3 . Series

3.1 . Summability of sequences

We can extend the operation of addition of two numbers a_1+a_2 to cover the addition of terms of any n-tuple of real numbers $a_1+a_2+\cdots+a_n$ owing to associative rule. This is possible not only for n-tuples but also for the first n terms of any (infinite) sequence called **partial sum**

$$A_n := a_1+a_2+\cdots+a_n\,.$$

We obtain the new sequence A_n which is usually called **series** corresponding to sequence a_n in literature. This is an analogy of antiderivative (or primitive function). The limit of A_n is usually called **sum of the series** a_n. But this terminology is rather misleading. [1] By another approach we can comprehend the partial sum of sequence as a mapping $\sum\limits_{k=1}^{n}$ which for any sequence yields the corresponding partial sum

$$\sum_{k=1}^{n} : \{a_n\}_{n=1}^{\infty} \mapsto a_1+a_2+\cdots+a_n \in \mathbb{R}\,.$$

By generalization we obtain a sum of all terms of sequence. We shall consider it as a mapping $\sum\limits_{k=1}^{\infty}$ which for any given sequence yields the value

$$\sum_{k=1}^{\infty} : \{a_n\}_{n=1}^{\infty} \mapsto \lim_{n\to\infty}(a_1+a_2+\cdots+a_n) \in \mathbb{R}^*\,,$$

(iff the limit exists). Domain of mapping $\sum\limits_{k=1}^{\infty}$ is all sequences for which the $\lim\limits_{n\to\infty}(a_1+a_2+\cdots+a_n)$ exists (finite or $\pm\infty$).

In the beginning of the last chapter we compared the notion of sequence (mapping $\mathbb{N}\longrightarrow\mathbb{R}$) with the notion of function (mapping $\mathbb{R}\longrightarrow\mathbb{R}$), now there is another obvious analogy. Sum of terms of sequence $\sum\limits_{n=1}^{\infty} a_n$ is similar to integral of function f (defined for instance on $[1;\infty[$) $\int_1^{\infty} f$. The integral can also be considered as an operator or a mapping which for any defined functions yields a real number or $\pm\infty$.

[1] For instance $\{a_n\}_{n=1}^{\infty}$ is a sequence, $\{A_n\}_{n=1}^{\infty}$ is a corresponding series, limit of $\{A_n\}_{n=1}^{\infty}$ is a sum of series corresponding to $\{a_n\}_{n=1}^{\infty}$, so the sum of sequence is $\{A_n\}_{n=1}^{\infty}$. Does this mean that the sum of $\{A_n\}_{n=1}^{\infty}$ is tantamount to the limit of $\{A_n\}_{n=1}^{\infty}$?

And so in the chapter with superscription "series" we shall quite avoid this notation.

partial sum and sum of sequence

def. 17

a_n sequence :
$$\sum_{k=1}^{n} a_k \overset{\text{def.}}{=} a_1 + a_2 + \cdots + a_n \quad \text{and for } n > m \quad \sum_{k=m+1}^{n} a_k \overset{\text{def.}}{=} a_{m+1} + a_{m+2} + \cdots + a_n$$
$$\sum_{k=1}^{\infty} a_k \overset{\text{def.}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n)$$
$$\sum_{k=m+1}^{\infty} a_k \overset{\text{def.}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} a_{m+k} = \lim_{n\to\infty} (a_{m+1} + a_{m+2} + \cdots + a_n)$$

We introduced two basic types of sequences in the last chapter. Now we can construct two tables of their sums.

Arithmetical sequence (with difference $d \in \mathbb{R}$ and opening term $a \in \mathbb{R}$):

$$\sum_{n=1}^{\infty} (a + (n-1)d)$$

$\sum$	$a > 0$	$a = 0$	$a < 0$
$d > 0$	∞	∞	∞
$d = 0$	∞	0	∞
$d < 0$	$-\infty$	$-\infty$	$-\infty$

Geometrical sequence (with quotient $q \in \mathbb{R}$ and opening term $a \in \mathbb{R}$):

$$\sum_{n=1}^{\infty} \left(a \cdot q^{n-1}\right)$$

$\sum$	$a > 0$	$a = 0$	$a < 0$
$q \geq 1$	∞	0	$-\infty$
$-1 < q < 1$	$\frac{a}{1-a}$	$\frac{a}{1-a}$	$\frac{a}{1-a}$
$q \leq -1$	*doesn't ex.*	0	*doesn't ex.*

To deduce the formula for sum of geometrical sequence for $-1 < q < 1$, we can prove by mathematical induction that
$$\left(1 + q + q^2 + \cdots + q^{n-1}\right)(1-q) = (1 - q^n) \ .$$
Then we have
$$1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q} \to \frac{1}{1-q} \, ,$$
therefore
$$\sum_{k=1}^{\infty} a q^{k-1} = \lim_{n\to\infty} a \left(1 + q + q^2 + \cdots + q^{n-1}\right) = \frac{a}{1-q} \quad \text{as } n \to \infty \, . \tag{5}$$

st. 59

$$(\forall m \in \mathbb{N}) \sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_m + \sum_{k=m+1}^{\infty} a_k$$

proof. A consequence of statement about limit of addition and equality $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_m + \sum_{k=m+1}^{n} a_k$ for $m < n$. ■

summability of sequence

def. 18 $\boxed{\{a_n\}_{n=1}^{\infty} \text{ summable } \iff \sum_{n=1}^{\infty} a_n \in \mathbb{R}}$

We sometimes say a sequence has finite sum (or series of a_n is convergent).

The fact that a sequence has finite sum, does not depend on a finite number of its first terms.

st. 60
$$\boxed{\begin{array}{ll} \sum_{n=1}^{\infty} a_n \text{ finite} & \implies (\forall m \in \mathbb{N}) \sum_{n=1}^{\infty} a_{m+n} \text{ finite} \\ (\exists m \in \mathbb{N}) \sum_{n=1}^{\infty} a_{m+n} \text{ finite} & \implies \sum_{n=1}^{\infty} a_n \text{ finite} \end{array}}$$

proof. Consequence of last statement. ■

Bolzano - Cauchy theorem

st. 61
$$\boxed{\sum_{n=1}^{\infty} a_n \text{ finite } \iff (\forall \epsilon > 0)\, (\exists n_0 \in \mathbb{N})\, (\forall m, n \geq n_0, n > m)\ |\sum_{k=m+1}^{n} a_k| < \epsilon}$$

proof. For partial sum $A_n = a_1 + \cdots + a_n$ of sequence a_n we have $A_n - A_m = \sum_{k=m+1}^{n} a_k$. Therefore the condition in the statement means the sequence A_n is Cauchy and we can use statement 52.

st. 62
$$\boxed{\sum_{n=1}^{\infty} a_n \text{ finite } \iff (\forall \epsilon > 0)\, (\exists n_0 \in \mathbb{N})\ |\sum_{k=n_0+1}^{\infty} a_k| < \epsilon}$$

proof. ($\implies$): Given $\epsilon > 0$ arbitrary. As the sequence of partial sums converge $a_1 + a_2 + \cdots + a_n \to a \in \mathbb{R}$ for all n from some n_0 onwards we have $|a - (a_1 + \cdots + a_n)| < \epsilon$. The inequality holds also for n_0 and by statement 59 we obtain $a - (a_1 + \cdots + a_{n_0}) = \sum_{k=1}^{\infty} a_k - (a_1 + \cdots + a_{n_0}) = \sum_{k=n_0+1}^{\infty} a_k$.

($\implies$): It follows from statement 61. ■

st. 63 $\boxed{\sum_{n=1}^{\infty} a_n \text{ finite } \implies a_n \to 0}$

proof. For the sequence of partial sums converges, $A_n := a_1 + a_2 + \ldots a_n \to a \in \mathbb{R}$ and so $A_{n+1} \to a$. As $a_n = A_n - A_{n-1}$ for $n > 1$ we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} A_n - \lim_{n\to\infty} A_{n-1} = a - a = 0\,. ■$$

The consequence of this statement is that a sequence with non zero limit has no finite sum.

st. 64 $\boxed{a_n \to a \Longrightarrow \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - a}$

proof. We can express the partial sum of the sequence $\{a_n - a_{n+1}\}_{n=1}^{\infty}$ in the form

$$(a_1 - a_2) + (a_2 - a_3) + \dots (a_n - a_{n+1}) = a_1 - a_{n+1} \to a_1 - a\,. \blacksquare$$

3.2 . Tests for positive sequences

In the following section we shall consider the sequences with non negative or positive terms. The sequence of partial sums of non negative sequences is increasing and its limit exists always (finite or ∞).

st. 65 $\boxed{(\forall n \in \mathbb{N})\ a_n \geq 0 \Longrightarrow \text{ exists } \sum_{n=1}^{\infty} a_n \geq 0}$

proof. Monotone sequences have limits. $\blacksquare$

We shall introduce a very important sequence.

harmonic sequence :

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

We can prove it also by the Bolzano - Cauchy theorem, with $\epsilon := \frac{1}{2}$ for any $k \in \mathbb{N}$ there are two numbers $m := k$ and $n := 2k$ such that

$$\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} \geq k \cdot \frac{1}{2k} = \frac{1}{2} = \epsilon\,.$$

So we know this sequence is not summable. According to the last statement 65, its sum must be equal to ∞.

st. 66 $\boxed{\begin{array}{c}(\forall n \in \mathbb{N})\ a_n \geq 0,\ A_n := a_1 + a_2 + \cdots + a_n : \\ \sum_{n=1}^{\infty} a_n < \infty \Longleftrightarrow A_n \text{ bounded}\end{array}}$

proof. The increasing sequence A_n is convergent iff it is bounded according to the statements 26 and 2. $\blacksquare$

comparison test

st. 67 $\boxed{\begin{array}{l}(\forall n \in \mathbb{N})\ a_n \geq 0,\, b_n \geq 0 : \\ (\forall n \in \mathbb{N})\ a_n \leq b_n \text{ and } \sum_{n=1}^{\infty} b_n = b \in \mathbb{R} \Longrightarrow (\exists a \in \mathbb{R}) \sum_{n=1}^{\infty} a_n = a \text{ and } a \leq b \\ (\forall n \in \mathbb{N})\ a_n \leq b_n \text{ and } \sum_{n=1}^{\infty} a_n = \infty \Longrightarrow \sum_{n=1}^{\infty} b_n = \infty\end{array}}$

proof. The inequality $0 \le a_n \le b_n$ induces the inequality $0 \le a_1 + a_2 + \cdots + a_n \le b_1 + b_2 + \cdots + b_n$. Thus this statement is a direct consequence of statements 3 and 5. ■

st. 68

$$\boxed{\begin{array}{l} (\forall n \in \mathbb{N})\ a_n > 0,\ b_n > 0 : \\ \frac{a_n}{b_n} \to c \in \mathbb{R} \text{ and } \sum\limits_{n=1}^{\infty} b_n < \infty \Longrightarrow \sum\limits_{n=1}^{\infty} a_n < \infty \end{array}}$$

proof. The convergent sequence $\frac{a_n}{b_n}$ is bounded, so there exists $M \in \mathbb{R}$ such that $0 \le \frac{a_n}{b_n} \le M$, so $0 \le a_n \le M \cdot b_n$. Then we use the last statement. ■

st. 69

$$\boxed{\begin{array}{l} (\forall n \in \mathbb{N})\ a_n > 0,\ b_n > 0 : \\ \frac{a_n}{b_n} \to c \in \mathbb{R} \text{ and } c \neq 0 \Longrightarrow \left(\sum\limits_{n=1}^{\infty} b_n < \infty \Longleftrightarrow \sum\limits_{n=1}^{\infty} a_n < \infty \right) \end{array}}$$

proof. This is a consequence of the last statement. It is only necessary to realize that for $c \neq 0$, the sequence $\frac{b_n}{a_n}$ also has a finite limit $\frac{1}{c}$. ■

comparison test

st. 70

$$\boxed{\begin{array}{l} (\forall n \in \mathbb{N})\ a_n > 0,\ b_n > 0 : \\ (\forall n \in \mathbb{N})\ \frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n} \text{ and } \sum\limits_{n=1}^{\infty} b_n < \infty \Longrightarrow \sum\limits_{n=1}^{\infty} a_n < \infty \\ (\forall n \in \mathbb{N})\ \frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n} \text{ and } \sum\limits_{n=1}^{\infty} a_n = \infty \Longrightarrow \sum\limits_{n=1}^{\infty} b_n = \infty \end{array}}$$

proof. For any $n \in \mathbb{N}$ we have

$$\frac{a_n}{a_1} = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \cdots \cdot \frac{a_3}{a_2} \cdot \frac{a_2}{a_1} \le \frac{b_n}{b_{n-1}} \cdot \frac{b_{n-1}}{b_{n-2}} \cdot \cdots \cdot \frac{b_3}{b_2} \cdot \frac{b_2}{b_1} = \frac{b_n}{b_1}\ .$$

I.: As $a_n < \frac{a_1}{b_1} b_n$ and $\sum\limits_{n=1}^{\infty} \frac{a_1}{b_1} b_n = \frac{a_1}{b_1} \sum\limits_{n=1}^{\infty} b_n < \infty$, the statement 67 gives us $\sum\limits_{n=1}^{\infty} a_n < \infty$.
II.: Similarly. ■

Maclaurin integral test

st. 71

$$\boxed{\begin{array}{l} f : [1;\infty[\longrightarrow \mathbb{R},\ f \ge 0 \text{ decreasing},\ (\forall n \in \mathbb{N})\ a_n = f(n) : \\ \qquad \sum\limits_{n=1}^{\infty} a_n < \infty \Longleftrightarrow \text{ there exists } (R) \int_1^{\infty} f < \infty \end{array}}$$

proof. We shall use knowledge of integral calculus that any monotone function on $[1;\infty[$ has (Riemann) integral $\int_1^{\infty} f$ being finite or infinite. We establish two functions $g : [1;\infty[\longrightarrow \mathbb{R}$ and $h : [1;\infty[\longrightarrow \mathbb{R}$

$$g(x) = \begin{cases} a_1 \text{ for } x \in [1;2[\\ a_2 \text{ for } x \in [2;3[\\ \ldots \\ a_{n-1} \text{ for } x \in [n-1;n[\\ \ldots \end{cases} \qquad h(x) = \begin{cases} a_2 \text{ for } x \in [1;2[\\ a_3 \text{ for } x \in [2;3[\\ \ldots \\ a_n \text{ for } x \in [n-1;n[\\ \ldots \end{cases}\ .$$

These functions are monotone so their integrals exist from 1 to ∞, too. Let us also take note of the relation $0 \le g \le f \le h$ on $[1;\infty[$.
$(\Longrightarrow)$: Given $t > 0$ arbitrary, there exists $n \ge t$ (for instance $n := [t] + 1$, $[t]$ denotes integral part of t.) We have

$$\int_1^t f \le \int_1^t h \le \int_1^n h = \sum_{k=2}^n a_k \le \sum_{k=1}^\infty a_k - a_1 \text{ and } \int_1^\infty f = \lim_{t\to\infty} \int_1^t f \le a - a_1$$

and so this integral is finite.
$(\Longleftarrow)$: Given $n \in \mathbb{N}$ arbitrary. We have

$$\sum_{k=1}^n a_k = \int_1^{n+1} g \le \int_1^{n+1} f \le \int_1^\infty f \text{ and so } \sum_{k=1}^\infty a_k = \lim_{n\to\infty} \sum_{k=1}^n a_k \le \int_1^\infty f$$

and sequence a_n has a finite sum. ■

d'Alembert ratio test

st. 72

$$\boxed{\begin{array}{lll} (\forall n \in \mathbb{N})\ a_n > 0 : & & \\ (\exists q \in \mathbb{R},\ 0 \le q < 1)\,(\forall n \in \mathbb{N}) & \frac{a_{n+1}}{a_n} \le q & \Longrightarrow \quad \sum\limits_{n=1}^\infty a_n < \infty \\ (\forall n \in \mathbb{N}) & \frac{a_{n+1}}{a_n} \ge 1 & \Longrightarrow \quad \sum\limits_{n=1}^\infty a_n = \infty \end{array}}$$

proof. I.: We shall step by step put $n = 1, 2, \ldots$ into the hypothesis and obtain

$$a_2 \le qa_1,\ a_3 \le qa_2 \le q^2 a_1,\ a_4 \le qa_3 \le q^3 a_1, \ldots, a_n \le q^{n-1} a_1\,.$$

The geometric sequence $\left\{a_1 q^{k-1}\right\}_{n=1}^\infty$ has finite sum $\sum\limits_{k=1}^\infty a_1 q^{k-1} = \frac{a_1}{1-q}$, therefore $\sum\limits_{n=1}^\infty a_n$ is finite by comparison test (statement 67).
II.: We again step by step put $n = 1, 2, \ldots$ into the hypothesis and obtain $a_n \ge a_1$ for all n. Then we have for partial sums $a_1 + a_2 + \cdots + a_n \ge na_1 \to \infty$ and so $\sum\limits_{n=1}^\infty a_n = \lim\limits_{n\to\infty} (a_1 + a_2 + \cdots + a_n) = \infty$. ■

st. 73

$$\boxed{\begin{array}{lll} (\forall n \in \mathbb{N})\ a_n > 0,\ c \in \mathbb{R}^*,\ c \ge 0 : & & \\ \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = c < 1 & \Longrightarrow & \sum\limits_{n=1}^\infty a_n < \infty \\ \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = c > 1 & \Longrightarrow & \sum\limits_{n=1}^\infty a_n = \infty \end{array}}$$

proof. I.: We put $\epsilon := \frac{1-c}{2}$ into the definition of limit and obtain from some n_0 onwards

$$c - \frac{1-c}{2} < \frac{a_{n+1}}{a_n} < c + \frac{1-c}{2} = \frac{1+c}{2}\,.$$

Then the sequence $a_{n_0+1}, a_{n_0+2}, \ldots$ fulfil the hypothesis of the first part of the ratio test (statement 72) for $q := \frac{1+c}{2} > 0$, therefore $\sum\limits_{k=n_0+1}^\infty a_k < \infty$. By the statement 60 also $\sum\limits_{k=1}^\infty a_k < \infty$.
II.: We put $\epsilon := c - 1$ into the definition of limit and obtain from some n_0 onward

$$1 = c - (c-1) < \frac{a_{n+1}}{a_n} < c + (c-1)\,.$$

Then the sequence $a_{n_0+1}, a_{n_0+2}, \ldots$ fulfill the hypothesis of the second part of the ratio test (statement 72), therefore $\sum\limits_{k=1}^{\infty} a_k \geq \sum\limits_{k=n_0+1}^{\infty} a_k = \infty$. ■

We can generalize the last statement 73 replacing the hypothesis in the first part with $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = c < 1$ and in the second, by $\liminf\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = c > 1$, because we can use the characterization of upper and lower limit in the statements 40, 41 and 43.

Cauchy root test

st. 74

$$\boxed{\begin{array}{llcl} (\forall n \in \mathbb{N})\ a_n \geq 0 : & & & \\ (\exists q \in \mathbb{R},\, 0 \leq q < 1)\,(\forall n \in \mathbb{N}) & \sqrt[n]{a_n} \leq q & \Longrightarrow & \sum\limits_{n=1}^{\infty} a_n < \infty \\ \text{for infinite number of } n \in \mathbb{N} & \sqrt[n]{a_n} \geq 1 & \Longrightarrow & \sum\limits_{n=1}^{\infty} a_n = \infty \end{array}}$$

proof. I.: As $a_n \leq q^n$ for all $n \in \mathbb{N}$ we can again use the comparison test for $\{a_n\}_{n=1}^{\infty}$ and the geometric sequence $\{q^n\}_{n=1}^{\infty}$. Then sum $\sum\limits_{n=1}^{\infty} a_n$ is finite because the sum of geometric sequence $\sum\limits_{n=1}^{\infty} q^n = \frac{q}{1-q}$ with quotient $q \in \langle 0; 1)$ is finite, too.

II.: The inequality $a_n \geq 1$ holds for infinite number of terms a_n. So for $\epsilon_0 := 1$ and for any n there is $k_n \geq n$ such that $|a_{k_n}| \geq 1$. According to the definition $a_n \to 0$ is not true and $\sum\limits_{n=1}^{\infty} a_n \in \mathbb{R}$ is not true. Therefore with regard to $a_n \geq 0$ we have $\sum\limits_{n=1}^{\infty} a_n = \infty$. ■

st. 75

$$\boxed{\begin{array}{lcl} (\forall n \in \mathbb{N})\ a_n \geq 0,\, c \in \mathbb{R}^*,\, c \geq 0 : & & \\ \lim\limits_{n\to\infty} \sqrt[n]{a_n} = c < 1 & \Longrightarrow & \sum\limits_{n=1}^{\infty} a_n < \infty \\ \lim\limits_{n\to\infty} \sqrt[n]{a_n} = c > 1 & \Longrightarrow & \sum\limits_{n=1}^{\infty} a_n = \infty \end{array}}$$

proof. I.: We put $\epsilon := \frac{1-c}{2}$ into the definition of limit and obtain from some n_0 onwards

$$c - \frac{1-c}{2} < \sqrt[n]{a_n} < c + \frac{1-c}{2} = \frac{1+c}{2}\,.$$

Then the sequence $a_{n_0+1}, a_{n_0+2}, \ldots$ fulfil the hypothesis of the first part of the root test (statement 74) for $q := \frac{1+c}{2} > 0$, therefore $\sum\limits_{k=n_0+1}^{\infty} a_k < \infty$. By the statement 60 also $\sum\limits_{k=1}^{\infty} a_k < \infty$.

II.: We put $\epsilon := c - 1$ into the definition of limit and obtain from some n_0 onward

$$1 = c - (c-1) < \sqrt[n]{a_n} < c + (c-1)\ .$$

Then the sequence $a_{n_0+1}, a_{n_0+2}, \ldots$ fulfill the hypothesis of the second part of the root test (statement 74), therefore $\sum\limits_{k=1}^{\infty} a_k \geq \sum\limits_{k=n_0+1}^{\infty} a_k = \infty$. ■

We can again generalize the last statement 75 replacing the $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ in hypothesis of both parts with $\limsup\limits_{n\to\infty} \sqrt[n]{a_n}$ and again use the characterization of only upper limit in the statement 40.

Rhabe test

st. 76

$$\boxed{\begin{array}{lll} (\forall n\in\mathbb{N})\ a_n>0: & & \\ (\exists p\in\mathbb{R},\, p>1)\,(\forall n\in\mathbb{N}) & \left(1-\frac{a_{n+1}}{a_n}\right)n\ge p & \Longrightarrow\ \sum\limits_{n=1}^{\infty} a_n<\infty \\ (\forall n\in\mathbb{N}) & \left(1-\frac{a_{n+1}}{a_n}\right)n\le 1 & \Longrightarrow\ \sum\limits_{n=1}^{\infty} a_n=\infty \end{array}}$$

proof. I.: We transform the condition into a form $(a_n-a_{n+1})\,n\ge pa_n$ or

$$\underbrace{\underbrace{(n-1)\,a_n}_{=b_n}-\underbrace{na_{n+1}}_{=b_{n+1}}}_{=c_n}\ge\underbrace{(p-1)}_{>0}a_n>0 \tag{6}$$

and define sequences $b_n:=(n-1)\,a_n$ and $c_n:=b_n-b_{n+1}$. From (6) we deduce that sequence b_n is decreasing. As it is also bounded below $b_n=(n-1)\,a_n\ge 0$ by statement 30 has finite limit $b_n\to b\in\mathbb{R}$. It follows from statement 64 that sequence $c_n=b_n-b_{n+1}$ has finite sum $\sum\limits_{n=1}^{\infty} c_n=b_1-b$. According to (6) we have $a_n\le\frac{1}{p-1}c_n$ for all n and $\sum\limits_{n=1}^{\infty}\frac{c_n}{p-1}=\frac{b_1-b}{p-1}<\infty$, therefore $\sum\limits_{n=1}^{\infty} a_n<\infty$, too, by the comparison test (statement 67).
II.: We transform the condition into a form

$$\underbrace{(n-1)\,a_n}_{=b_n}\le\underbrace{na_{n+1}}_{=b_{n+1}}$$

and again define sequence $b_n:=(n-1)\,a_n$. This sequence b_n is increasing and for all $n\ge 2$ we have $b_n\ge b_2=a_2>0$ and also $a_n=\frac{b_n}{n-1}\ge\frac{a_2}{n-1}$. As $\sum\limits_{n=2}^{\infty}\frac{a_2}{n-1}=a_2\sum\limits_{n=1}^{\infty}\frac{1}{n}=\infty$ we obtain $\sum\limits_{n=1}^{\infty} a_n=\infty$ by the comparison test (statement 67). ■

st. 77

$$\boxed{\begin{array}{ll} (\forall n\in\mathbb{N})\ a_n>0,\ c\in\mathbb{R}^*,\ c\ge 0: & \\ \lim\limits_{n\to\infty}\left(1-\frac{a_{n+1}}{a_n}\right)n=c>1 & \Longrightarrow\ \sum\limits_{n=1}^{\infty} a_n<\infty \\ \lim\limits_{n\to\infty}\left(1-\frac{a_{n+1}}{a_n}\right)n=c<1 & \Longrightarrow\ \sum\limits_{n=1}^{\infty} a_n=\infty \end{array}}$$

proof. It is the same as in the statement 75. ■

The notion of replacing the limits in the hypothesis by upper and lower limits can similarly be accepted in this statement as in the case of statement 73.

Now we state a quite general test from which other tests can be obtained.

Kummer test

st. 78

$$(\forall n\in\mathbb{N})\ a_n>0, b_n>0 \text{ and } \sum_{n=1}^{\infty} b_n=\infty:$$

$$(\exists p\in\mathbb{R},\ p>0)\,(\forall n\in\mathbb{N})\qquad \frac{a_n}{a_{n+1}}\frac{1}{b_n}-\frac{1}{b_{n+1}}\ge p \implies \sum_{n=1}^{\infty} a_n<\infty$$

$$(\forall n\in\mathbb{N})\qquad \frac{a_n}{a_{n+1}}\frac{1}{b_n}-\frac{1}{b_{n+1}}\le 0 \implies \sum_{n=1}^{\infty} a_n=\infty$$

proof. It is similar to that of the statement 76.
I.: We define sequences $d_n := \frac{a_n}{b_n}$ and $c_n := d_n - d_{n+1}$ and we can again transform the condition into a form

$$\underbrace{\underbrace{\frac{a_n}{b_n}}_{=d_n}-\underbrace{\frac{a_{n+1}}{b_{n+1}}}_{=d_{n+1}}}_{=c_n}\ge pa_{n+1}\ge 0\,. \tag{7}$$

Sequence d_n has positive terms and it is decreasing. Therefore by statement 30 it has finite limit $d_n\to d\in\mathbb{R}$. It follows from statement 64 that sequence $c_n=d_n-d_{n+1}$ has finite sum $\sum_{n=1}^{\infty} c_n=\frac{a_1}{b_1}-d$. According the (7) $a_n\le\frac{1}{p-1}c_n$ for all n and $\sum_{n=1}^{\infty}\frac{c_n}{p-1}<\infty$, therefore also $\sum_{n=1}^{\infty} a_n<\infty$ by the comparison test (statement 67).
II.: We transform the condition into a form

$$\underbrace{\frac{a_n}{b_n}}_{=d_n}\le\underbrace{\frac{a_{n+1}}{b_{n+1}}}_{=d_{n+1}}$$

and define sequence $d_n:=\frac{a_n}{b_n}$. This sequence d_n is increasing and for all $n\in\mathbb{N}$ we have $d_n\ge\frac{a_1}{b_1}>0$. Therefore it has limit $A\in\mathbb{R}^*$ such that $A>0$. So there exists $K\in\mathbb{R}$, $K>0$ such that $a_n\ge Kb_n$ (for $A\in\mathbb{R}$ we have $K:=\frac{A}{2}$, for $A=\infty$ we can take arbitrary K, for instance $K:=1$). As $\sum_{n=1}^{\infty} b_n=\infty$ we obtain $\sum_{n=1}^{\infty} a_n=\infty$ by the comparison test (statement 67). ■

3.3 . Tests for positive decreasing sequences

Now we go on to consider the sequences with positive or non-negative terms and we shall more over suppose they are decreasing. For any sequence with finite sum we have $a_n\to 0$, for decreasing sequences with positive terms in addition $na_n\to 0$.

st. 79

$$(\forall n\in\mathbb{N})\ a_n>0,\ a_{n+1}\le a_n \text{ and } \sum_{n=1}^{\infty} a_n\in\mathbb{R}\implies na_n\to 0$$

proof. Given $\epsilon>0$ arbitrary. By Bolzano-Cauchy theorem (statement 61) there is some n_1 such that for all $k,l\ge n_1$, $k>l$ we have

$$a_{l+1}+\cdots+a_k<\frac{\epsilon}{2}\,. \tag{8}$$

Let $n_0 := 2n_1 + 4$ and let $n \geq n_0$ is arbitrary. There exists $m \in \mathbb{N}$ such that $n \geq 2m > n-2$ (we can put $m := \left[\frac{n}{2}\right]$, $[\cdot]$ denotes integral part of real number). We put $l := m-1 \geq n_1$ and $k := 2m \geq n_1$ into the relation (8) and we obtain

$$a_m + \cdots + a_{2m} < \frac{\epsilon}{2}\,.$$

Because sequence a_n is monotone and inequalities $n \geq 2m$ and $m+1 > \frac{n}{2}$ holds we have the valid estimation below

$$\underbrace{\underbrace{a_m}_{\geq a_{2m}} + \cdots + a_{2m}}_{m+1 \text{ terms}} \geq \underbrace{(m+1)}_{>\frac{n}{2}} \underbrace{a_{2m}}_{\geq a_n} > \frac{n}{2} a_n\,.$$

Then for arbitrary $n \geq n_0$ we have $\frac{na_n}{2} < a_m + \cdots + a_{2m} < \frac{\epsilon}{2}$ and $|na_n| < \epsilon$. ■

The condition a_n is a decreasing (not strictly) sequence is necessary in the statement 79.

For instance the sequence

$$1,\ 0,\ 0,\ \frac{1}{4},\ 0,\ 0,\ 0,\ 0,\ \frac{1}{9},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{16},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{25},\ 0,\ 0,\ \ldots$$

$$a_n = \begin{cases} \frac{1}{n} & \text{for } n = k^2 \\ 0 & \text{for } n \neq k^2 \end{cases}$$

is summable but $n \cdot a_n \not\to 0$ as $n^2 \cdot a_{n^2} = 1$.

Cauchy accumulation test

st. 80 $\boxed{(\forall n \in \mathbb{N})\ a_n > 0,\ a_{n+1} \leq a_n : \quad \sum_{n=1}^{\infty} a_n < \infty \Longleftrightarrow \sum_{n=1}^{\infty} 2^n a_{2^n} < \infty}$

proof. We shall denote $A_n := a_1 + a_2 + \cdots + a_n$ and $B_n := 2a_2 + 4a_4 + 8a_8 + \cdots + 2^n a_{2^n}$.
(I.$\Longrightarrow$:) For any $n \in \mathbb{N}$ we have

$$A_{2^n} = a_1 + a_2 + \underbrace{\underbrace{a_3}_{\geq a_4} + a_4}_{\geq 2a_4} + \underbrace{\underbrace{a_5}_{\geq a_8} + \underbrace{a_6}_{\geq a_8} + \underbrace{a_7}_{\geq a_8} + a_8}_{\geq 4a_8} + \cdots + a_{2^{n-1}} + \underbrace{\underbrace{a_{2^{n-1}+1}}_{\geq a_{2^n}} + \cdots + a_{2^n}}_{\geq 2^{n-1} a_{2^n}}$$

and

$$A_{2^n} - a_1 \geq a_2 + 2a_4 + 4a^8 + \cdots + 2^{n-1} a_{2^n} = \frac{1}{2} B_n\,.$$

Therefore $B_n \leq 2\left(A_{2^n} - a_1\right) \leq 2\left(\sum_{n=1}^{\infty} a_n - a_1\right)$. As B_n is increasing and bounded it converges by statement 28.
(I.$\Longleftarrow$:) Similarly for any $n \in \mathbb{N}$ we have

$$A_{2^n} = a_1 + \underbrace{a_2 + \underbrace{a_3}_{\leq a_2}}_{\leq 2a_2} + \underbrace{a_4 + \underbrace{a_5}_{\leq a_4} + \underbrace{a_6}_{\leq a_4} + \underbrace{a_7}_{\leq a_4}}_{\leq 4a_4} + \cdots + a_{2^{n-1}-1} + \underbrace{a_{2^{n-1}} + \cdots + \underbrace{a_{2^n+1}}_{\leq a_{2^{n-1}}}}_{2^{n-1} a_{2^{n-1}}} + \underbrace{a_{2n}}_{\leq 2^n a_{2^n}}\,.$$

Therefore $A_{2^n} \le B_n \le \sum_{n=1}^{\infty} 2^n a_{2^n}$ and as A_n is increasing and bounded it converges again by statement 28. ■

We can also state similar test which is more general.

st. 81

$$\boxed{\begin{array}{l} (\forall n \in \mathbb{N})\ a_n > 0,\ a_{n+1} \le a_n, \\ p : \mathbb{N} \longrightarrow \mathbb{N} \text{ increasing and } (\exists M > 0)\,(\forall n \in \mathbb{N}, n \ge 2)\, p(n+1) - p(n) \le M\,(p(n) - p(n-1)) : \\ \qquad \sum_{n=1}^{\infty} a_n < \infty \Longleftrightarrow \sum_{n=1}^{\infty} (p(n+1) - p(n))\, a_{p(n)} < \infty \end{array}}$$

proof. Proof is very similar to that of last statement 80.
We again denote $A_n := a_1 + a_2 + \cdots + a_n$ and

$$B_n := (p(2) - p(1))\, a_{p(1)} + (p(3) - p(2))\, a_{p(2)} + \cdots + (p(n+1) - p(n))\, a_{p(n)}\,.$$

(I.$\Longrightarrow$:) For any $n \in \mathbb{N}$ we have

$$A_{p(n)} = a_1 + \cdots + a_{p(1)-1} + \underbrace{a_{p(1)}}_{\ge \frac{a_{p(1)}}{M}(p(2)-p(1))} +$$

$$+ \underbrace{\underbrace{a_{p(1)+1}}_{\ge a_{p(2)}} + \cdots + a_{p(2)}}_{\ge a_{p(2)} \underbrace{(p(2)-p(1))}_{\ge \frac{1}{M}(p(3)-p(2))}} + \underbrace{\underbrace{a_{p(2)+1}}_{\ge a_{p(3)}} + \cdots + a_{p(3)}}_{\ge a_{p(3)} \underbrace{(p(3)-p(2))}_{\ge \frac{1}{M}(p(4)-p(3))}} + \cdots + a_{p(n-1)} + \underbrace{\underbrace{a_{p(n-1)+1}}_{\ge a_{p(n)}} + \cdots + a_{p(n)}}_{a_{p(n)} \underbrace{(p(n)-p(n-1))}_{\ge \frac{1}{M}(p(n+1)-p(n))}}$$

and

$$A_{p(n)} \ge a_1 + \cdots + a_{p(1)-1} + \frac{1}{M} B_n \ge \frac{1}{M} B_n\,.$$

Therefore $B_n \le M A_{p(n)} \le M \sum_{n=1}^{\infty} a_n$ and as B_n is increasing and bounded it is convergent according to statement 28.
(I.$\Longleftarrow$:) Let us denote $B_n \to B \in \mathbb{R}$. Then for any $n \in \mathbb{N}$ we have

$$A_{p(n)} = a_1 + \cdots + a_{p(1)-1} +$$

$$+ \underbrace{a_{p(1)} + \cdots + \underbrace{a_{p(2)-1}}_{\le a_{p(1)}}}_{\le a_{p(1)}(p(2)-p(1))} + \underbrace{a_{p(2)} + \cdots + \underbrace{a_{p(3)-1}}_{\le a_{p(2)}}}_{\le a_{p(2)}(p(3)-p(2))} + a_{p(3)} + \cdots + a_{p(n-1)-1} + \underbrace{a_{p(n-1)} + \cdots + \underbrace{a_{p(n)-1}}_{\le a_{p(n-1)}}}_{a_{p(n-1)}(p(n)-p(n-1))} + a_{p(n)}$$

and

$$A_{p(n)} \le a_1 + \cdots + a_{p(1)-1} + B_n \le a_1 + \cdots + a_{p(1)-1} + B\,.$$

As A_n is increasing and bounded it converges again by statement 28. ■

The condition a_n is a decreasing (not strictly) sequence is necessary in the condensation test 80.

For instance the sequence

$$1,\ \frac{1}{4},\ 1,\ \frac{1}{16},\ 1,\ 1,\ 1,\ \frac{1}{64},\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \frac{1}{196},\ 1,\ 1,\ \ldots$$

$$a_n = \begin{cases} \frac{1}{n^2} & \text{for } n = 2^k \\ 1 & \text{for } n \neq 2^k \end{cases}$$

is not summable but $2^n \cdot a_{2^n} = 2^n \cdot \frac{1}{(2^n)^2 = \frac{1}{2^n}}$ is summable.

3.4 . Tests for other sequences

The following statement is similar to integration by parts in integral calculus.

Abel partial summation

st. 82 $\boxed{\begin{aligned} &a_1 b_1 + \cdots + a_n b_n = \\ &= a_1 (b_1 - b_2) + (a_1 + a_2)(b_2 - b_3) + \cdots + (a_1 + a_2 + \cdots + a_{n-1})(b_{n-1} - b_n) + (a_1 + \cdots + a_n) b_n \end{aligned}}$

proof. We shall consider the following list

$$\begin{array}{llllll}
a_1 (b_1 - b_2) & & & & & \\
a_1 (b_2 - b_3) & a_2 (b_2 - b_3) & & & & \\
a_1 (b_3 - b_4) & a_2 (b_3 - b_4) & a_3 (b_3 - b_4) & & & \\
 & \ldots & & & & \\
a_1 (b_{n-1} - b_n) & a_2 (b_{n-1} - b_n) & a_3 (b_{n-1} - b_n) & \ldots & a_{n-1} (b_{n-1} - b_n) & \\
a_1 b_n & a_2 b_n & a_3 b_n & \ldots & a_{n-1} b_n & a_n b_n
\end{array}$$

Upon adding the rows and columns we obtain exactly and respectively the right and left sides of the statement. ■

We can denote $A_k := a_1 + \cdots + a_k$ (something like an integral) and $b'_k := b_k - b_{k-1}$ for $k = 1, \ldots, n$ (something like a derivative). Then the Abel partial summation can be written in the form $\sum\limits_{k=1}^{n} a_k b_k = A_n b_n - \sum\limits_{k=1}^{n-1} A_k b'_{k+1}$, which suggests integration by parts indeed.

Abel test

st. 83 $\boxed{a_n \text{ summable }, b_n \text{ bounded and decreasing } \Longrightarrow a_n b_n \text{ summable}}$

proof. As sequence b_n is bounded there is some $M \geq 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. As it is decreasing we have $b_k - b_{k+1} \geq 0$.
I. First we suppose $b_n \geq 0$ for any $n \in \mathbb{N}$. Given $\epsilon > 0$ arbitrary. As a_n is summable according to the Bolzano - Cauchy statement there is $n_0 \in \mathbb{N}$ such that $|a_{m+1} + \cdots + a_k| < \frac{\epsilon}{M}$ for any $k, m \geq n_0$ $k > m$. Given $m, n \geq n_0$

arbitrary. We use the Abel partial summation to estimate

$$\begin{aligned}
&|a_{m+1}b_{m+1}+\dots+a_nb_n| = \\
&\quad = |a_{m+1}(b_{m+1}-b_{m+2}) + (a_{m+1}+a_{m+2})(b_{m+2}-b_{m+3}) + \dots \\
&\qquad \dots + (a_{m+1}+\dots+a_{n-1})(b_{n-1}-b_n) + (a_{m+1}+\dots+a_n)\,b_n| \le \\
&\quad \le \underbrace{|a_{m+1}|}_{<\frac{\epsilon}{M}}(b_{m+1}-b_{m+2}) + \underbrace{|a_{m+1}+a_{m+2}|}_{<\frac{\epsilon}{M}}(b_{m+2}-b_{m+3}) + \dots \\
&\qquad \dots + \underbrace{|a_{m+1}+\dots+a_{n-1}|}_{<\frac{\epsilon}{M}}(b_{n-1}-b_n) + \underbrace{|a_{m+1}+\dots+a_n|}_{<\frac{\epsilon}{M}}\,b_n < \\
&\qquad < \frac{\epsilon}{M}\underbrace{\Big(\sum_{k=m+1}^{n-1}(b_k-b_{k+1}) + b_n\Big)}_{=b_{m+1}\le M} \le \epsilon
\end{aligned}$$

II. For any (not only positive) sequence b_n we use the part I. with sequence $b_n+M \ge 0$ and can conclude that $\sum_{n=1}^{\infty} a_nb_n = \sum_{n=1}^{\infty} a_n(b_n+M) - M\sum_{n=1}^{\infty} a_n$ is finite, too. ■

Dirichlet test

st. 84 $\boxed{(\exists M>0)\,(\forall n\in\mathbb{N})\,|a_1+a_2+\dots+a_n|\le M, b_n \text{ decreasing and } b_n\to 0 \Longrightarrow a_nb_n \text{ summable}}$

proof. Sequence b_n is decreasing and tends to 0 hence $b_k - b_{k+1} \ge 0$ and $b_n \ge 0$. Given $\epsilon > 0$ arbitrary. As $b_n \to 0$ there is $n_0 \in \mathbb{N}$ such that $|b_n| < \frac{\epsilon}{2M}$ for any $n \ge n_0$. Given $m, n \ge n_0$ arbitrary. We shall again use the Abel partial summation to estimate

$$\begin{aligned}
&|a_{m+1}b_{m+1}+\dots+a_nb_n| = \\
&\quad = |a_{m+1}(b_{m+1}-b_{m+2}) + (a_{m+1}+a_{m+2})(b_{m+2}-b_{m+3}) + \dots \\
&\qquad \dots + (a_{m+1}+\dots+a_{n-1})(b_{n-1}-b_n) + (a_{m+1}+\dots+a_n)\,b_n| \le \\
&\quad \le \underbrace{|a_{m+1}|}_{\le 2M}(b_{m+1}-b_{m+2}) + \underbrace{|a_{m+1}+a_{m+2}|}_{\le 2M}(b_{m+2}-b_{m+3}) + \dots \\
&\qquad \dots + \underbrace{|a_{m+1}+\dots+a_{n-1}|}_{\le 2M}(b_{n-1}-b_n) + \underbrace{|a_{m+1}+\dots+a_n|}_{\le 2M}\,b_n \le \\
&\qquad \le 2M\underbrace{\sum_{k=m+1}^{n-1}(b_k-b_{k+1}) + b_n}_{=b_{m+1}<\frac{\epsilon}{2M}} < \epsilon\,.
\end{aligned}$$ ■

Leibniz test

st. 85 $\boxed{b_n \text{ decreasing and } b_n\to 0 \Longrightarrow (-1)^n\,b_n \text{ summable}}$

proof. Consequence of the last statement for sequence $a_n := (-1)^n$. This sequence $-1, 1, -1, 1, \dots$ has bounded partial sums $-1, 0, -1, 0, \dots$. ■

The condition a_n is a decreasing (not strictly) sequence is necessary in the Leibniz test 85.
For instance the sequence

$$\underbrace{\frac{1}{2}-\frac{1}{3}}_{=\frac{1}{6}}+\underbrace{\frac{1}{4}-\frac{1}{6}}_{=\frac{1}{12}}+\underbrace{\frac{1}{6}-\frac{1}{9}}_{=\frac{1}{18}}+\underbrace{\frac{1}{8}-\frac{1}{12}}_{=\frac{1}{24}}+\underbrace{\frac{1}{10}-\frac{1}{15}}_{=\frac{1}{30}}+\cdots=\frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{n}=\infty$$

$$a_n=\begin{cases}\frac{1}{2^n} & \text{for } n=2k-1\\ \frac{1}{3^n} & \text{for } n=2k\end{cases}$$

is not summable but $a_n \to 0$.

3.5 . Summability in average

st. 86 $\boxed{c_n \to 0 \Longrightarrow \frac{c_1+\cdots+c_n}{n} \to 0}$

proof. Given $\epsilon > 0$ arbitrary. There is $n_1 \in \mathbb{N}$ such that $|c_n| < \frac{\epsilon}{2}$ for any $n \geq n_1$. We can choose some $n_2 > \frac{2}{\epsilon}(|c_1|+\cdots+|c_{n_1}|)$ and put $n_0 := \max(n_1, n_2)$. For arbitrary $n \geq n_0$ we have

$$\left|\frac{c_1+\cdots+c_n}{n}\right| \leq \underbrace{\frac{|c_1|+\cdots+|c_{n_1}|}{n}}_{\leq\frac{|c_1|+\cdots+|c_{n_1}|}{n_2}<\frac{\epsilon}{2}}+\underbrace{\frac{|c_{n_1+1}|+\cdots+|c_n|}{n}}_{(n-n_1)\frac{\epsilon}{2}\frac{1}{n}<\frac{\epsilon}{2}}<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\,. \blacksquare$$

st. 87 $\boxed{a_n \to a \in \mathbb{R} \Longrightarrow \frac{a_1+\cdots+a_n}{n} \to a}$

proof. It is a consequence of the last statement with $c_n := a_n - a \to 0$. $\blacksquare$

st. 88 $\boxed{c_n \to 0 \text{ and } d_n \to 0 \Longrightarrow \frac{c_1d_n+c_2d_{n-1}+\cdots+c_nd_1}{n} \to 0}$

proof. As the sequences c_n and d_n are convergent they are also bounded, so for some constant $M > 0$ it holds $|c_n|, |d_n| \leq M$ for any $n \in \mathbb{N}$. Given $\epsilon > 0$ arbitrary. There is $n_1 \in \mathbb{N}$ such that $|c_n| < \frac{\epsilon}{2M}$ for any $n \geq n_1$ and similarly for all n from a certain n_2 onwards $|d_n| < \frac{\epsilon}{2M}$. Let $n_0 := \max(n_1, n_2)$ and $n \geq 2n_0$ be arbitrary. Then $n - n_0 + 1 > n_0$ too, and we can estimate

$$\left|\frac{c_1d_n+\cdots+c_nd_1}{n}\right| \leq \underbrace{\frac{|c_1||d_n|+\cdots+|c_{n_0}||d_{n-n_0+1}|}{n}}_{\leq\frac{(|c_1|+\cdots+|c_{n_0}|)\frac{\epsilon}{2M}}{n}<\frac{n_0}{n}M\frac{\epsilon}{2M}\leq\frac{\epsilon}{2}}+\underbrace{\frac{|c_{n_0+1}||d_{n-n_0}|+\cdots+|c_n||d_1|}{n}}_{\leq\frac{(|d_1|+\cdots+|d_{n-n_0}|)\frac{\epsilon}{2M}}{n}<\frac{n-n_0}{n}M\frac{\epsilon}{2M}\leq\frac{\epsilon}{2}}<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\,. \blacksquare$$

st.89 $\boxed{a_n \to a \in \mathbb{R} \text{ and } b_n \to b \in \mathbb{R} \Longrightarrow \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} \to ab}$

proof. It is a consequence of the last statement with $c_n := a_n - a \to 0$ and $d_n := b_n - b \to 0$. ■

st. 90 $\boxed{c_n \to 0,\ \sum_{n=1}^{\infty} b_n = \infty \text{ and } (\forall n \in \mathbb{N})\ b_n > 0 \Longrightarrow \frac{c_1 b_1 + c_2 b_2 + \cdots + c_n b_n}{b_1 + b_2 + \cdots + b_n} \to 0}$

proof. It is similar to that of statement 86. Given $\epsilon > 0$ arbitrary. There is $n_1 \in \mathbb{N}$ such that $|c_n| < \frac{\epsilon}{2}$ for any $n \geq n_1$. We can choose some n_2 such that $\sum_{n=1}^{n_2} b_n > \frac{2}{\epsilon}(|c_1|b_1 + \cdots + |c_{n_1}|b_{n_1})$ and set $n_0 := \max(n_1, n_2)$. For arbitrary $n \geq n_0$ we have

$$\left|\frac{c_1 b_1 + \cdots + c_n b_n}{b_1 + \cdots + b_n}\right| \leq \underbrace{\frac{|c_1|b_1 + \cdots + |c_{n_1}|b_{n_1}}{b_1 + \cdots + b_n}}_{<\frac{\epsilon}{2}} + \underbrace{\frac{|c_{n_1+1}|b_{n_1+1} + \cdots + |c_n|b_n}{b_1 + \cdots + b_n}}_{\frac{\epsilon}{2}\frac{b_{n_1+1}+\cdots+b_n}{b_1+\cdots+b_n} \leq \frac{\epsilon}{2}} < \epsilon\,. \blacksquare$$

st. 91 $\boxed{\frac{a_n}{b_n} \to a \in \mathbb{R},\ \sum_{n=1}^{\infty} b_n = \infty \text{ and } (\forall n \in \mathbb{N})\ b_n > 0 \Longrightarrow \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \to a}$

proof. It is a consequence of the last statement with $c_n := \frac{a_n}{b_n} - a \to 0$. ■

The next statement requires little recall of the l'Hospital rule.

Stolz theorem

st. 92 $\boxed{\frac{a_n - a_{n-1}}{b_n - b_{n-1}} \to a \in \mathbb{R},\ b_n \to \infty \text{ and } b_n \text{ strictly increasing } \Longrightarrow \frac{a_n}{b_n} \to a}$

proof. It is a consequence of the last statement with sequences $a'_1 := a_1$, $a'_n := a_n - a_{n-1}$ and $b'_1 := b_1$, $b'_n := b_n - b_{n-1}$. ■

ex. Example

$$\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \ldots \sqrt{n}}{n\sqrt{n}} = \frac{2}{3}$$

we can use Stolz theorem 92 for sequences $a_n = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$ and $b_n = n\sqrt{n} \to \infty$ increasing, lets calculate

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \frac{\sqrt{n}}{n\sqrt{n} - (n-1)\sqrt{n-1}} = \frac{\sqrt{n}(n\sqrt{n} - (n-1)\sqrt{n-1})}{n^3 - (n-1)^3} =$$

$$= \frac{n^2 + (n-1)\sqrt{n(n-1)}}{3n^2 - 3n + 1} = \frac{1 + (1 - \frac{1}{n})\sqrt{1 - \frac{1}{n}}}{3 - \frac{3}{n} + \frac{1}{n^2}} \to \frac{2}{3}$$

C - summability of sequence

def. 19 $\boxed{\{a_n\}_{n=1}^{\infty} \text{ C - summable } \iff \frac{1}{n}\sum_{k=1}^{n}(a_1+\cdots+a_k) \to a \in \mathbb{R}}$

We usually denote $s_n := a_1 + a_2 + \cdots + a_n$ and

$$\sigma_n := \frac{1}{n}\left(a_1 + (a_1+a_2) + \cdots + (a_1+\cdots+a_n)\right) = \frac{1}{n}\sum_{k=1}^{n} s_k = \sum_{k=1}^{n}\left(1 - \frac{k-1}{n}\right)a_k \,.$$

st. 93 $\boxed{a_n \text{ summable and } \sum_{n=1}^{\infty} a_n = a \in \mathbb{R} \Longrightarrow a_n \text{ C - summable and } \frac{1}{n}\sum_{k=1}^{n}(a_1+\cdots+a_k) \to a}$

proof. We keep the notations of s_n and σ_n. Given $\epsilon > 0$ arbitrary. As $s_n \to a$ there is $n_0 \in \mathbb{N}$ such that $a - \epsilon < s_n < a + \epsilon$ for any $n \geq n_0$. Let $n \geq n_0$ be arbitrary. Then we can estimate and use limits so that

$$\sigma_n = \frac{1}{n}\left(s_1 + \cdots + s_{n_0} + s_{n_0+1} + \cdots + s_n\right) > \frac{1}{n}\left(s_1 + \cdots + s_{n_0}\right) + \frac{n-n_0}{n}(a-\epsilon) \to a - \epsilon \text{ and}$$
$$\sigma_n = \frac{1}{n}\left(s_1 + \cdots + s_{n_0} + s_{n_0+1} + \cdots + s_n\right) < \frac{1}{n}\left(s_1 + \cdots + s_{n_0}\right) + \frac{n-n_0}{n}(a+\epsilon) \to a + \epsilon \,.$$

These inequalities are true for any $\epsilon > 0$, hence $\sigma_n \to a$. ■

st. 93a $\boxed{\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} \leq \limsup_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} \leq \limsup_{n\to\infty} a_n}$

proof.
I. Lets denote $a = \liminf\limits_{n\to\infty} a_n$ and $\epsilon > 0$ arbitrary. For any n from some n_0 onward it holds $a - \epsilon < a_n$. Then for any $n \geq n_0$ we have

$$\frac{a_1+a_2+\cdots+a_n}{n} = \underbrace{\frac{a_1+\cdots+a_{n_0}}{n}}_{\to 0} + \underbrace{\frac{a_{n_0+1}+\cdots+a_n}{n}}_{\geq (n-n_0)\frac{a-\epsilon}{n} \to a-\epsilon}$$

and so

$$\liminf_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} \geq a - \epsilon \,.$$

Because $\epsilon > 0$ was arbitrary the inequality

$$\liminf_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} \geq \liminf_{n\to\infty} a_n$$

has been proved.
II. The second inequality is a consequence of statement .
III. The proof of the third inequality is the same like at case I. ■

Statement 87 is a straight consequence of this inequalities.

st. 93b $\boxed{\begin{array}{l}(\forall n \in \mathbb{N})\ a_n > 0: \\ \liminf\limits_{n\to\infty} a_n \le \liminf\limits_{n\to\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot a_n} \le \limsup\limits_{n\to\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot a_n} \le \limsup\limits_{n\to\infty} a_n\end{array}}$

proof. It is the consequence of the last statement 93a for the sequence $b_n = \ln a_n$. ■

st. 93c $\boxed{\begin{array}{l}(\forall n \in \mathbb{N})\ a_n > 0: \\ \liminf\limits_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf\limits_{n\to\infty} \sqrt[n]{a_n} \le \limsup\limits_{n\to\infty} \sqrt[n]{a_n} \le \limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n}\end{array}}$

proof. It is the consequence of the statement 93a for the sequence $b_1 = lna_1$, $b_n = \ln \frac{a_n}{a_{n-1}}$ for $n \ge 2$. Then

$$\liminf_{n\to\infty} \ln \frac{a_n}{a_{n-1}} \le \liminf_{n\to\infty} \frac{\ln a_1 + \ln \frac{a_2}{a_1} + \cdots + \ln \frac{a_n}{a_{n-1}}}{n} = \liminf_{n\to\infty} \ln \sqrt[n]{a_n}$$

and

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \liminf_{n\to\infty} \ln \frac{a_n}{a_{n-1}} \le \liminf_{n\to\infty} \sqrt[n]{a_n}\ .$$

■

3.6 . Absolute summability

absolute summability

def. 20 $\boxed{a_n \text{ absolutely summable } \overset{\text{def.}}{\Longleftrightarrow} |a_n| \text{ summable}}$

st. 94 $\boxed{a_n \text{ absolutely summable } \Longrightarrow a_n \text{ summable}}$

proof. It is based on Bolzano - Cauchy statement and inequality $|a_{n+1} + \cdots + a_m| \le |a_{n+1}| + \cdots + |a_m|$. For any $\epsilon > 0$ there is some $n_0 \in \mathbb{N}$ such that $|a_{n+1}| + \cdots + |a_m| < \epsilon$ for all $m > n \ge n_0$. Therefore $|a_{n+1} + \cdots + a_m| < \epsilon$, too. ■

Note that $\sum\limits_{n=1}^{\infty} |a_n|$ and $\left|\sum\limits_{n=1}^{\infty} a_n\right|$ can assume different values.

st. 95 $\boxed{a_n \text{ absolutely summable and } a_{p(n)} \text{ subsequence of } a_n \Longrightarrow a_{p(n)} \text{ absolutely summable}}$

proof. As p is strictly increasing and by the statement 35 injective, too, the set of different indices $\{p(1), p(2), \ldots, p(n)\}$ is a subset of $\{1, 2, \ldots, p(n)\}$. Hence

$$B_n := \sum_{k=1}^{n} |a_{p(k)}| \le \sum_{k=1}^{p(n)} |a_k| \le \sum_{k=1}^{\infty} |a_k| < \infty$$

is bounded and we can use the statement 66.

def. 21 $\boxed{a^+ \overset{\text{def.}}{=} \max(a,0) \qquad a^- \overset{\text{def.}}{=} \max(-a,0)}$

Obviously $|a| = a^+ + a^-$ and $a = a^+ - a^-$.

st. 96 $\boxed{a_n \text{ absolutely summable } \Longrightarrow a_n^+, a_n^- \text{ summable}}$

proof. As $0 \le a_n^+ \le |a_n|$ (the same for a_n^-) it is obvious. ■

Clear that $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} a_n^+ - \sum\limits_{n=1}^{\infty} a_n^-$ and $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} a_n^+ + \sum\limits_{n=1}^{\infty} a_n^-$.

The commutative law permits "reordering" of the sum of a finite collection of numbers, for instance

$$a_1 + a_2 + a_3 = a_3 + a_1 + a_2\,.$$

Now we ask ourselves whether something similar holds for infinite sums. First it is necessary to define the term "reordering".

rearrangement

def. 22 $\boxed{\text{sequence } \left\{a_{p(n)}\right\}_{n=1}^{\infty} \text{ , where } p : \mathbb{N} \longrightarrow \mathbb{N} \text{ bijective, is rearrangement of } \left\{a_n\right\}_{n=1}^{\infty}}$

For instance we can describe the rearrangement "two odd; one even" by the map

$$p(n) := \begin{cases} \frac{4n-1}{3} = 4k-3 \text{ for } n = 3k-2 \\ \frac{4n+1}{3} = 4k-1 \text{ for } n = 3k-1 \\ \frac{2}{3}n = 2k \text{ for } n = 3k \end{cases} \tag{9}$$

The map $p : \mathbb{N} \longrightarrow \mathbb{N}$ is bijective.

ex. Example **F**or $a_n := (-1)^{n+1}\frac{1}{n}$ and map $p : \mathbb{N} \longrightarrow \mathbb{N}$ described by (9) we have $\sum\limits_{n=1}^{\infty} a_n \le \frac{10}{12}$ and $\sum\limits_{n=1}^{\infty} a_{p(n)} \ge \frac{11}{12}$.

$$\sum_{n=1}^{\infty} a_n = \underbrace{1 - \frac{1}{2} + \frac{1}{3}}_{=\frac{5}{6}} \underbrace{- \frac{1}{4} + \frac{1}{5}}_{=-\frac{1}{20}} \underbrace{- \frac{1}{6} + \frac{1}{7}}_{=-\frac{1}{42}} - \ldots \underbrace{- \frac{1}{2k} + \frac{1}{2k+1}}_{=-\frac{1}{2k(2k+1)}} - \cdots \le \frac{5}{6}$$

$$\sum_{n=1}^{\infty} a_{p(n)} = \underbrace{1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4}}_{=\frac{389}{420} \ge \frac{11}{12}} + \underbrace{\frac{1}{9} + \frac{1}{11} - \frac{1}{6}}_{=\frac{7}{198} > \frac{1}{20}} + \underbrace{\frac{1}{13} + \frac{1}{15} - \frac{1}{8}}_{=\frac{29}{1560} > \frac{1}{42}} + \cdots + \underbrace{\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k}}_{=\frac{8k-3}{2k(4k-3)(4k-1)}} + \cdots \ge \frac{11}{12}$$

This example shows that summation in a different order gives a different sum in general.

Even, if a sequence is summable and not absolutely summable then for any $A \in \mathbb{R}^*$ there exists rearrangement such that the sum of rearranged sequence is equal to A.

Riemann theorem

st. 97 $\boxed{\sum\limits_{n=1}^{\infty} a_n < \infty,\ \sum\limits_{n=1}^{\infty} |a_n| = \infty,\ A \in \mathbb{R}^* \Longrightarrow \text{there exists rearrangement } p : \mathbb{N} \longrightarrow \mathbb{N},\quad \sum\limits_{n=1}^{\infty} a_{p(n)} = A}$

proof. The proof is similar to the process at last example. Because $a_n^+ = \frac{1}{2}(|a_n| + a_n)$ and $a_n^- = \frac{1}{2}(|a_n| - a_n)$ it holds $\sum\limits_{n=1}^{\infty} a_n^+ = \infty$, $\sum\limits_{n=1}^{\infty} a_n^- = \infty$ and partial sums of positive terms a_n^+ and negative terms a_n^- are not bounded.

Suppose $a_1 \geq A < \infty$ first. We gain the rearrangement p by following way. First, we will add the non-negative terms of sequence a_n until their sum does not get over number A, then we add the negative terms od sequence a_n until the whole sum does not get below A and so on. The distance of partial sums of this newly rearranged sequence $\sum\limits_{k=1}^{n} a_{p(k)}$ from A is less than $|a_{p(n)}| \to 0$. Hence $\sum\limits_{n=1}^{\infty} a_{p(n)} = A$.

The same for $-\infty < A < a_n$.

We proceed similarly for $A = \infty$. First we add the positive terms a_n^+ until the sum does not get over 1, then we add one negative term and again we add another positive terms until the sum does not get over 2 and so on. The same for $A = -\infty$. ■

Even there exists rearrangement p such that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_k$ does not exists.

The next statement shows that the order does not matter in summation of absolutely summable sequences.

about rearrangement

st. 98 $\boxed{\sum\limits_{n=1}^{\infty} = a \in \mathbb{R}, a_n \text{ absolutely summable and } a_{p(n)} \text{ rearrangement of } a_n \Longrightarrow \sum\limits_{n=1}^{\infty} a_{p(n)} = a}$

proof.
I. $a_{p(n)}$ is summable:
Given $\epsilon > 0$ arbitrary. According to the statement 62 there is $n_1 \in \mathbb{N}$ such that

$$\sum_{k=n_1}^{\infty} |a_k| < \epsilon$$

as the sequence a_n is absolutely summable. As p is bijective there is $n_2 \in \mathbb{N}$ such that $p\{1, 2, \dots, n_2\} = \{p(1), p(2), \dots, p(n_2)\} \supset \{1, 2, \dots, n_1\}$, (it is possible to choose $\max\{p_{-1}(1, \dots, n_1)\}$, $p_{-1}(K)$ denotes pre-image of set K).

Let $n \geq n_2$ arbitrary. It holds $p(k) \geq n_1$ for any $k \geq n \geq n_2$, then

$$\left| \sum_{k=n}^{\infty} a_{p(k)} \right| \leq \sum_{k=n}^{\infty} \left| a_{p(k)} \right| \leq \sum_{l=n_1}^{\infty} |a_l| < \epsilon$$

and according to the statement 62 $a_{p(n)}$ is summable.
II. $\sum\limits_{k=1}^{\infty} a_{p(k)} = a$:
Given $\epsilon > 0$ arbitrary. According to the statement 62 there is $n_1 \in \mathbb{N}$ such that

$$\sum_{k=n_1}^{\infty} |a_k| < \frac{\epsilon}{2}$$

as the sequence a_n is absolutely summable. As p is bijective there is $n_2 \in \mathbb{N}$ such that $p\{1, 2, \dots, n_2\} = \{p(1), p(2), \dots, p(n_2)\} \supset \{1, 2, \dots, n_1\}$.

We denote partial sum $A_n = \sum_{k=1}^{n} a_k$ and $S_n = \sum_{k=1}^{n} a_{p(n)}$. Let $n \geq n_0 = \max(n_1, n_2)$ arbitrary. Then we can write

$$|S_n - A_n| = \left|\sum_{k=1}^{n} a_{p(n)} \; - \; \sum_{l=1}^{n} a_l\right| = \left| \sum_{\substack{1\leq k\leq n \\ k\leq n_2}} a_{p(k)} + \sum_{\substack{1\leq k\leq n \\ n_2<k}} a_{p(k)} - \sum_{1\leq l\leq n_1} a_l - \sum_{n_1<l\leq n} a_l \right| \leq$$

$$\leq \left|\sum_{k=n_2}^{n} a_{p(k)}\right| + \left|\sum_{k=n_1}^{n} a_k\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,,$$

the first and third terms in the second part of expression above are equal because $n \geq n_2$ and

$$\begin{aligned} \{1,\dots,n_1\} \subset \{p(1),\dots,p(n_2)\} \subset \{p(1),\dots,p(n)\} \\ \{1,\dots,n_1\} \subset \{p(1),\dots,p(n)\} \cap \{1,\dots,n_1\} \subset \{1,\dots,n_1\} \\ \{p(k); 1 \leq k \leq n, p(k) \leq n_1\} = \{1,\dots,n_1\}\,. \end{aligned} \tag{1}$$

Then $S_n - A_n \to 0$ and

$$\sum_{n=1}^{\infty} a_{p(n)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} A_n = a$$

and proof is complete. ■

3.6 . Product of series

It is possible to calculate the product

$$(a_1 + a_2)(b_1 + b_2) = a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2\,.$$

Now we ask ourselves whether something similar holds for infinite sums. We have several possibilities how to create such product - **Cauchy product**

$$\sum_{n=1}^{\infty} c_n \quad \text{where } c_n \;=\; a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$$

and **Dirichlet product**

$$\begin{aligned} \sum_{n=1}^{\infty} d_n \quad \text{where } d_n \; &= a_nb_1 + a_nb_2 + \cdots + a_nb_{n-1} + a_nb_n + \\ &+ a_{n-1}b_n + \cdots + a_2b_n + a_1b_n\,\,. \end{aligned}$$

st. 99 $\boxed{\sum_{k=1}^{\infty} a_k = a \in \mathbb{R}, \sum_{k=1}^{\infty} b_k = b \in \mathbb{R} \Longrightarrow \sum_{k=1}^{\infty} (a_nb_1 + a_nb_2 + \cdots + a_nb_n + \cdots + a_2b_n + a_1b_n) = ab}$

proof. We denote partial sums $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$ and d_n like above. We have

$$\begin{aligned}\sum_{k=1}^{n} d_k \; &= a_1b_1 + a_1b_2 + \cdots + a_1b_n + \\ &+ a_2b_1 + a_2b_2 + \cdots + a_2b_n + \\ &+ \cdots + \\ &+ a_nb_1 + a_nb_2 + \cdots + a_nb_n = a_1B_n + a_2B_n + \cdots + a_nB_n = A_nB_n \to ab\,. \blacksquare\end{aligned}$$

Mertens

st. 100 $\boxed{\sum_{k=1}^{\infty} a_k = a \in \mathbb{R}, \sum_{k=1}^{\infty} b_k = b \in \mathbb{R} \text{ and } a_n \in \mathbb{R} \text{ absolutely summable } \Longrightarrow \sum_{k=1}^{\infty} (a_1b_k + a_2b_{k-1} + \cdots + a_kb_1) = ab}$

proof. We introduce new notations (β_n and ω_n) based on the relation

$$\begin{aligned}&\sum_{k=1}^{n} (a_1b_k + a_2b_{k-1} + \cdots + a_kb_1) = \\ &= a_1(\underbrace{b_1 + \cdots + b_n}_{:=b-\beta_n}) + a_2(\underbrace{b_1 + \cdots + b_{n-1}}_{:=b-\beta_{n-1}}) + \cdots + a_{n-1}(\underbrace{b_1 + b_2}_{:=b-\beta_2}) + a_n \underbrace{b_1}_{:=b-\beta_1} = \\ &= (a_1 + \cdots + a_k)b - (\underbrace{a_1\beta_n + a_2\beta_{n-1} + \cdots + a_n\beta_1}_{:=\omega_n})\,,\end{aligned}$$

so $\omega_n := a_1\beta_n + a_2\beta_{n-1} + \cdots + a_n\beta_1$ and $\beta_n := \sum_{l=n+1}^{\infty} b_l \to 0$ by Bolzano - Cauchy statement 61 and 52. We denote $M := \sum_{k=1}^{\infty} |a_k|$. Given $\epsilon > 0$ arbitrary. As $\beta_n \to 0$ there is $N \in \mathbb{N}$ (depending on ϵ) such that $|\beta_n| < \frac{\epsilon}{M}$. Let $n \geq N$ be arbitrary. We can split ω_n into two parts

$$\omega_n = a_1\beta_n + \cdots + a_{n-N+1}\beta_N + a_{n-N+2}\beta_{N-1} + \cdots + a_n\beta_1$$

and estimate the absolute value of the first part

$$|a_1\beta_n + \cdots + a_{n-N+1}\beta_N| \leq |a_1||\beta_n| + \cdots + |a_{n-N+1}||\beta_N| \leq \frac{\epsilon}{M}\left(|a_1| + \cdots + |a_{n-N+1}|\right) \leq \frac{\epsilon}{M}M = \epsilon\,.$$

For any $n \geq N$ we have

$$|\omega_n| \leq \epsilon + |a_{n-N+2}||\beta_{N-1}| + \cdots + |a_n||\beta_1|$$

therefore

$$\lim_{n\to\infty} |\omega_n| \leq \epsilon + \underbrace{\lim_{n\to\infty} |a_{n-N+2}|}_{=0} |\beta_{N-1}| + \cdots + \underbrace{\lim_{n\to\infty} |a_n|}_{=0} |\beta_1| = \epsilon\,.$$

As ϵ was arbitrary $\omega_n \to 0$. $\blacksquare$

The condition of absolute convergence is necessary in Mertenz theorem.

ex. Examples **F**or $a_k = b_k := \frac{(-1)^k}{\sqrt{k}}$, $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} b_k$ are finite, but $(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1)$ is not summable.
We can use Leibniz test for $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} b_k$. We have inequality $\sqrt{k}\sqrt{n-k+1} \leq n$ and $\frac{1}{\sqrt{k}\sqrt{n-k+1}} \geq \frac{1}{n}$ for any $1 \leq k \leq n$. Therefore for sequence

$$c_n := (a_1b_n + a_2b_{n-1} + \cdots + a_nb_1) = (-1)^{n+1} \underbrace{\left(\frac{1}{1}\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n-1}} + \cdots + \frac{1}{\sqrt{k}\sqrt{n-k+1}} + \cdots + \frac{1}{\sqrt{n}}\frac{1}{1}\right)}_{\geq 1}$$

we have $|c_n| \geq 1$, hence $\lim\limits_{n\to\infty} c_n \neq 0$ and c_n is not summable by statement 63.

But we have the following statement.

Abel theorem

st. 101 $\boxed{\sum\limits_{n=1}^{\infty} a_n = a \in \mathbb{R},\ \sum\limits_{n=1}^{\infty} b_n = b \in \mathbb{R} \text{ and } \sum\limits_{n=1}^{\infty} (a_1b_n + \cdots + a_nb_1) = c \in \mathbb{R} \Longrightarrow c = ab}$

proof. We shall denote $A_n := \sum\limits_{k=1}^{n} a_k$, $B_n := \sum\limits_{k=1}^{n} b_k$ and $C_n := \sum\limits_{k=1}^{n} (a_1b_k + \cdots + a_kb_1)$. We obtain by simple calculation

$$C_r = \sum_{\substack{k+l\leq r\\ k,l\geq 1}} a_kb_l = \sum_{k=1}^{r} a_k \sum_{l=1}^{r-k+1} b_l = \sum_{k=1}^{r} a_kB_{r-k+1} \text{ and}$$
$$C_1 + \cdots + C_n = A_1B_n + \cdots + A_nB_1\,. \tag{2}$$

We can divide this equation by n and in limiting use the statements 87 and 89

$$\underbrace{\frac{1}{n}(C-1+\cdots+C_n)}_{\to c} = \underbrace{\frac{1}{n}A_1B_n + \cdots + A_nB_1}_{\to ab}\,. \blacksquare$$

It is possible to prove this statement using power series (for instance in).

3.6 . Double series

Given a finite table of numbers, we can add all the rows to obtain a last column and then add the last column. Or first add all the columns to obtain a last row and then add the last row. For instance

$$\begin{array}{ccccccc}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \dots & a_{1,n} & a_{1\cdot} = \sum_{l=1}^{n} a_{1,l} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \dots & a_{2,n} & a_{2\cdot} = \sum_{l=1}^{n} a_{2,l} \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \dots & a_{3,n} & a_{3\cdot} = \sum_{l=1}^{n} a_{3,l} \\
\dots & & & & & & \\
a_{m,1} & a_{m,2} & a_{m,3} & a_{m,4} & \dots & a_{m,n} & a_{m\cdot} = \sum_{l=1}^{n} a_{ml} \\
a_{\cdot 1} = \sum_{k=1}^{m} a_{k,1} & a_{\cdot 2} = \sum_{k=1}^{m} a_{k,2} & a_{\cdot 3} = \sum_{k=1}^{m} a_{k,3} & a_{\cdot 4} = \sum_{k=1}^{m} a_{k,4} & \dots & a_{\cdot n} = \sum_{k=1}^{m} a_{k,n} & a_{\cdot\cdot} = \sum_{k=1}^{m}\sum_{l=1}^{m} a_{k,l}
\end{array}$$

Now we can ask ourselves if something similar holds for infinite tables called double series.

$$\begin{array}{cccccc}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \dots & s_1 = \sum_{l=1}^{\infty} a_{1,l} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \dots & s_2 = \sum_{l=1}^{\infty} a_{2,l} \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \dots & s_3 = \sum_{l=1}^{\infty} a_{3,l} \\
\dots & & & & & \\
t_1 = \sum_{k=1}^{\infty} a_{k,1} & t_2 = \sum_{k=1}^{\infty} a_{k,2} & t_3 = \sum_{k=1}^{\infty} a_{k,3} & t_4 = \sum_{k=1}^{\infty} a_{k,4} & \dots & \sum_{n=1}^{\infty} t_n \stackrel{?}{=} \sum_{n=1}^{\infty} s_n
\end{array}$$

This cold be be linked to above question by changing the order of limits.

$$\begin{array}{ccccccc}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & \dots & \to & S_1 \\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & \dots & \to & S_2 \\
A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & \dots & \to & S_3 \\
\dots & & & & & & \\
\downarrow & \downarrow & \downarrow & \downarrow & & & \downarrow \\
T_1 & T_2 & T_3 & T_4 & \dots & \to & \lim_{n\to\infty} T_n \stackrel{?}{=} \lim_{n\to\infty} S_n
\end{array}$$

But this is not true in general. For instance if $A_{m,n} := \frac{m}{m+n}$ we have $\lim_{m\to\infty} A_{m,n} = 1$ and $\lim_{n\to\infty} A_{m,n} = 0$. Also for sums we have counterexample

$$\begin{array}{cccccc}
-1 & 0 & 0 & 0 & \dots & -1 \\
\frac{1}{2} & -1 & 0 & 0 & \dots & -\frac{1}{2} \\
\frac{1}{4} & \frac{1}{2} & -1 & 0 & \dots & -\frac{1}{4} \\
\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \dots & -\frac{1}{8} \\
\dots & & & & & \\
0 & 0 & 0 & 0 & \dots & 0 \neq -2
\end{array}$$

But it is possible for limits of increasing sequences or sums of sequences with positive terms.

st. 102

$$\boxed{\begin{array}{l}
(\forall k,l \in \mathbb{N})\ A_{k,l} \le A_{k+1,l} \text{ and } A_{k,l} \le A_{k,l+1}, \\
(\forall l \in \mathbb{N})\ \lim_{k\to\infty} A_{k,l} = S_l \in \mathbb{R} \text{ and } \lim_{l\to\infty} S_l = S \in \mathbb{R} \Longrightarrow \\
\qquad \Longrightarrow (\forall k \in \mathbb{N})\,(\exists T_k \in \mathbb{R})\ \lim_{l\to\infty} A_{k,l} = T_k \text{ and } \lim_{k\to\infty} T_k = S
\end{array}}$$

proof. I.$(\forall k \in \mathbb{N})\,(\exists T_k \in \mathbb{R}) \lim_{l\to\infty} A_{k,l} = T_k$: From hypothesis for any $k, l, m \in \mathbb{N}$, $l < m$ we have $A_{k,l} \leq A_{k,m}$ therefore $S_l \leq S_m$ and $S_l \leq S$. Let $k \in \mathbb{N}$ be arbitrary. Also $A_{k,l} \leq S_l \leq S$, hence $\{A_{k,l}\}_{l=1}^{\infty}$ is an increasing sequence bounded above by S. According to the statement 28 it has finite limit $\lim_{l\to\infty} A_{k,l} \leq S$ and we shall denote it by T_k.
II. $T_k \to S$: Given ϵ arbitrary. As $S_l \to S$ there is $n_0 \in \mathbb{N}$ such that $S - \frac{\epsilon}{2} \leq S_l \leq S$ for any $l \geq n_0$. We can use $\lim_{k\to\infty} A_{k,n_0} = S_{n_0}$ similarly and from some $m_0 \in \mathbb{N}$ (depending on n_0) onwards $S_{n_0} - \frac{\epsilon}{2} \leq A_{k,n_0} \leq S_{n_0}$. Let $k \geq m_0$ and $l \geq n_0$ be arbitrary. Then

$$A_{kn_0} \leq A_{kl} \leq T_k \leq S, \quad S_{n_0} - \frac{\epsilon}{2} \leq A_{kn_0} \quad \text{and} \quad S - \frac{\epsilon}{2} S_{n_0}\,.$$

Hence

$$S - \epsilon \leq A_{kl} \leq T_k \leq S \quad \text{and} \quad S - \epsilon \leq \lim_{k\to\infty} T_k \leq S\,.$$

As $\epsilon > 0$ was arbitrary $S = \lim_{k\to\infty} T_k$. ■

st. 103
$$\boxed{\begin{array}{l}(\forall k, l \in \mathbb{N})\ A_{k,l} \leq A_{k+1,l} \text{ and } A_{k,l} \leq A_{k,l+1},\\ (\forall l \in \mathbb{N}) \lim\limits_{k\to\infty} A_{k,l} \in \mathbb{R} \text{ and } (\forall k \in \mathbb{N}) \lim\limits_{l\to\infty} A_{k,l} \in \mathbb{R} \Longrightarrow \\ \qquad\qquad \Longrightarrow \lim\limits_{k\to\infty}\lim\limits_{l\to\infty} A_{k,l} = \lim\limits_{l\to\infty}\lim\limits_{k\to\infty} A_{k,l}\end{array}}$$

proof. Keeping denotation from the last statement T_k, S_l are increasing, hence they have limits. If one such limit is finite we can use the last statement, if not both limits are ∞. ■

st. 104
$$\boxed{\begin{array}{l}(\forall k, l \in \mathbb{N})\ a_{k,l} \geq 0,\ (\forall l \in \mathbb{N}) \sum\limits_{k=1}^{\infty} a_{k,l} = S_l \in \mathbb{R} \text{ and } \sum\limits_{l=1}^{\infty} S_l = S \in \mathbb{R} \Longrightarrow \\ \qquad\qquad \Longrightarrow (\forall k \in \mathbb{N})\,(\exists T_k \in \mathbb{R}) \sum\limits_{l=1}^{\infty} a_{k,l} = T_k \text{ and } \sum\limits_{k=1}^{\infty} T_k = S\end{array}}$$

proof. We can use the statement 101 with $A_{k,l} := \sum_{i=1}^{k}\sum_{j=1}^{l} a_{i,j}$. ■

st. 105
$$\boxed{(\forall k, l \in \mathbb{N})\ A_{k,l} \geq 0,\ (\forall l \in \mathbb{N}) \sum_{k=1}^{\infty} a_{k,l} \in \mathbb{R} \text{ and } (\forall k \in \mathbb{N}) \sum_{l=1}^{\infty} a_{k,l} \in \mathbb{R} \Longrightarrow \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{k,l} = \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} a_{k,l}}$$

proof. It is a consequence of the statement 102. ■

st. 106
$$\boxed{\begin{array}{l}(\forall k, l \in \mathbb{N})\ a_{k,l} \leq 0,\ (\forall l \in \mathbb{N}) \sum\limits_{k=1}^{\infty} a_{k,l} = S_l \in \mathbb{R} \text{ and } \sum\limits_{l=1}^{\infty} S_l = S \in \mathbb{R} \Longrightarrow \\ \qquad\qquad \Longrightarrow \sum\limits_{r=2}^{\infty} (a_{1,r} + a_{2,r-1} + \cdots + a_{r,1}) = S\end{array}}$$

proof. Given $\epsilon > 0$ arbitrary. There is $N \in \mathbb{N}$ such that for any $m \geq N$

$$\left|\sum_{l=1}^{m} S_l - S\right| = \left|\sum_{l=m+1}^{\infty} S_l\right| < \frac{\epsilon}{3}\,. \tag{10}$$

We also have for $n \in \mathbb{N}$ from some n_1 onwards

$$\left|\sum_{k=1}^{n} a_{k,1} - S_1\right| < \frac{\epsilon}{3N}\,, \tag{}$$

from some n_2 onwards

$$\left|\sum_{k=1}^{n} a_{k,2} - S_2\right| < \frac{\epsilon}{3N} \tag{}$$

and so on, from some n_N onward

$$\left|\sum_{k=1}^{n} a_{k,N} - S_N\right| < \frac{\epsilon}{3N}\,. \tag{}$$

Let $n_0 := \max(n_1, n_2, \ldots, n_N, N-1)$ and $n \geq 2n_0$ be arbitrary. Because $\sum\limits_{r=1}^{n}(a_{1r} + \cdots + a_{r1}) = \sum\limits_{l=1}^{n}(a_{1l} + \cdots + a_{n-l+1,l})$, we can estimate

$$\begin{aligned}
&\left|\sum_{r=1}^{n}(a_{1,r} + a_{2,r-1} + \cdots + a_{r,1}) - S\right| = \left|\sum_{l=1}^{n}(a_{1,l} + a_{2,l} + \cdots + a_{n-l+1,l}) - S\right| = \\
&\quad = \left|\sum_{l=1}^{N}(a_{1,l} + a_{2,l} + \cdots + a_{n-l+1,l}) + \sum_{l=N+1}^{n}(a_{1,l} + a_{2,l} + \cdots + a_{n-l+1,l}) - S\right| \leq \\
&\quad \leq \underbrace{\left|\sum_{l=1}^{N}(a_{1,l} + \cdots + a_{n-l+1,l} - S_l)\right|}_{:=I} + \underbrace{\left|\sum_{l=1}^{N} S_l - S\right|}_{:=II} + \underbrace{\left|\sum_{l=N+1}^{n}(a_{1,l} + \cdots + a_{n-l+1,l})\right|}_{:=III} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

using the following inequalities. For $1 \leq l \leq N$ and $n \geq 2n_0$ we have $n - l + 1 \geq n_0 \geq \max(n_1, \ldots, n_N)$, and

$$|a_{1,l} + \cdots + a_{n-l+1,l} - S_l| < \frac{\epsilon}{3N}\,,$$

hence

$$I = \left|\sum_{l=1}^{N}(a_{1,l} + \cdots + a_{n-l+1,l} - S_l)\right| < \frac{\epsilon}{3}\,.$$

Also

$$II = \left|\sum_{l=1}^{N} S_l - S\right| < \frac{\epsilon}{3}\,,$$

according to the inequality (10). We shall also use (10) in the third estimate

$$III = \left|\sum_{l=N+1}^{n}(a_{1,l} + \cdots + a_{n-l+1,l})\right| \leq \left|\sum_{l=N+1}^{n} S_l\right| \leq \left|\sum_{l=N+1}^{\infty} S_l\right| < \frac{\epsilon}{3}$$

and proof is complete. ■

For limits we have relations

$$\lim_{n\to\infty}(a_n+b_n)=\lim_{n\to\infty}a_n+\lim_{n\to\infty}b_n \quad \text{or} \quad \liminf_{n\to\infty}(a_n+b_n)\ge\liminf_{n\to\infty}a_n+\liminf_{n\to\infty}b_n$$

Let us consider the situation where something similar holds for infinite sums. We obtain three statements, which are similar to those in theory of Lebesgue integral.

Levi theorem

st. 107
$$\boxed{\begin{array}{l}(\forall k,n\in\mathbb{N})\ a_{k,n}\ge 0,\ a_{k,n}\le a_{k,n+1},\\ \lim\limits_{n\to\infty}a_{k,n}\in\mathbb{R} \text{ and } \sum\limits_{k=1}^{\infty}a_{k,n}\in\mathbb{R}\Longrightarrow\\ \qquad\qquad\Longrightarrow\sum\limits_{k=1}^{\infty}\lim\limits_{n\to\infty}a_{k,n}=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{\infty}a_{k,n}\end{array}}$$

We use the statement 102 with $A_{m,n}:=\sum\limits_{k=1}^{m}$. Then $A_{m,n}\le A_{m+1,n}$ and $A_{m,n}\le A_{m,n+1}$ and as

$$A_{m,n}=\sum_{k=1}^{\infty}a_{k,n}<\infty \quad \text{and} \quad \lim_{n\to\infty}A_{m,n}=\sum_{k=1}^{m}\lim_{n\to\infty}a_{k,n}<\infty$$

we can conclude that

$$\sum_{k=1}^{\infty}\lim_{n\to\infty}a_{k,n}=\lim_{m\to\infty}\lim_{n\to\infty}A_{m,n}=\lim_{n\to\infty}\lim_{m\to\infty}A_{m,n}=\lim_{n\to\infty}\sum_{k=1}^{\infty}a_{k,n}\,.\ \blacksquare$$

Fatou theorem

st. 108
$$\boxed{\begin{array}{l}(\forall k,n\in\mathbb{N})\ a_{k,n}\ge 0,\ \lim\limits_{n\to\infty}a_{k,n}\in\mathbb{R} \text{ and } \sum\limits_{k=1}^{\infty}a_{k,n}\in\mathbb{R}\Longrightarrow\\ \qquad\qquad\Longrightarrow\sum\limits_{k=1}^{\infty}\liminf\limits_{n\to\infty}a_{k,n}\le\liminf\limits_{n\to\infty}\sum\limits_{k=1}^{\infty}a_{k,n}\end{array}}$$

proof. We shall use the statement 106 with $b_{k,n}:=\inf(a_{k,n},a_{k,n+1},\dots)$. We have $a_{k,n}\ge b_{k,n}\ge 0$, $b_{k,n+1}\ge b_{k,n}$ $\sum\limits_{k=1}^{\infty}b_{k,n+1}\ge\sum\limits_{k=1}^{\infty}b_{k,n}$ and by the definition of lower limit $\liminf\limits_{n\to\infty}a_{k,n}=\lim\limits_{n\to\infty}b_{k,n}$. Therefore

$$\sum_{k=1}^{\infty}\liminf_{n\to\infty}a_{k,n}=\sum_{k=1}^{\infty}\lim_{n\to\infty}b_{k,n}=\lim_{n\to\infty}\sum_{k=1}^{\infty}b_{k,n}=\liminf_{n\to\infty}\sum_{k=1}^{\infty}b_{k,n}\le\liminf_{n\to\infty}\sum_{k=1}^{\infty}a_{k,n}\,,$$

we also used that monotonic sequences are convergent (statement 28, 29) and statement 48. $\blacksquare$

Lebesque theorem

st. 109
$$\boxed{\begin{array}{l}(\forall k,n\in\mathbb{N})\ |a_{k,n}|\le b_k,\ \sum\limits_{k=1}^{\infty}b_k\in\mathbb{R} \text{ and } \lim\limits_{n\to\infty}a_{k,n}\in\mathbb{R}\Longrightarrow\\ \qquad\qquad\Longrightarrow\sum\limits_{k=1}^{\infty}\lim\limits_{n\to\infty}a_{k,n}=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{\infty}a_{k,n}\end{array}}$$

proof. We can use the last statement for $a_{k,n} + b_k \geq 0$ and then for $b_k - a_{k,n} \geq 0$. We shall also use $\liminf\limits_{n\to\infty} (-a_{k,n}) = -\limsup\limits_{n\to\infty} a_{k,n}$. ■

ex. Examples

(1) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum\limits_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

We can use formula (5) .

(2) $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots = \sum\limits_{n=1}^{\infty} \frac{n}{2^n}$ is finite (it means sequence $\frac{n}{2^n}$ is summable)

We can use the ratio test for instance: $\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1$.

(3) $2 + \frac{3}{4} + \frac{4}{27} + \frac{5}{64} + \cdots = \sum\limits_{n=1}^{\infty} \frac{n+1}{n^n}$ is finite (it means sequence $\frac{n+1}{n^n}$ is summable)

We can use the root test for instance: $\sqrt[n]{a_n} = \frac{\sqrt[n]{n+1}}{n} \to 0 < 1$.

(4) $\left(1 - \frac{1}{2}\right)^2 + \left(1 - \frac{1}{3}\right)^3 + \left(1 - \frac{1}{4}\right)^4 + \cdots = \sum\limits_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$ is not finite (it means sequence $\left(1 - \frac{1}{n}\right)^n$ is not summable)

If this sum is finite then $\lim\limits_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = 0$, but it is not true, indeed $\left(1 - \frac{1}{n}\right)^n \to \frac{1}{e}$.

(5) $\frac{1}{2} + \frac{1\cdot 3}{2^2\cdot 2!} + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!} + \cdots = \sum\limits_{n=1}^{\infty} \frac{1\cdot 3\cdot \cdots \cdot (2n-1)}{2^n n!}$ is not finit (it means sequence $\frac{1\cdot 3\cdot \cdots \cdot (2n-1)}{2^n n!}$ is not summable)

We can use the Rhabe test:

$$\left(1 - \frac{a_{n+1}}{a_n}\right) n = \left(1 - \frac{1\cdot 3\cdot \cdots \cdot (2(n+1)-1)}{2^{n+1}(n+1)!} \frac{2^n n!}{1\cdot 3\cdot \cdots \cdot (2n-1)}\right) n =$$

$$= \left(1 - \frac{2n+1}{2n+2}\right) n = \frac{n}{2n+2} \to \frac{1}{2} < 1 .$$

(6) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ is not finite (it means sequence $\frac{1}{n}$ is not summable)

We can use the integral test $\int\limits_1^{\infty} \frac{1}{x}\, dx = [\ln x]_1^{\infty} = \infty$.

(7) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{2} + \cdots = \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is not finite (it means sequence $\frac{1}{\sqrt{n}}$ is not summable)

(8) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is finit (it means sequence $\frac{1}{n^2}$ is summable)

Similarly by integral test as $\int\limits_1^{\infty} \frac{1}{\sqrt{x}}\, dx = [2\sqrt{x}]_1^{\infty} = \infty$ and $\int\limits_1^{\infty} \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1$.

(8) $\frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \cdots = \sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ is not finite (it means sequence $\frac{1}{n\ln n}$ is not summable)

We can use for instance Cauchy accumulation test ($p(n) := 2^n$) and inquire sum $\sum\limits_{n=2}^{\infty} 2^n \frac{1}{2^n \ln 2^n} = \sum\limits_{n=2}^{\infty} \frac{1}{n \ln 2}$, this sum is not finit by example .

(We can also use integral test.) (9) $\sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^{\sqrt{n}}$ is finite (it means sequence $\left(\frac{1}{2}\right)^{\sqrt{n}}$ is summable)

We can use generalized Cauchy accumulation test for $p(n) := n^2$ and inquire sum $\sum\limits_{n=1}^{\infty} \left((n+1)^2 - n^2\right) \frac{1}{2}^n =$

$\sum\limits_{n=1}^{\infty} \frac{2n+1}{2^n}$, This sum is finite according to the ratio test for instance.

(10) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ is finite (it means sequence $\frac{(-1)^n}{\sqrt{n^2+1}}$ is summable)

We can use Leibniz test as sequence with positive terms $\frac{1}{\sqrt{n^2+1}}$ is decreasing and tends tends to 0.

(11) $\sum\limits_{n=1}^{\infty} \frac{\cos n}{n}$ is finite (it means sequence $\frac{\cos n}{n}$ is summable)

We can use Dirichlet test for sequence $a_n := \cos n$ and decreasing sequence with positive terms $b_n := \frac{1}{n} \to 0$. To prove that a_n has bounded partial sums we use $\cos 1 + \cos 2 + \cos 3 + \cdots + \cos n = \frac{\cos \frac{n+1}{2} \sin \frac{n}{2}}{\sin \frac{1}{2}}$ and we have $|a_1 + \cdots + a_n| \leq \frac{1}{\sin \frac{1}{2}}$.

(12) $\sum\limits_{n=1}^{\infty} n^{\frac{1}{n}-2}$ is finite (it means sequence $n^{\frac{1}{n}-2}$ is summable)

We can use Abel test for sequence with positive terms $b_n := \sqrt[n]{n}$ (it is decreasing from the third term onwards) and $a_n := \frac{1}{n^2}$ as $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is finite.