1.1.3.6. Expressive Boundaries – Transcendent Numbers

Even if algebraists were able to explain why nobody succeeded in solving the problem of the trisection of an angle, the problem of quadrature of a circle resisted algebraic methods. Gradually a suspicion arose that this problem is insoluble as well. Nevertheless, its insolubility is not for algebraic reasons. This suspicion found an exact expression in the distinction between algebraic and transcendental numbers. Transcendental numbers are numbers that cannot be characterized using the language of algebra. The first example of a transcendental number was given by Joseph Liouville in 1851. It is the number:

$$
\begin{aligned}
l &= \sum_{n=1}^{\infty} 10^{-n!} = 10^{-1!} + 10^{-2!} + 10^{-3!} + 10^{-4!} + 10^{-5!} + 10^{-6!} + 10^{-7!} + \dots \\
&= 10^{-1} + 10^{-2} + 10^{-6} + 10^{-24} + 10^{-120} + 10^{-720} + \dots \\
&= 0,1100010\dots(17 zeros)\dots 010\dots(96 zeros)\dots 010\dots(600 zeros)\dots 010\dots
\end{aligned}
$$

In the decimal expansion of this number the digit 1 is in the $n!^{th}$ places.[10] All other digits are zeros. This means that the digit one is on the first, second, sixth, twenty-fourth, ... decimal places. Even though this number is relatively easy to define, it does not satisfy any algebraic equation. This means that it is a transcendental number – it transcends the expressive power of the language of algebra (see Courant and Robbins 1941, p. 104–107). Liouville's number l illustrates the expressive boundaries of the language of algebra. In 1873 Charles Hermite proved the transcendental nature of e (the basis of natural logarithms) and in 1882 Ferdinand Lindemann proved the transcendental nature of π. Thus l, e, and π are quantities about which we can say nothing using the language of algebra.

1.1.4. Analytic Geometry

Analytic geometry originated from the union of several ideas or even traditions of thought that existed independently of each other for many centuries. The first of them was the idea of *co-ordinates*. In geography co-ordinates have been used since antiquity. One of the highlights of ancient geography was the *Introduction to geography* (*Geógrafiké*

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[10] The symbol $n!$ represents the product of the natural numbers from 1 to n (thus $4! = 1.2.3.4 = 24$, while $6! = 1.2.3.4.5.6 = 720$).

Hyfégésis) written by Ptolemy around 150 AD. It contained the longitudes and latitudes of more than 8 000 geographical locations, many of which were in India or China. At the beginning of the fifteenth century we can witness a revival of cartography due to the expansion of sea trade. At that time Giacomo d'Angli discovered in a Byzantine bookshop a copy of the Greek manuscript of Ptolemy's *Geography*, which he translated into Latin. It was published in 1477 at Bologna together with the charts drawn by Italian cartographers. Ptolemy's method based on the use of the co-ordinate system has thus spread through all of Europe. From a geometrical point of view it is fascinating to realize that the cartographers were able to draw faithful outlines of whole continents on the basis of the data collected from sailors. Today, of course, everybody knows the form of Africa but only few realize that before space flights nobody could really *see* these outlines. Thus shapes like those of Italy or Africa, obtained by cartographers, are a very special kind of shape. These shapes were very different from ordinary geometrical forms because nobody could see them prior to their construction by plotting the positions of several hundreds of points of the coastline on the chart using a co-ordinate system. Just as in cartography, so also in analytic geometry we create shapes that were unseen before. Instead of sailors, nevertheless, we seek the help of algebra.

So we come to a second tradition that played an important part in the creation of analytic geometry – *algebra*. As we mentioned in the previous chapter, algebra succeeded in overcoming the boundaries of three-dimensional space. When al-Khwárizmí introduced the terms for the powers of the unknown (his *shai*, *mal* and *kab*) he used geometrical analogies. The word *kab* means in Arabic a cube. But in contrast to the ancient Greeks, who stopped after the third power, the Arabic mathematician went further and introduced higher powers. Thus for instance *mal-mal*, *kab-mal* and *mal-mal-mal* stood for the fourth, fifth and sixth power. In contrast to the first three powers, for which we have a geometrical interpretation in the form of a line segment, of a square and of a cube, the fourth, sixth and all higher powers of the unknown lack any intuitive geometrical meaning. But this did not prevent al-Khwárizmí from introducing algebraic operations for these quantities. The works of al-Khwárizmí were in the twelfth century translated into Latin (in 1126 by Abelard of Bath and in 1145 by Robert of Chester). These translations initiated the development of algebra which led to the creation of algebraic symbolism. From the point of view of analytic geometry the creation of polynomial forms (i.e., expressions such as

$x^5 + 24x^3 - 4x + 2$) was of particular interest. It was these forms which "replaced the data of the sailors" in the creation of the shapes of analytic geometry.

The third idea that entered the creation of analytic geometry made it possible to unite the two previous ones. It consisted in a new interpretation of algebraic operations. This idea was introduced by René Descartes. Ever since Euclid the product $x \cdot y$ of two magnitudes (which were represented by line segments) was interpreted as an oblong with the sides x and y, that is, as a magnitude of a different kind than x and y. It is interesting to notice that in his *Regulae ad directionem ingenii* (written between 1619 and 1628) Descartes still interpreted the product of two line segments as the area of the oblong formed by them (see Regula XVIII). Nevertheless, in the definition of the product of three segments $a \cdot b \cdot c$ he wrote that it is better to take the product $a \cdot b$ in the form a line segment of the length a times b. Thus the idea of interpreting the product of two lines not as an area but again as a line segment was present already in this early work of Descartes, even though its importance was not yet fully grasped. Descartes mentioned it only in passing, only as a trick that made it possible to interpret the product $a \cdot b \cdot c$ as an oblong (with one side $a \cdot b$ and the other c) and not as a prism (with the sides a, b, and c). Thus the product of two line segments leads Descartes to a magnitude of a higher dimension and the ingenious trick is used only to prevent the occurrence of volumes and magnitudes of even higher dimension. When in 1637 Descartes published his *Géométrie* as a supplement to the *Discours de la Méthode*, the importance of the new interpretation of the product was fully recognized. The product of two (and of any higher number of) line segments was understood as a line segment. The product $a \cdot b$ was simply the number that indicated the length of the segment. Thus the product and similarly also the quotient of line segments did not lead to a change of dimensionality. In this way Descartes created for the first time in the history of mathematics a system of quantities that was closed under the four arithmetical operations (addition, subtraction, multiplication, and division) and thus, with slight anachronism we can say that he created the first *algebraic field*. In this way he succeeded in overcoming the barrier of dimensionality also in geometry. The critical step in the creation of analytic geometry was that Descartes interpreted the product of two straight lines a and b as a straight line of the length $a \cdot b$. Thanks to this interpretation it became possible to transfer to geometry the fundamental expressive advantage of algebra, its ability to form and combine magnitudes of any degree.

Analytic geometry, i.e., the iconic language with a new way of generating geometrical objects, was created from a combination of the above three ideas. As a starting point we take a polynomial, that is an *algebraic expression*, for instance x^5-4x^3+3x+2. From the algebraic point of view it is a purely symbolic object without any geometrical interpretation. It was precisely in order to transcend the boundaries laid on Euclidean geometry by three-dimensional space and to become able to form higher powers of the unknown that the algebraists had to give up the possibility of any visual representation of their formalism. They knew how to calculate with polynomials, but they never associated any geometrical form with them. In the second step of the generation of the new geometrical objects we apply to this purely symbolic algebraic expression Descartes' *geometrical interpretation* of the algebraic operations. Thus we take for instance $x = 1$, and the above polynomial will represent a line segment of the length $y = 1 - 4 + 3 + 2$, that is 2 units. If we take $x = 2$, we obtain the corresponding line segment $y = 32 - 32 + 6 + 2$, that is 8, and so forth. In the third step we plot the pairs of values of x and y that we obtained in the previous step, in a *co-ordinate system*. When we plot enough of them, before our eyes something radically new appears; a new curve which before 1637 nobody could see. In a similar way as the shapes of the continents emerged before the eyes of cartographers, a totally new universe of forms emerged before the eyes of mathematicians.

The polynomial, as introduced by the algebraists, was a purely symbolic object, without any geometrical interpretation. Fortunately this loss of visual representation did not last long. In the seventeenth century analytic geometry was developed. In analytic geometry with every polynomial there is associated a curve. In this way all algebraic concepts such as root, degree, etc. acquire geometric interpretation. For instance the degree of a curve can be geometrically interpreted as the maximal number of its intersections with a straight line. The idea of associating a curve to any algebraic polynomial resembles in many aspects the Pythagorean idea of visualization of numbers, which associated geometric forms to arithmetic properties with the help of figurate numbers. *In a way similar to the Pythagorean visualization of arithmetic, analytic geometry visualizes algebra.* In both cases we have to deal with creation of a new iconic language, which incorporates some features of the particular symbolic language.

1.1.4.1. Logical Power – Proof of the Fundamental Theorem of Algebra

Carl Friedrich Gauss in his doctoral dissertation *Demonstratio nova Theorematis omnem Functionem algebraicam rationalem integram unis Variabilis in Factores reales primi et secundi Gradus resolvi posse* from 1799 proved the fundamental theorem of algebra. In his proof Gauss used the plane as a model of complex numbers. The fundamental theorem of algebra says that for every polynomial $p(x)$ there is a complex number α such that $p(\alpha) = 0$. The zero on the right-hand side of this equation is only for historical reasons; in principle we could put there 7 or any other complex number. The zero is there only because before starting to solve a polynomial equation, the algebraists first transformed it to a form having on the right-hand side a zero. But the fundamental theorem of algebra would hold also if we decided to put any other complex number instead of zero on the right-hand side of the polynomial equation $p(x) = 0$. This indicates that the true meaning of the fundamental theorem of algebra is that a polynomial $p(x)$ defines a surjective transformation $z = p(x)$ of the complex planes x onto the complex plane z. Thus this theorem is rather geometrical (or topological) than purely algebraic.

Gauss gave four different proofs of the fundamental theorem of algebra, the first of them in his doctoral dissertation. An elementary proof of this theorem can be found in (Courant and Robbins 1941, pp. 269–271). The fundamental theorem of algebra is an existential theorem that was a precursor of a whole series of similar existential theorems for differential equations, integral equations and for problems in calculus of variations. What is essential for such proofs is firstly the existence of some analytical formalism (polynomial algebra, differential equations, or calculus of variations) by means of which we define the object, the existence of which we are going to prove. The second component of such existential proofs is some domain (complex numbers, smooth functions, etc.) in the realm of which the existence of the particular object is asserted. This domain has usually some topological property of completeness, closeness, or compactness.

It seems that analytic geometry was the first language that contained both these ingredients necessary for a successful existential proof. On one hand it contained the algebraic formalism in the framework of which we can formulate a polynomial equation. On the other hand it contained the complex plane, i.e., a geometrical continuum which was connected with algebraic formalism in a standard way. All previous existential proofs must have happened inside the formalism itself. Basically they had to be constructive; they had to produce a formal expression which represents the object, the existence of which was at stake. In contrast to this in an existential poof like that of Gauss, we proceed on two levels, one symbolic and the other geometrical. Of course, after the nineteenth century many geometrical existential proofs were considered not sufficiently stringent. Nevertheless, it seems that the language of analytical geometry, by building a tight connection between the formalism and the geometrical structure of the complex plane, was the first language that made it possible to prove the existence of solutions for such a wide class of problems as the class of all polynomial equations.

1.1.4.2. Expressive Power – Ability to Represent Algebraic Curves of Any Degree

Analytic geometry brought a new method of generating geometrical pictures. The configuration is constructed point by point, using a coordinate system. This is something qualitatively new in comparison with Euclid. Euclid generated the picture (a term of the iconic language of geometry) with ruler and compasses. This means that Euclid has some basic "mechanical" forms, which he locates on paper. In contrast to this, analytic geometry breaks every configuration into points and plots the independent points separately, point by point. A form can be associated with every polynomial in this way. Descartes invented this new method of constructing curves by solving a problem stated by Pappus. He writes:

> "If then we should take successively an infinite number of different values for the line y, we should obtain an infinite number of values for the line x, and therefore an infinity of different points, such as C, by means of which the required curve could be drawn." (Mancosu 1992, p. 89)

In this way, using the language of algebra, a much richer universe of forms is disclosed, unknown to the Greeks. Looking back from the Cartesian point of view on Euclidean geometry, we can say that with a few exceptions (such as the *quadratrix* of Hippias, the *spiral* of Archimedes, the *conchoid* of Nicomedes, and the *cisoid* of Diocles (Heath 1921, pp. 226, 230, 238, and 264)) the whole of Euclidean geometry deals with quadratic curves only (i.e., curves whose equations are of second degree). The universe of the analytic geometry is qualitatively richer; it contains qualitatively more curves, than the Euclidean universe. Almost every important mathematician of the seventeenth century came up with a new curve; let us just mention Descartes' *folium*, Bernoulli's *lemniscate*, Pascal's *shell*, the *cardioid*, the *astroid*, and the *strophoid*. So the expressive power of the new geometrical language is greater. It is sufficient to take a polynomial expression, choose its coefficients in an appropriate way and a new form emerges, a form that Euclid could never possibly see. Nevertheless, it is important to realize that within the analytic universe we can reconstruct a region which will correspond to the Euclidean: the universe of quadratic curves.

In algebra a formula represents the order of the particular steps of a calculation. This corresponds to construction of separate points of analytic curves. For each point we have to calculate the values of its co-ordinates using an algebraic formula. So far we use the formula in the same way as it is used in algebra: as a relation between two particular numbers. Nevertheless, analytic geometry goes further. By plotting the independent points separately, point after point, a new form becomes visible. None of the separate points itself gives rise to the form. Only if they are all together can we see the form. The algebraic formula itself determines only each single point; putting them all together is the new step taken by analytic geometry. If we gradually change one co-ordinate, and for each of its values we calculate the second co-ordinate according to the algebraic formula, we obtain a curve. Thus the curve not only expresses a *relation* between isolated values of the variables x and y (the ability to express this relation constitutes the logical power of the language of algebra) but the curve also discloses the *dependence* between the two co-ordinates. This is not a functional dependence yet (i.e., dependence of a function on its argument). The concept of function was introduced by Leibniz. In analytic geometry we have just dependence between co-ordinates, which means that the dependence is geometric in nature. Nevertheless, this geometric way of representing dependence was an important step towards the concept of function itself.

The geometric visualization of dependence as dependence between variables resembles the Pythagorean visualization of arithmetical properties using figurate numbers. Just as the line segment of indefinite length, which the Pythagoreans used in their proofs, was a precursor of the concept of variable, the dependence between variables, as used in analytic geometry, is a precursor of the concept of a function. The line segment of indefinite length, and the dependence between variables are part of the iconic languages, while the concepts of variables and functions are constituents of the symbolic language. Nevertheless, the role of the geometrical intermediate states in the formation of the concepts of a variable or function is clearly visible.

1.1.4.3. Explanatory Power – Ability to Explain the Casus Irreducibilis

Analytical geometry explains why algebraic formulas can lead to paradoxes. The idea stems from Newton. If solving an algebraic equation means determining the intersection points of a particular curve with the x-axis, the universal solvability of all equations would automatically mean that each curve would have to intersect this axis. This is clearly nonsense. Therefore there must be a way in which an algebraic equation does not give rise to intersection points. The appearance of negative numbers under the sign of the square root can prevent an algebraic equation from having a solution.

Thus the *casus irreducibilis* is not a failure of algebra or of the algebraist – on the contrary. Since algebraic expressions determine analytical curves, the formulas giving the solutions of algebraic equations must fail, in some cases, to give curves the freedom to intersect. In this way the failure of the formulas, which might look like a weakness of the algebraic language from a purely algebraic point of view, is no weakness at all. Neither is it an exceptional case. It must be a systematic feature of all algebraic formulas, in order to give analytic geometry the necessary freedom. Thus the language of analytical geometry explains the failure of the language of algebra, which looked rather odd from a purely algebraic point of view.[11] Here again we have to do with an explanation similar to the explanation given by synthetic geometry of the insolubility of some arithmetic problems. In both cases the geometrical language disclosed the richness of possible situations responsible for the failure of a particular symbolic language. Thus these explanations are not examples of the skill of some mathematicians. They rather disclose an epistemological feature of the language itself, namely its

[11] The situation is not so simple in the case of Cardano's *casus irreducibilis* because the cubic parabola has three real roots. Thus here we have to deal with a more delicate question. The formula expressing the solutions of a cubic equation represents its real roots as the sum of complex quantities. Nevertheless, this does not contradict the basic fact that the representation of a polynomial by a curve makes it possible to understand phenomena, which from the purely algebraic point of view appear rather puzzling.

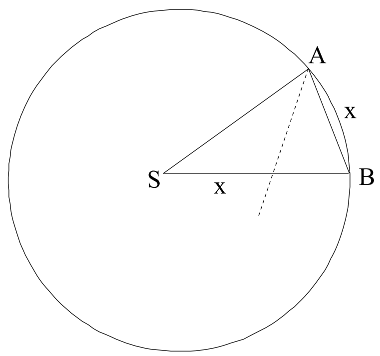
explanatory power. Who, when and under what circumstances, discovered the explanatory power of a particular language is a historical question. But the explanatory power itself is an epistemological fact, requiring philosophical rather than historical analysis.

1.1.4.4. Integrative Power – Integration of Geometric Methods

Descartes published his *Géométrie* in 1637 as an appendix to his *Discours de la méthode*. It comprises three books. The first book, *Problems the construction of which requires only straight lines and circles* opens with a bold claim:

> "Any problem in geometry can easily be reduced to such a term that a knowledge of the lengths of certain straight lines is sufficient for its construction." (Descartes)

Descartes showed that any ruler-and-compasses construction is equivalent to a construction of the root of a second-degree equation. The core of this part of *Géométrie* is a general strategy for solving all geometrical problems. It consists of three steps: *naming*, *equating* and *constructing*. In the first step we assume that the problem is already solved, and give names to all the lines which seem needed to solve the problem. In the second step we ignore the difference between the known and the unknown, find the relations that hold among the lines, and express them in the form of algebraic equations. In the third step we solve the equations and construct their roots. Thus Descartes managed to introduce the universal analytical methods of algebra into geometry. As an illustration of this can be taken the construction of the *decagon*:



> "Suppose that a regular decagon is inscribed in a circle with radius 1, and call its side x. Since x will subtend an angle 36^o at the centre of the circle, the other two angles of

> the large triangle will each be 72^o, and hence the dotted line which bisects the angle A divides triangle SAB into two isosceles triangles, each with equal sides of length x. The radius of the circle is thus divided into two segments, x and $1-x$. Since SAB is similar to the smaller isosceles triangle, we have $1/x = x/(1-x)$. From this proportion we get the quadratic equation $x^2 + x - 1 = 0$, the solution of which is $x = (\sqrt{5} - 1)/2$." (Courant and Robbins 1941, p. 122)

To construct a line segment of this length is easy. Then we just mark off this length ten times as a chord of the circle and the decagon is produced. We do not need to memorize any construction trick – the constructive part of the problem is trivial. Classical constructive geometry was difficult because to construct any object a specific procedure had to be remembered. In analytic geometry we do not construct objects. Properties of an object are rewritten as algebraic equations, these are solved, and only the line segments with lengths corresponding to the solutions of the equations are constructed. So instead of constructing a regular decagon we have to construct a line segment of the length $(\sqrt{5} - 1)/2$. Thus Descartes brought the universal methods of algebra into geometry.[12]

Please verify) in $\sqrt{5} - 1)/2$

It is possible that a classical geometer would be able to construct a decagon making fewer steps as we did in the above construction. His construction may seem much more elegant. The advantage of the analytic method is that it is in a sense unrelated to the particular object that we are constructing – in our case the pentagon. With any other object the procedure would be basically the same, only the particular equation would be different. That procedure is based on Descartes' insight that beneath the visible surface of geometry, on which the tricks of the geometers take place, there is a deeper layer of structural relations, which make these tricks possible. The language of algebra is able to grasp this deeper layer of geometry, the structural relations among the mag-

[12] Here I have to apologize to the reader. I could not resist the temptation and I have chosen an illustration that is rather elegant. But precisely because of its elegance it obscures the point. In the construction of the regular pentagon I have used a trick – its replacement by the regular decagon. If I were not to do this, I would obtain a similar equation, only not so quickly. But what is important is not how quickly we obtain the result, but the fact that the whole ensuing construction is absolutely trivial. All the difficult work was transferred onto the shoulders of algebra. Algebra led us from the equations to the formula for its solution. For the geometrical part it remains only to construct the roots of the algebraic equations, which is trivial.

nitudes that occur in a particular geometrical construction. The language of analytical geometry discloses a deeper unity which is hidden beneath the apparent diversity of the particular geometrical problems. All problems of classical geometry consisted in the construction of particular line segments, the length of which was determined indirectly by a set of relations to other line segments. The relations determining the particular line segments can be expressed using algebraic equations. Therefore the integrative force of the language of analytic geometry can be characterized by its ability to integrate all the different construction methods of synthetic geometry into one universal scheme.

1.1.4.5. Logical Boundaries – The Impossibility of Squaring the Circle

The transcendence of π was demonstrated in 1882 by Carl Lindemann. His proof used methods of the theory of functions of complex variables and it can be found in many books (see Stewart 1989, p. 66; Dörrie 1958 p. 148; or Gelfond 1952, p. 54). The transcendence of π means that π is not a root of any algebraic equation. In the problem of the quadrature of the circle the number π plays a crucial role and the transcendence of π means that this problem is insoluble with the methods of analytic geometry. The insolubility of the problem of the quadrature, however, cannot be expressed by means of the language of analytic geometry; it only shows itself in the failure of all attempts to solve it.

1.1.4.6. Expressive Boundaries – The Inexpressibility of the Transcendent Curves

Soon after the discovery of analytical geometry it turned out that the language of algebra is too narrow to deal with all the phenomena encountered in the world of analytic curves. First of all, two kinds of curves are not algebraic, the exponential curves and the goniometric curves. They cannot be given with the help of a polynomial. These curves transcend the language of classical analytic geometry just as the transcendental numbers transcend the language of algebra.

1.1.5. The Differential and Integral Calculus

The differential and integral calculus is, just like arithmetic and algebra, a symbolic language, i.e., a formal language designed for manipulation