tion of the convergence of Fourier series transcends the logical boundaries of the language of iterative geometry. The logical power of this language is insufficient to answer that question. Furthermore, between the theory of Fourier series and set theory there is also an interesting historical connection. It was the study of convergence of Fourier series that led Cantor to the discovery of set theory. As was noticed by Zermelo, the editor of Cantor's collected works, in his commentary to the paper *Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen*:

> "We see in the theory of Fourier series the birthplace of Cantor's set theory." (Cantor 1872)

1.1.6.6. Expressive Boundaries – Non-Measurable Sets

Even if the richness of the universe of fractals may seduce one to believing that the language of iterative geometry is strong enough to express any subset on the real line, nevertheless, there are sets of real numbers that cannot be expressed by means of this language. These are, for example, the non-measurable sets, the existence of which is based on the axiom of choice. It is precisely the non-constructive character of the axiom of choice that is the reason why the different sets, which can be defined by means of this axiom, cannot be constructed using an iterative process. Thus the existence of non-measurable sets illustrates the expressive boundaries of the language of iterative geometry.

1.1.7. Predicate Calculus

The history of logic started in ancient Greece, where there were two independent logical traditions. One of them has its roots in Plato's Academy and was codified by Aristotle in his *Organon*. From the contemporary point of view the Aristotelian theory can be characterized as a *theory of inclusion of classes* (containing also elements of quantification theory). Thanks to Aristotle's influence during the late Middle Ages, this logical tradition had a dominant influence on the development of logic in early modern Europe. The second tradition was connected with the Stoic school and from the contemporary point of view it can be characterized as *basic propositional calculus* (first of all a theory of logical connectives). In antiquity, probably as a result of the antagonism between the Peripatetic and the Stoic schools, these two logical traditions were considered incompatible. Nevertheless, it is closer to the truth that they complemented each other and together covered a substantial part of elementary logic. The period of intensive development of logic during classical antiquity was followed by a period of relative stagnation, which lasted during the middle ages and early modern period. This stagnation was only partial because, for instance in the field of modal logic, remarkable results were achieved (Kneale and Kneale 1962). Nevertheless, most of the achievements and innovations of medieval logic were almost completely lost as a result of rejection of scholastic philosophy. The authority of Aristotle in the field of logic had as its consequence that even Boole considered the theory of syllogisms to be the last word in logic and saw his contribution only as a rewriting of Aristotelian logic in a new algebraic form.

But even if Boole did not dispute the authority of Aristotelian logic, the importance of his *The Mathematical Analysis of Logic* (Boole 1847) can be seen in bringing into mutual contact classical logic and the symbolic language of algebra. Of course, there were some attempts in this direction earlier (by Leibniz and Euler), but these did not go further than some general proclamations and a few simple examples. It was Boole who created the first functioning logical calculus that is still in use under the name *Boolean algebra*. From the ideas of Boole a whole new logical tradition emerged. This tradition was called the algebra of logic and is represented by names such as de Morgan, Venn, Jevons, and Schröder. In this way, logic that was traditionally considered to be a philosophical discipline, came into close contact with mathematics. This contact turned out to be fruitful not only from the point of view of philosophy but also of mathematics. It took place in a favorable moment of time, which is often called the crisis of the foundations of mathematics. The discovery of the pathological functions (that we discussed in the previous chapter) shattered confidence in the intuitive content of such notions as function, curve, or derivative and led to the conviction that all intuitive reasoning should be removed from the foundations of mathematics. As a replacement for intuitive reasoning, on which until then all of mathematics rested, logic came more and more to the front. But if logic wanted to cope with the new role of securing the foundations of mathematics, it was necessary to change the relation of logic to mathematics.

Logicians like Boole or Schröder considered Aristotelian logic as a correct articulation of the fundamental logical principles. Their aim was only to recast this traditional logic into a new algebraic symbolism and so to make it accessible to mathematical investigations. Thus they wanted to *use the language of mathematics to advance logic*. A radically different view on logic is connected with the names of Frege, Dedekind, and Peano. These mathematicians reversed the above relation when they wanted to *use logic as a tool to advance the foundations of mathematics*. This change of perspective led to a surprising discovery: Aristotelian logic is not only insufficient for construction of the foundations of mathematics (this is not so surprising, because Aristotle lived long before the problems in the foundations of mathematics emerged), but Aristotelian logic is unable to give an analysis of even the most simple propositions of elementary arithmetic, such as

$$2 + 3 = 5\,.$$

The reason for this is that according to Aristotle every judgment has a subject-predicate structure. It turns out, however, that the above equation does not have an unequivocal decomposition into a subject and a predicate. There are at least 6 ways to do it:

1. The subject is the number 2 and we assert that after adding 3 to it we obtain 5.
2. The subject is the number 3 and we assert that after it is added to 2 we obtain 5.
3. The subject is the number 5 and we assert that it is the sum of 2 and 3.
4. The subject is the sum 2 + 3 and we assert that it is equal to 5.
5. The subject is the addition + and we assert that applied to 2 and 3 it gives 5.
6. The subject is the equality = and we assert that it holds between 2 + 3 and 5.

This ambiguity indicates that the decomposition of a proposition into a subject and a predicate is not an issue of logic but it has rather to do with rhetorical emphasis. Each of our six decompositions lays emphasis on a different aspect of the proposition, which from the logical point of view make no difference. The subject-predicate decomposition of propositions is closely related to the Aristotelian quantification theory. According to Aristotle, the quantification determines the scope within which the predicate is asserted about the subject. Aristotle divides propositions into universal, particular, and singular depending on whether the predicate is asserted about all, some, or a single subject. Therefore in each proposition only one "variable" can be quantified, namely the one that is in the role of the subject. This, of course, cannot be sufficient for the needs of mathematics, where for instance the notion of continuity requires the quantification of three variables.

In 1879 Gottlob Frege published his *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, which contained a new formal language, that we call *predicate calculus*. While arithmetic manipulates with numbers, algebra with "letters" and mathematical analysis with differentials, Frege created a calculus for symbolic manipulation with concepts and propositions. Perhaps the most important innovation of Frege in comparison with Aristotelian logic was replacement of the subject-predicate structure of a proposition by the argument-function structure. This made it possible to broaden fundamentally the scope of quantification theory. While Aristotle could quantify only one argument (the subject of the proposition), Frege can quantify as many arguments as the proposition contains. Frege constructed his predicate calculus in an axiomatic way. The system of axioms that he proposed is remarkable, because as was shown by Gödel half a century later, it is complete. And, with the exception of a small blemish that was detected by Lukasiewicz (the axiom 3 can be derived from axioms 1 and 2), Frege's axioms were also independent.

1.1.7.1. Logical Power – the Proof of Completeness of Predicate Calculus

In classical mathematics the notion of a proof was understood intuitively as a convincing and valid argument. Nevertheless, occasionally it happened that a proof that seemed correct to one generation of mathematicians was rejected by the next. This happened for instance with proofs based on manipulations with infinitesimals. Many of Leibniz's or Euler's arguments were some generations later met with suspicion, and mathematicians like Bolzano, Cauchy, and Weierstrass replaced them by totally different argumentation. A similar destiny met even Euclid, who was considered for many centuries a model of logical precision. Pasch noticed that when Euclid constructed the middle point of a line segment, he *used* the fact that the two circles described from the endpoints of the segment with radiuses equal to its length do intersect. Nevertheless, among the postulates and axioms of Euclid there is no one that would guarantee the existence of such an intersection point.[14] The existence of this point is thus a hidden assumption on which, besides its postulates and axioms, Euclidean geometry is erected. Later it turned out that it is by far not the only such assumption.

The manipulation with objects of doubtful status (such as the infinitesimals), or the use of hidden assumptions (such as the assumption of the existence of intersection of circles), are two extremes between which lies a whole range of intermediate positions. Therefore a classical mathematician could be never absolutely sure that the propositions, which he "proved" were really proven. To exclude such doubts was Frege's main objective in the creation of his calculus. In his *Begriffsschrift* he formalized the language of mathematics to such a degree that it was possible to turn logical argumentation into manipulation with symbols. Therefore, in contrast to his predecessors, when Frege proved a proposition he could be sure that nobody would ever discover some mistake in his proof.[15] Frege turned the notion of a proof, which generations of mathematicians used more or less intuitively, into an exact mathematical notion. So we can take as illustration of the logical power of the language of predicate calculus the proof of completeness of the propositional calculus given by Bernays in 1926, and the proof of completeness of the predicate calculus given by Gödel in 1930.

1.1.7.2. Expressive Power – Formalization of the Fundamental Concepts of Classical Mathematics

The expressive power of the language of predicate calculus can be seen in its ability to define the fundamental concepts of classical mathematics. This often required making rather subtle distinctions. So for instance the difference between the pointwise and uniform convergence of a functional series consists only in the order of two quantifiers. Of

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[14] Pasch's argumentation is based on the achievements of iterative geometry; first of all on Dedekind's paper *Stetigkeit und Irrationale Zahlen*. Dedekind discovered the fundamental difference between the system of all real and the system of all rational numbers with respect to completeness. Pasch realized that a plane from which all points with irrational co-ordinates were be eliminated would still serve as a model of Euclidean geometry (through any two points with rational co-ordinates passes exactly one straight line consisting of rational points, etc.). In that model, nevertheless, the two mentioned circles would not intersect, because the point of their intersection has one irrational co-ordinate and so does not belong to the model. Thus the existence of the intersection of circles cannot be a consequence of Euclid's postulates, because the postulates are satisfied by the model, while the intersection of the circles does not exist there.

[15] It is an irony of fate that when Frege abandoned the system of his *Begriffsschrift* and turned to his *Grundgesetze der Arithmetik*, the new system turned out to be contradictory.

course, mathematicians used these notions several decades before the creation of formal logic. But Cauchy's "proof" of the false theorem that the limit of a convergent series of continuous functions must be continuous shows that even mathematicians of Cauchy's standing were not always able to handle these fine differences correctly (see Edwards 1979, p. 312). Many fundamental notions of the theory of functions of a real variable and functional analysis are rather difficult to master without their, at least basic, formalization. Therefore we are now used to define the basic notions of our theories by means of formal logic. In order to see the advantages of this approach, it is sufficient to look at the history of a sufficiently complex mathematical theory. The development of such theories was often hindered by ambiguity of their fundamental concepts, which were defined in a loose and non-formal way. For this reason A. F. Monna gave his book on the history of the Dirichlet principle the subtitle "*a mathematical comedy of errors*" (Monna 1975).

1.1.7.3. Explanatory Power – Formulation of the Fundamental Questions of Philosophy of Mathematics

In the nineteenth century mathematicians were confronted with different philosophical questions. The problem of Euclid's fifth postulate and the discovery of non-Euclidean geometries raised questions about the relation of mathematical theories to reality, about their truth and necessity. An exposition of this debate can be found in (Russell 1897) and its historical summary in (Torretti 1978). The birth of formal logic made it possible to cast these philosophical questions into a precise mathematical form as the questions of *consistency*, *independence*, and *completeness* of a system of axioms. Similarly in the course of the arithmetization of mathematical analysis the question arose whether arithmetic itself can be further reduced to logic or whether natural numbers form a system of objects that is independent from formal logic, and have to be therefore characterized by a set of extra-logical axioms, just as we characterize the basic objects of elementary geometry. Formal logic made it possible to formulate such questions in an exact mathematical manner and it offered tools for their formal investigation. Our understanding of many problems in philosophy of mathematics was deepened considerably thanks to the methods of formal logic. This advance made in the philosophy of mathematics can thus be seen as an illustration of the explanatory power of the language of predicate calculus.

1.1.7.4. Integrative Power – Programs of the Foundations of Mathematics

The formalization of the language of mathematics is closely connected with three programs of the foundations of mathematics which developed towards the end of the nineteenth century. These programs were originally formulated only for arithmetic as attempts to answer the question of what are numbers. Frege formulated in his *Foundations of Arithmetic* (Frege 1884) the *logicist program*, according to which arithmetic can be reduced to logic. Frege believed that if we formulated the fundamental laws of logic in a sufficiently complete form, it would be possible to derive from them all principles of arithmetic. A few years later Peano in his *Arithmetices principia nova methodo exposita* (Peano 1889) formulated the *formalist program*, according to which numbers are abstract objects independent of logic. For their characterization, just as for the characterization of objects of elementary geometry, it is necessary to make recourse to extra-logical axioms. The third program, initiated by Dedekind in his *Was sind und was sollen die Zahlen?* (Dedekind 1888), can be called the *set-theoretical program*. Dedekind presented a middle position between Frege and Peano. He tried to reduce arithmetic to set theory and therefore defined numbers as cardinalities of sets. Dedekind was in agreement with Peano in characterizing sets as abstract objects by a system of special axioms. On the other hand, he agreed with Frege in that he did not consider numbers to be primitive objects, but reduced them to a more fundamental level, represented by set theory, which he called "*theory of systems*" and considered a part of logic.

Whatever is our opinion on the above programs in the foundations of arithmetic, we cannot deny that all three of them are manifestations of the integrative power of the language of predicate calculus. It was this language which enabled Frege to embark on the project of reduction of arithmetic to logic, and which allowed Peano to formulate his axioms. While working on their programs Frege and Peano made substantial contributions to formal logic. When it later turned out that the three programs cannot be realized in their original version because of the paradoxes discovered by Russell in 1901, all three programs were revised and broadened to include the whole of mathematics. Whitehead and Russell revised and broadened Frege's logicist program in their *Principia Mathematica*. Peano's formalist program was revised and broadened by Hilbert's school at Göttingen – they proposed a formulation of Peano's axioms which was immune to the paradoxes. And

Dedekind's set theoretical program was incorporated by Zermelo into axiomatic set theory. In this way the three programs of the foundations of arithmetic were extended into universal programs including the whole of mathematics. But either in their original version limited to arithmetic or in their advanced and broadened version, the three foundational programs illustrate the integrative power of the language of predicate calculus. The unity that logicism, formalism, and set theory reveal in mathematics is the unity brought to the fore by the means of formal logic.

1.1.7.5. Logical Boundaries – Logical Paradoxes

The principle of nested intervals that Cauchy used as the foundation of mathematical analysis belongs to iterative geometry. It says that under certain conditions an iterative process always has a limit and that this limit is a unique point. Thus in 1821 Cauchy shifted the problem of the foundations of the differential and integral calculus onto the shoulders of iterative geometry. Later, when the "pathological" functions were discovered, mathematicians realized the complexities involved in the notion of an iterative process. Thus even if there were no particular reasons to doubt the principle of nested intervals itself, an effort arose to base mathematical analysis on more solid foundations than this principle can offer. In 1872 Dedekind, Weierstrass, and Cantor independently of each other offered a construction of real numbers and so initiated the arithmetization of mathematical analysis. Their aim was to offer an explicit construction of the real numbers and so to build a foundation on which Cauchy's principle could be proven. These constructions, nevertheless, had one common weakness. They assumed the existence of some actually infinite system of objects (in Dedekind's it was the set of all rational numbers, in Cantor's and Weierstrass' construction it was the set of all sequences of rational numbers). Even though these assumptions seem innocuous, some mathematicians considered the existence of actually infinite systems of objects as not sufficiently clear. Therefore Dedekind (in 1888), Peano (in 1889) and Frege (in 1893), independently of each other, offered three alternative constructions of the system of all natural numbers as a canonical actually infinite set of objects. For a short time it seemed that the project of arithmetization started by Cauchy had reached its definitive and successful end. But soon the logical paradoxes emerged and the foundations of mathematics crumbled.

Russell informed Frege about the paradox in Frege's theory in a letter in 1901 (van Heijenoort 1967 p. 124). Frege was surprised by the paradox, but realized immediately that the same paradox can be formulated also in the system of Dedekind, and it is not difficult to see that the system of Peano has a similar fault (see Gillies 1982, pp. 83–93). This shows that the paradox is not the consequence of some mistake of the particular author. It is not probable that Frege, Dedekind, and Peano would make the same mistake. The conceptual foundations of their systems are so different that the occurrence of the same paradox in all of them can be explained only as a feature of the language itself. The logical paradoxes are not individual mistakes but they rather reveal the logical boundaries of the language. When we characterized the logical power of the language of the differential and integral calculus, we mentioned that the basic logical innovation of this language was the introduction of functions of the second degree. And the paradoxes stem exactly from this source. They are caused by the careless use of second-order functions and predicates. In this respect these paradoxes are analogous to the paradoxes appearing in algebra. In algebra the main logical innovation was the introduction of the (implicit) first order functions (power, square root, etc.) enabling one to express the solution of an algebraic problem in the form of a formula. The paradoxes in algebra (the *casus irreducibilis*) were caused by the careless use of these first-order functions.

The three programs of the foundations of mathematics – the logicist, formalist, and set-theoretical – extricated themselves from the crisis. Nevertheless, this extrication was achieved by means of a stronger language. This resembles algebra, where the extrication from the paradoxes was achieved by means of the stronger language of analytic geometry. This language made it possible to construct a model of complex numbers in the form of the complex plane. Mathematicians used this model to build a semantics for the paradoxical algebraic expressions and to learn to use them safely. In case of the logical paradoxes the situation is similar. Here again a stronger language was invented, the language of axiomatic set theory (or any other equally powerful extensional language), that made it possible to distinguish the paradoxical expressions from the correct ones.

1.1.7.6. Expressive Boundaries – The Incompleteness of Arithmetic

Despite the fact that large parts of mathematics can be formalized in the framework of logical calculi, it turned out that even this symbolic representation has its limits. These limits were discovered in 1931 by Gödel, who after proving the completeness of the predicate calculus turned to arithmetic and attempted to prove the completeness also of this theory. These attempts resulted in the perhaps most surprising discovery of mathematics in the twentieth century – the discovery of the incompleteness of arithmetic and the improvability of its consistency. Nevertheless, the tools by means of which Gödel achieved his results transcend the language of predicate calculus. Gödel used a new kind of symbolic language, the theory of recursive functions. Therefore the proofs of incompleteness and improvability of consistency do not belong to the framework of predicate calculus. These results can be seen as an illustration of the expressive boundaries of the language of predicate calculus. The situation here is similar to the previous cases. The language of the next stage (in this case the language of the theory of recursive functions, computability, and algorithms) makes it possible to draw the boundaries of the given language. Similarly as the language of algebra made it possible to prove the non-constructability of the regular heptagon, and thus delineated the boundaries of the language of synthetic geometry; or as the language of the differential and integral calculus made it possible to prove the transcendence of π, and so to draw the boundaries of the language of algebra; also Gödel had to use a stronger language than the one, the boundaries of which he succeeded in drawing. In the language itself its boundaries are inexpressible. They only display themselves in the fact that all the attempts to prove, for instance, the consistency of arithmetic undertaken by Hilbert's school were unsuccessful. The language of the predicate calculus, however, did not make it possible to understand the reason for this systematic failure. Only when Gödel developed his remarkable method of coding and laid the foundations of the theory of recursive functions, did he create the linguistic tools necessary for demarcation of the expressive boundaries of the language of predicate calculus.

1.1.8. Set Theory

Infinity fascinated mankind from the earliest times. The distance of the horizon or the depths of the sea filled the human soul with a feeling of awe. When mathematics created a paradigm of exact, precise, and unambiguous knowledge, infinity because of its incomprehensibility and ambiguousness found itself beyond the boundaries of mathematics. The ancient Greeks could not imagine that the infinite (called