nitudes that occur in a particular geometrical construction. The language of analytical geometry discloses a deeper unity which is hidden beneath the apparent diversity of the particular geometrical problems. All problems of classical geometry consisted in the construction of particular line segments, the length of which was determined indirectly by a set of relations to other line segments. The relations determining the particular line segments can be expressed using algebraic equations. Therefore the integrative force of the language of analytic geometry can be characterized by its ability to integrate all the different construction methods of synthetic geometry into one universal scheme.

1.1.4.5. Logical Boundaries – The Impossibility of Squaring the Circle

The transcendence of π was demonstrated in 1882 by Carl Lindemann. His proof used methods of the theory of functions of complex variables and it can be found in many books (see Stewart 1989, p. 66; Dörrie 1958 p. 148; or Gelfond 1952, p. 54). The transcendence of π means that π is not a root of any algebraic equation. In the problem of the quadrature of the circle the number π plays a crucial role and the transcendence of π means that this problem is insoluble with the methods of analytic geometry. The insolubility of the problem of the quadrature, however, cannot be expressed by means of the language of analytic geometry; it only shows itself in the failure of all attempts to solve it.

1.1.4.6. Expressive Boundaries – The Inexpressibility of the Transcendent Curves

Soon after the discovery of analytical geometry it turned out that the language of algebra is too narrow to deal with all the phenomena encountered in the world of analytic curves. First of all, two kinds of curves are not algebraic, the exponential curves and the goniometric curves. They cannot be given with the help of a polynomial. These curves transcend the language of classical analytic geometry just as the transcendental numbers transcend the language of algebra.

1.1.5. The Differential and Integral Calculus

The differential and integral calculus is, just like arithmetic and algebra, a symbolic language, i.e., a formal language designed for manipulation with symbols. It was discovered independently by Gottfried Wilhelm Leibniz and Isaac Newton in the seventeenth century. Nevertheless, the roots of this language reach into ancient Greece, to Archimedes who developed a *mechanical method* for calculating areas and volumes of geometrical figures (Heath 1921, vol. 2, pp. 27–34). This rather ingenious method resembled in many respects our differential and integral calculus. In his calculations Archimedes placed the geometrical object, the area or the volume of which he wanted to determine, on one side of a lever, cut it into thin slices and these slices he then counterbalanced with similar slices of some other geometrical figures. For instance in the calculation of the volume of the sphere he counterbalanced the slices of a cylinder with slices of a sphere and a cone. From the condition of the equilibrium of the lever he then determined the ratio of the areas or volumes of the involved objects. Since in each calculation only one geometrical object had a hitherto unknown area or volume, from the ratio of this unknown area or volume to the known ones Archimedes finally determined the result.

This method is fascinating because in an embryonic form it contains all the ingredients from which the notion of the definite integral will be constructed. First of all, in the method of Archimedes just as in the definite integral, the object, the area or the volume of which we want to determine is cut up into thin slices. Thus in both cases the calculation is understood as a *summation* of slices. This summation happens on an *interval* that Archimedes determined by the projection of the particular geometrical object onto the arm of the lever. The lever itself is also interesting, because it is in a sense the germ of the notion of a *measure*. The different slices of the geometrical object are placed on the arm of the lever not in a haphazard way, but at precise distances (which may be seen to correspond to the elements of $\mathrm{d}x$). Thus with the advantage of hindsight we can say that in Archimedes all the basic ingredients of the notion of definite integral were present.

But equally important as the similarities are also the differences. The first difference is that Archimedes *lacked the notion of a function*. He was not integrating an area delimited by functions, but rather he calculated the volume of objects that were defined geometrically. The language of synthetic geometry is much poorer than the language of functions. Therefore Archimedes calculated only areas and volumes of *single isolated objects*. He was forced to calculate the area or volume of each geometrical object from scratch. For each particular calculation he had to find a special way of cutting it into slices and of counterbalancing these slices with slices of some other geometrical objects. In his calculations he could not use the results of the previous calculations. Therefore the method of Archimedes was a collection of ingenious tricks. In contrast to the geometrical language of Archimedes, the language of the differential and integral calculus is based on the notion of a function. This makes it rich enough for the calculations of different integrals to be tied together by means of particular substitutions and the rule of *per partes*. The result of one integration is automatically the starting point of others. Instead of areas or volumes of isolated geometrical objects we have to do with broad classes of functions, the integrals of which are closely interconnected. We are not forced to start each integration from scratch. The integrals form a systematically constructed calculus that contains entire classes of functions, for which standard methods of integration exist.

The reason for this fundamental difference between the method of Archimedes and the modern integral calculus lies in the language. One could say that Archimedes was forced to invent his tricks to compensate for the weak expressive power of the language of geometry. Only when algebraic symbolism was developed with its formal substitutions, and when on its basis in analytic geometry a much broader realm of geometric forms was made accessible, could Newton and Leibniz introduce the notion of a function and the linguistic framework in which the ideas and insights of Archimedes could be transformed into a systematic technique, which we call differential and integral calculus.

1.1.5.1. Logical Power – Ability to Solve the Problem of Quadrature

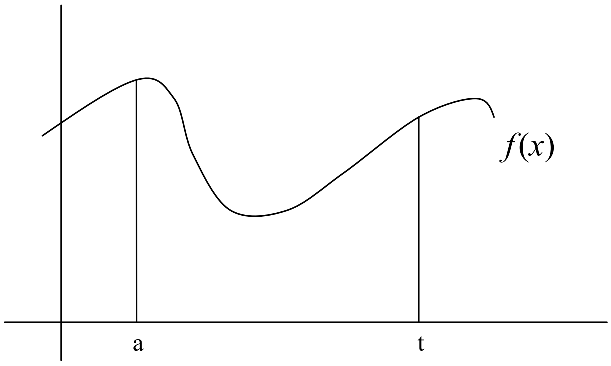
Even if analytic geometry brought decisive progress in geometry, it was not able to solve many geometrical problems. Two problems – the problem of tangents, i.e., of finding a tangent to a given curve, and the problem of quadratures, i.e., of finding the area under a given curve – gave birth to technical methods exceeding the language of algebra. These new technical methods, developed in the context of the problem of quadratures, were based on the idea of dividing the given object into infinitely many infinitesimal parts. After an appropriate transformation these parts could be put together again so that a new configuration would be formed, whose area or volume could be determined more easily. Kepler, Cavalieri, and Torricelli were great masters of the new infinitesimal methods. They found the areas or volumes of many geometrical configurations. But for every configuration it was necessary to find a special trick for dividing it into infinitesimal parts and then for summing these parts again. In some cases a regular division led to success, in others the parts had to obey some special rule. Every trick worked only for the particular object, or for some similar ones at best. It lacked a universal language that would enable the discovery of more general techniques.

The basic idea of Leibniz was in many respects similar to the *regula della cosa* of the early algebraists – to create a symbolic language, allowing manipulation with letters (more precisely groups of letters, namely the differentials dx, dy, etc.) mimicking Kepler's, Cavalieri's, and Torricelli's manipulation with infinitesimals. The differential and integral calculus, like algebra or arithmetic, is a symbolic language. It introduces formal rules for manipulation with linear strings of symbols, by means of which it quickly and elegantly gives answers to questions arising in the universe of analytic curves. The central point of differential and integral calculus is the connection, discovered by Newton and Leibniz, between the definite and the indefinite integral:

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \qquad \text{where} \qquad F'(x) = f(x). \tag{1.2}$$

This formula makes it possible to *replace the difficult geometrical problem* of quadrature (expressed by the definite integral on the left-hand side, and consisting of the division of the area beneath the curve $f(x)$ into infinitesimally small parts and their rearrangement so that it is possible to determine their sum) by a much *easier computational problem* of formal integration (expressed by the difference of the two values on the right-hand side, consisting of finding for a given function $f(x)$ its primitive function $F(x)$ such that $F'(x) = f(x)$). Actually the formula (1.2) entails that instead of calculating the area given by the definite integral $\int_a^b f(x)\mathrm{d}x$, we can first find the primitive function $F(x)$ and then just calculate the difference $F(b) - F(a)$. Therefore if we wish for instance to calculate the area enclosed beneath the curve $y = x^3$ between the boundaries $x = 3$ and $x = 5$, it is not necessary to calculate the integral $\int_3^5 x^3\mathrm{d}x$. It is sufficient to take the function $\frac{1}{4}x^4$, which is the primitive function of x^3 and to calculate the difference $(\frac{5^4}{4} - \frac{3^4}{4}) = \frac{625-81}{4} = 136$. Thus instead of complicated infinitesimal techniques it is sufficient to perform some elementary operations. In most cases finding the primitive function is not so easy, but nevertheless the whole calculation is even then much simpler than geometrical methods developed by Kepler or Cavalieri (see Edwards 1979, pp. 99–109).

The basic epistemological question is: what made possible this fundamental progress in the solution of the problem of quadrature? To answer this question we have to analyze the way which led Newton to the discovery of the formula (1.2). Newton's basic idea was to consider the area below the curve $f(x)$, i.e., the definite integral $\int_a^t f(x)\mathrm{d}x$, not as a fixed quantity but as a variable one, i.e., as a function of the upper bound which he identified with the time t.



Let us therefore imagine the area beneath a curve $f(x)$, enclosed between $x = a$ and $x = t$, and let the parameter t grow gradually. This means that the right side of the figure will move slowly to the right and thus its area will slowly increase. In order to capture the changing area beneath the curve $f(x)$ Newton introduced the *function* $F(t)$ which expresses this area for any particular moment t:

$$F(t) = \int_a^t f(x)\mathrm{d}x \,. \tag{1.3}$$

In this way Newton incorporated the problem of the calculation of the area of *one single object* (namely the figure depicted above, but taken with a fixed left side $x = b$) into a whole class of similar problems, one for each value of the variable t. Now instead of trying to determine for a particular value of t the corresponding area, we can analyze the nature of the dependence of the area on the variable t. Among other things we can try to determine the velocity of the growth of the area. Of course, this velocity *depends* on the value of the function $f(x)$ at the particular moment $x = t$. If $f(t)$ is small, then the area will increase slowly, while if $f(t)$ is great, then the area will increase rapidly. We can try to determine this dependence. The velocity of any change is the magnitude of the increment divided by the time in which this increment was achieved. Thus the velocity of the change of area is

$$\frac{\int_{t}^{t+\Delta t} f(x)\mathrm{d}x}{\Delta t} \approx \frac{f(t)\cdot\Delta t}{\Delta t} = f(t)\,. \tag{1.4}$$

That means that the velocity of the growth of the area beneath the curve at a given moment t is equal to the value of the function $f(t)$ at this moment. Thus to calculate the area beneath the curve it is sufficient to find a function such that its velocity of growth is precisely $f(t)$. The velocity is given by differentiation. Thus we can forget about areas. It is sufficient to find a function $F(t)$, whose derivative is $f(t)$. This fact is represented by the formula (1.2).

We see that the decisive step in the discovery of the formula (1.2) was a new view of the area under a curve. Newton looked on this geometrical quantity as a *function*, namely a function of the upper boundary of the figure whose area we calculate. It is important to notice that the concept of the area beneath a curve is already given with the help of a function, namely the function $f(t)$ which determines the particular curve. This is the way in which functions are present in the iconic language of analytic geometry. What is new in the calculus is that the area, which is already determined with the help of the function $f(t)$, is considered to be a function of the upper boundary. Thus we have to deal here with the implicit concept of a function of a function, that is a function of the second degree, as Frege described the calculus in the quoted passage. To be able to deal with the second-order functions (like integrals or derivatives), the ordinary functions had to become explicit. The introduction of an explicit notation for functions is thus a characteristic feature of the symbolic language of the differential and integral calculus, which is fully parallel to the introduction of explicit notation for variables, which occurred in the previous stage of the development of symbolic languages, in algebra.

1.1.5.2. Expressive Power – Ability to Represent Transcendental Functions

The differential and integral calculus is a language of higher expressive power than was the language of algebra on which analytic geometry was based. Functions like ln(x), cos(x), and elliptic functions are not polynomial. They transcend the boundaries of the language of algebra. For the differential and integral calculus they present no problem. For instance the logarithmic function can be expressed as an infinite series

$$\ln\big(1+x\big) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - + \dots , \qquad (1.5)$$

or as an integral

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\mathrm{d}t , \qquad (1.6)$$

or as a solution of a differential equation

$$y' = \frac{1}{y} , \qquad y(1) = 0 . \qquad (1.7)$$

The logarithmic function is a rather simple example. It would be possible to present more complicated examples, for instance Euler's Γ-function, Bessel's functions, Riemann's ζ-function, elliptic functions and a number of special functions occurring in physics or in technical applications. The differential and integral calculus is a symbolic language which can express many functions absolutely inconceivable in the framework of the language of algebra. Nevertheless, the definitions of these new functions use infinite series, derivatives, or integrals, that is, functions of the second degree.

The language of algebra can be embedded in the new language of the calculus. If in an infinite series (expressing for instance ln(x)) we restrict ourselves only to a finite number of initial terms, we obtain a polynomial. In the universe of polynomials, differentiation and integration can be defined by explicit rules, and this universe is closed under these rules. We can consider these restricted operations (prescribing derivative and integral of any polynomial) as new unary algebraic operations.

1.1.5.3. Explanatory Power – Ability to Explain the Insolubility of the Quadrature of the Circle

Algebra can explain why the trisection of an angle is impossible. Nevertheless, algebra is not able to explain why nobody succeeded in squaring the circle. As we already mentioned, the reason for this is the transcendental nature of the number π. In 1873 Hermite proved the transcendence of the number *e* and in 1882 Lindemann succeeded, using the ideas of Hermite, in proving the transcendence of π (see Gelfond 1952, p. 54–66). These proofs are based on the language of the differential and integral calculus and so they illustrate its explanatory power. A further interesting fact obtained by these methods is that with the exception of the point $(0, 0)$ at least one of the two co-ordinates (x,sin(x)), that determine a point of the sinusoid, is transcendental. In other words, with the exception of the point $(0, 0)$ none of the points of the sinusoid can be constructed using the methods of analytic geometry. Thus already such a simple curve as the sinusoid totally defies the language of analytic geometry.

1.1.5.4. Integrative Power – Mathematical Physics

The differential and integral calculus is the language that makes it possible for modern physics to unite all its particular branches into an integrated system and so to demonstrate the fundamental unity of nature. While the idea of a universal mathematical description of nature stemmed from Descartes, his technical tools, based on the language of analytic geometry, did not have sufficient integrative power to accomplish his project. The Cartesian polynomials had to be replaced by functions and algebraic equations by differential equations, to integrate all natural phenomena into the universal picture.

Descartes expressed the unity of nature on a metaphysical level. One of the basic purposes of his mechanistic world-view was to unify all natural phenomena. Thus his metaphysics had to fulfil the function which his formal language could not fulfil – to integrate nature into a unified theory. Modern mathematical physics does not require any special metaphysical position in order to see the unity of nature. The unity of physics is fully formal. It is provided by the language and not by metaphysics. Maxwell's equations of an electromagnetic field remained valid after we abandoned the theory of the ether. The ether served for Maxwell as an ontological foundation of his theory, but later it became obvious that the theory could do without this or any other ontological basis. This shows that the unity of physics is a formal unity independent of any ontology. Its source is the integrative power of the language of differential and integral calculus.

1.1.5.5. Logical Boundaries – Crisis of the Foundations

The first criticism of the foundations of the differential and integral calculus appeared soon after its discovery. In his famous book *The Analyst, or a discourse addressed to an Infidel Mathematician*, which appeared in Dublin in 1734, George Berkeley expressed the view that the whole calculus is based on a series of errors. He criticized the way of reasoning, typical in differential and integral calculus, by which one makes calculations with some quantity assuming that its value is different from zero (in order to be able to divide by it), and then at the end one equates this quantity with zero. Berkeley correctly stressed that if a quantity is zero at the end of some calculation, it must have been zero also at its beginning. Thus all the reasoning is incorrect. According to Berkeley the correct results of calculus are due to compensation of different errors.

Various attempts were presented to rebuild the foundations of the calculus to save it from Berkeley's criticism. Perhaps the most important of them was put forward by Cauchy in his *Cours de l'Analyse* published in 1821. Cauchy tried to base the whole calculus on the concept of limit. Nevertheless, in the rebuilding of the calculus Cauchy left the language of the differential and integral calculus; he left the realm of formal manipulations with symbols and as a fundamental principle by means of which he proved the existence of limits and on which he built the whole theory, he chose the principle of nested intervals. This principle is a geometrical principle. All later attempts to build the foundations of the calculus follow Cauchy in this respect; they leave the language of the differential and integral calculus and erect the building of the calculus on some variant of the theory of the continuum. Thus the crisis of the foundations of the calculus can be seen as a manifestation of the logical boundaries of the language of the differential and integral calculus. An indication of this interpretation is that all attempts to solve this crisis turned their backs on symbolic language.

1.1.5.6. Expressive Boundaries – Fractals

The successes of the differential and integral calculus justified the belief that all functions can be described with the help of infinite series, integrals, derivatives, and differential equations. Therefore it was a real surprise when the first "monstrous" functions started to appear. Bolzano's function does not have a derivative at any point, Dirichlet's function is discontinuous at each point, and Peano's function fills the unit square. These functions gave rise to a considerable refinement of the basic concepts of differential and integral calculus and gave rise to a whole new branch of mathematics – the theory of functions of a real variable. In the course of their study it turned out that the methods of the calculus can be applied only to a rather narrow class of "decent" functions. The rest of the functions lie beyond the expressive boundaries of the language of the differential and integral calculus.

1.1.6. Iterative Geometry

Differential and integral calculus were born in very close connection to analytical geometry. Perhaps this was one of the reasons why mathematicians for a long time considered Descartes' way of generating curves (i.e., point by point, according to a formula) to be the correct way of visualizing the universe of mathematical analysis. They thought that it would be enough to widen the realm of formulas used in the process of generation, and to accept also infinite series, integrals or perhaps other kinds of analytical expressions instead of polynomials. They believed that the symbolic realm of functions and the iconic realm of curves were in coherence. Leibniz expressed this conviction with the following words:

> "Also if a continuous line be traced, which is now straight, now circular, and now of any other description, it is possible to find a mental equivalent, a formula or an equation common to all the points of this line by virtue of which formula the changes in the direction of the line must occur. There is no instance of a face whose contour does not form part of a geometric line and which can not be traced entire by a certain mathematical motion. But when the formula is very complex, that which conforms to it passes for irregular." (Leibniz 1686, p. 3)

First doubts about the possibility of expressing of an arbitrary curve by an analytical expression occurred in the discussion between Euler and d'Alembert on the vibrating string. The vibrations of a string are described by a differential equation that was derived in 1715 by Taylor. In 1747 d'Alembert found a solution of this equation in the form of travelling waves. Nevertheless, as the differential equation describing the string is an analytical formula, d'Alembert assumed that the initial shape of the string must be given in an explicit form of an analytical