are exceptional. This change is an illustration of the explanatory power of the language of set theory.

1.1.8.4. Integrative Power – The Ontological Unity of Modern Mathematics

Even though the first foundationalist program in mathematics was Frege's logicism, while set theory did not have at the beginning such broad ambitions, the truth is that the overwhelming majority of contemporary mathematics is done in the framework of set theory. Therefore while the axiomatic method unifies mathematics on the methodological level, set theory unifies it on the ontological one. If we take some mathematical object – be it a number, a space, a function, or a group – contemporary mathematics studies this object by means of its set-theoretical model. It considers natural numbers as cardinalities of sets, spaces as sets of points, functions as sets of ordered pairs, and groups as sets with a binary operation. Thanks to this viewpoint, mathematics acquired an unprecedented unity. We are used to it and so we consider it as a matter of course, but a look into history reveals the radical novelty of this unity of the whole of mathematics. We are justified in seeing the ontological unity of modern mathematics as an illustration of the integrative power of the language of set theory.

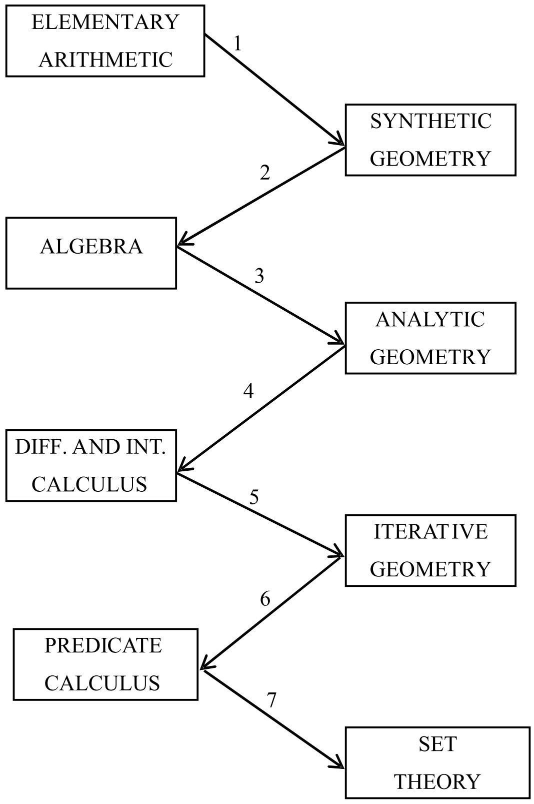
1.1.8.5. Logical and Expressive Boundaries

Set theory is one of the last re-codings which were created in the history of mathematics and so today a substantial part of all mathematical work is done in its framework. Therefore it is difficult to determine the logical and expressive boundaries of its language. The logical and expressive boundaries of a particular language can be most easily determined by means of a stronger language, which transcends these boundaries and so makes it possible to draw them. This is so, because the stronger language makes it possible to express things that were in the original language inexpressible. Nevertheless, it seems that mathematics has not surpassed the boundaries posed by the language of set theory. Therefore to characterize these boundaries remains an open problem for the future. What we can say today is that from the contemporary point of view the expressive, logical, explanatory, and integrative force of the language of set theory is total. The boundaries of the language of set theory are the boundaries of the world of contemporary mathematics and as such they are inexpressible.

1.2. Philosophical Reflections on Re-Codings

Analysis of the development of the symbolic language of mathematics from arithmetic and algebra through the differential and integral calculus to the predicate calculus, presented in the previous chapter, can be seen as an unfolding of the idea from Frege's paper *Funktion und Begriff*, quoted on page 15. Nevertheless, our exposition differs from Frege in two respects. First is terminological – we do not subsume algebra or mathematical analysis under "*arithmetic*", but consider them as independent languages. More important, however, is that we show how the "*development of arithmetic*", described by Frege, interplayed with the development of geometry. Frege separated arithmetic from geometry and connected the different phases in the "*development of arithmetic*" only loosely, using phrases as "*then they went on*", "*the next higher level*", or "*the next step forward*". But the question why '*they went on*' or where "*the next higher level*" came from remained unanswered. From the point of view of the logicist program this is perhaps unavoidable, because logicism makes a sharp distinction between the context of discovery and the context of justification. Even though we do not want to question this distinction, we believe that it is interesting to try to understand the dynamics of the transitions described by Frege. It turns out that in these transitions a remarkable tie between the symbolic and the iconic languages appears. The transition to "*the next higher level*" in development of the symbolic language happens by means of an iconic intermediate level. Thus for instance the notion of a variable, the introduction of which represented the transition from arithmetic to algebra, was created in two steps. The first of them was geometrical and consisted in the creation of the notion of a segment of indefinite length. The second step occurred when for this embryonic idea of a variable an adequate symbolic representation was found. An analogous intermediate geometrical step can be found also in the creation of the notion of function. This notion too was created in two steps; the first of them being the birth of an embryonic idea of function in analytic geometry as the dependence between two variables represented by means of a curve. The second step occurred when for this embryonic geometrical notion an adequate symbolic representation was created. I believe that this interplay between symbolic and iconic languages deserves some philosophical reflection.

The following table contains an overview of the symbolic and iconic languages in the development of mathematics, presented in the histori-



cal order of their appearance. The transitions between these languages consisted in the discovery of a new universe of objects. Four of them, represented by the arrows 1, 3, 5, and 7, correspond to the construction of new iconic languages, each of which opens a door into a new universe of geometric forms. These four transitions consist in the *visualization* of a particular notion that had formerly purely a symbolic meaning. The *Pythagorean visualization* of number by means of figural numbers (represented by the arrow 1) changed in a fundamental way our understanding of the phenomenon of quantity. Numbers that were originally accessible only through counting entered, thanks to the Pythagorean visualization, into a whole range of new relations which led to the creation of a new branch of mathematics called number theory. Number theory is something very different from counting. Instead of an operational relation between the operands and the result of an arithmetical operation (as for instance "*Take away 1/9 of 10, namely 1 1/9; the remainder is 8 2/3 1/6 1/18*", known from the Rhind papyrus) in number theory we have structural relations (as for instance that the sum of two even numbers is even). Similarly in the *Cartesian visualization* of polynomial forms by means of analytic geometry (represented by the arrow 3), to every polynomial that was until then interpreted purely as a formula and was subjected only to symbolic manipulations, a geometric form was associated. Thanks to this notion, polynomials enter into a whole range of new relations, which were until then unimaginable. The solution of a polynomial, which was traditionally one of the main purposes of algebra, obtained a geometrical interpretation. Thanks to this interpretation it became comprehensible why certain polynomials have no solution in the domain of real numbers – the corresponding curve simply does not cut the x-axis. In this way the modal predicate of insolubility of a polynomial was transformed into an extensional predicate of the curve (the non-existence of a particular intersection). Similarly, iterative geometry brought the *visualization of the limit transition* (represented by the arrow 5). If we understand the properties of objects such as the Peano curve or the Mandelbrot set, we acquire a better insight into many problems that can occur on the transition to a limit. And it is possible to see also the birth of set theory as a "visualization", or perhaps in this case it would be more appropriate to speak about the *Cantorian extenzionalization* of the predicate calculus (represented by the arrow 7).[19]

[19] The reader may wonder why set theory appears in the right-hand column of the diagram, alongside with synthetic, analytic, and iterative geometry. It may seem more appropriate to place set theory between the two columns because it is not so closely connected to spatial intuition as the other three iconic languages. I have several reasons for this placement. *First of all* there is an analogous relation between set theory and iterative geometry to that between algebra and the differential and integral calculus. Just as an infinite series can be interpreted as a prolongation of a polynomial to "infinity", so also Cantor arrived at his transfinite sets by prolonging the iterative process used in iterative geometry beyond the set of indexes (or steps of iteration) numbered by natural numbers. Therefore set theory is related to iterative geometry in the same way as subsequent languages of the same kind (iconic or symbolic) are related to each other, namely by *using an infinite number of operations of the previous language as one step of the new language*. In an analogous way a curve in analytic geometry is constructed point by point according to a formula. This, from the viewpoint of synthetic geometry, would require an infinite number of construction steps, and therefore it would be impossible to do. In analytic geometry we take this infinite number of construction steps as one step of the new language and take the curve as if it had already been constructed. *The second reason* is that when we define new sets of the form $\{x \in A; \phi(x)\}$, using the axiom of separation, where $\phi(x)$ is a formula of a particular fragment of the predicate calculus, this move is fully analogous to Descartes who defined a curve as a set of points that fulfill a particular polynomial form $p(x)$. In both cases formulas of a symbolic language (predicate calculus, algebra) are used to *define a new object as the system of elements that satisfy the particular formula*. Thus also the relation of set theory to the previous symbolic language follows a rather standard pattern. *And finally* the other iconic languages are also not so directly connected to intuition as it may seem. To believe that one is able really to see a

The three remaining transitions represented by the arrows 2, 4, and 6 correspond to the creation of new symbolic languages, each of which opens a door into a new universe of formulae. These transitions can be characterized as symbolizations in which a particular aspect of the iconic language acquires a symbolic representation. Thus in the birth of algebra (represented by the arrow 2) the *symbolization of the notion of a variable* occurred. An embryonic idea of a variable was already present in synthetic geometry in the form of a line segment of indefinite length. Euclid used this idea in his proofs in number theory. The idea of a line segment of indefinite length is not a fully fledged notion of a variable, because it is closely related to geometrical constructions. The operations which can be performed by such line segments are limited by the three dimensions of space (thus only first, second, and third powers of it can be formed) and by the principle of homogeneity (thus different powers cannot be added or subtracted). Only when the line segments were replaced by letters did the generality of reference which is the core of the notion of a variable obtain its full strength. Similarly the creation of the differential and integral calculus (represented by the arrow 4) brought a *symbolization of the notion of a function*. In an embryonic form the idea of a function is present already in analytic geometry in the form of a curve associated with a polynomial. But this idea lacked an adequate symbolic representation and thus the range of operations which it was possible to do with these curves was rather restricted. Only after Leibniz created a new symbolic language which contained a universal symbolic representation of the notion of a function did it become possible to perform symbolic operations with functions, as for instance to form a function of a function, to perform integration by the method of *per partes*, or integration by the method of substitution. Both of these integration methods were, of course, unimaginable when functions were represented only geometrically. As the last symbolization I would like to mention the *symbolization of quantification and of logical derivation* (represented by the arrow 6) in the predicate calculus. This step followed after a radical increase of precision of the mathematical language that was enforced by problems in analysis. The $\varepsilon - \delta$ analysis of Weierstrass contained in an implicit form many aspects of the quantification theory, and its fine

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fractal is naïve. And analytic geometry studies also curves in many-dimensional complex or projective spaces. Thus set theory is surely not the first iconic language which does not have a straightforward visualization.

distinctions between different kinds of convergence gave rise to many notions, which found their explicit expression in the predicate calculus.

The changes of languages that have the form of *visualization* or *symbolization* are closely related and I would like to subsume them under the common notion of *re-codings*. Our analysis of re-codings revealed a remarkable regularity of alternation of the symbolic and the iconic languages, which is important, both for the philosophy of mathematics and for mathematics education.

1.2.1. Relation between Logical and Historical Reconstructions of Mathematical Theories

Understanding of the role of the iconic intermediate levels in the development of mathematical symbolism could shed new light on the relation between the analytic and the historical approach to various questions in the philosophy of mathematics. Until now these two approaches developed independently, and occasionally there existed tensions between them. One reason for these tensions could be the fact that the proponents of the historical approach to philosophy of mathematics, as for instance Lakatos and some participants of the debate on revolutions in mathematics, such as Crowe or Dauben, understood history primarily as a space for the clash of ideas and for the conflict of opinions, i.e., as an area for social negotiation about norms and values of rational discourse. To logically oriented analytic philosophers, historical reconstructions of such a sociological sort appeared to be missing the (logical) core of the philosophical problems. Therefore they took a negative stance not only on the particular historical reconstructions, but on the very possibility of a historical approach to problems in the philosophy of mathematics.

Nevertheless, our analyses have shown that a broad scale of *historical* changes in mathematics can be studied using categories such as logical or expressive power of language, which are rather close to the *analytic* approach. The point is that historicity must not necessarily mean a historicity of social values and norms. A rather important kind of historicity is the historic nature of our linguistic tools. It would be obviously naïve to believe that a historical reconstruction of the development of the language of mathematics will remove all conflicts between the historical and logical approaches to the philosophy of mathematics. Nevertheless, it could contribute to their rapprochement.

Notions such as *logical, expressive, explanatory*, or *integrative power of* a language are sufficiently exact to be acceptable to an analytic philosopher. In the course of history these aspects of the language of mathematics underwent fundamental changes (as was recognized also by Frege). Thus the historical development of mathematics deals not only with changes of values and norms. There were also changes that are much more accessible to logical description and analysis. The problems posed by development of the different aspects of the language of mathematics could engage also the logically oriented philosopher of mathematics and invite him or her to see in history more than a tangle of psychological and social contingences. On the other hand the fact that, in such important ruptures in the history of mathematics as the birth of algebra or of set theory, the logical aspects of language played such an important role could bring proponents of a historical approach to the philosophy of mathematics to the idea to include among factors influencing the historical process also the different aspects of language. These general theses can be supported by two concrete examples.

1.2.1.1. Revolutions in Mathematics

As most of the reconstructions presented in Chapter 1.1 illustrated the importance of history for understanding of the logical structure of the language of mathematics, I will try to illustrate at least by one example the opposite point, i.e., the importance of logical analysis for the proper understanding of the history of mathematics. For this I have chosen the debate on revolutions in mathematics, initiated by the paper *Ten "laws" concerning patterns of change in the history of mathematics* published by Michael Crowe in 1975. The debate was summarized in the collection *Revolutions in Mathematics* (Gillies 1992). From the ten case studies discussed in the collection as candidates for revolutions in mathematics, five were *re-codings*. Dauben discussed the Pythagorean visualization of number and the birth of Cantorian set theory (Dauben 1984), Mancosu discussed Descartes' discovery of analytic geometry (Mancosu 1992), Grosholz discussed Leibniz (Grosholz 1992), and Gillies discussed the birth of predicate calculus (Gillies 1992a). Let us leave aside for a while the question whether these authors considered the particular re-codings as revolutions in mathematics or not. Let us simply take these papers as an indication of the kind of changes in mathematics which historians consider as sufficiently fundamental to try to explain them using Kuhn's theory of scientific revolutions. We have seen that these changes (whether we consider them revolutions or not) can be reconstructed as changes of the logical, expressive, explanatory, and integrative power of the language of mathematics. This indicates that even if the notion of a scientific revolution is a sociological one, it is correlated with something that can be analyzed by purely analytical means.

When we accept this correlation between revolutions in mathematics and re-codings, it can remind us that two important cases were omitted by the authors of the collection. These two cases were the birth of algebra and the birth of iterative geometry. So we see that a historian can get from the analytic approach a notion of completion and well-arrangement of his material. The logician can draw the historian's attention to certain cases which escaped him and also to arrange the material in a transparent way (for instance according to the growing logical and expressive power of the language). Further, the analytic approach can offer the historian a certain criterion of consistency. If a historian wants to pronounce one or two particular *re-codings* as revolutions (as Dauben did), the obvious question arises, why not also pronounce as revolutions the remaining ones? Of course, the historian has here the final word (whether a change in mathematics is or is not a revolution is a historical and not a logical question). But the analysis of re-codings in terms of the logical, expressive, explanatory, and integrative power of language makes it possible to discuss these historical questions in a more systematic way. Another problem is that two case studies presented in *Revolutions in Mathematics* are changes of a different kind, namely *relativizations* (changes that will be analyzed in Part 2 of the present book): Zheng discusses non-Euclidean geometry (Zheng 1992) and Gray discusses changes of ontology in algebraic number theory (Gray 1992). This leads to the question whether it is acceptable to extend the notion of revolution in mathematics also to the changes of this second kind, and if yes, then whether it would not be more appropriate to differentiate between different kinds of revolutions.

Coming back to the question whether re-codings are genuine revolutions in mathematics, we cannot deny that our reconstruction of the development of mathematics in Chapter 1.1 has an anti-Kuhnian flavor. The regularity of the alternation of symbolic and iconic languages, together with a cumulative growth of the logical, expressive, explanatory, and integrative force of the language of mathematics raises doubts about the presence of incommensurability in the Kuhnian sense in the development of mathematics. The symbolic and the iconic languages cannot be mutually translated one to the other; there is a genuine non-translatability between them. Nevertheless, great fragments of them *can be translated*, thus this non-translatability surely is not incommensurability. Thus instead of the controversies whether there are revolutions in mathematics, it would be perhaps easier first to try to characterize in a precise analytic way the different degrees of non-translatability between its symbolic and iconic languages and so to put these controversies on a more solid basis.

1.2.1.2. The Historicity of Logic

Robinson's discovery of non-standard analysis is an illustration of the logical power of the language of set theory. Nevertheless, besides its importance for mathematics proper, non-standard analysis is important also from the philosophical point of view. It shows that the rejection or acceptance of certain mathematical methods (in cases, when there is no specific error discovered in them) is determined by the linguistic framework on the background of which we judge these methods. In the times of Cauchy, Dirichlet, and Weierstrass the framework of model theory was not available and so *their* rejection of methods based on manipulations with infinitesimals was justified. Nevertheless, by means of *our* model theory it is possible to give to the methods of Leibniz and Euler solid and convincing foundations and so to rehabilitate them against the criticism of Cauchy, Dirichlet, and Weierstrass. So it is possible that also other mathematical theories, which were in the past rejected, could be brought to new life by means of modern logic and set theory.

A shift in the opposite direction occurred in the case of Euclid, who was considered for a long time the paradigm of logical rigor. Pasch, thanks to new mathematical achievements (in the field of real analysis) discovered that many of Euclid's proofs cannot be correct. If we take the plane consisting of points both co-ordinates of which are rational numbers, we obtain a model of the Euclidean axioms. Nevertheless, some of Euclid's constructions cannot be performed in the case of this plane, because several points used in these constructions do not exist there. But as this model fulfills all the axioms, it is clear that the *existence* of the points used by Euclid in his constructions cannot follow logically from his axioms. Pasch's criticism led to attempts to improve Euclid and to find a system of axioms from which all the propositions that Euclid believed to have proven, would really logically follow. The most successful of these new systems of axioms for Euclidean geometry was Hilbert's *Grundlagen der Geometrie* (Hilbert 1899).

The examples of Pasch and Robinson illustrate the historicity of our view of mathematical rigor. The question whether Euclid or Euler did prove certain propositions or not depends on the linguistic framework on the background of which we interpret their proofs. Before the construction of real numbers it was natural to consider Euclid's proofs as logically rigorous, just as before the advent of non-standard analysis it was natural to consider Euler's methods as problematic. Thus the question whether a proposition is correctly proven is historically conditioned. Nevertheless, this historicity of the notion of logical rigor has nothing to do with historicism or relativism. Before Pasch it was objectively correct to hold Euclid's proof to be rigorous, just as after Robinson it is objectively correct to view Euler's calculations with more respect. The relation between logic and history is an objective one; it has nothing to do with the subjective preferences of mathematicians, historians, or philosophers. The point is that the one side (a particular proof of Euclid or Euler) as well as the other side (the mathematician, historian, or philosopher who interprets the particular proof) are historically situated. They are situated in a particular linguistic framework, which decides whether it is possible to reconstruct Euler's arguments or whether it is possible to find a counterexample to a particular proof of Euclid's. Dedekind's construction of a plane containing only points with rational co-ordinates, just like Robinson's construction of the hyper-real numbers, forced us to change our view of certain periods of the history of mathematics.

The discovery of counterexamples to Euclid's constructions was made possible thanks to the linguistic framework of the theory of real functions (which we included in iterative geometry). As Hintikka and Friedman pointed out, in the times of Euclid there simply did not exist the necessary logical tools that would make it possible to realize that two circles that obviously converge can have nevertheless no point in common (as is the case on the plane with only rational points for circles that intersect in a point with irrational co-ordinates – this point simply does not exist on that plane). Obviously we cannot blame Euclid for not knowing the modern theory of the continuum. Therefore Pasch's criticism and Hilbert's axioms are not so much improvements over Euclid's shortcomings, but rather transpositions of Euclid's theory into a new linguistic background. A similar line of thought is possible also in the case of Euler. Even though non-standard analysis made it possible to vindicate several results obtained by Euler, we must admit that in the linguistic framework of eighteenth century mathematics it was impossible to justify Euler's methods. Therefore the critical reaction of the nineteenth century was fully justified. The fact that some later linguistic framework makes it possible to find some counterexamples, or to prove the consistency of certain methods, does not mean that these counterexamples or consistency proofs were possible (or even in some sense did exist) before the new linguistic framework was introduced. Therefore, at least from the *historical* point of view it seems proper to interpret and evaluate every theory against the background of the linguistic framework in which it was formulated. That means Euclid should be discussed against the background of the framework of synthetic geometry and Euler against the background of the differential and integral calculus.

Nevertheless, from a *philosophical* point of view such a requirement seems to be rather too restrictive. We cannot deny that Pasch and Robinson brought something very important. It would be a great loss to give up the new depth of understanding of the work of Euclid or Euler that was attained by Pasch and Robinson because of some doubtful historical correctness. If we give up the role of a historian who understands himself as a judge that must obey justice, a much more interesting possibility opens up for the history of mathematics. We can fully use the intellectual richness of modern mathematics (as we already did when we characterized the expressive and the logical boundaries of a particular language using the tools of the later periods) in order to see the theories of the past from as many points of view as possible. Thus we can try to do systematically what Robinson did in the case of Euler – to see the possibilities and boundaries of the mathematical theories of the past. It may be the case that also other theories of the past were discarded for the wrong reasons (sometimes even before they were fully elaborated). Modern logic and set theory give the historian strong tools which he or she can use to really understand what happened in history; not only on the social or institutional level, but also from the logical point of view.

1.2.2. Perception of Shape and Motion

In the chapter on iterative geometry we quoted a passage from the book *Fractal Geometry of Nature* (Mandelbrot 1977) in which the author compares the universe of fractals with that of Euclidean geometry. Mandelbrot characterized the Euclidean universe as cold; as a universe in which there is no room for the shape of a cloud, for the coastline of an island, or for the form of the bark of a tree. In contrast to this conception, in iterative geometry clouds, coastlines, or the barks of trees start to be shapes in the strict geometrical sense, i.e., shapes that can be generated by the means of geometric language. If we look at the three iconic languages described in Chapter 1.1 – the language of synthetic, of analytic, and of iterative geometry – we can interpret them as three different ways of grasping the phenomenon of form; as three ways of drawing the boundary between form and the formless.

Synthetic geometry tries to grasp the phenomenon of form by means of static objects such as a circle, a square, a cube, or a cone. To this approach, as Mandelbrot noticed, the form of many natural objects remains hidden. Synthetic geometry is suitable first of all for the planning of artifacts such as buildings, bridges, or dams. The second approach to the phenomenon of form is by analytic geometry, which grasps this phenomenon by means of a co-ordinate system and some analytic formulas (polynomials, infinite series, differential equations, and the like). This approach broadened the realm of objects and processes, the form of which can be constructed geometrically. So for instance in the suspended chain we discovered the catenarian curve; in the trajectory of an arrow we found a parabola; in the vibrating membrane we detected Bessel's functions. Even these few illustrations indicate that analytic geometry brought us much nearer to nature. Everywhere where the form is the result of a simple law or of a small number of determining factors, analytic geometry is able to describe it and so make it accessible to further investigation. Nevertheless, Mandelbrot's criticism still remains valid. Objects that are not created at once but which are the result of erosive or evolutionary processes (as the relief of a mountain or the form of a tree), objects which are the result not of one or ten determining factors but of millions of random influences (as a coastline or a cloud), remain formless even from the point of view of analytic geometry. The third approach to the phenomenon of form is by iterative geometry, which grasps this phenomenon as a limit of successive iterations of a transformation. Because the growth of plants and animals happens due to repeated cell division and the different processes of erosion are the result of repeated periodic influences of the environment, iterative geometry is able to grasp many of the natural forms that emerge through growth or erosion. The development along the line of synthetic, analytic, and fractal geometry changed thus the way of our perception of form and shifted the boundary between form and the formless.[20]

From the philosophical point of view this development is interesting, because at first glance it might appear that the perception of form, as well as the boundary between what has form and what we perceive as formless, is something given by our biological or cognitive make-up. Even though I do not deny the importance of biological and psychological factors in perception, it is interesting to notice, that perception has also a linguistic dimension. Changes of the language of mathematics; the birth of analytic and of iterative geometry made it possible to create a radically new universe of forms. These new forms appeared first in the minds of a few mathematicians, later they entered into the theories of physicists and other scientists, and finally, due to the progress of technology, the new forms percolated also into the socially constructed reality of our everyday lives. Thus mathematics also belongs to the factors that influence our perceptional function. Anyone who has studied fractal geometry perceives the leaf of a fern or the shape of a mountain in a new way. Therefore it is probable that modern man perceives forms differently from the ancient Greeks, and that at least part of this difference in perception is caused by mathematics. It is probable that mathematics shapes not only the way we think, but it determines also how we perceive the boundary between form and the formless.

If we look from this point of view at Aristotle's theory of natural motions, the fact that he recognized only rectilinear and circular motions as being natural, can be interpreted as the "influence" of Euclid.[21] In Euclidean geometry the circle and the straight line are privileged forms, and Aristotle in his theory of natural motion simply repeats this geometrical distinction. Thus Aristotle found in nature rectilinear and circular motions not for some physical reasons, and not even because of some metaphysical preferences, but he simply copied them from ge-

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[20] It is interesting that in ordinary language there is no word for the opposite of formless; a word that would signify all objects that have form. It seems that the language perceives form as something that constitutes the object as such; as something that the object cannot be deprived of. It seems as if form is not an ordinary predicate (like color). It seems as if form would belong to the essence of an object, and therefore we need no special word to indicate that objects have form. It is sufficient that we have a word for existence, because to exist and to have form is the same.

[21] Of course I am aware that Euclid lived after Aristotle, therefore if there was an influence, it had to have the opposite direction. Nevertheless, Euclid's *Elements* are a culmination of a particular tradition. When I speak about Euclid's influence on Aristotle, I have in mind the influence of this tradition (represented by Eudoxus, Theaetetus and other mathematicians whose discoveries are contained in the *Elements*). Euclid can be seen as the embodiment of this tradition.

ometry. When in the seventeenth century analytic geometry opened the door into a new universe of forms, the circle and the straight line lost their privileged position. At the same time physicists started to select the trajectories of mechanical motions from a much wider range of possible curves. They discovered that a stone thrown in the air follows a parabola, that the planets orbit the sun on ellipses, and that the trajectory of fastest descent (the brachystochrona) is a cycloid. Against the background of this broader universe of forms Aristotle's theory appeared as an artificial restriction of the possible trajectories. The view that rectilinear and circular motions are natural became obscure. These motions were natural only against the background of Euclidean geometry, but against the background of analytic geometry they are not more natural than any other polynomial curve of a low degree. Similarly when in the nineteenth century the first fractals appeared, they slowly found their way into physics (among their first application was the theory of Brownian motion developed by Norbert Wiener). Thus the different geometrical languages were important not only from the point of view of the perception of *form*, but because they influenced in a fundamental way also our perception of *motion*. Just like geometrical shapes, the different physical processes can also be represented using synthetic, analytic, or iterative geometry.

Perhaps the most interesting aspect of our reconstruction of the development of the iconic languages is that this development is not confined to the world of geometry. The transitions from the synthetic through analytic to fractal geometry did not happen inside of the geometric universe. The new geometrical languages appeared always thanks to a symbolic intermediate level. Descartes was able to break the narrow barriers of the Euclidean universe and to open the door into the universe of analytic curves only thanks to the language of algebra. Similarly for the discovery of the universe of fractal geometry the language of the differential and integral calculus played a decisive role. This interplay between the development of geometry and the development of symbolic languages sheds new light on philosophical theories that base mathematics on intuition (Kant's transcendental idealism or Husserl's transcendental phenomenology). These philosophies can deepen our understanding of the visual aspect of mathematics; they can explain how our perception of shape is constituted and what its structure is. Nevertheless, they are unable to understand its changes, because the changes of our perception of form do not take place in the geometrical realm alone. Thus unless the philosophy of intuition is completed by a theory of symbolic languages, it cannot understand the differences between intuition in synthetic and in analytic or fractal geometry. Therefore such philosophies can at best produce a philosophical reflection of the Euclidean geometry, but they cannot be extended into a philosophy of the whole realm of geometry, let alone of the whole of mathematics.

1.2.3. Epistemic Tension and the Dynamics of the Development of Mathematics

Every period in the history of mathematics (with the exception of ancient Egypt and Babylonia) has a symbolic and an iconic language. These two languages determine the universe of the mathematics of the respective period. For instance the mathematics of the Renaissance was based on the symbolic language of algebra and the iconic language of synthetic geometry. An interesting aspect of these two languages is their untranslatability. The iconic and the symbolic languages are never coextensive. There is always an *epistemic overlap* between them. The diagram on page 87 makes it possible to determine this overlap more precisely. The language which is placed lower in the table (i.e., which is historically younger) is stronger than the language that is placed higher (i.e., which is older). Thus in the case of the Renaissance the symbolic language of algebra had an *epistemic overlap* over the iconic language of synthetic geometry. This epistemic overlap of the language of algebra over the language of synthetic geometry means that in the language of algebra there were expressions which could not be interpreted geometrically. Therefore in the Renaissance it was possible to calculate more than it was possible to represent geometrically.

Such a situation causes a tension; it creates a need for change of the iconic language so that it becomes possible to find some geometrical representation also for those algebraic expressions for which synthetic geometry is unable to give an interpretation (such as the fourth power of the unknown). When Descartes created analytic geometry, which made it possible to represent geometrically any power of the unknown, a new epistemic overlap occurred, but this time on the side of the iconic language. Analytic geometry enables us to draw not only curves that are defined by means of a polynomial, but also curves such as the logarithmic curve or the goniometric curves, i.e., curves which cannot be represented algebraically. This epistemic overlap on the iconic side was reduced by Leibniz who laid the foundations of the differential and integral calculus. But as it later turned out, the new symbolic language based on the notion of the limit transition had an epistemic overlap over the language of analytic geometry. In order to reduce this epistemic overlap, iterative geometry was created.

This shows that the epistemic overlap is irreducible. A perfect harmony between the symbolic and the iconic poles of mathematics seems to be impossible. And precisely the irreducibility of the epistemic tension between the symbolic and the iconic languages is the basis of our reconstruction of the development of mathematics. We call the diagram, which we presented on page 87 as a summary of our reconstruction, the *bipolar diagram*. It represents the development of mathematics as a process of oscillations between two poles. In our view it is a more appropriate scheme than the classical view of the development of mathematics as a cumulative process. It captures not only the growth and differentiation of knowledge (which can be obtained from our scheme if we restrict ourselves to one of the poles, as for instance Frege, who described only the development of the symbolic pole), but it captures also the epistemic tension, which drives this growth. The two poles, even though they are mutually irreducible (it is impossible to construct a symbolic language that would have *exactly* the same logical, expressive, explanatory, and integrative power as a particular iconic language), do not exclude each other. On the contrary, they complement each other and together provide mathematics with the necessary expressive and logical means. The tension between the two poles drives mathematics to create always new symbolic and iconic languages. For the description of the relation of these two poles perhaps Bohr's notion of *complementarity* is most suitable.

1.2.4. Technology and the Coordination of Activities

We have shown that the development of geometry influenced our perception of form. The question arises whether also the development of the symbolic languages of mathematics from elementary arithmetic and algebra to the predicate calculus changes somehow our perception of our surroundings. Unfortunately, it is not easy to answer this question unequivocally. In the case of geometry it was clear from the very beginning that all three iconic languages have something to do with the perception of form. On the other hand in the case of the symbolic languages it is not clear what that something should be, the perception of which is changing when we pass from arithmetic to algebra. Therefore first of all we have to answer the question what should be the analogy of the phenomenon of form, on the changing perception of which we could follow the development of the symbolic language of mathematics. One of the possibilities is to turn to technology. The algebraic symbols were created in order to be able to represent particular (mathematical) operations. Thus the levels in the development of the symbolic languages could correspond to changes of the principles of coordination of human activities, and thus with the fundamental changes of technology. Nevertheless, this correspondence between the symbolic languages and types of technology is much more tentative than the one between iconic languages and the perception of form.

The *handicraft technology* of Antiquity and the Middle Ages can be interpreted as the technology that is based on the same schemes of coordination, on which the calculative procedures of elementary arithmetic were based. The craftsman manipulates with particular objects similarly as the Egyptian reckoner manipulated with numbers following the instructions of the Rhind papyrus. Thus, just like Frege characterized elementary arithmetic, we can say about handicraft technology that it consists in the manipulation of constant objects.

Machine technology consists in dividing the technological process into its elementary components and letting each worker perform only one or a few these elementary operations. Thus the technological process is divided into its elementary components just like an algebraic calculation is divided into elementary steps which form the constituents of an algebraic formula. Similarly as the formula represents a particular numerical quantity (for instance the root of an equation) in the form of a series of algebraic operations, the technological process "represents" (or produces) the particular product in a series of technological operations which gained a relative independence and thus can be performed by different workmen. The generality which in algebra is due to the use of variables, is present in the technological process in that a given workman performs the particular technological operation in its generality, i.e., with all objects that pass him on the production line. In contrast with the craftsman, who makes all the operations of the technological process with one object from the beginning to the end (and so his operations are manipulations with that constant object), the workman performs his particular operation with all objects (and so his object is "variable"). Thus just as Frege characterized the language of algebra, we can say about machine technology that it consists of manipulation with variables (or variable objects).

The technology with schemes analogous to the structure of the language of differential and integral calculus could be *chemical technology*. The technological processes in chemistry do not have the form of a series of separate elementary operations, which have to be performed one after the other, but rather it is a continuous process that has to be controlled. Mixing some reagents is not an operation parallel to those that we know from algebra. Mixing, heating, or adding some reagents are continuous processes by means of which we can control the chemical reactions.[22] Thus in analogy with Frege's characterization of the language of the calculus, we can also say about the technological processes in chemistry that they consist of manipulations with functions of second degree.

When we look for technologies that would be analogous to the language of the predicate calculus we can take the *analog controlled technology*. The control by means of an analog computer makes it possible to realize almost any technological process. When we construct the respective logical circuit, it will control the technological process with the required precision. Thus in analogy to Frege's characterization of the language of the predicate calculus we can say also about the schemes of analog control in technology that they consist of manipulations with arbitrary second-level functions.

This short account does not aspire to be a universal history of technology. Its aim was rather to offer a new view on the nature of technological innovations. The history of technology is usually brought into connection with the development of physics. This, of course, is correct. Nevertheless, we would like to call attention to the fact that many changes in the general structure of technology can have close relations also with the development of mathematics. The creation and spread of algebraic symbolism in western mathematics happened shortly before the birth of machine technology. It is possible that this was not a mere coincidence. The symbolic thought that was cultivated in algebra could

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[22] If the analogy between the differential and integral calculus and chemical technology is legitimate, it opens a new perspective on one peculiar aspect of Newton's works—Newton's alchemy. Newton's works on alchemy were for a long period ignored by historians. John Maynard Keynes made the first attempt to understand this part of Newton's work (Keynes 1947). If the interpretation of chemical technology as the area in which the character of coordination of activities is analogous to the character of operations in differential and integral calculus is correct, then Newton's penchant for alchemy comes into new light. It is possible that between alchemy and the differential and integral calculus there was a deep analogy that fascinated Newton. Thus his predilection for alchemy was not just an extravagancy of a genius but maybe had a rational core.

contribute to a change in the perception of technological process. And it cannot be excluded that the development of the symbolic languages of mathematics influenced the advancement of technology also in other cases. Mathematics offers tools which enable us to perceive the coordination of technological operations in an efficient way. Therefore one of the contributions of mathematics to the development of western society could lie in the cultivation of our perception of algorithms.

1.2.5. The Pre-History of Mathematical Theories

When we look at the development of a particular mathematical discipline from the point of view of its content (and not its language, as we did in Chapter 1.1), i.e., when we look at the concepts, methods, and propositions that form this discipline, we will find that mathematicians discovered the main facts about this new domain *long before an adequate language was created*, which enabled them to express these findings in a precise way and to prove them. Perhaps the best illustration of this is the integral calculus. The first mathematical results in the field that later was called integral calculus were achieved by Archimedes almost two thousand years before Newton and Leibniz created the language which allowed one to calculate integrals by means of purely formal operations. Archimedes worked in the framework of synthetic geometry, which is not suitable for the calculation of areas and volumes of curvilinear objects. Therefore he used the mechanical model of a lever in his calculations as a heuristic tool. From the two arms of the lever he suspended parts of different geometrical objects and then from the conditions of equilibrium of the lever he was able to derive relations among the areas or volumes of the suspended objects. The main disadvantage of this method consisted in the fact that for each geometrical object he had to invent an ingenious way of cutting it into pieces and of balancing them by other geometrical figures in order to obtain equilibrium on the lever. Thus the greatest part of Archimedes's ingenuity was consumed by the difficulties posed by the language of synthetic geometry that is unsuitable for the calculation of areas and volumes of curvilinear figures.

Every further language in the development of mathematics enriched the arsenal of methods that were at disposal for the calculations of areas and volumes. Thus the birth of algebraic symbolism made it possible to generalize the calculations of the area of a square and of the volume of a cube to the arbitrary power x^n. Even though this problem lacked any geometrical interpretation, the language of algebra enabled Cavalieri using his method of "*summing the powers of lines*" to find the relations, which correspond to our integrals $\int_0^a x^n dx = \frac{1}{n+1}a^{n+1}$ for $n = 1, 2, 3, \ldots, 9$ (Edwards 1979, p. 106–109). For instance for $n = 5$ he formulated his result as: "*all the quadrato-cubes have to be in the ratio of* 6 : 1". The *quadrato-cubes* represented the fifth powers of the unknown and the ratio 6:1 gives the reciprocal value of the fraction that stands on the right-hand side of our formula for the integral of x^5. Even though it was not clear what he was calculating, the results of Cavalieri were absolutely correct.

The birth of analytic geometry brought a new interpretation of algebraic operations. Thus x^3 was interpreted not as a cube, as this expression was interpreted by Cavalieri, but simply as a line segment of the length of x^3. This change of interpretation together with the idea of a co-ordinate system enabled Fermat to find a new method for calculating Cavalieri's cubatures. Fermat did not sum "*powers of lines*", and his cubature was not a calculation of the volume of some four-dimensional object, as it was for Cavalieri. Analytic geometry enabled Fermat to interpret this cubature as the calculation of the area below the curve $y = x^3$. Thanks to a suitable division of the interval $(0, a)$ by points forming a geometric progression, this integral is easy to find. And perhaps what is even more interesting, Fermat's method is universal, it can be made with n as parameter. Thus while Cavalieri had to find for each case a special trick to sum his powers, the method of dividing the interval by points that form a geometric progression, works uniformly for all n. This example nicely illustrates the increase of the integrative power of the language.

The growth of the expressive and integrative power of language by passing from synthetic geometry, which was used by Archimedes, through algebra that was employed by Cavalieri, to analytic geometry used by Fermat, enriched in a fundamental way the methods of integration. This enrichment enabled Newton and Leibniz to discover a fundamental unity of all these methods and to incorporate it into the syntax of the newly created integral calculus. The creation of a language that makes it possible by means of manipulation with symbols to solve the problem of quadratures and cubatures was a decisive turn in the development of mathematics. From the point of view of the integral calculus all results obtained by mathematicians of the past seem to be no more than simple exercises. The tricks of an Archimedes, Cavalieri, or Fermat are impressive, but a second-year university student manages in a few hours what represented the apex of their scientific achievements.

In mathematics education this organic growth of linguistic tools is often ignored and the teaching of a particular mathematical discipline starts with the *introduction of the language*, i.e., with the introduction of the symbols and syntactic rules by means of which it became possible to formulate the discipline's basic notions and results in a precise way. So the teaching of algebra starts with the introduction of the symbol x for the unknown, and mathematical analysis starts with the introduction of the syntactic distinction between function and argument. Thus the students do not develop their own cognitive process of formalization; they do not learn how to formalize their own intuitive concepts and how to create a symbolic representation for them. Only if we keep in mind the centuries of gradual changes and the dozens of innovations that separate the quadratures of Archimedes from the brilliant technique of integration, say of Euler, can we realize the complexity of the difficulties involved in mastering the linguistic means used in a mathematical domain such as the calculus. The historical reconstructions show that it is inappropriate to start the teaching of a mathematical discipline by the introduction of its linguistic tools. Only when the linguistic tools are brought into a relation with the cognitive contents that they have to represent, only then can they be adequately mastered.

When we write a formula, we are usually unaware of the centuries of mathematical experience that are present in it in a condensed form. Only when we dissect the language into its historical layers does it become obvious how many innovations must have taken place in order to make it possible to write for instance the principle of mathematical induction

$$\Big\{\varphi(0) \wedge (\forall n)\big[\varphi(n) \Rightarrow \varphi(n+1)\big]\Big\} \Rightarrow (\forall n)\varphi(n)\,.$$

Here the Cossist invention of the representation of the unknown by a letter (in this case with the letter n), is combined with Leibniz's distinction between a function and an argument (in this case $\varphi(n)$), and Frege's invention of the quantification of parts of a formula. And here we mention only the most important of these innovations, because for instance Leibniz's distinction between a function and an argument was the consummation of a long process starting with Archimedes (a few moments of this process, connected with Cavalieri and Fermat, were touched on above). Only if we keep in mind all these changes, can we appreciate the problems that mathematical formalism presents to our students.