CHAPTER 1

Re-coding as the First Pattern of Change in Mathematics

The roles of geometry and of arithmetic in contemporary philosophy of mathematics are rather asymmetric. While arithmetic plays a central role in foundational approaches and therefore its logical structure is thoroughly studied and well understood (see Shapiro 2005), geometry is the central topic of the antifoundational approaches (see Boi, Flament, and Salanskis 1992). This of course does not mean that there are no foundational studies of geometry; it is sufficient to mention the work of Alfred Tarski (see Tarski 1948 and 1959). Nevertheless, in these cases geometry is just another illustration of the methods developed for the analysis of arithmetic. The visual aspect of geometry, the very fact that geometry has something to do with space and spatial intuition, is totally ignored in these studies. These studies are just exceptions, and they do not change the basic difference that the philosophy of arithmetic is dominated by the foundational approach, while the philosophy of geometry is mainly antifoundational.

The reason for this asymmetry lies in the different attitudes to the languages of these two main parts of mathematics. Since the works of Frege, Peano, and Russell, the language of arithmetic has been fully formalized, and so the formulas of arithmetic are considered to be a constitutive part of the theory. On the other hand, in geometry the geometrical pictures are considered only as heuristic aids, which can help us to understand the theory but strictly speaking do not belong to the theory itself. Since Hilbert, the content of a geometrical theory is independent of any pictures and is determined by its axioms. Thus in the formalization of arithmetic the specific symbols like "+" or "⩽", as well as the rules which they obey, were considered part of the language. In geometry the process of formalization took rather the opposite direction: all the special symbols, like "." or "—", were excluded from the language. An interesting analysis of the reasons for this exclusion of diagrams and of diagrammatic reasoning from the foundations of mathematics is given in (Graves 1997).

Although there were good reasons for such a development of the foundations of mathematics, we believe it might be interesting to try to bridge the gap between the philosophy of arithmetic and the philosophy of geometry. For this purpose it is necessary to do in geometry what Frege did in arithmetic. This means, first of all, formalizing its language and thus turning the pictures from mere heuristic aids into integral parts of the theories themselves. A picture is not just the physical object formed by spots of graphite on the more or less smooth surface of the paper. We understand the picture as an expression (a term) of the iconic language with its own meaning and reference. We follow here an analogy with arithmetic or algebra, where a formula is understood not as a physical object, i.e., not as spots of ink on a sheet of paper.[1] If we succeeded in the incorporation of pictures into the language of mathematics it would enable us to deepen our understanding of those periods of the history of mathematics where geometric pictures and argumentation based upon them played a crucial role (i.e., practically the whole history of geometry before Pasch and Hilbert).

For our purposes it is enough to give a short characterization of the iconic language of geometry. We interpret a picture as a term of the iconic language. Then a geometrical construction becomes a generating sequence of the resulting expression (picture). In this way the

[1] The aim of interpreting geometrical figures as a language may be challenged by posing the problem of how to represent propositions. Nevertheless, it is important to realize that also in the case of symbolic languages the representation of propositions was not introduced overnight. Viète at the end of the sixteenth century, i.e., more than a century after Regiomontanus, did not have a symbol for the expression of identity and so he was unable to express a proposition solely by symbols. Therefore he connected fragments of his symbolic language into propositions by means of ordinary language. Thus also in the case of the language of geometry it is not necessary in a single move to solve all the problems of its syntax. In the following we will restrict ourselves to the representation of terms of the iconic language. Maybe later someone will find a way of forming propositions from terms of the iconic language similarly to the way in which, by means of the symbol of identity, we form propositions from the terms of a symbolic language.

Euclidean postulates ("*To draw a straight line from any point to any point.*" or "*To describe a circle with any centre and distance.*") become formation rules of this language, analogous to the Fregean rules for symbolic languages, which prescribe, how from an n-ary functional symbol F and n terms $t_1, t_2, \ldots, t_n$, a new term $F(t_1, t_2, \ldots, t_n)$ is formed. A picture is called a "well-formed expression" if each construction step is performed in accordance with the formation rules (axioms). If we rewrite the Euclidean postulates 1, 2, 3, and 5 (see Euclid, p. 154) as formation rules, we obtain a general description of the language.[2] The questions when two terms are equal or how we can introduce predicates into the pictorial language, and what its propositions look like, are subtle questions, which we don't want to raise now. They would require more detailed investigations, which would lead us rather far from the subject of our book. For our present purposes it is sufficient to realize that the iconic language of geometry can be treated with the same strength and precision as that which Frege introduced for arithmetic. Seen from this position, mathematics for us will consist no longer of an exact symbolic language supplemented by some heuristic pictures, but rather of two languages of the same rank, one of them symbolic and the other iconic.

Of course the pictures of Euclidean (synthetic) geometry are not the only pictures used in geometry. There are pictures also in analytic (algebraic and differential) geometry as well as in iterative (fractal) geometry. If we complement the language of mathematics by pictures of synthetic, analytic, and fractal geometry, it will in a radical way increase the capacity of our linguistic approach to the study of changes in the development of mathematics. We will describe an interesting periodic motion in the history of mathematics, consisting in alternation of its symbolic and geometrical periods. Thus in ancient Egypt and Babylonia the symbolic approach was dominant. Later, in Ancient Greece, geometry came to the fore and dominated mathematics until the sixteenth century, when a revival of the symbolic approach took place in

[2] From this point of view Pasch's discovery, that Euclidean geometry contains no postulate that would guarantee that two circles drawn from the opposite ends of a given straight line with radii equal to the length of that line will intersect, could be interpreted as the discovery of the fact that the corresponding figure is not a well-formed term of the language of synthetic geometry. We, of course, see the points of intersection, but if we realize that a plane containing only points with rational co-ordinates is a model of Euclid's postulates, it becomes obvious that the existence of the points of intersection cannot be proven from the postulates. This example illustrates what it means to view figures as terms of the language of synthetic geometry and to interpret Euclid's axioms as formation rules of this language.

algebra. In the next century, due to the discovery of analytic geometry, the iconic language came again to the fore, while the nineteenth century witnessed a return of the dominance of the symbolic language in the form of the arithmetization of analysis. This phenomenon has not yet been sufficiently understood. Nobody has tried to undertake a serious analysis of it. The reason could be that the alternation of symbolic and geometrical periods in mathematics has a vague and obscure nature, which dissuades people from undertaking its serious analysis. Nevertheless, I am convinced that this vagueness and obscurity is only a result of insufficient understanding of the logical and epistemological aspects of geometrical pictures. As long as pictures are considered vague and obscure, their alternation with formal languages, which is clearly visible in the history of mathematics, must remain vague and obscure as well. By interpreting pictures as iconic language, we create a framework which makes it possible to understand the relations between the symbolic and iconic periods in the history of mathematics. In their regular alterations we will discover the first pattern of changes in the development of mathematics.

1.1. Historical Description of Re-codings

In reconstruction of the linguistic innovations in the history of mathematics that we labeled re-codings we will follow Gottlob Frege. Frege published in 1879 his *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* in which he presented the modern quantification theory and an axiomatic system of the predicate calculus. In the closing chapter of his book Frege introduced definitions of several notions of the theory of infinite series. He devoted the next fourteen years to further elaboration of ideas from this chapter and in 1893 he published the first volume of his opus magnum: *Grundgesetze der Arithmetik, Begriffsschriftlich abgeleitet*. If we compare the logical framework of these books, we find several important changes. The *Begriffsschrift* is based on a syntactic approach to logic – Frege chose a particular system of formulas which served as axioms and derived from them the whole system of the predicate calculus. The *Grundgesetze* contained several semantic extensions of the logical system of the *Begriffsschrift,* among which perhaps the most important was the introduction of *concepts* as functions whose possible values are *the True* and *the False,* and the introduction of *extensions of concepts* as course-of-values of these functions. Frege's thoughts motivating these changes can be found in the papers published between 1891 and 1892: *Funktion und Begriff, Über Begriff und Gegendstand* and *Über Sinn und Bedeutung.*

The first of these papers contains an idea which we will take as the starting point of our reconstruction of re-codings in mathematics. In his paper Frege described the evolution of the symbolic language of mathematics from elementary arithmetic through algebra and mathematical analysis to predicate calculus:

> " If we look back from here over the development of arithmetic, we discern an advance from level to level. At first people did calculations with individual numbers, 1, 3, etc.
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> $$2 + 3 = 5 \qquad\qquad 2 \cdot 3 = 6$$
>
> are theorems of this sort. *Then they went on* to more general laws that hold good for all numbers. What corresponds to this in symbolism is the transition to the literal notation. A theorem of this sort is
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> $$(a + b) \cdot c = a \cdot c + b \cdot c\,.$$
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> At this stage they had got to the point of dealing with individual functions; but were not yet using the word, in its mathematical sense, and had not yet formed the conception of what it now stands for. *The next higher level* was the recognition of general laws about functions, accompanied by the coinage of the technical term 'function'. What corresponds to this in symbolism is the introduction of letters like f, F, to indicate functions indefinitely. A theorem of this sort is
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> $$\frac{\mathrm{d}F(x)\cdot f(x)}{\mathrm{d}x} = F(x)\cdot\frac{\mathrm{d}f(x)}{\mathrm{d}x} + f(x)\cdot\frac{\mathrm{d}F(x)}{\mathrm{d}x}\,.$$
>
> Now at this point people had particular second-level functions, but lacked the conception of what we have called second-level functions. By forming that, we make *the next step forward*." (Frege 1891, p. 30; English translation p. 40)

Our interpretation of this development of the symbolic language will differ from Frege's in two respects. The first is terminological: we will not subsume algebra or mathematical analysis under the term "arithmetic" but will rather consider them as independent languages. More important, however, is the fact that we will show how this “development of arithmetic” described by Frege interplayed with geometrical intuition. In order to achieve this we need to complement Frege’s analysis of the “development of arithmetic” with a similar “development of geometry”. Frege identified the main events in the development of the symbolic language as the introduction of the concept of “individual functions” in algebra, then of the “particular second-level functions” in the calculus and finally the general concept of “second-level functions” of the predicate calculus. In a similar way we will try to identify the crucial events of the development of the iconic language of geometry. It will turn out that the events parallel to those described by Frege are the creation of analytic geometry, fractal geometry and set theory.

The unification of the symbolic and iconic languages enables us to consider the development of mathematics as the evolution of its language. We will study the development of the language of mathematics from the following six aspects:

1. *Logical power* – how complex formulas can be proven in the language,

2. *Expressive power* – what new things can the language express, which were inexpressible in the previous stages,

3. *Explanatory power* – how the language can explain the failures which occurred in the previous stages,

4. *Integrative power* – what sort of unity and order the language enables us to conceive there, where we perceived just unrelated particular cases in the previous stages,

5. *Logical boundaries* – marked by occurrences of unexpected paradoxical expressions,

6. *Expressive boundaries* – marked by failures of the language to describe some complex situations.

The evolution of the language of mathematics consists in the growth of its logical and expressive power – the later stages of development of the language make it possible to prove more theorems and to describe a wider range of phenomena. The explanatory and the integrative power of the language also gradually increases – the later stages of development of the language enable deeper understanding of its methods and offer a more unified view of its subject. To overcome the logical and expressive boundaries, more and more sophisticated and subtle techniques are developed. We will illustrate the growth of the logical, expressive, explanatory, and integrative power, as well as the shifts of the logical and expressive boundaries of the language of mathematics by some suitable examples.[3]

Mathematics has a tendency to improve its languages by addition. So, for instance, we are used to introducing the concept of variable into the language of arithmetic (enabling us to write equations in this language) and often we choose the field of real numbers as a base (so that the language is closed with respect to limits). This is very convenient from the pragmatic point of view, because it offers us a strong language in which we can move freely without any constraints. But, on the other hand, it makes us insensitive to historically existing languages. The old languages do not appear to us as independent systems with their own logical and expressive powers. They appear only as fragments of our powerful language. Since our aim is the epistemological analysis of the language of mathematics, we try to characterize every language as closely as possible to the level on which it was created. We ignore later emendations which consist of incorporation of achievements of the later development into the former languages (for instance of the concept of variable into arithmetic). In this book the language of arithmetic will be a language without variables. We think that such a stratification of the language of mathematics into different historical layers will be interesting also for logical investigations, showing the order in which different logical tools appeared.

1.1.1. Elementary Arithmetic

Counting is as old as mankind. In every known language there are special words expressing at least the first few numbers. For instance the Australian tribes around Cooper bay call 1 – *guna*, 2 – *barkula*, 3 – *barkula guna*, and 4 – *barkula barkula* (Kolman 1961, p. 15). With the development of society it became necessary to count greater quantities of goods and so different aids were introduced in counting: fingers, pebbles, or strings with knots. A remarkable tool was found in Moravia

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[3] The logical and expressive boundaries of a particular language can be expressed only by means of a later language; a language that is strong enough to enable us to construct a situation by the description of which the original language fails. Therefore for characterization of the logical and expressive boundaries of a particular language we will use later languages.