+1}}{1-x} = 1 - x^{n+1}$$

for $0 \leq x < 1$ and $s_n(1) = 0$. Partial sums $s_n \to f$ pointwise on $[0;1]$, where

$$f(x) = \begin{cases} 0 & \text{for } x = 0,\ 1 \\ 1 & \text{otherwise} \end{cases}.$$

Functions f_n are continuous on $[0;1]$. If $\sum_{n=0}^{\infty} f_n = f$ uniformly on $[0;1]$ according to the statement f must be continuous on $[0;1]$, too. But it is not.

ex. **(summability)**

(1.) $\frac{\ln^n x}{n}$ is summable iff $x \in [\frac{1}{e}; e[$
For $x > 1$ we can use ratio test $\frac{f_{n+1}(x)}{f_n(x)} = \frac{n}{n+1} \ln x \to \ln x$ amd we have $f_n(x)$ is summable for $x < e$ and is not summable for $x > e$.
Similarly we have for $0 < x < 1$ sequence $\frac{\ln^n x}{n} = \frac{(-1)^n \ln^n \frac{1}{x}}{n}$ as well as $\frac{\ln^n \frac{1}{x}}{n}$ is summable for $\frac{1}{x} < e$ i.e. $x > \frac{1}{e}$. and is not summable For $x < \frac{1}{e}$ sequence is not summable as $\ln \frac{1}{x} > 1$ and $\frac{\ln^n \frac{1}{x}}{n} \not\to 0$. It remain to inquire summability only for $x := e, 1, \frac{1}{e}$ i.e. $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, $\sum_{n=1}^{\infty} 0 = 0$ and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} < \infty$.

ex. **(calculation of sums)**

(1.3) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ pointwise on $[-1;1[$ (and uniformly on $[-1;\beta]$ for any $\beta < 1$)
It is geometrical sequence. We can do derivative on $]-1;1[$ and by useing statement $\sum_{n=0}^{\infty} nx^{n+1} = \sum_{n=0}^{\infty} (x^n)' = \left(\sum_{n=0}^{\infty} x^n\right)' = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$ obtain next result. 3 $\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)}$ pointwise on $]-1;1[$
Similarly we can integrate it for $x \in [-1;1[$ and by using statement $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \int_0^x t^n\,dt = \int_0^x \sum_{n=0}^{\infty} t^n\,dt = \int_0^x \frac{1}{1-t}\,dt = -\ln(1-x)$ obtain next result similar one of statement 134.

(1.2) $\sum\limits_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$ pointwise on $[-1;1[$

These formulas can be used for instance in following examples.

(5) $\sum\limits_{n=0}^{\infty} \frac{n}{3^n} = \frac{1}{3}$

(5) $\sum\limits_{n=1}^{\infty} \frac{1}{n3^n} = \ln 3 - \ln 2$

(5) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n} = -\ln 2$

(5) $\sum\limits_{n=1}^{\infty} \frac{x^{2n+1}}{n2^n} = x\left(\ln 2 - \ln\left(2-x^2\right)\right)$ pointwise on $[-\sqrt{2};\sqrt{2}[$

We can use formula () $x \sum\limits_{n=1}^{\infty} \frac{1}{n}(()\,\frac{x^2}{2})^n = x\left(-\ln\left(1-\frac{x^2}{2}\right)\right)$.

(1.) $\int\limits_0^{\infty} \sum\limits_{n=1}^{\infty} \frac{n}{3^n(n^2+x^2)}\,dx = \frac{\pi}{4}$

Sequence $\frac{n}{3^n(n^2+x^2)}$ is uniformly summable on $\mathbb{R}$ by the Weierstrass test. Therefore we can change sumation and integration an calculate (using substitution $t := \frac{x}{n}$)

$$\int_0^{\infty} \sum_{n=1}^{\infty} \frac{n}{3^n\left(n^2+x^2\right)}\,dx = \sum_{n=1}^{\infty} \frac{n}{3^n} \int_0^{\infty} \frac{dx}{n^2+x^2} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{3^n} \int_0^{\infty} \frac{dt}{1+t^2} = \sum_{n=1}^{\infty} \frac{1}{3^n}\left[\arctan t\right]_0^{\infty} = \sum_{n=1}^{\infty} \frac{\pi}{2}\frac{1}{3^n} = \frac{\pi}{2}\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\pi}{4} \quad (7)$$

ex. **(expansions)**

(1.) $\int\limits_0^{x} e^{-\xi^2}\,d\xi = x\sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)n!}$

We can expand $e^t = \sum\limits_{n=0}^{\infty} \frac{t^n}{n!}$ for any $t \in \mathbb{R}$, so for $t := -x^2$ we have $e^{-x^2} = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$. We can integrate this power series term by term

$$\int_0^{x} e^{-\xi^2}\,d\xi = \int_0^{x} \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{2n}}{n!}\,d\xi = \sum_{n=0}^{\infty} \int_0^{x} (-1)^n \frac{\xi^{2n}}{n!}\,d\xi = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\,n!}\,.$$

(1.) $\lim\limits_{x\to 0} \frac{\sin x - x}{x^3} = \frac{1}{6}$

As $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$ we have $\frac{\sin x - x}{x^3} = -\frac{1}{6} + \frac{x^2}{120} - \ldots \to \frac{1}{6}$.

Schwartz inequality

st.137

$f,g : [\alpha;\beta] \longrightarrow \mathbb{R}$ integrable on $[\alpha;\beta]$:

$$\int_\alpha^\beta |fg| \le \sqrt{\int_\alpha^\beta f^2}\sqrt{\int_\alpha^\beta g^2}$$

proof. From the existence of integrals of f and g it follows by the properties of Riemann integral the existence of integrals 154 and 155 of f^2, g^2, fg and $(|f|+\gamma|g|)^2$ for any $\gamma \in \mathbb{R}$. As

$$0 \le \int_\alpha^\beta (|f|+\gamma|g|)^2 = \int_\alpha^\beta f^2 + 2\gamma \int_\alpha^\beta |fg| + \gamma^2 \int_\alpha^\beta g^2$$

discriminant of this quadratic equation must be non-positive. Therefore

$$\left(2\int_\alpha^\beta |fg|\right)^2 - 4\int_\alpha^\beta f^2 \int_\alpha^\beta g^2 \le 0 \,. \blacksquare$$

Kronecker delta

def.28 $(\forall k,l = 0,1,2\dots) \quad \delta_{kl} \stackrel{\text{def.}}{=} \begin{cases} 0 & \text{for } k \ne l \\ 1 & \text{for } k = l \end{cases}$

orthonormal system of functions

def.29 $(\forall n = 0,1,2,\dots) \quad v_n : [\alpha;\beta] \longrightarrow \mathbb{R},\ \alpha,\beta \in \mathbb{R}:$

$v_0, v_1, v_2, \dots, v_n, \dots$ orthonormal system of functions on $[\alpha;\beta] \stackrel{\text{def.}}{\Longleftrightarrow} \int_\alpha^\beta v_k v_l = \delta_{kl}$

Fourier coefficients

def.30 $v_0, v_1, v_2, \dots$ orthonormal system of functions on $[\alpha;\beta]$, $f : [\alpha;\beta] \longrightarrow \mathbb{R}$, integrable on $[\alpha;\beta]$:

$c_k \stackrel{\text{def.}}{=} \int_\alpha^\beta f v_k$ Fourier coefficients of f in $v_0, v_1, v_2, \dots$

Existence of them follows from Schwartz inequality.

st.138 $f :]\alpha;\beta] \longrightarrow \mathbb{R}$ integrable on $]\alpha;\beta]$

c_k Fourier coefficients of f in orthonotmal system $v_0, v_1, v_2, \dots$:

$(\forall \gamma_0, \gamma_1, \gamma_2, \dots \gamma_n \in \mathbb{R}) \quad \int_\alpha^\beta \left(f - \sum_{k=0}^n \gamma_k v_k\right)^2 \le \int_\alpha^\beta \left(f - \sum_{k=0}^n c_k v_k\right)^2$

proof. Let us calculate right site of inequality

$$(R) := \int_\alpha^\beta \left(f - \sum_{k=0}^n c_k v_k\right)^2 = \int_\alpha^\beta \left(f - \sum_{k=0}^n c_k v_k\right)\left(f - \sum_{l=0}^n c_l v_l\right) =$$

$$= \int_\alpha^\beta f^2 - \sum_{k=0}^n c_k \underbrace{\int_\alpha^\beta v_k f}_{=c_k} - \sum_{l=0}^n c_l \underbrace{\int_\alpha^\beta f v_l}_{=c_l} + \sum_{k=0}^n c_k \underbrace{\sum_{l=0}^n c_l \underbrace{\int_\alpha^\beta v_k v_l}_{=\delta_{kl}}}_{=c_k} = \int_\alpha^\beta f^2 - \sum_{k=0}^n c_k^2 \quad (8)$$

and now left side of it

$$(L) := \int_\alpha^\beta \left(f - \sum_{k=0}^n \gamma_k v_k\right)^2 = \int_\alpha^\beta f^2 + \underbrace{\sum_{k=0}^n \gamma_k^2 - 2\sum_{k=0}^n \gamma_k c_k + \sum_{k=0}^n c_k^2}_{=\sum_{k=0}^n (\gamma_k - c_k)^2} - \sum_{k=0}^n c_k^2 =$$

$$= (R) + \sum_{n=0}^n (c_k - \gamma_k)^2 \geq (R)\,. \blacksquare \quad (9)$$

Bessel inequality

st.139

$f :]\alpha;\beta] \longrightarrow \mathbb{R}$ integrable on $]\alpha;\beta]$
c_k Fourier coefficients of f in orthonotmal system $v_0, v_1, v_2, \cdots$:
$\int_\alpha^\beta f^2 \geq \sum_{k=0}^\infty c_k^2$

proof. Consequence of the last statement. ■

Parseval equality

st.140

$f : [\alpha;\beta] \longrightarrow \mathbb{R}$ integrable on $[\alpha;\beta]$, $f(\alpha) = f(\beta)$
c_k Fourier coefficients of f in orthonotmal system $v_0, v_1, v_2, \cdots$:
$(\forall \epsilon > 0)\,(\exists n \in \mathbb{N}, \gamma_0, \gamma_1, \ldots \gamma_n \in \mathbb{R}) \sup_{x\in[\alpha;\beta]} \|f(x) - \sum_{k=0}^n \gamma_k v_k(x)\| < \epsilon \Longrightarrow \int_\alpha^\beta f^2 = \sum_{k=0}^\infty c_k^2$

proof. Given $\epsilon > 0$ arbitrary. For $\epsilon_1 := \sqrt{\frac{\epsilon}{\beta-\alpha}}$ there are $\gamma_0, \gamma_1, \gamma_2, \ldots \gamma_n \in \mathbb{R}$ such that $\sup_{x\in]\alpha;\beta]} |f(x) - \sum_{k=0}^n \gamma_k v_k(x)| < \epsilon_1 = \sqrt{\frac{\epsilon}{\beta-\alpha}}$. Therefore

$$\epsilon = \epsilon_1^2(\beta - \alpha) > \int_\alpha^\beta \left(f - \sum_{k=0}^n \gamma_k v_k\right)^2 \geq \int_\alpha^\beta f^2 - \sum_{k=0}^n c_k^2 \geq \int_\alpha^\beta f^2 - \sum_{k=0}^\infty c_k^2$$

and also $\int_\alpha^\beta f^2 - \sum_{k=0}^\infty c_k^2 \leq 0$ because $\epsilon > 0$ was arbitrary. ■

trigonometric system

st.141

$v_0 := \frac{1}{\sqrt{2\pi}},\ v_1 := \frac{1}{\sqrt{\pi}}\sin x,\ v_2 := \frac{1}{\sqrt{\pi}}\cos x,\ v_3 := \frac{1}{\sqrt{\pi}}\sin 2x,\ v_4 := \frac{1}{\sqrt{\pi}}\cos 2x, \ldots$ $\ldots, v_{2k-1} := \frac{1}{\sqrt{\pi}}\sin kx,\ v_{2k} := \frac{1}{\sqrt{\pi}}\cos kx, \ldots$ is an orthonormal system of functions on $[-\pi;\pi]$

proof. We have to enumerate integrals $\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{2\pi}}\right)^2 dx = 1$, for all k $\int_{-\pi}^{\pi} \sin kx\,dx = 0$, $\int_{-\pi}^{\pi} \cos kx\,dx = 0$, $\int_{-\pi}^{\pi} \sin^2 kx\,dx = 1$, $\int_{-\pi}^{\pi} \cos^2 kx\,dx = 1$ for all $k \in \mathbb{N}$ and $\int_{-\pi}^{\pi} \sin kx \sin lx\,dx = 0$, $\int_{-\pi}^{\pi} \cos kx \cos lx\,dx = 0$ and $\int_{-\pi}^{\pi} \sin kx \cos lx\,dx = 0$for all $k, l \in \mathbb{N}$, $k \neq l$. ■

Also so called Legendre polynomials $v_0 := 1$, $v_1 := \sqrt{3}\,(2x-1)$, $v_2 := \sqrt{5}\left(6x^2-6x+1\right), \ldots$ create orthonormal system of functions on $[0;1]$.

st.142 $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$

proof. We shall use Parseval equality with trigonometric system (see the consequence of Stone - Weierstrass theorem) for function $f(x) := \begin{cases} -\frac{x}{2} - \frac{\pi}{2} & \text{for } x \in [-\pi; 0[\\ -\frac{x}{2} + \frac{\pi}{2} & \text{for } x =\in [0; \pi] \end{cases}$. We can calculate the Fourier coefficients $c_0 = 0$, $c_{2k} = 0$ and

$$\begin{aligned} c_{2k-1} &= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \sin kx \, dx = \frac{1}{\sqrt{\pi}} \left(\int_{-\pi}^{0} \frac{-x-\pi}{2} \sin kx \, dx + \int_{0}^{\pi} \frac{-x+\pi}{2} \sin kx \, dx \right) = \\ &= \frac{2}{\sqrt{\pi}} \int_{0}^{\pi} \frac{\pi - x}{2} \sin kx \, dx = \frac{1}{k\sqrt{\pi}} \left(\pi \int_{0}^{k\pi} \sin t \, dt - \frac{1}{k} \int_{0}^{k\pi} t \sin t \, dt \right) = \\ &= \frac{1}{k\sqrt{\pi}} \left(-\pi \left[\cos t\right]_0^{k\pi} - \frac{1}{k} \left[\sin t - t \cos t\right]_0^{k\pi} \right) = \frac{1}{k\sqrt{\pi}} \left(\pi \left(1 - (-1)^k\right) + \frac{1}{k} k\pi \, (-1)^k \right) = \frac{\sqrt{\pi}}{k} . \end{aligned} \tag{10}$$

Parseval inequality gives

$$\pi \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=0}^{\infty} c_k^2 = \int_{-\pi}^{\pi} f^2(x) \, dx = 2 \int_{0}^{\pi} \left(\frac{x-\pi}{2} \right)^2 dx = \frac{1}{2} \int_{0}^{\pi} t^2 \, dt = \frac{1}{6} \left[t^3\right]_0^{\pi} = \frac{\pi^3}{6} . \blacksquare$$

st.143 $\boxed{\cos a \sin b = \frac{1}{2} \left(\sin(a+b) - \sin(a-b)\right)}$

proof. We shal substract following two equalities

$$\begin{aligned} \sin(a+b) &= \sin a \cos b + \cos a \sin b \\ \sin(a-b) &= \sin a \cos b - \cos a \sin b . \blacksquare \end{aligned} \tag{11}$$

Dirichlet kernel

def.31 $\boxed{D_n(x) \overset{\text{def.}}{=} \begin{cases} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin \frac{x}{2}} & \text{for } x \neq 0, \pm 2\pi, \pm 4\pi, \ldots \\ 2n+1 & \text{for } x = 0, \pm 2\pi, \pm 4\pi, \ldots \end{cases}}$

st.144 $\boxed{D_n(x) = 1 + 2 \sum_{k=1}^{n} \cos kx}$

proof. We shall use the statement 143 for $a := kx$ and $b := \frac{x}{2}$

$$2\sin\frac{x}{2}\cos kx = \sin\left(k+\frac{1}{2}\right)x - \sin\left(k-\frac{1}{2}\right)x$$

and add over $k = 1, \ldots, n$

$$2\sin\frac{x}{2}\sum_{k=1}^{n}\cos kx = \sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x\,. \blacksquare$$

st.145 $\boxed{\int\limits_{-\pi}^{\pi} D_n(t)\,dt = 2\pi}$

proof. Easy calculation. ■

st.146
$f : [-\pi;\pi] \longrightarrow \mathbb{R}$ integrable on $[-\pi;\pi]$, $f(-\pi) = f(\pi)$,
$c_0, c_1, c_2, \ldots$ Fourier coefficients of f
in orthonormal system $v_0 := \frac{1}{\sqrt{2\pi}}, v_{2k-1} := \frac{1}{\sqrt{\pi}}\cos kx,\ v_{2k} := \frac{1}{\sqrt{\pi}}\sin kx, k = 1, 2, \cdots :$
$(\forall x \in [-\pi;\pi[)\ \sum\limits_{i=0}^{2n} c_i v_i(x) = c_0\frac{1}{\sqrt{2\pi}} + \sum\limits_{k=1}^{n}\left(c_{2k-1}\frac{\cos kx}{\sqrt{\pi}} + c_{2k}\frac{\sin kx}{\sqrt{2\pi}}\right) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} D_n(t)f(x-t)\,dt$

proof. We can wide function f to whole $\mathbb{R}$ periodically and calculate by using substitution $s := x - t$

$$\int\limits_{-\pi}^{\pi} D_n(t)f(x-t)\,dt = \int\limits_{x-\pi}^{x+\pi} D_n(x-s)f(s)\,ds\,.$$

As both D_n and f are periodical functions with perion 2π we obtain the same integral with limits of integration changed from $[x-\pi; x+\pi]$ to $[-\pi;\pi]$ and

$$\int\limits_{-\pi}^{\pi} D_n(x-s)f(s)\,ds = \int\limits_{-\pi}^{\pi} f(s)\,ds + 2\int\limits_{-\pi}^{\pi}\sum_{k=0}^{n}\cos k(x-s)f(s)\,ds =$$

$$= \int\limits_{-\pi}^{\pi} f(s)\,ds + 2\int\limits_{-\pi}^{\pi}\sum_{k=1}^{n}\left(\cos kx\cos ks + \sin kx\sin ks\right)f(s)\,ds =$$

$$= \underbrace{\int\limits_{-\pi}^{\pi} f(s)\,ds}_{=\sqrt{2\pi}c_0} + 2\sum_{k=1}^{n}\cos kx\underbrace{\int\limits_{-\pi}^{\pi} f(s)\cos ks\,ds}_{=\sqrt{\pi}c_{2k-1}} + 2\sum_{k=1}^{n}\sin kx\underbrace{\int\limits_{-\pi}^{\pi} f(s)\sin ks\,ds}_{=\sqrt{\pi}c_{2k}} = 2\pi\sum_{i=0}^{2n}c_i v_i(x)\,. \blacksquare \quad (12)$$

Lipschitz function

def.32
$K \subset \mathbb{R}, I \subset K, f : K \longrightarrow \mathbb{R}$:
f Lipschitz function $\iff (\exists M > 0)\,(\forall x_1, x_2 \in I)\,|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$

For instance any function f which have finite derivative f' on set $K \subset \mathbb{R}$ is Lipschitz on any $[\alpha;\beta] \subset K$. Function $f(x) := |x|$ is an example of that which is Lipschitz on $[-1;1]$ and has no derivative at 0. It is easy seen that any function Lipschitz on $]\alpha;\beta[$ is continuous on $]\alpha;\beta[$. Function $f(x) := \sqrt[3]{x}$ can be presented as an example of that which is continuous on $[-1;1]$ and is not Lipschitz on $[-1;1]$.

st.147

$f : [-\pi;\pi] \longrightarrow \mathbb{R}$ integrable on $[-\pi;\pi]$, $f(-\pi) = f(\pi)$, $a \in]-\pi;\pi[$
$c_0, c_1, c_2, \ldots$ Fourier coefficients of f
in orthonormal system $v_0 := \frac{1}{\sqrt{2\pi}}, v_{2k-1} := \frac{1}{\sqrt{\pi}}\cos kx,\ v_{2k} := \frac{1}{\sqrt{\pi}}\sin kx, k = 1, 2, \cdots :$
$(\exists\Delta > 0)\, f$ Lipschitz on $]a-\Delta; a+\Delta[\Longrightarrow \sum\limits_{i=0}^{\infty} c_i v_i(a) = c_0\frac{1}{\sqrt{2\pi}} + \sum\limits_{k=1}^{\infty}\left(c_{2k-1}\frac{\cos ka}{\sqrt{\pi}} + c_{2k}\frac{\sin ka}{\sqrt{2\pi}}\right) = f(a)$

proof. By hypothesis for some $M > 0$ we have $|f(a+x) - f(x)| \le M|x|$ on $]-\Delta;\Delta[$. We can again wide function f periodically to $\mathbb{R}$ and use

$$\sum_{i=0}^{2n} c_i v_i(a) - f(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(a-x) - f(a))\, D_n(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(a-x) - f(a))\,\frac{\sin\left(n+\frac{1}{2}\right)x}{sin\frac{x}{2}}\, dx =$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{(f(a-x) - f(a))}_{\text{integrable on } [-\pi;\pi]}\cot\frac{x}{2}\sin nx\, dx}_{\to 0} + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{(f(a-x) - f(a))}_{\text{integrable on } [-\pi;\pi]}\cos nx\, dx}_{\to 0} \to 0\,. \quad (13)$$

We can use statement 156 as $\cot\frac{x}{2}$ and also $(f(a-x) - f(x))\cot\frac{x}{2}$ is integrable on $[\delta;\Delta]$ for any $\delta > 0$ and bounded on $[0;\Delta]$ (no matter function value at 0) because

$$\left|(f(a-x) - f(x))\cot\frac{x}{2}\right| = \underbrace{\frac{|f(a-x) - f(x)|}{|x|}}_{\le M}\ \underbrace{\left|\frac{x}{\sin\frac{x}{2}}\right|}_{\to 2 \text{ around } 0}\ \underbrace{\left|\cos\frac{x}{2}\right|}_{\le 1}\,. \ \blacksquare$$

ex. **Example** $\sum\limits_{n=1}^{\infty}\frac{1}{2n-1}\sin\left[(2n-1)\,a\right] = \frac{\pi}{4}$ for any $a \ne 0, \pm$

We can use the statement 148 for function $fx := \begin{cases} 1 & \text{for } x \in [0;\pi] \cup \{-\pi\} \\ -1 & \text{for } x \in]-\pi;0[\end{cases}$ and $0 < a < \pi$. Fourier coefficients are $c_0 = 0$, $c_{2k-1} = 0$ and $c_{2k} = \frac{2}{k\sqrt{\pi}}\left(1 - (-1)^k\right)$ and so

$$1 = f(a) = \sum_{k=1}^{\infty}\frac{2}{k\sqrt{\pi}}\left(1 - (-1)^k\right)\frac{\sin ka}{\sqrt{\pi}} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left[(2n-1)\,a\right]}{2n-1}\,.$$