The Blackwell Guide to Philosophica Logic. Ed. L. Gobble. Blackwell, Oxford. 2001, 224–257.

Intuitionistic Logic

Dirk van Dalen

1 Basic principles

There are basically two ways to view intuitionistic logic: (i) as a philosoph ical–foundational issue in mathematics, (ii) as a technical discipline within mathematical logic.

Let us deal with the philosophical aspects first, they will provide the motivation for the subject. In doing so, we will follow L.E.J. Brouwer, the founding father of intuitionism. Brouwer did contribute little to intuitionistic logic as we know it from text books and papers, but he pointed the way for his successors.

Logic in Brouwer's intuitionism takes a secondary place, the first place is reserved for mathematics. Mathematics has to be understood in the widest possible sense, it is the constructional mental activity of the individual (for a survey of the Brouwerian global philosophy of man and his mathematical enterprise, see [Dalen 1999]). The role of logic is to note and systematically study certain regularities in the mathematical constructional process. Contrary to traditional views, logic thus is dependent on mathematics and not vice versa.

Mathematical practice has taught us that relatively few logical connectives suffice for an efficient treatment of arguments. In the case of intuitionism, the meaning of the connectives has to be explained in terms of the basic mathematical notion: construction. A fact A is established by means of a construction. An easy example is $3 + 2 = 5$, this is established by the following construction: construct 3, construct 2 and compare the outcome with the result of the construction of 5. The outcome is a confirmation of the above equation.

The construction criteria for 'truth' also yields an interpretation of the connectives. Let us write "$a : A$" for "a is a construction that establishes A", we call this a a *proof* of A.

A proof of $A \wedge B$ is simply a pair of proofs a and b of A and B. For convenience we introduce a a notation for the pairing of constructions, and for the inverses (projections); (a, b) denotes the pairing of a and b, and $(c)_0$,$(c)_1$, are the first and second projection of c. Now, the proof of a disjunction $A \vee B$ is a pair (p, q) such that p carries the information, which disjunct is correct, and q is the proof of it. We stipulate that $p \in \{0, 1\}$. So $p = 0$ and $q : A$ or $p = 1$ and $q : B$. Note that this disjunction is *effective*, in the sense that the disjunct is specified, this in contrast with classical logic, where one does not have to know which disjunct holds.

Negation is also defined by means of proofs: $p : \neg A$ says that each proof a of A can be converted by the construction p into a proof of an absurdity, say $0 = 1$. A proof of $\neg A$ thus tells us that A has no proof!

The most interesting propositional connective is the implication. The classical solution, i.e. $A \to B$ is true if A is false or if B is true, cannot be used because here the classical disjunction is used; moreover, it assumes that the truth values of A and B are known before one can settle the status of $A \to B$.

Heyting showed that this is asking too much. Consider A = "there occur twenty consecutive 7's in the decimal expansion of π", and B = "there occur nineteen consecutive 7's in the decimal expansion of π". Then $\neg A \vee B$ does not hold constructively, but on the interpretation following here, the implication, $A \to B$ is obviously correct.

The intuitionistic approach, based on the notion of proof, demands a definition of a proof a of the implication $A \to B$ in terms of (possible) proofs of A and B. The idea is quite natural: $A \to B$ is correct if we can show the correctness of B as soon as the correctness of A has been established.

In terms of proofs: $p : A \to B$ if p transforms each proof $q : A$ into a proof $p(q) : B$. The meaning of the quantifiers is specified along the same lines. Let us assume that we are dealing with a given domain D of mathematical objects. A proof p of $\forall x A(x)$ is a construction which yields for every object $d \in D$ a proof $p(d) : A(d)$. A proof p of $\exists x A(x)$ is a pair (p_0, p_1) such that $p_1 : A(p_0)$. Thus the proof of an existential statement requires an instance plus a proof of this instance.

$a : A$	conditions
$a :\perp$	false
$a : A \wedge B$	$a = (a_1, a_2)$, where $a_1 : A$ and $a_2 : B$
$a : A \vee B$	$a = (a_1, a_2)$, where $a_2 : A$ if $a_1 = 0$ and $a_2 : B$ if $a_1 = 1$
$a : A \rightarrow B$	for all p with $p : A \quad a(p) : B$
$a : \exists x A(x)$	$a = (a_1, a_2)$ and $a_2 : A(a_1)$
$a : \forall x A(x)$	for all $d \in D\ \ a(d) : A(d)$ where D is a given domain
$a : \neg A$	for all $p : A\ \ a(p) :\perp$

Observe that an equivalent characterization of the disjunction can be given: $a = (a_1, a_2)$, where $a_1 = 0$ and $a_2 : A$ or $a_1 = 1$ and $a_2 : B$. The above formulation has the technical advantage that only $\wedge$ and $\rightarrow$ are required, and they belong to the 'negative fragment'. The interpretation of the connectives in terms of proofs was made explicit by Heyting in [Heyting 1934]. Kolmogorov had given a similar interpretation in terms of *problems* and *solutions*, [Kolmogorov 1932]. The formulations are, up to terminology, virtually identical.

We will demonstrate the *proof interpretation* for a few statements.

1. $A \rightarrow (B \rightarrow A)$.
   We want an operation p that turns a proof $a : A$ into a proof of $B \rightarrow A$. But if we already have a proof $a : A$, then there is a simple transformation that turns a proof $q : B$ into a proof of A, i.e. the constant mapping $q \mapsto a$, which is denoted by $\lambda q \cdot a$. And so the construction that takes a into $\lambda q \cdot a$ is $\lambda a \cdot (\lambda q \cdot a)$, or in an abbreviated notation $\lambda a \lambda q \cdot a$.
   Hence $\lambda a \lambda q . a : A \rightarrow (B \rightarrow A)$.

2. $(A \vee B) \rightarrow (B \vee A)$.
   Let $p : A \vee B$, then $p_0 = 0$ and $(p)_1 : A$ or $(p)_0 = 1$ and $(p)_1 : B$. By interchanging A and B we get, looking for $q : B \vee A$: $(q)_0 = 1$ and $(q)_1 : B$ or $(q)_1 = 0$ and $(q)_1 : A$, which comes to $\overline{sg}((p)_0) = (q)_0$ and $(p_1) : B$ or $\overline{sg}((p)_0) = (q)_0$ and $(q)_1 : A$ that is $(\overline{sg}((p_0)), (p)_1) : B \vee A$. And so $\lambda p \cdot (\overline{sg}((p_0), (p)_1) : A \vee B \rightarrow B \vee A$.

3. $A \rightarrow \neg\neg A$
   A proof q of $\neg\neg A$ is a proof of $\neg A \rightarrow \perp$. Assume $p : A$, and $q : \neg A$. Then $q(p) :\perp$, so $\lambda q \cdot q(p) : \neg\neg A$. Hence $\lambda p \lambda q \cdot q(p) : A \rightarrow \neg\neg A$.

4. $\neg\neg A \rightarrow A$
   This turns out to be unprovable. For assume that $a : \neg\neg A$, then we

know that $A \to\perp$ has no proof, and that is all. No more information can be extracted. Hence we cannot claim that there is a proof of A.

5. $A \vee \neg A$
   $p : A \vee \neg A \Leftrightarrow (p)_0 = 0$ and $(p)_1 : A$ or $(p)_0 = 1$ and $(p)_1 : \neg A$.
   However, for an arbitrary proposition A we do not know whether A or $\neg A$ has a proof, and hence $(p)_0$ cannot be computed. So, in general there is no proof of $A \vee \neg A$.

6. $\neg\exists x A(x) \to \forall x \neg A(x)$
   $p : \neg\exists x A(x) \Leftrightarrow p(a) :\perp$ for a proof $a : \exists x A(x)$. We have to find $q : \forall x \neg A(x)$, i.e. $q(d) : A(d) \to\perp$ for any $d \in D$.
   So pick an element d and let $r : A(d)$, then $(d, r) : \exists x A(x)$ and so $p((d, r)) :\perp$. Therefore we put $(q(d))(r) = p((d, r))$, so $q = \lambda r \lambda d \cdot p((d, r))$ and hence $\lambda p \lambda r \lambda d \cdot p((d, r)) : \neg\exists x A(x) \to \forall x \neg A(x)$.

Brouwer handled logic in an informal way, he showed the untenability of certain classical principles by a reduction to unproven statements. His technique is goes by the name of 'Brouwerian counterexample'. A few examples are shown here. Brouwer considered certain open problems, usually formulated in terms of the decimal expansion of π. In spite of considerable computational power, there are enough open questions concerning the occurrence of specific sequences of decimals.[1]

Let us illustrate the technique: we compute simultaneously the decimals of π and the members of a Cauchy sequence. We use $N(k)$ as an abbreviation for "the decimals $p_{k-89}, \ldots, p_k$ of π are all 9". Now we define:

$$a_n = \begin{cases} (-2)^{-n} & \text{if } \forall k \leq n \neg N(k) \\ (-2)^{-k} & \text{if } \ k \leq n \text{ and } N(k) \end{cases}$$

a_n is an oscillating sequence of negative powers of -2 until a sequence of 90 nines in π occurs, from then onwards the sequence is constant:
$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \ldots (-2)^{-k}, (-2)^{-k}, (-2)^{-k}, (-2)^{-k}, \ldots$.

The sequence determines a real number a, in the sense that it satisfies the Cauchy condition. The sequence is well-defined, and $N(n)$ can be checked for each n (at least in principle).

For this particular a we cannot say that it is possitive, negative or zero.

$a > 0 \Leftrightarrow N(k)$ holds for the first time for an even number
$a < 0 \Leftrightarrow N(k)$ holds for the first time for an odd number

[1] Brouwer considered various sequences, for one then the occurrence has been established: 01234567890 does indeed occur among the decimals of π, cf. [Borwein 1998].

$a = 0 \Leftrightarrow N(k)$ holds for no k.

Since we have no effective information on the status of the occurence of $N(k)$'s, we cannot affirm the trichotomy law, $\forall x \in \mathbb{R}(x < 0 \vee x = 0 \vee x > 0)$, cannot be said to have a proof.

The above number a cannot be irrational, for then $N(k)$ would never apply, and hence $a = 0$. Contradiction. Hence we have shown ($\neg\neg a$ is rational). But there is no proof that a is rational. So $\neg\neg A \to A$ fails. One also easily sees that $a = 0 \vee a \neq 0$ fails to have proof.

Brouwer used his counter examples to show that some classical mathematical theorems are not intuitionistically valid, cf.[Brouwer 1923].

The Brouwerian counterexamples are *weak* in the sense that they show that some proposition has as yet no proof, but it is not excluded that eventually a proof may be found. In formal logic there is a similar distinction: $\not\vdash A$ and $\vdash \neg A$. The Brouwerian counter examples are similar to the first case, strong counterexamples cannot always be expected. For example, although there are instances of the Principle of the Excluded Third (PEM) where no proof has been provided, the negation cannot be proved. For $\neg(A \vee \neg A)$ is equivalent to $\neg A \wedge \neg\neg A$, which is a contradiction! We will get to some strong refutations of classical principles in the following sections.

Brouwer did not develop logic for its own sake, but he was the first to establish a non-trivial result: $\neg A \leftrightarrow \neg\neg\neg A$ ([Brouwer 1919b]).

Suggested further reading: [Heyting 1934], [Heyting 1956], [**?**], [Dummett 1977] Ch. 1, 7, [Bridges-Richman 1987] Ch. 1, Philosophia Mathematica, vol 6 – Special Issue: Perspectives on Intuitionism (ed. R. Tieszen), p. 129 – 226.

In 1928 the Dutch Mathematics Society posed in its traditional prize contest a problem asking for a formalization of Brouwer's logic. Arend Heyting sent in an essay, with the motto 'stones for bread', in which he provided a formal system for intuitionistic predicate logic. The system was presented in 'Hilbert style' (as it is called now). That is to say, a system with only two derivation rules and a large number of axioms. Heyting had patiently checked the system of Russell's Principia Mathematica, and isolated a set of intuitionistically acceptable axioms. His system was chosen in such a way that the additon of PEM yields full classical logic. We give here a list of axioms taken from [Troelstra–van Dalen 1988], p. 86.

$A \wedge B \to A, \quad A \wedge B \to B,$
$A \to (B \to (A \wedge B),$

$$A \to A \vee B, \quad B \to A \vee B,$$
$$(A \to (B \to C)) \to ((A \vee B \to C)),$$

$$A \to (B \to A),$$
$$(A \to (B \to C) \to ((A \to B) \to (A \to C)),$$

$$\perp \to A,$$

$$A(t) \to \exists x A(x),$$
$$\exists x(A(x) \to B) \to (\exists x A(x) \to B),$$

$$\forall x A(x) \to A(t),$$
$$\forall x(B \to A(x)) \to (B \to \forall x A(x)).$$

There are two derivation rules: the well-known *modus ponens* (i.e. $\to E$, see below) and the rule of generalization: $\vdash_i A(x) \Rightarrow \vdash_i \forall x A(x)$, where $\vdash_i$ stands for intuitionistic derivability. When no confusion arises, we will simply write $\vdash$. The $\forall I$-rule (see below) would have done just as well.

Heyting's formalization was published in 1930, Glivenko (1929) and Kolmogorov (1925) had already published similar formalizations, which, however, did not cover full intuitionistic logic, cf. [Troelstra 1978].

In 1934 Gerhardt Gentzen introduced two new kinds of formalization of logic. Both of these are eminently suited for the investigation of formal deriations as objects in their own right.

In the system of *Natural Deduction* there are introduction and elimination rules for the logical connectives that reflect their meanings. The rules are given below in an abbreviated notation.

$$\wedge I \quad \frac{A \quad B}{A \wedge B} \qquad\qquad \wedge E \quad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$\vee I \quad \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad \vee E \quad \frac{A \vee B \quad \begin{array}{c}[A]\\ \mathcal{D}\\ C\end{array} \quad \begin{array}{c}[B]\\ \mathcal{D}'\\ C\end{array}}{C}$$

$$\to I \quad \frac{\begin{array}{c}[A]\\ \mathcal{D}\\ B\end{array}}{A \to B} \qquad\qquad \to E \quad \frac{A \quad A \to B}{B}$$

$$\bot\, E \quad \frac{\bot}{A}$$

$$\forall I \quad \frac{\begin{array}{c}\mathcal{D}\\ A(x)\end{array}}{\forall x A(x)} \qquad\qquad \forall E \quad \frac{\forall x A(x)}{A(t)}$$

$$\exists I \quad \frac{A(t)}{\exists x A(x)} \qquad\qquad \exists E \quad \frac{\exists x A(x) \qquad \begin{array}{c}[A(x)]\\ \mathcal{D}\\ C\end{array}}{C}$$

There are a few conventions for the formulation of the natural deduction system. (i) A hypothesis of a derivation between square brackets is cancelled, that is to say, it no longer counts as a hypothesis. Usually all of the hypotheses are cancelled simultaneously. This is not really required, and it may even be be necessary to allow for 'selective' cancellation (cf. [Troelstra–van Dalen 1988] p. 559, 568). For most practical purposes it is however a convenient convention. (ii) For the quantifier rules there are some natural conditions on the free variables of the rule. In $\forall I$ the variable x may not occur free in the hypotheses of $\mathcal{D}$. In $\forall E$ and $\exists I$ the term t must be free for x in A. Finally in $\exists E$ the variable x may not occur in C or in the remaining hypotheses of $\mathcal{D}$. For an explanation see [Dalen 1997].

The rules are instructive for more than one reason. In the first place, they illustrate the idea of the proof-interpretation. Recall that $p : A \to B$ stands for "for every $a : A$ $(p(a) : B)$".

Now $\to E$ tells us that if we have a derivation $\mathcal{D}$ of A and a derivation

$\mathcal{D}'$ of $A \to B$, then the combination

$$\to E \qquad \frac{\begin{array}{cc}\mathcal{D} & \mathcal{D}' \\ A & A \to B\end{array}}{B} = \frac{\mathcal{D}''}{B}$$

is a derivation of B. So we have an automatic procedure that, given $\mathcal{D}'$, converts $\mathcal{D}$ into a derivation $\mathcal{D}''$ of B.

The introduction rule also illustrates the transformation character of the implication.

Let $\begin{array}{c} A \\ \mathcal{D} \\ B \end{array}$ be given, then $\to I$ yields $\dfrac{\begin{array}{c}[A] \\ \mathcal{D} \\ B\end{array}}{A \to B}$.

The first derivation shows that if we add a proof $\mathcal{D}'$ of A, we automatically get a proof of B. So the rules tell us that there is a particular construction, converting proofs of A into proofs of B. This is exactly the justification for the derivation of $A \to B$.

For, say, the conjunction rules the analogy is even more striking. We will see below that the analogy can be made even more explicit in the Curry-Howard isomorphism.

Martin-Löf has designed in the seventies systems of type theory that embodies many of the features of Gentzen's natural deduction. He remarked that the introduction and elimination rules determine the meaning of the connectives 'in use'. E.g. $\wedge I$ tells us that if we know (have evidence for) A and B, then we also know $A \wedge B$. So $\wedge I$ specifies what one has to require for $A \wedge B$. The elimination rule tells us what we may claim on the grounds of evidence for $A \wedge B$. If we know $A \wedge B$ then we also know A and likewise B.

The introduction and elimination rules have been chosen so that they are 'in harmony'. The evidence required in $\wedge I$ is exactly the evidence one can derive in $\wedge E$.

Michael Dummett used the same feature of Gentzen's natural deduction to support his claim that intuitionistic logic fits the requirement of 'meaning is use', that is to say, the meaning of the logical operations can be concluded from their use in introduction and elimination rules. Dummett insists on demonstrative knowledge of mathematical notions, 'the grasp of the meaning of a mathematical statement must, in general, consist of a capacity to use that statement in a certain way, or to respond in a certain way to its use by others'([Dummett 1975]). As a consequence the traditonal, Platonistic, notion of truth has to be replaced by something more palpable; the notion of proof is exactly what will fill the need for communicability and observability. Hence the slogan "a grasp of the meaning of a statement consists in a capacity to recognize a proof of it when one is presented to us".

This, of course, is in complete accord with intuitionistic practice. The rejection of the Platonistic notion of truth is indeed an aspect of Dummett's anti-realism. Brouwer had always denied the realistic thesis, that there is an outer world independent of us.

The second derivation system of Gentzen is his *sequent calculus*. It differs from the natural deduction system in various ways. The most striking is the extra memory the derivation rules carry along. In natural deduction the rules are (more or less) *local*, the rules handle the formulas involved in the inference, but the assumptions (hypotheses) largely remain unnoticed (of course, in rules with discharge, some hypotheses that play a role in the derivation step play a role). Furthermore the introduction-elimination feature is replaced by left-right technique.

We give an example:

$$\frac{\Gamma, A \vdash_i \triangle}{\Gamma, A \wedge B \vdash_i \triangle}$$

$$\frac{\Gamma_1 \vdash_i \triangle_1, A \qquad \Gamma_2 \vdash_i \triangle_2, B}{\Gamma_1 \cup \Gamma_2 \vdash_i \triangle_1 \cup \triangle_2\ A \wedge B}$$

$\Gamma \vdash_i \triangle$ should be thought of as 'the conjunction of all propositions in Γ yields the disjunction of the proposition in $\triangle$' (where Γ and $\triangle$ are finite sets of propositions). The 'right'-rule tells us what we have to know in order to conclude $A \wedge B$, the 'left-rule tells us that in order to draw a conclusion from $A \wedge B$, it suffices to draw the conclusion from A (or B, for that matter).

For a treatment of the sequent calculus we refer the reader to [Kleene 1952], [Girard 1987], [Girard 1989], [Troelstra–van Dalen 1988].

Suggested further reading: [**?**], [Dalen 1997] Ch. 5, [Dragalin 1988] Part 1, [Dummett 1977] Ch. 4, [Kleene 1952] Part II, [Negri–von Plato], [Troelstra-Schwichtenberg 1996] Ch. 1,2,3, [Troelstra–van Dalen 1988] Ch. 2.

Formal logic has two faces, the syntactic, combinatorial part, and the denotational part. The syntactic part may be viewed as a pure game with symbols, according to given rules. It goes without saying that this position is usually softened by an informal meaning concept. After all, one has to know why to play this game and not that. Proof theory, traditionally, is placed in the combinatorial part of logic.

The study of interpretations is called semantics, from this viewpoint formulas denote certain things. Frege already pointed out that propositions denote truth values, namely 'true' and 'false', conveniently denoted by 1 and 0.

The interpretations of logic under this two-valued semantics is handled by the well-known truth tables (cf. [Dalen 1997]).

The method has a serious drawback: all propositions are supposed to be true or false, and PEM automatically holds (some might perhaps see this as an advantage rather than a drawback). It certainly is too much of a good thing for intuitionistic logic. By our choice of axioms (or rules) intuitionistic logic is a subsystem of classical logic, so the two-valued semantics obliterates the distinction between the two logics: too many propositions become true!

One might say that we should not worry about the problem of semantics, since we already have the intended proof interpretation. This certainly is the case, however, the proof-interpretation is too little specific to yield sharp decisive results, as one would like in model theory. One would need more assumptions about 'constructions', before technical problems can be settled.

All of the formal semantics discussed below are strongly complete for intuitionistic predicate and propositional logic, in the sense that $\Gamma \vdash_i A \Leftrightarrow \Gamma \models A$, where $\models$ is the semantical consequence relation in the particular semantics.

1.1 The topological interpretation

In the mid-thirties a number of systematic semantics were introduced that promised to do for intuitionistic logic what the ordinary truth tables did for classical logic.

Heyting had already introduced many valued truth tables in his formalization papers, e.g. to establish non-definability of the connectives. Jaskowski presented in 1936 a truth table family that characterized intuitionistic propositional logic, [Jaskowski 1936]. Gödel had dispelled the expectation that intuitionistic logic was the logic of some specific finite truthvalue system, [Gödel 1932].

An elegant interpretation was introduced by Tarski, [Tarski 1938]. It was, in fact, a generalization of the Boolean valued interpretation of classical logic. Ever since Boole it was known that the laws of logic correspond exactly to those of Boolean algebra (think of the powerset of a given set with $\cap, \cup$

and ^{c} as operations, corresponding to $\wedge, \vee$ and $\neg$). Now we want a similar algebra with the property that $U^{cc} \neq U$(for $\neg\neg A$ and A are not equivalent). By Brouwer's theorem ($\neg\neg\neg A \leftrightarrow \neg A$), we expect $U^{ccc} = U^c$. The remaining laws of logic demand that $(U \cap V)^{cc} = U^{cc} \cap V^{cc}$ and $(U \cup V)^{cc} \subseteq U^{cc} \cup V^{cc}$. This suggests that the operator ^{cc} behaves as a closure operator in topology. It turns out that it is a good choice to let the open sets in a topological space X play the role of arbitrary sets in the power set of X. So the family $\mathcal{O}(X)$ plays the role of $\mathcal{P}(X)$.
Here is the notation: $[\![A]\!]$ is the open set of X assigned to A. The valuation $[\![\cdot]\!] : PROP \to \mathcal{O}(X)$ is defined inductively for all propositions; let $[\![A]\!]$ be given for all atomic A, where $[\![\bot]\!] = \emptyset$, then

$$\begin{array}{rcl} [\![A \wedge B]\!] & = & [\![A]\!] \cap [\![B]\!] \\ [\![A \vee B]\!] & = & [\![A]\!] \cup [\![B]\!] \\ [\![A \to B]\!] & = & Int([\![A]\!]^c \cup [\![B]\!]) \\ [\![\neg A]\!] & = & Int([\![A]\!]^c) \end{array}$$

Here Int(K) is the interior of the set K (i.e. the largest open subset of K). Note that this looks very much like the traditional Venn diagrams, with the extra requirement that negation is interpreted by the interior of the complement. This is necessary if we want to get open sets all the way. A simple calculation shows that $\underset{i}{\vdash} A \Rightarrow [\![A]\!] = X$ (we use $\vdash_i$, resp. $\vdash_c$, for derivability in intuitionistic, resp. classical, logic). Since X is the largest open set in $\mathcal{O}(X)$, it is plausible to call A true in $\mathcal{O}(X)$ if $[\![A]\!] = X$.

This interpretation can be used to show underivability of propositions. Consider, for example, $\mathcal{O}(\mathbb{R})$, the open sets of real numbers. Assign to the atom A the set $\mathbb{R} - \{0\}$. Then $[\![A]\!] = \mathbb{R} - \{0\}, [\![\neg A]\!] = Int(\{0\}) = \emptyset, [\![\neg\neg A]\!] = \mathbb{R}$. So $[\![\neg\neg A \to A]\!] = Int[\![\neg\neg A]\!]^c \cup A = \emptyset \cup A = A \neq \mathbb{R}$.
Hence $\underset{i}{\nvdash} \neg\neg A \to A$. Similarly $\underset{i}{\nvdash} A \vee \neg A$, for $[\![A \vee \neg A]\!] = A \neq \mathbb{R}$.

The topological interpretation can be extended to predicate logic. Let a domain D be given, then

$$[\![\exists x A(x)]\!] = \bigcup\{[\![A(d)]\!] | d \in D\}$$
$$[\![\forall x A(x)]\!] = Int(\bigcap\{[\![A(d)]\!] | d \in D\})$$

The topological interpretation is indeed *complete* for intuitionistic logic. Let us say that A is true in $\mathcal{O}(X)$ if $[\![A]\!] = X$ for all assignments of open sets to atoms. *A is true* if A is true under all topological interpretations.

Completeness can now be formulated as usual: $\vdash_i A \leftrightarrow A$ is true. The theory of topological interpretations is treated in [Rasiowa-Sikorski 1963]. Classical logic appears as a special case when we provide a set X with the trivial topology: $\mathcal{O}(X) = \mathcal{P}(X)$.

The algebra of open subsets of a topological space is a special case of a *Heyting algebra*. Heyting algebra's are defined, much like Boolean algebra's, by axioms for the various operations.

A Heyting algebra has binary operations $\wedge, \vee, \rightarrow$, a unary operation $\neg$ and two constants, $0, 1$.

The laws are listed below:

$$
\begin{array}{rclrcl}
a \wedge b & = & b \wedge a & 0 \wedge a & = & 0 \\
a \vee b & = & b \vee a & 0 \vee a & = & a \\
a \wedge (b \wedge c) & = & (a \wedge b) \wedge c & a \rightarrow a & = & 1 \\
a \vee (b \vee c) & = & (a \vee b) \vee c & a \wedge (a \rightarrow b) & = & a \wedge b \\
a \wedge (b \vee c) & = & (a \wedge b) \vee (a \wedge c) & b \wedge (a \rightarrow b) & = & b \\
a \vee (b \wedge c) & = & (a \vee b) \wedge (a \vee c) & a \rightarrow (b \wedge c) & = & (a \rightarrow b) \wedge (a \rightarrow c) \\
1 \wedge a & = & a & \neg a & = & a \rightarrow 0 \\
1 \vee a & = & 1 & & &
\end{array}
$$

Since it is necessary for the interpretation of predicate logic to allow infima and suprema of collections of elements, we often consider *complete* Heyting algebra's. That is, algebra's with the property that for a collection $\{a_i | i \in I\}$ of elements there is a supprenum $\bigvee\limits_{i \in I} a_i$ (the unique least element majorizing all a_i), and an infimum $\bigwedge\limits_{i \in I} a_i$

For a complete Heyting algebra (i.e. an algebra in which all sups and infs exist), the laws can be simplified somewhat, we adopt the standard axioms for a latice and add

$$a \wedge \bigvee S = \bigvee \{a \wedge s | s \in S\}$$

1.2 Beth–Kripke semantics

Elegant as the topological interpretation may be, it is not as flexibled as two later interpretations introduced by E.W. Beth and Saul Kripke. Both semantics have excellent heuristics. We will consider here a hybrid semantics, Beth–Kripke models, introduced in [Dalen 1984].

The basic idea is to mimick the mental activity of Brouwer's individual, who creates all of mathematics by himself. This idealized mathematician, also called *creating subject* by Brouwer, is involved in the construction of mathematical objects, and in the construction of proofs of statements. This process takes place in time. So at each moment he may create new elements, and at the same time he observes the basic facts that hold for his universe so far. In passing from one moment in time to the next, he is free how to continue his activity, so the picture of his possible activity looks like a partially ordered set (even like a tree). At each moment there is a number of possible next stages. These stages have become known as *possible worlds.*

Let us just consider, for the moment, the first-order case, that is to say we consider elements of one and the same sort, furthermore there is a finite number of relations and functions (as in a standard first-order language). The stages for the individual form a partially ordered set $\langle K, \leq \rangle$. We view $k \leq \ell$ as "k is before ℓ or coincides with ℓ. For each $k \in K$ there is a local domain of elements created so far, denoted by $D(k)$. It is reasonable to assume that no elements are destroyed later, so $k \leq \ell \Rightarrow D(k) \subseteq D(\ell)$.

A path in the poset K is a maximal ordered subset. For a node k in the poset K a bar $\mathbf{B}$ is a subset with the property that very path through k intersects $\mathbf{B}$. Now we will stipulate how the individual arrives at the atomic facts. He does not necessarily establish an atomic fact A 'at the spot', but he will, no matter how he pursues his research, establish A eventually. This means that there is a bar $\mathbf{B}$ for k such that for all nodes ℓ in $\mathbf{B}$ the statement A holds at ℓ. Note that the individual does not (have to) observe composite states of affairs.

The next step is to interprete the connectives. Let us write "$k \Vdash A$" for "A holds at k". The technical terminology is "k forces A". For atomic A $k \Vdash A$ is already given, $\perp$ is never forced.

$k \Vdash A \wedge B$	$k \Vdash A$ and $k \Vdash B$
$k \Vdash A \vee B$	$\exists \mathbf{B} \forall \ell \in \mathbf{B} \ell \Vdash A$ or $\ell \Vdash B$
$k \Vdash A \rightarrow B$	$\forall \ell \geq k(\ell \Vdash A \Rightarrow \ell \Vdash B)$
$k \Vdash \neg A$	$k \Vdash A \rightarrow \perp$ $\forall \ell \geq k(\ell \Vdash A \Rightarrow \ell \Vdash \perp$ $\forall \ell \geq k(\ell \nVdash A)$
$k \Vdash \exists x A(x)$	$\exists \mathbf{B} \forall \ell \in \mathbf{B} \exists a \in D(\ell)\ \ell \Vdash A(a)$
$k \Vdash \forall x A(x)$	$\forall \ell \geq k \forall a \in D(\ell)\ \ell \Vdash A(a)$

Observe that the 'truth' at a node k essentially depends on the future. This is an important feature in intuitionism (and in constructive mathematics, in general). The dynamic character of the universe demands that the future is taken into account. This is particularly clear for $\forall$. If we claim that "all dogs are friendly", then *one* unfriendly dog in the future may destroy the claim.

A Beth–Kripke model with the property that the bar $\mathbf{B}$ in all the defining clauses is precisely the node k itself, is called a *Kripke model.* And if all the local domains $D(k)$ are identical, we have a *Beth model.*

An individual model over a poset K is denoted by $\mathcal{K}$.

We say that "A is true in a Beth–Kripke model $\mathcal{K}$" if for all $k \in K$ $k \Vdash A$. "A is true" if A is true in all Beth–Kripke models. 'Semantical consequence' is defined as $\Gamma \Vdash A$ iff for all Beth–Kripke models $\mathcal{K}$ and all $k \in K$ $k \Vdash C$ for all $C \in \Gamma \to k \Vdash A$

Beth–Kripke- , Kripke- and Beth- semantics are strongly complete for intuitionistic logic: $\Gamma \vdash_i A \Leftrightarrow \Gamma \Vdash A$ (special case: $\underset{i}{\vdash} A \Leftrightarrow A$ is true).

There are a number of simple propertiwes that can easily be shown, e.g.: forcing is monotone, i.e. $k \Vdash A, k \leq \ell \Rightarrow \ell \Vdash A$,
in Beth–Kripke and Beth models $k \Vdash A \Leftrightarrow \exists \mathbf{B} \forall \ell \in \mathbf{B} \ell \Vdash A$.

In practice, Kripke models are the more flexible tools for metamathematical purposes. A large number of applications have been given since their introduction. We will discuss them first.

Since Kripke models are given by partially ordered sets, we can indicate them by means of diagrams. By labelling the set with propositions, it is easy to see what the logical properties are. We give some examples.

Note that $k \Vdash \neg A$ if $\forall \ell \geq k(\ell \not\Vdash A)$, i.e. A is not forced after k. $k \not\Vdash \neg A$ iff for some $\ell \geq k$ $\ell \Vdash A$

So $k \Vdash \neg\neg A$ if for each $\ell \geq k$ there is an $m \geq \ell$ with $m \Vdash A$.

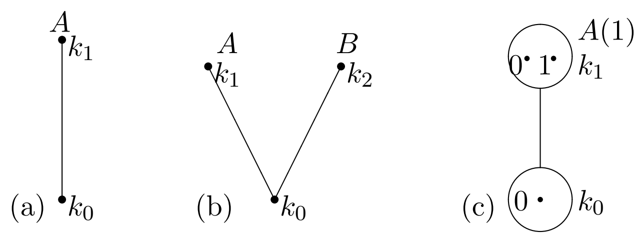
(a) Consider an atomic A and let $k_1 > k_0, k_1 \Vdash A$, but $k_0 \not\Vdash A$. By the above remark, we see that $k_0 \Vdash \neg\neg A$. Hence $k_0 \not\Vdash \neg\neg A \to A$.
Furthermore $k_0 \not\Vdash \neg A$, so $k_0 \not\Vdash A \vee \neg A$.

(b) $k_1 > k_0, k_2 > k_0, k_1 \Vdash A$ and $k_2 \Vdash B$. We see that $k_0 \not\Vdash \neg A, k_0 \not\Vdash \neg B$, so $k_0 \not\Vdash \neg A \vee \neg B$. $k_1 \not\Vdash A \wedge B, k_2 \not\Vdash A \wedge B$, so $k_0 \Vdash \neg(A \wedge B)$.

Hence $k_0 \not\Vdash \neg(A \wedge B) \to (\neg A \vee \neg B)$ (De Morgan's law fails).

(c) $k_1 > k_0, k_0 \not\Vdash A(0), k_1 \not\Vdash A(0), k_1 \Vdash A(1)$. Clearly $k_1 \not\Vdash \forall x A(x)$, so $k_0 \Vdash \neg\forall x A(x)$. If $k_0 \Vdash \exists x \neg A(x)$ then $k_0 \Vdash \neg A(1)$. But this contradicts $k_1 \Vdash A(1)$. Hence $k_0 \not\Vdash \exists x \neg A(x)$. So $k_0 \not\Vdash \neg\forall x A(x) \to \exists x \neg A(x)$.

There is an extensive modeltheory for Kripke semantics. In particular there are a large number of results on the structure of the partially ordered sets. For example, ipredicate logic is complete for Kripke models over trees,



and for propositional logic this can even be strengthened to finite trees (the so-called *finite model property*: $\not\vdash A \Rightarrow A$ is false in a Kripke model over a finite tree).

The completeness over tree models can be used to prove the *disjunction property*

$$(DP) \ \ \vdash_i A \vee B \Rightarrow \vdash_i A \text{ or } \vdash_i B$$

The proof uses reductio ad absurdum. Suppose $\not\vdash_i A$ and $\not\vdash_i B$, then there is a tree model $\mathcal{K}_1$ where A is not forced, similarly a tree model $\mathcal{K}_2$ where B is not forced. The two models are glued together as follows: put the two models side by side and place a new node k below both. In this new node no proposition is forced. The result is a correct Kripke model. Since $\vdash_i A \vee B$, we have $k \Vdash A \vee B$, and hence $k \Vdash A$ or $k \Vdash B$. But that contradicts the fact that A and B are not forced in $\mathcal{K}_1$ and $\mathcal{K}_2$. Therefore $\vdash_i A$ or $\vdash_i B$.

There is a corresponding theorem for existential sentences, the *existence property*

$$(EP) \ \ \vdash_i \exists x A(x) \Rightarrow \vdash_i A(t) \text{ for a closed term } t$$

These theorems lend a pleasing support to the intuitionistic intended meaning of 'existence': if you have established a disjunction, you know that you can established one of the disjuncts. Similarly for existence: if you have shown the existence of something, you can indeed point to a specific instance.

The disjunction and existence property hold for a number of prominent theories, the most important being arithmetic. DP and EP are often considered the hallmark of constructive logic; one should, however, not overestimate the significance of a technical result like this.

There is a snag in the conclusions drawn above. As the reader has seen, the proof made use of reductio ad absurdum, i.e. the result has not been established constructively. What one would like is a method that extracts from a proof of $\exists x A(x)$ a proof of $A(d)$ for some d. Fortunately there are proof theoretical devices that provide exactly this kind of information, see

28. Smorynski has shown that in a fairly large number of cases 'semantic' proofs can be made constructive, [Smorynski 1982].

For more information on Kripke model theory see [Dalen 1997], [Fitting 1969], [Troelstra–van Dalen 1988].

Finite Beth models are not interesting for intuitionistic logic. For in each leaf (maximal node) A or $\neg A$ holds, so in each node $A \vee \neg A$ holds, i.e. the logic is classical. Beth's original models are slightly more special, he considered *constant domains*, i.e. $D_k = D_\ell$ for all k, ℓ. This is a certain drawback when compared to Kripke models. The combinatorics of Beth models is just more complicated than that of Kripke models.

Kripke showed in 1963 that Kripke and Beth models can be converted into each other [Kripke 1965], see also [Schütte 1968].

Beth models are easily convertible into topological models. For convenience we consider Beth models over tree. On a tree one can define a topology as follows: open sets are those subsets U of the tree that are closed under successors, i.e. $k \in U$ and $k \leq \ell \Rightarrow \ell \in U$. It is easily seen that these sets form a topology. One also sees that $\{k | k \Vdash A\}$ is an open set, let us therefore put $[\![A]\!] = \{k | k \Vdash A\}$. Now it is a matter of routine to show that the function $[\![.]\!]$ is a valuation as defined in section 2.1.

Beth semantics also turn out to be a convenient tool in completeness proofs, see [Troelstra–van Dalen 1988], Ch. 13, [Dummett 1977]. In particular, it is useful for rendering completeness proofs in an intuitionistic metamathematics, Veldman was in 1976 the first to consider a modified Kripke semantics, for which he could give an intuitionistically correct completeness proof, [Veldman 1976]. Since then De Swart, Friedman, Dummett, Troelstra have given alternative versions for Beth semantics.

Beth models happen to be better adapted to second-order arithmentic with function variables, the so-called *intuitionistic analysis*. They allow for a very natural interpretations of choice sequences. The use of Beth models for second-order systems can be found in [Dalen 1978], [Dalen 1984].

1.3 Super Semantics.

We have seen a number of interpretations of intuitionistic theories, but by no means all of them. There is, for example, a totally different interpretation of intuitionistic logic (and arithmetic) in Kleene's realizability interpretation. This is based on algorithms, and on the face of it, one could not find a similarity with the above semantics. One might wonder if these semantics are totally unrelated, or whether there is some common ground. The obvious common ground is the logic that they are modelling, but that would not be sufficient to link, say Kripke models and realizability.

Fortunately there is a general kind of semantics based on category theory. The pioneer in this area was Lawvere, who saw the possibilities for treating logical notions in a categorical setting. He, in particular, discovered the significance of adjointness for logic.

Historically these newer interpretations grew out of existing semantics, e.g. Dana Scott showed in his extension of the topological interpretation to second-order systems, how one could capture strikingly intuitionistic features, [Scott 1968, Scott 1970]. Subsequently Joan Moschovakis and Van Dalen used the same technique for interpreting the theory of choice sequences over natural numbers, and second order arithmetic, **HAS**, [Moschovakis 1973], [Dalen 1974].

The proper formulation of this kind of semantics is given in the so-called *sheaf-semantics*.

In order to avoid complications we will look at Scott's original model, adapted to the sheaf semantics.

Consider the topology of the real line, R. The open sets will be our truth values. We will interprete the intuitionistic theory of reals. The objects are partial continuous real-valued functions with an open domain. This domain 'measures' the existence of the object: $Ea \in \mathcal{O}(\mathbb{R})$.

$$DRAWING$$

The interpretation forces us to consider the notion of partial object and the notions of 'equivalence' and 'strict equality'. All objects are partial, and we say that a is total if $Ea = \mathbb{R}$. Equality has to be reconsidered in this light: $[\![a = b]\!] = Int\{t \in \mathbb{R} | a(t) = b(t)\}$

$$DRAWING$$

Observe that $[\![a = a]\!] = Ea$, and in general $[\![a = b]\!] \subseteq E(a) \cap E(b)$.

In addition to the notion of equality there is a convenient notion of *equivalence*: "a and b coincide where they exist", in symbols $a \simeq b := Ea \vee Eb \to a = b$. According to the above definition we have $[\![a \simeq a]\!] = \mathbb{R}$.

The operation on the partial elements are defined pointwise:

$$\begin{array}{rcl} (a+b)(t) & = & a(t) + b(t). \\ (a \cdot b)(t) & = & a(t) \cdot b(t) \end{array}$$

The definition of the inverse is, however, problematic. One can define it pointwise for values distinct from 0, but then there seem to be problems with the invertibility of non-zero elements.

The fact is that 'non-zero' is not good enough to have an inverse. Let $a(t) = t$, then $[\![a = 0]\!] = Int\{t|a(t) = t = 0\} = Int\{0\} = \emptyset$.

So $[\![a \neq 0]\!] = [\![\neg a = 0]\!] = Int[\![a = 0]\!]^c = \mathbb{R}$

This a is non-zero in the model. There is however no b such that $[\![a.b = 1]\!] = \mathbb{R}$. This is where apartness comes in. It is well-known that a real number has an inverse iff it is apart from 0. So let us interprete # in the model:

$$[\![a\#b]\!] = \{t|a(t) \neq b(t)\}.$$

The condition for an inverse now becomes

$$a\#0 \rightarrow \exists b(a \cdot b = 1).$$

In the model we now get an inverse for a. Take $b(t) = t^{-1}$ for $t \neq 0$, then $[\![a\#0]\!] = \mathbb{R} - \{0\} = \bigcup c[\![a \cdot c = 1]\!] \subseteq [\![a \cdot b = 1]\!] = \mathbb{R} - \{0\}$.

The introduction of an existence predicate, and partial equality, of course, carries obligations for our quantifiers. In particular

$$\exists x A(x) \leftrightarrow \exists x(Ex \wedge A(x)) \text{ and}$$
$$\forall x A(x) \leftrightarrow \forall x(Ex \rightarrow A(x)).$$

The logical axioms, or rules, those have to be revised. It suffices to consider the quantifier rules and the equality rules:

$$\dfrac{\begin{array}{c}[Ex]\\ \mathcal{D}\\ A(x)\end{array}}{\forall x A(x)} \qquad \dfrac{\forall x A(x) \quad Et}{A(t)}$$

$$\dfrac{A(t) Et}{\exists x A(x)} \qquad \dfrac{\exists x A(x) \qquad \begin{array}{c}[A(x)][Ex]\\ \mathcal{D}\\ C\end{array}}{C}$$

The axiomatization of the equality aspects is simple if we use the equivalence relation instead of identity itself. We add the axioms:

$$x \simeq y \wedge A(x) \rightarrow A(y)$$
$$\forall z(x \simeq z \leftrightarrow y \simeq z) \rightarrow x \simeq y.$$

That is, we take '$\simeq$' as a primitive, '$=$' is then regained by defining $t = s := Et \wedge Es \wedge t \simeq s$.

For the details of the logic can be found in [Troelstra–van Dalen 1988] and Scott's original [Scott 1979].

One may add also the ordering to the model of $\mathbb{R}$:

$$[\![a < b]\!] := \{t | a(t) < b(t)\}.$$

Scott extended his model to a higher-order theory of reals, in which he could show that Brouwer's continuity theorem holds, see [Scott 1970].

Joan Moschovakis applied the methods of Scott's semantics to intuitionistic analysis in the style of Kleene's **FIM**, [Moschovakis 1973]. Her objects were continuous mappings from Baire space to Baire space (a model with total elements). Van Dalen's model for analysis is based on Beth's original semantics. It can be translated straightforwardly into a topological model, but its formulations in a Beth model allows for some extra fine tuning so that techniques from set theory can be applied. E.g. a model for lawless sequences is constructed by means of forcing. The model also interprets the theory of the creating subject, [Dalen 1978]. Grayson used the same approach in his version of the permutation model of Krol, [Grayson 1981].

Sheaf models were further presented by Scott, Fourman, Hyland and Grayson, [Fourman-Hyland 1979], [Fourman-Scott 1979], [Grayson 1979, Grayson 1981, Grayson 1982, Grayson 1984].

Sheaf models allow an interpretation of higher-order logic, they are natural examples of a *topos*. Moerdijk and Van der Hoeven used a special sheaf model to interprete the theory of lawless sequences, [Hoeven-Moerdijk 1984]. Since Kripke models and Beth models are built over trees, they carry a natural topology; as a result they can be viewed as sheaf models.

The next generalisation after sheaf models is that of categorical model. Certain categories are powerful enough to interprete higher-order intuitionistic logic. In that sense topos models can be viewed as intuitionistic universes. Categorical models also turned out to be important for typed lambda calculus. The reader is referred to the literature for details

It should be mentioned that the topos semantics managed to capture most of the known semantics. E.g. Hyland showed that the realizability interpretation fitted into his effective topos, [Hyland 1982], [Troelstra–van Dalen 1988], Ch. 13 §8.

Suggested further reading: [Dragalin 1988] Part3, [Dummett 1977] Ch. 5, [Fitting 1969], [Fourman-Scott 1979], [Friedman 1977], [Smorynski 1973a], [Gabbay 1982], [Lambek–Scott 1986], [MacLane–Moerdijk 1992], [McLarty 1995], [Rasiowa-Sikorski 1963], [Troelstra–van Dalen 1988] Ch. 2, §5, 6, Ch. 13,13,15.
There is no special foundational bias towards first-order theories, but it is an observed fact that large parts of mathematics lend themselves to formu-

lation in a first-order language. We will look in this section at a number of first-order theories and point out some salient properties.

Predicate logic itself is, just like its classical counterpart, undecidable. Fragments are, however, decidable. E.g. the class of prenex formulae is decidable (and as a corollary, not every formula has a prenex normal form in **IQC**), cf. [Kreisel 1958], on the other hand, monadic predicate logic is decidable in classical, but undecidable in intuitionistic logic, [Kripke 1968], [Maslov 1965] Lifschitz showed that the theory of equality is undecidable, [Lifschitz 1967]. An extensive treatment of first-order theories is given in papers of Gabbay and the dissertation of Smorynski, see [Gabbay 1982], [Smorynski 1973b], [Smorynski 1973a].

In propositional logic one may impose restrictions on the underlying trees of Kripke models, in predicate logic one may also put restrictions on the domains of the models. A well-known instance is the *constant domain theory* of S. Görneman (Koppelberg). She showed that predicate logic plus the axiom $\forall x(A(x) \vee B) \to \forall x A(x) \vee B$ is complete for Kripke models with constante domains, [Goernemann 1971].

In general Kripke semantics is a powerful tool for meta-mathematical purposes. Many examples can be found in the cited works of Smorynski and Gabbay. We have demonstrated the usefulness in section 3. in the case of the disjunction property. The key method in the proof was the joining of a number of disjoint Kripke models by placing one extra node (the root of the tree) below the models. This technique has become known as gluing. Following Smorynski we will apply it to a few theories.

First we point out that, although a number of intuitionistic first-order theories share their axioms with their classical counterparts, in general theories are sensitive to the formulation of the axioms, notably in the absence of PEM. We consider a few basic theories below.

The theory of equality has the usual axioms – reflexivity, symmetry, transitivity.

One can strengthen the theory in many ways, for example, the theory of stable equality is given by

$$\mathbf{EQ}^{st} = \mathbf{EQ} + \forall xy(\neg\neg x = y \to x = y)$$

And the decidable theory of equality is axiomatized by

$\mathbf{EQ}^{dec} = \mathbf{EQ} + \forall xy(x = y \vee x \neq y)$, where $x \neq y$ abbreviates $\neg x = y$.

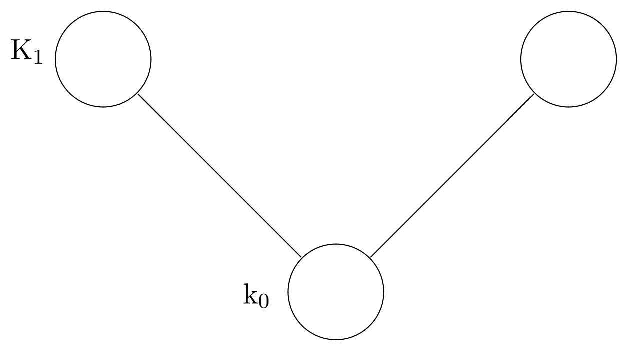
In intuitionistic mathematics there is a strong notion of inequality: *apartness*. Introduced by Brouwer in [Brouwer 1919a] and axiomatized by Heyting in [Heyting 1925]. The symbol for apartness is $\#$.

The axioms of **AP** are given by **EQ** and the following list

$$\begin{cases} \forall xyx'y'(x\#y \wedge x = x' \wedge y = y' \to x'\#y') \\ \forall xy(x\#y \to y\#x) \\ \forall xy(\neg x\#y \leftrightarrow x = y) \\ \forall xyz(x\#y \to x\#z \vee y\#z) \end{cases}$$

We will now use the gluing technique to show that **AP** has the disjunction and existence property.

Let $\mathbf{AP} \vdash A \vee B$ and assume $\mathbf{AP} \nvdash A$ and $\mathbf{AP} \nvdash B$. Then, by the strong completeness theorem, there are models $\mathcal{K}_1$ and $\mathcal{K}_2$ of **AP** such that $\mathcal{K}_1 \not\Vdash A$ and $\mathcal{K}_2 \not\Vdash B$



We consider the disjoint union of $\mathcal{K}_1$ and $\mathcal{K}_2$ and place the one-point world below it. That is to say, we designate points a and b in $\mathcal{K}_1$ and $\mathcal{K}_2$ which are identified with the point 0. The new model obviously satisfies the axiom of **AP**. Hence $k_0 \Vdash A \vee B$ and so $k_0 \Vdash A$ or $k_0 \Vdash B$. Both are impossible on the grounds of the choice of $\mathcal{K}_1$ and $\mathcal{K}_2$. Contradition. Hence $\mathbf{AP} \vdash A$ or $\mathbf{AP} \vdash B$.

For EP it is convenient to assume that the theory has a number of constants, say $\{c_i | i \in I\}$.

Now let $\mathbf{AP} \vdash \exists x A(x)$ and $\mathbf{AP} \nvdash A(c_i)$ for all i. Then for each i there is a model $\mathcal{K}_i$ with $\mathcal{K}_i \not\Vdash A(c_i)$.

As we did above, we glue the models $\mathcal{K}_i$ by means of a bottom world $\mathcal{K}^*$ with a domain consisting of just the elements c_i. No non-trivial atoms are forced in k_0 (i.e. only the trivial identities $c_i = c_i$). The identification of the c_j with elements in the models $\mathcal{K}_i$ is obvious. Again, it is easy to check that the new model satisfies **AP**. Hence $k_0 \Vdash \exists x A(x)$, i.e. $k_0 \Vdash A(c_i)$ for some i. But then also $\mathcal{K}_i \Vdash A(c_i)$, contradiction. Hence $\mathbf{AP} \vdash A(c_i)$ for some i.

The gluing operation demonstrates that there are interesting operations in Kripke model theory that make no sense in traditional model theory.

The apartness axioms have consequences for the equality relations. In particular we get a *stable equality*: $\neg\neg x = y \to x = y$. For, $\neg x \# y \leftrightarrow x = y$, so $x = y \leftrightarrow \neg\neg\neg x \# y \leftrightarrow \neg\neg x = y$.

In fact the equality fragment is axiomatized by an infinite set of quasi-stability axioms.

$$
\begin{array}{lrcl}
\text{Put} & x \neq_0 y & := & \neg x = y \\
 & x \neq_{n+1} y & := & \forall z(z \neq_n x \vee z \neq_x y)
\end{array}
$$

For these "approximations to apartness" we formulate quasi-stability axioms: $Sn := \forall xy(\neg x \neq_n y \to x = y)$. The S_n axiomatize the equality fragment of **AP**. To be precise: **AP** is conservative over $\mathbf{EQ} + \{S_n | n \geq 0\}$.

The result was first established by proof theoretic means in [Dalen 1978], and subsequently an elegant model theoretic proof was given in [Smorynski 1973c]. This shows that even a relatively simple theory like equality is incomparably richer than the classical theory.

Apartness and linear order are closely connected. The theory **LO** of linear order has axioms:

$$
\begin{array}{l}
\forall xyz(x < y \wedge y < z \to x < z) \\
\forall xyz(x < y \to z < y \vee x < z) \\
\forall xyz(x = y \leftrightarrow \neg x < y \wedge \neg y < z)
\end{array}
$$

The second axiom is interesting because it tells, so to speak, that $a < b$ means that a is "far" to the left of b, in the sense that if we pick an arbitrary third point, it has to be to right of a or to the left of b. The relation with apartness is given by $\quad \mathbf{LO} + \mathbf{AP} \vdash x < y \vee y < x \leftrightarrow x \# y$

One can also use $x < y \vee y < x$ to define apartness. In a way this gives the best possible result since **LO**+**AP** is conservative over **LO** ([Smorynski 1973c], [Dalen 1997]).

It is important to note that the atoms, or in general the quantifier free part of a theory does not yet determine whether it is classical or not. There are cases where a decidable equality results in the fact that the theory is classical, e.g. the theory of algebraically closed fields; on the other hand, for arithmetic we can prove $\forall xy(x = y \vee x \neq y)$, but the theory is not classical. There are quite a number of surprising results on first-order theories, cf. [Gabbay 1972], [Gabbay 1973], [Gabbay 1982], [Smorynski 1973a], [Smorynski 1978].

The best known first-order theory is, of course, arithmetic. Intuitionistic arithmetic, **HA**, called after Heyting, [Heyting 1930b, Heyting 1930a], is axiomatized by exactly the same axioms as Peano's arithmetic. The difference is the underlying logic: $\mathbf{PA} = \mathbf{HA} + PEM$.

The gluing technique also works for intuitionistic arithmetic, **HA**. In this case one has to do some extra work to check the induction axiom. The result is that for **HA** we have the existence property for numerals:

$\mathbf{HA} \vdash \exists x A(x) \Rightarrow \mathbf{HA} \vdash A(n)$ for some n.

The disjunction property is an obvious consequence of the existence property, as $\vee$ is definable in terms of $\exists$:

$\mathbf{HA} \vdash (A \vee B) \leftrightarrow \exists x((x = 0 \rightarrow A) \wedge (x \neq 0 \rightarrow B))$

It would seem rather unlikely that the existence property is a consequence of the disjunction property; Harvey Friedman proved however, to the surprise of the insiders, that in the case of **HA** (and a number of related systems) this is indeed the case, [Friedman 1975].

Intuitionistic arithmetic is, of course, even more incomplete than classical arithmetic because it is a subsystem of **PA**. In fact **PA** is an unbounded extension of **HA** (Smorynski).

There are a number of interesting extensions of intuitionistic arithmetic, one may add principles that have a certain constructive motivation. One of such principles is, Markov's Principle:

MP $\quad\quad \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x A(x) \rightarrow \exists x A(x)$

The principle is a generalization of the original formulation of Markov, [Markov 1971]; Markov considered the halting of a Turing machine, see also [Dragalin 1988]. Such a machine is an abstract computing device that operates on a potentyially infinite tape. The key question for Turing machines is, does the machine, when presented with an input on the tape, eventually halt (and hence produce an output)? Suppose now that someone tells us that it is impossible that the machine never halts, do we know that it indeed halts? Markov argued 'yes'. The decision procedure for the halting in this case consists of turning on the machine and waiting for it to halt. An intuitionist would not buy the argument. When somebody claims that a Turing machine will stop, the natural question is 'when?' We want an actual bound on the computation time. Reading 'the machine halts at time x' for $A(x)$, the above formulation exactly covers Markov's argument. In fact, in Markov's case $A(x)$ is primitive recursive.

A simple Kripke model shows that $\mathbf{HA} \nvdash MP$. Consider a model with two nodes k_0, k_1, where $k_0 < k$. In the bottom node we put the standard model of natural numbers, in the top node we put a (classical) non-standard model $\mathcal{M}$ in which the negation of Gödel's sentence "I am not provable in **PA**", i.e. the proper sentence of the form $\exists x A(x)$, is true.

So $k_1 \Vdash \exists x A(x)$ and hence $k_0 \Vdash \neg\neg\exists x A(x)$. But $k_0 \Vdash \exists x A(x)$ would ask for an instance $A(n)$ to be true in the standard model, and hence would yield a conflict with the independence of the Gödel sentence from **PA**. Since, clearly, $k_0 \Vdash \forall x(A(x) \vee \neg A(x))$, the model refutes MP.

Markov's principle may be unprovable in **HA**, but its companion, *Markov's rule*, has a stronger position.

Given a statement of the form $A \to B$, one may formulate a corresponding rule, $\Gamma \vdash A \Rightarrow \Gamma \vdash B$, which is in general weaker than $\Gamma \vdash A \to B$. For example: Markov's rule MR with respect to **HA** says:

$$\mathbf{HA} \vdash \forall x(A(x) \vee \neg A(x) \wedge \neg\neg\exists x A(x) \Rightarrow \mathbf{HA} \vdash \exists A(x)$$

We say that **HA** is *closed under Markov's rule*. The heuristic argument is that we have more information in this case, we actually have a proof of $\neg\neg\exists x A(x)$, from this extra evidence we may hope to draw a stronger conclusion. Markov's rule is indeed correct, cf. [Troelstra–van Dalen 1988], p.129, p. 507, [Beeson 1976], [Beeson 1985], p. 397. Note that $A(x)$ may contain more free variables. For closed $\exists x A(x)$, the proof of closure under Markov's rule is particularly simple: if $\mathbf{HA} \vdash \neg\neg\exists x A(x)$, then $\mathbf{PA} \vdash \exists x A(x)$, and hence $A(n)$ is true in the standard model for some n. Now $\mathbf{HA} \vdash A(n) \vee \neg A(n)$ and by DP $\mathbf{HA} \vdash A(n)$ or $\mathbf{HA} \vdash \neg A(n)$. The letter is impossible, hence $\mathbf{HA} \vdash A(n)$, and a fortiori $\mathbf{HA} \vdash \exists x A(x)$ ([Smorynski 1973a], p. 366.

Since the theory of natural numbers is at the very heart of mathematics, it is no surprise that a great deal of research has been devoted to the subject. In the early days of intuitionism people wondered to what extent **HA** was safer that **PA**. This was settled in 1933 by Gödel (and independently by Gentzen), who showed that **PA** can be translated into **HA**. Gödel defined a translation, which from the intuitionistic viewpoint weakened statements. This was basically done by a judicious distribution of negations. Here is the formal definition:

$$\begin{aligned}
A^\circ &= \neg\neg A \text{ for atomic } A \\
(A \wedge B)^\circ &= A^\circ \wedge B^\circ \\
(A \vee B)^\circ &= \neg(\neg A^\circ \wedge \neg B^\circ) \\
(A \rightarrow B)^\circ &= A^\circ \rightarrow B^\circ \\
(\forall x A(x))^\circ &= \forall x A^\circ(x) \\
(\exists x A(x))^\circ &= \neg\forall x \neg A^\circ(x)
\end{aligned}$$

For this Gödel translation we get $\quad \mathbf{PA} \underset{c}{\vdash} A \Leftrightarrow \mathbf{HA} \underset{i}{\vdash} A^\circ$.
Hence $\quad \mathrm{PA} \underset{c}{\vdash} 0 = 1 \Leftrightarrow \mathbf{HA} \underset{i}{\vdash} \neg\neg 0 = 1 \Leftrightarrow \mathbf{HA} \underset{i}{\vdash} 0 = 1$

So **PA** is consistent if and only if **HA** is so. In other words, no deep philosophical insight can be expected here.

It is an easy consequence of the Gödel translation theorem that the universal fragment of **PA** is conservative over **HA**.

The Gödel translation, of course, also works for predicate logic:

$$\Gamma \underset{c}{\vdash} A \Leftrightarrow \Gamma^\circ \underset{i}{\vdash} A^\circ,$$

where $\Gamma^\circ = \{B^\circ | B \in \Gamma\}$ (cf. [Dalen 1997], p. 164). Kolmogorov had already in 1925 proposed a similar translation (unknown to Gödel in 1933), he added a double negation in front of every subformula (cf. [Troelstra–van Dalen 1988], p. 59). Kolmogorov had in fact axiomatized a fragment of intuitionistic predicate logic, but the translation and the proof of the corresponding theorem immediately extends to full predicate logic.

The result of the Gödel translation may be improved for certain simple formulas. Kreisel showed that **PA** and **HA** prove the same Π^0_2 sentences, $\mathbf{PA} \vdash_c \forall x \exists y A(x, y) \Leftrightarrow \mathbf{HA} \vdash_i \forall x \exists y A(x, y)$, where $A(x, y)$ is quantifier free.

The proof has some quite interesting features. First we note that a quantifier free formula $A(x, y)$ is equivalent in **HA** to an equation $t(x, y) = 0$ (where we assume that **HA** has defining equations for the primitive recursive functions).

Next we introduce the *Friedman translation*: for a given formula F we obtain A^F from A by replacing all atoms P by $P \vee F$.

It is a routine exercise to show that

$\Gamma \underset{i}{\vdash} A \Rightarrow \Gamma^F \underset{i}{\vdash} A^F$
$A \underset{i}{\vdash} A^F$
$\mathbf{HA} \underset{i}{\vdash} A \Rightarrow \mathbf{HA} \underset{i}{\vdash} A^F$.

Now consider a term t and let $\mathbf{HA} \vdash \neg\neg\exists x(t(x, y) = 0)$. We apply the Friedman translation with respect to $F := \exists x(t(x, y) = 0)$.

$((\exists x(t(x,y)=0) \to \perp) \to \perp))^{\exists x(t(x,y)=0} = ((\exists x(t(x,y)=0) \vee \exists x(t(x,y)=0)) \to (\perp \vee \exists x(t(x,y)=0)) \to (\perp \vee \exists x(t(x,n)=0)))$.
This formula is equivalent to $\exists x(t(x,y)=0)$. Hence $\mathbf{HA} \vdash_i \exists x(t(x,y)=0)$

Observe that we now have closure under Markov's rule with numerical parameters.

It is just one step now to get to Kreisel's theorem. We know from (formalized) recursion theory that a function f is provably recursive in **HA** with index e if $\mathbf{HA} \vdash \forall x \exists y T(e,x,y)$, where T is Kleene's T-predicate, which formalizes the notion y is a (halting) computation on input x (strictly speaking 'y is the code of a computation ...').

Now $\mathbf{PA} \vdash \forall x \exists y T(e,x,y) \Leftrightarrow \mathbf{PA} \vdash \exists y T(e,x,y) \Leftrightarrow$ (Gödel translation) $\mathbf{HA} \vdash \neg\neg\exists y T(e,x,y) \Leftrightarrow$ (closure under Markov's Rule) $\mathbf{HA} \vdash \exists y T(e,x,y)$ $\Leftrightarrow \mathbf{HA} \vdash \forall x \exists y T(e,x,y)$.

Friedman showed that this fact also holds in intuitionistic set theory, **IZF**, ([Friedman 1977])

Finally we mention one more principle that has a metamathematical use in Gödel's Dialectica translation (cf. [Troelstra 1973]): the *independence of premiss principle*, IP.

$$IP \qquad (\neg A \to \exists x B(x)) \to \exists x(\neg A \to B(x))$$

IP is not provable in **HA**. One can see that it strengthens the intended meaning as follows: in the left hand formula, the existence of x is concluded on the basis of a proof of $\neg A$, so this proof may enter into the computation of x. Whereas in the right hand formula, x must be computed straightaway. However, **HA** is closed under the Independence of Premiss Rule:

$\mathbf{HA} \vdash (\neg A \to \exists x B(x)) \Rightarrow \mathbf{HA} \vdash \exists x(\neg A \to B(x))$ (cf. [**?**], p. 296).

1.4 Set Theories

Following the tradition in set theory, the various intuitionistic modifications of **ZF** are also first-order. For most classical theories one can consider one or more corresponding intuitionistic theories. In some cases it suffices to omit PEM from the logical axioms, one has to be careful however, when some non-logical axioms are by themselves strong enough to imply PEM.

Here is an example: the full axiom of choice implies PEM. Let A be a statement, define

$P = \{n \in \mathbb{N} | n = 0 \vee (n = 1 \wedge A)\}$

$Q = \{n \in \mathbb{N} | n = 1 \vee (n = 0 \wedge A)\}$,

$S = \{X, Y\}$.

Obviously $\forall X \in S \exists x \in \mathbb{N}(x \in X)$ AC would yield a choice function F such that $\forall X \in S(F(x) \in X)$. Observe that $F(P), F(Q) \in \mathbb{N}$, so $F(P) = F(Q) \vee F(P) \neq F(Q)$. If $F(P) = F(Q)$, then A holds. If, on the other hand $F(P) \neq F(Q)$, then $\neg A$. So we get $A \vee \neg A$.

The above fact was discovered by Diaconescu ([Diaconescu 1975]), the proof given here is Goodman and Myhill's ([Goodman-Myhill 1978]).

Harvey Friedman studied **IZF**, an intuitionistic version of **ZF**. For a formulation see ([Beeson 1985], Ch. 8, §1). The axioms for sets are modifications of the traditional **ZF** ones. The theory is very sensitive to the formulation, a wrong choice of axiom may introduce unwanted logical principles. E.g. $\in$-induction is used rather than Foundation, and similarly the Axiom of Collection takes the place of the replacement axiom.

Friedman has shown that the Gödel translation theorem works for a suitable formulation of set theory ([Friedman 1973], [Beeson 1985], p. 174).

For more information on the topic the reader is referred to the literature. (see [Beeson 1985]).

Peter Aczel considered another version of constructive set theory, **CZF**. This set theory has the attractive feature that it is interpretable in a particular type theory of Martin-Löf, cf. [Troelstra–van Dalen 1988], p. 624.

Set theory under the assumption of Church's Thesis was extensively studied by David McCarty, [McCarty 1984], [McCarty 1986], [McCarty 1991]. He built a model of cumulative constructive set theory in which a number of interesting phenomena can be observed, e.g. in Kleene's realizability universe sets with apartness are subcountable (i.e. the range of a function on $\mathbb{N}$).

Suggested further reading: [Kleene 1952], [Troelstra–van Dalen 1988], [Dummett 1977], [Gabbay 1982].

Hilbert designed his proof theory for the purpose of consistency proofs, Gentzen on the other hand considered proofs as an object for study. In his system the structure of derivations is the object of investigation.

The fundamental theorem of the natural deduction system is the so-called *normalization theorem*. Gentzen noted that there are obvious superfluous steps in proofs that should be avoided. Here is one: $\dfrac{\dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ A & B\end{matrix}}{A \wedge B}}{A}$.

We first introduced $A \wedge B$ and subsequently eliminated it. The derivation

$\begin{array}{c}\mathcal{D}\\ A\end{array}$ would have accomplished the same.

Introductions, immediately followed by eliminations are called *cuts*. Gentzen showed how to eliminate cuts from natural deduction derivations (in intuitionistic or classical prediate logic). A *cut free derivation* is also called a *normal derivation*, and the process of eliminating cuts is called *normalization.*

The normalization theorem says that each derivation can be converted by cut elimination steps to a normal derivation. The theorem has a strong and a weak form. The weak form states that a derivation can be converted into a normal form. The strong form says that each sequence of conversions will lead to a normal form (and actually one and the same normal form).

There is a structure theorem for normal derivations that tells us that in normal proofs one can run in a systematic way through derivation (starting at the top land going from major assumption to conclusion) so that all the elimination steps precede the introduction steps (with possible $\perp$-steps in between).

From the normal form theorem many corrollaries can be drawn. For example:

(i) *Intuitionistic proposition calculus is decidable*, i.e. there is a decision method for $\vdash_i A$.

(ii) *The disjunction property* for **IQC**. If $\vee$ does not occur positively in any formula in $\Gamma \vdash_i A \vee B$, the $\Gamma \vdash_i A$ or $\Gamma \vdash_i B$.

(iii) *The existence property* for **IQC**. If $\exists$ and $\vee$ do not occur positively in any formula in Γ and $\Gamma \vdash_i \exists x A(x)$, then $\Gamma \vdash_i A(t)$ for some closed t.

(iv) $\vdash_i A$ is decidable for prenex formulas A.

Since full predicate logic is not decidable (Church's theorem for intuitionistic predicate calculus), this result shows that in **IQC** not every formula is equivalent to a prenex formula.

For details about normalization see [Gentzen 1935], [Girard 1987], [Girard 1989], [Dalen 1997], [Troelstra–van Dalen 1988], [Troelstra-Schwichtenberg 1996]. The revival of Gentzen's ideas is due to Prawitz's beautiful work, [Prawitz 1965], [Prawitz 1970], [Prawitz 1977].

Since natural deduction is so close in nature to the proof interpretation, it is perhaps not surprising that a formal correspondence between a term calculus and natural deduction can be established.

We will first demonstrate this for a small fragment, containing only the connective '$\to$'. Consider an $\to$ introduction:

$$\dfrac{\begin{matrix}[A]\\ \mathcal{D}\\ B\end{matrix}}{A \to B} \qquad \dfrac{\begin{matrix}[x:A]\\ \mathcal{D}\\ t:B\end{matrix}}{\lambda x \cdot t : A \to B}$$

We assign in a systematic way λ-term to formulas in the derivation. Since A is an assumption, it has a hypothetical proof term, say x. On discharGing the hypotheses; we introduce a λx in front of the (given) term t for B. By binding x, the proof term for $A \to B$ no longer depends on the hypothetical proof x of A.

The elimination runs as follows:

$$\dfrac{A \to B \qquad A}{B} \qquad\qquad \dfrac{t : A \to B \qquad s : A}{t(s) : B}$$

Observe the analogy to the proof interpretation. Let us consider a particular derivation.

$$\dfrac{\dfrac{[A]}{B \to A}}{A \to (B \to A)} \qquad \dfrac{\dfrac{[x:A]}{\lambda y \cdot x : B \to A}}{\lambda x \cdot \lambda y \cdot x : A \to (B \to A)}$$

Thus the proof term of $A \to (B \to A)$ is $\lambda xy.x$, this is Curry combinator K. Note that the informal argument of page 3 is faithfully reflected.

Let us now consider a cut elimination conversion.

$$\dfrac{\dfrac{\begin{matrix}x:B\\ \mathcal{D}\\ t:A\end{matrix}}{\lambda x \cdot t : B \to A} \qquad \dfrac{\mathcal{D}'}{s:B}}{(\lambda x \cdot t)(s) : A} \qquad \text{reduces to} \qquad \begin{matrix}\mathcal{D}'\\ s:B\\ \mathcal{D}[\int/\S]\\ t[s/x]A\end{matrix}$$

The proof theoretic conversion corresponds to the β-reduction of the* λ-calculus.

In order to deal with full predicate logic we have to introduce specific operations in order to render the meaning of the connectives and their derivation rules. Here is a list:

$$\begin{cases} p & -- \quad \text{pairing} \\ p_0, p_1 & -- \quad \text{projections} \end{cases}$$

$$\begin{cases} D & -- \quad \text{discriminator (“case dependency”)} \\ k & -- \quad \text{case obliteration} \end{cases}$$

E – witness extractor
$\perp$ – ex falso operator

$$\wedge\text{ I}\ \frac{t_0 : A_0 \ \ t_1 : A_1}{p(t_0, t_1) : A_0 \wedge A_1} \qquad \wedge\text{ E}\ \frac{t : A_0 \wedge A_1 (i \in \{0, 1\})}{p_i(t) : A_i}$$

$$\vee\text{ I}\ \frac{t : A_i \ (i \in \{0, 1\})}{k_i; A_0 \vee A_1} \qquad \vee\text{ E}\ \frac{t : A \vee B \ \ t_0[x^A] : C \ \ t_1[x^B] : C}{D_{u,v}(t, t_0[u], t_1[v]) : C}$$

$$\to\text{ I}\ \frac{t[x^A] : B}{\lambda y^A \cdot t[y^A] : A \to B} \qquad \to\text{ E}\ \frac{t : A \to B \ \ t' : A}{t(t') : B}$$

$$\forall\text{ I}\ \frac{t[x] : A(x)}{\lambda y \cdot t[y] : \forall y A(y)} \qquad \forall\text{ E}\ \frac{t : \forall x A(x)}{t(t') : A(t')}$$

$$\exists\text{ I}\ \frac{t_1 : A(t_0)}{p(t_0, t_1) : \exists x A(x)} \qquad \exists\text{ E}\ \frac{t : \exists x A(x) \ \ t_1[y, z^{A(y)}] : C}{E_{u,v}(t, t_1[u, v]) : C}$$

There are a number of details that we have to mention.

(i) In $\to I$ the dependency on the hypothesis has to be made explicit in the term. We do this by assigning to each hypothesis its own variable. E.g. $x^A : A$.

(ii) In $\vee E$ (and similarly $\exists E$) the dependency on the particular (auxilliary) hypotheses A and B disappears. This is done by a variable binding technique. In D_u, v the variables u and v are bound.

(iii) In the falsum rule the result, of course, depends on the conclusion A. So A has its own ex falso operator $\perp_A$.

Now the conversion rules for the derivation automatically suggest the conversion for the term.

Normalization and cut-elimination for second-order logic is far more complicated because of the presence of the comprehension rule, which introduces a impredicativity. The proof of a normalization theorem was a spectacular breakthrough, the names to mention here are Girard, Prawitz and Martin-Löf, see their papers in [Fenstad 1971].

Suggested further reading: [Troelstra-Schwichtenberg 1996], [Troelstra–van Dalen 1988] Ch. 10, [Dummett 1977] Ch. 4, [Girard 1989], [Buss 1998], [Beeson 1985], [Kleene 1952] Ch. XV, [Troelstra 1973].

We have seen in the previous section that the term calculus corresponds with the natural deduction system. This suggests a correspondence between proofs and propositions on the one hand and elements (given by the terms) and types (the spaces where these terms are to be found). This correspondence was first observed for a simple case (the implication fragment) by Haskell Curry, [Curry-Feys 1958], ch. 9, § E, and extended to full intuitionistic logic by W. Howard, [Howard 1980]. Let us first look at a simple case, the one considered by Curry.

Since the meaning of proposition is expressed in terms of possible proofs — we know the meaning of A if we know what things qualify as proofs — one may take an abstract view and consider a proposition as its collection of proofs. From this viewpoint there is a striking analogy between propositions and sets. A set has elements, and a proposition has proofs. As we have seen, proofs are actually a special kind of constructions, and they operate on each other. E.g. if we have a proof $p : A \to B$ and a proof $q : A$ then $p(q) : B$. So proofs are naturally typed objects.

Similarly one may consider sets as being typed in a specific way. If A and B are typed sets then the set of all mappings from A to B is of a higher type, denoted by $A \to B$ or B^A. Starting from certain basic sets with types, one can construct higher types by iterating this 'function space'-operation. Let us denote 'a is in type A' by $a \in A$.

Now there is this striking parallel.

Propositions	Types
$a : A$	$a \in A$
$p : A \to B, q : A$ $\Rightarrow p(q) : B$	$p \in A \to B, q \in A$ $\Rightarrow p(q) \in B$
$x : A \Rightarrow t(x) : B$ then $\lambda x \cdot t : A \to B$	$x : A \Rightarrow t(x) \in B$ then $\lambda x \cdot t \in A \to B$

It now is a matter of finding the right types corresponding to the remaining connectives. For $\wedge$ and $\vee$ we introduce a product type and a disjoint sum type. For the quantifiers generalizations are available. The reader is referred to the literature, cf. [Howard 1980], [Gallier 1995].

The main aspect of the Curry-Howard isomorphism, (also knowns as "proofs as types"), is the faithful correspondence:

$$\frac{\text{proofs}}{\text{propositions}} = \frac{\text{elements}}{\text{types}}$$

with their conversion and normalization properties.

Per Martin-Löf was the first logician to see the full importance of the connection between intuitionistic logic and type theory. Indeed, in his approach the two are so closely interwoven, that they actually merge into one master system. His type systems are no mere technical innovations, but they intend to catch the foundational meaning of intuitionistic logic and the corresponding mathematical universe. Expositions can be found in e.g. [Martin-Löf 1975], [Martin-Löf 1984].

Suggested further reading: [Girard 1989], [Gallier 1995], [Troelstra–van Dalen 1988].
Whereas first-order intuitionistic logic is a subsystem of classical logic, higher-order logic may contain rules or axioms that are constructively justified, but which contradict classical logic.

In practice there are two ways to formulate second-order logic: with set variables and with function variables.

1.5 $\mathbf{IQC}^2$

Second-order logic with set variables is a straightforward adaptation of the classical formulation; cf. [Dalen 1997], [Troelstra-Schwichtenberg 1996], CH. 11. It is a surprising fact that in $\mathbf{IQC}^2$ the connectives are not independent as in first-order logic. Prawitz showed that one can define the connectives in $\forall^1, \forall^2$ and $\to$, [Prawitz 1965].

$$\begin{array}{rcl} \perp & \leftrightarrow & \forall^2 X.X \\ A \wedge B & \leftrightarrow & \forall^2 X((A \to (B \to X)) \to X) \\ A \vee B & \leftrightarrow & \forall^2 X((A \to X) \wedge (B \to X) \to X) \\ \exists^1 xA & \leftrightarrow & \forall^2 X(\forall^1 y(A \to X) \to X) \\ \exists^2 XA & \leftrightarrow & \forall^2 X(\forall Y((A \to X) \to X) \end{array}$$

(where X is a 0-ary predicate)

Classically, sets and functions are interdefinable as each set has a characteristic function. But intuitionistically S has a characteristic function of its membership is decidable: $\begin{cases} a \in S \Leftrightarrow k_S(a) = 1 \\ a \notin S \Leftrightarrow k_S(a) = 0 \end{cases}$

Since $k_S(a) = 1 \vee k_S(a) = 0$, we get $a \in S \vee a \notin S$. The moral is that there are lots of sets without characteristic functions. We see that the "set"-approach and the "function"-approach to second-order logic (arithmetic) yield diverging theories. Generally speaking, total functions, with their 'input–output' behaviour are more tractable than sets. That is a good reason to study second-order arithmetic with function variables. Another reason is that this formulation is a natural framework for treating Brouwer's choice sequences.

For the proof theory of $\mathbf{IQC^2}$, see [Prawitz 1970], [Troelstra-Schwichtenberg 1996], [Troelstra 1973].

1.6 The theory of choice sequences

For the practice of intuitionistic mathematics, second-order arithmetic with function variables is even more significant than the version with set variables. The reason is that this theory allows us to capture the properties of Brouwer's choice sequences. Since this survey is about logic, the topic is not quite within its scope. Let us therefore just briefly note the main points, for more information the reader is referred to the literature.

Brouwer's chief contribution to this part of intuitionistic logic is that he realized that particular quantifier combinations get a specific reading. Choice sequences of, say, natural numbers are infinite sequences, α, of natural numbers, chosen more or less arbitrarily. That is to say, in general there is no law that determines future choices. Suppose now that we have shown $\forall\alpha\exists n A(\alpha, n)$ for some formula A, this means that when we generate a choice sequence α, we will eventually be able to compute the number n, such that $A(\alpha, n)$ holds. Roughly speaking, this tells us that at some stage in the generation of α, we have all the information we need for the computation of n, but then we need no further information, and any β that coincides so far with α will yield the same n. This sketch is somewhat simplified, for an extensive analysis cf. [van Atten–van Dalen ??]. This, in a nutshell, is *Brouwer's Continuity Principle*:

$$\forall\alpha\exists x A(\alpha, n) \to \forall\alpha\exists xy\forall\beta(\overline{\alpha}(y) = \overline{\beta}(y) \to A(\beta.x)$$

Here $\overline{\gamma}(y)$ stands for the initial segment of length y of a sequence γ

On the basis of this principle, formulated for the $\forall\alpha\exists!x$ in [Brouwer 1918], already a number of intuitionistic facts, conflicting with classical mathematics, can be derived. It allows, for example, a simple rejection of PEM.

In later papers Brouwer further investigated the continuity phenomenon. He added a powerful induction principle, which enabled him to show his famous *Continuity Theorem*: all total real functions on the continuum are locally uniformly continuous, [Brouwer 1924]. As a corollary the continuum cannot be decomposed into two inhabited parts. Another well-know consequence is the *Fan theorem* (basically the compactness of the Cantor space).

Brouwer introduced in the twenties an extra strengthening of analysis, the details of which were published only after 1946. The new idea, known by the name "the creating subject", was formulated by Kreisel in terms of a tensed modal operator. Subsequently Kripke simplified the presentation by introducing a choice sequence α that "witnesses" a particular statement A. α keeps track of the succes of the subject in establishing A; it produces zeros as long as the subject has not established A, and when A is proved, or experienced, α produces a single 'one' and goes on with zeros. The existence of such a α is the content of Kripke's schema:

$$KS \quad \exists\alpha(A \leftrightarrow \exists x\alpha(x) = 1)$$

Brouwer used the creating subject (and implicitly Kripke's schema) to establish strong refutations, which go beyond the already existing Brouwerian counterexamples.

He showed for example that (cf. [Brouwer 1949], [Hull 1969])

$$\neg\forall x \in \mathbb{R}(\neg\neg x > 0 \rightarrow x > 0) \text{ and}$$
$$\neg\forall x \in \mathbb{R}(x \neq 0 \rightarrow x\#0)$$

Kripke's schema has indeed consequences for the nature of the mathematical universe, e.g. it conflicts with $\forall\alpha\exists\beta$-continuity, ([Myhill 1966]), that is $\forall\alpha\exists\beta A(\alpha,\beta) \rightarrow \exists F\forall\alpha A(\alpha, F(\alpha)$, where F is a continuous operation.

Using KS, van Dalen showed that Brouwer's indecomposability theorem for $\mathbb{R}$ can be extended to all dense negative subsets of $\mathbb{R}(A \subseteq \mathbb{R}$ is *negative* if $\forall x(x \in A \leftrightarrow \neg\neg x \in A)$. So, for example, the set of not-not-rationals is indecomposable (cf. [Dalen 1999A]).

In order to demonstrate the technique of the creating subject, we show here that, under the assumption of Kripke's schema one can show a converse of Brouwer's indecomposability theorem: KS + $\mathbb{R}$ is indecomposable $\Rightarrow$ there are no discontinuous real functions.

Proof. Let f be discontinuous in 0, and $f(0) = 0$. So $\exists k\forall n\exists x(|f(x)| > 2^{-k})$. Hence there are x_n with $|f(x_n)| > 2^{-k}$.

Let α be the Kripke sequence for $r \in \mathbb{Q}$, and β for $r \notin \mathbb{Q}$. Put

$$\begin{cases} \gamma(2n) & = & \alpha(n) \\ \gamma(2n+1) & = & \beta(n) \end{cases} \qquad c_n = \begin{cases} x_n & if & \forall p \leq n\ \gamma(k) = 0 \\ x_p & if & p \leq n \text{ and } \gamma(p) = 1 \end{cases}$$

and $c = lim(c_n)$

Now, $f(c) < 2^{-k} \vee f(c) > 0$. If $f(c) < 2^{-k}$ then $f(0) = 0$, so $\forall p(\gamma(p) = 0)$. Contradiction. If $f(c) > 0$, then $r \in \mathbb{Q} \vee r \notin \mathbb{Q}$. Hence $\forall r(r \in \mathbb{Q} \vee r \notin \mathbb{Q})$. This contradicts the indecomposability of $\mathbb{R}$.

The theory of choice sequences has received a great deal of attention after Kleene and Kreisel formulated suitable formalizations. Some notions, such as "lawless sequence", have found important applications in metamathematics. C.f. [Brouwer 1981], [Brouwer 1992], [**?**], [Kleene–Vesley 1965], [Kreisel–Troelstra 1970], [Troelstra 1977], [Troelstra–van Dalen 1988] Ch. 12, [Beeson 1985], [Dummett 1977] Ch. 3.

There is an extensive literature on the semantics of second-order arithmetic, c.f. [Scott 1968], [Scott 1970] [Moschovakis 1973], [Dalen 1978], [Fourman-Hyland 1979], [Fourman-Scott 1979], [Hoeven-Moerdijk 1984].

References

[van Atten–van Dalen ??] M.van Atten, D. van Dalen. Arguments for Brouwer's continuity principle. Forthcoming.

[Beeson 1976] M. Beeson. The unprovability in intuitionistic formal systems of the continuity of effective operations on the reals. *J Symb Logic*, 41: 18–24.

[Beeson 1985] ——— *Foundations of Constructive Mathematics*. Springer, Berlin.

[Beth 1956] E.W. Beth. Semantic Construction of Intuitionistic Logic. *Kon. Nederlandse Ac. Wetenschappen afd. Letteren:Mededelingen*, 19/11: 357–388.

[Borwein 1998] J.M. Borwein. Brouwer-Heyting Sequences Converge. *Math. Intelligencer*, 20:14–15.

[Bridges-Richman 1987] F. Richman, Bridges, D. *Varieties of Constructive Mathematics*. Cambridge University Press, Cambridge.

[Brouwer 1918] L.E.J. Brouwer. Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Erster Teil, Allgemeine Mengenlehre. *Kon Ned Ak Wet Verhandelingen*, 5:1–43.

[Brouwer 1919a] ——— Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Zweiter Teil, Theorie der Punktmengen. *Kon Ned Ak Wet Verhandelingen*, 7:1–33.

[Brouwer 1919b] ——— Intuitionistische Mengenlehre . *Jahresber. Dtsch. Math. Ver.*, 28:203–208. Appeared in 1920.

[Brouwer 1923] ——— Über die Bedeutung des Satzes vom ausgeschlossenen Dritten in der Mathematik insbesondere in der Funktionentheorie. *J Reine Angew Math*, 154:1–8.

[Brouwer 1924] ——— Beweis dass jede volle Funktion gleichmässig stetig ist. *Nederl Ak Wetensch Proc*, 27:189–193.

[Brouwer 1949] ——— Essentieel negatieve eigenschappen. *Ind. Math.*, 10:322-323.

[Brouwer 1981] ——— *Brouwer's Cambridge Lectures on Intuitionism.* Cambridge University Press, Cambridge.

[Brouwer 1992] ——— (Ed. D. van Dalen) Intuitionismus. Bibliographisches Institut, Wissenschaftsverlag, Mannheim.

[Buss 1998] S. Buss. *handproof.* Elsevier, Amsterdam.

[Curry-Feys 1958] R. Feys Curry, H.B. *Combinatory Logic. Vol. I.* North–Holland, Amsterdam.

[Dalen 1974] ——— A model for HAS. A topological interpretation of second-order intuitionistic arithmetic with species variables. *Fund. Math.*, 82:167–174.

[Dalen 1978] ——— An interpretation of intuitionistic analysis. *Ann Math Log*, 13:1–43.

[Dalen 1984] ——— How to glue analysis models. *J Symb Logic*, 49:1339–1349.

[Dalen 1997] ——— *Logic and Structure* (3rd edition, revised 2008). Springer Verlag, Berlin.

[Dalen 1999] ——— *Mystic, Geometer, and Intuitionist: The Life of L.E.J. Brouwer Vol. 1. The dawning Revolution.* Oxford University Press, Oxford.

[Dalen 2005] ——— *Mystic, Geometer, and Intuitionist: The Life of L.E.J. Brouwer Vol. 2. Hope and Disillusion.* Oxford University Press, Oxford.

[Dalen 1999A] ——— From Brouwerian Counter Examples to the Creating Subject. *Studia Logica*, 62:305–314.

[Dalen 2002] ——— Intuitionistic Logic, In *Handbook of Philosophical Logic, 2nd edition, vol. 5*, eds. D. Gabbay, F, Günthner. pages 1–114. Kluwer, Dordrecht.

[Dalen 1978] R. Statman Dalen. Equality in the presence of apartness. In *Hintikka et al(eds.) Essays on Mathematical and Philosophical Logic*, pages 95–116,Reidel. Dordrecht.

[Diaconescu 1975] R. Diaconescu. Axiom of choice and complementation. *Proc. American Math. Soc.*, 51:176–178.

[Dragalin 1988] A. Dragalin. *Mathematical Intuitionism.* American Mathematical Society, Providence.

[Dummett 1975] M. Dummett. The philosophical basis of intiutionistic logic. In J. Shepherdson H.E. Rose, editor, *Logic Colloquium '73,*, pages 5–40, North-Holland. Amsterdam.

[Dummett 1977] ——— *Elements of Intuitionism.* Oxford University Press, Oxford.

[Fenstad 1971] J.E. Fenstad (ed.) *Proceedings of the Second Scandinavian Logic Symposium*, 405 pp. North–Holland, Amsterdam.

[Fitting 1969] M. Fitting. *Intuitionistic model theory, and forcing.* North-Holland Publ. Co., Amsterdam.

[Fourman-Scott 1979] Fourman, M.P. and D.S. Scott. *Sheaves and Logic*, pages 280–301. Springer, Berlin.

[Fourman-Hyland 1979] Fourman, M.P. and J.M.E. Hyland. *Sheaf models for analysis*, pages 280–301. Springer, Berlin.

[Friedman 1973] H. Friedman. Some applications of Kleene's methods for intuitionistic systems. *In [Mathias–Rogers 1973]*, 113–170.

[Friedman 1975] ——— The disjunction property implies the numerical existence property. *Proc. Nat. Acad. Science USA*, 72:2877–2878.

[Friedman 1977] ——— Set theoretic foundations for constructive analysis. *Ann Math*, 105:1–28.

[Gabbay 1972] D.M. Gabbay. Decidability of some intuitionistic theories. *J Symb Logic*, 37:579–587.

[Gabbay 1973] ——— The undecidability of intuitionistic theories of algebraically closed and real closed fields. *J Symb Logic*, 38:86–92.

[Gabbay 1982] ——— *Semantical Investigations in Heyting's intuitionistic logic*. Reidel, Dordrecht.

[Gallier 1995] J. Gallier. On the Correspondence between proofs and λ-terms. In Ph. de Groote, editor, *The Curry–Howard Isomorphism*, pages 55–138, Academia. Louvain-la-Neuve.

[Gentzen 1935] G. Gentzen. Untersuchungen über das logische Schliessen I, II. *Math. Zeitschr*, 39:176–210, 405–431.

[Girard 1987] J.-Y. Girard. *Proof Theory and Logical Complexity. Vol. I.* Bibliopolis, Napoli.

[Girard 1989] P. Taylor Girard, J.-Y., Y. Lafont. *Proofs and Types.* Cambridge University Press, Cambridge.

[Gödel 1932] K. Gödel. Zum intuitionistischen Aussagenkalkü l. *Anzeiger der Akademie der Wissenschaften in Wien*, 69:65–66, . (Coll.Works I, p. 222–225).

[Goernemann 1971] S. Goernemann. A logic stronger than intuitionism. *J Symb Logic*, 36:249–261.

[Goodman-Myhill 1978] Goodman, N.D. and J. Myhill. Choice implies excluded middle. *Z.Math.Logic Grundlagen Math.*, 24:461.

[Grayson 1979] R.J. Grayson. Heyting-valued models for intuitionistic set theory. In (Eds. M.P. Fourman, C.J. Mulvey, D.S. Scott, editor, *durham*, pages 402–414,Springer. Berlin.

[Grayson 1981] ——— Concepts of general topolog in constructive mathematics and in sheaves. *Ann Math Log*, 20:1–41.

[Grayson 1982] ——— Concepts of general topolog in constructive mathematics and in sheaves. II. *Ann Math Log*, 23:55–98.

[Grayson 1984] ——— Heyting–valued Semantics. In Lolli, G., G. Longo, A. Marcia, editor, *Proceedings of the Logic Colloquium '82*, pages 181–208, Amsterdam. Elsevier.

[Heyting 1925] A. Heyting. *Intuïtionistische axiomatiek der projectieve meetkunde*. PhD thesis, Amsterdam.

[Heyting 1930a] ——— Die formalen Regeln der intuitionistischen Logik. *Die Preussische Akademie der Wissenschaften. Sitzungsberichte. Physikalische-Mathematische Klasse.*, 42–56.

[Heyting 1930b] ——— Die formalen Regeln der intuitionistischen Mathematik II, III. *Die Preussische Akademie der Wissenschaften. Sitzungsberichte. Physikalische-Mathematische Klasse.*, 57–71, 158–169.

[Heyting 1934] ——— *Mathematische Grundlagenforschung. Intuitionismus, Beweistheorie*. Springer Verlag, Berlin.

[Heyting 1956] ——— *Intuitionism, an Introduction* . North–Holland, Amsterdam,.

[Hoeven-Moerdijk 1984] Hoeven, G..F. and Moerdijk, I. Sheaf models for choice sequences. *Ann Pure Appl Logic*, 27:63–107.

[Howard 1980] W.A. Howard. The Formulas-as-Types Notion of Construction. In J.R. Hindley J.P. Seldin, editor, *To H.B. Curry: Essays on Combinatory Logic, Lambda calculus and Formalism.*, pages 479–490, Academic Press. New York.

[Hull 1969] R.G. Hull Counterexamples in intuitionistic analysis using Kripke's schema. *Zeitschr. math. Logik*, 15:241–246.

[Hyland 1982] J.M.E. Hyland. The Effective Topos. In D. van Dalen A.S. Troelstra, editor, *The L.E.J. Brouwer Centenary Symposium*, pages 165–216,Elsevier. Amsterdam.

[Jaskowski 1936] Recherches sur le système de la logique intuitioniste, *Congr. intern. Phil. des Sciences*, 6: 58–61, Hermann. Paris.

[Kleene 1952] S.C. Kleene. *Introduction to Metamathematics.* North-Holland, Amsterdam..

[Kleene–Vesley 1965] Kleene, S.C. and R.E. Vesley. *The Foundations of Intuitionistic Mathematics especially in relation to Recursive Functions.* North–Holland, Amsterdam..

[Kolmogorov 1932] A.N. Kolmogorov. Zur Deutung der intuitionistischen Logik. *Math. Zeitschr*, 35:58–65..

[Kreisel 1958] G. Kreisel. Elementary completeness properties of intuitionistic logic with a note on negations of prenex formulae. *J Symb Logic*, 23:317–330..

[Kreisel–Troelstra 1970] Kreisel, G. and A.S. Troelstra. Formal systems for some branches of intuitionistic analysis. *Ann Math Log*, 1:229–387..

[Kripke 1965] S.A. Kripke. Semantical Analysis of Intuitionistic Logic. In M.A.E. Dummett J.N. Crossley, editor, *Formal Systems and Recursive Functions. Proc. of the eighth Logic Colloquium, Oxford 1963*, pages 92–130, North-Holland. Amsterdam.

[Kripke 1968] ——— Semantical Analysis of Intuitionistic Logic II. *unpublished*,

[Lambek–Scott 1986] Lambek, J. and P.J. Scott. *Introductioin to higher order categorical logic*. Cambridge University press, Cambridge.

[Lifschitz 1967] V. Lifschitz. The decision problem for some constructive theories of equality. *Seminar Math. Institute Steklov*, 4:29–31.

[MacLane–Moerdijk 1992] MacLane, S. and I. Moerdijk. *Sheaves in Geometry and Logic. A first Introduction to Topos Theory.* Springer, Berlin.

[Markov 1971] A.A. Markov. On Constructive Mathematics. *AMS Transl.*, 98:1–9. Original Trudy Mat. Inst. Steklov.67 (1962).

[Martin-Löf 1975] P. Martin-Löf. An Intuitionistic Theory of Types. In H.E. Rose and J.C. Shepherdson, editors, *Logic Colloquium '73*, pages 73–118,North–Holland. Amsterdam.

[Martin-Löf 1984] ——— *Intuitionistic Type Theory.* Bibliopolis, Napoli.

[Maslov 1965] V.P. Orevkov Maslov, S.Yu., G. Mints. Unsolvability in the constructive predicate calculus of certain classes of formulas containing only monadic predicate variables. *Soviet Mat. Dokl.*, 6:918–920.

[Mathias–Rogers 1973] Mathias,A. and H. Rogers (eds.) . *Cambridge Summer School in Mathematical Logic*. Springer Verlag, Berlin.

[McCarty 1986] C. McCarty. Realizability and Recursive Set Theory. *Ann Pure Appl Logic*, 32:153–183.

[McCarty 1984] ——— *Realizability and Recursive Mathematics.* PhD thesis, Oxford.

[McCarty 1991] ——— Polymorphism and Apartness. *Notre Dame Journal of Formal Logic*, 32:513–532.

[McLarty 1995] C. McLarty. *Elementary Categories, Elementary Toposes.* Clarendon Press, Oxford.

[Moschovakis 1973] J. Moschovakis. A Topological Interpretation of Intuitionistic Analysis. *Comp Math*, 26:261–275.

[Myhill 1966] J. Myhill, Notes towards an axiomatization of intuitionistic analysis, *Logique et Analyse* 9, 280-297.

[Negri–von Plato] S. Negri, J. von Plato. *Structured Proof Theory.* Forthcoming.

[Prawitz 1965] D. Prawitz. *Natural Deduction. A proof-theoretical study.* Almqvist & Wiksell, Stockholm.

[Prawitz 1970] ——— Some results for intuitionistic logic with second order quantification. In Myhill, J and A. Kino and R. Vesley, editor, *buffalo*, pages 259–269, North-Holland. Amsterdam.

[Prawitz 1977] ——— Meaning and Proofs: On the Conflict between Classical and Intuitionistic Logic. *Theoria*, 43:2–40.

[Rasiowa-Sikorski 1963] Rasiowa, H. and R. Sikorski. *The Mathematics of Metamathematics.* Panstowe Wydawnictwo Naukowe, Warsaw.

[Schütte 1968] K. Schütte. *Vollständige Systeme modaler und intuitionistischer Logik.* Springer, Berlin.

[Troelstra-Schwichtenberg 1996] A.S. Troelstra Schwichtenberg, H. *Basic proof theory.* Cambridge University Press, Cambridge.

[Scott 1968] D.S. Scott. Extending the topological interpretation to intuitionistic analysis. *Comp. Math.*, 20:194–210.

[Scott 1970] ——— Extending the topological interpretation to intuitionistic analysis II. In Myhill, J and A. Kino and R. Vesley, editor, *buffalo*, pages 235–255,North-Holland. Amsterdam.

[Scott 1979] ——— *Identity and Existence in Intuitionistic logic*, pages 660–696. Springer, Berlin.

[Smorynski 1973a] C.S. Smorynski. Applications of Kripke models. In A.S. Troelstra, editor, *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, pages 324–391, Springer. Berlin.

[Smorynski 1973b] ——— *Investigation of Intuitionistic Formal Systems by means of Kripke Models.* PhD thesis, Chicago.

[Smorynski 1973c] ——— On axiomatizing fragments. *J Symb Logic*, 42: 530–544.

[Smorynski 1978] ——— The axiomatizing problem for fragments. *Ann Math Log*, 14:193–221.

[Smorynski 1982] ——— Nonstandard models and constructivity. In D. van Dalen A.S. Troelstra, editor, *The L.E.J. Brouwer Centenary Symposium*, pages 459–464, Elsevier. Amsterdam.

[Tarski 1938] A. Tarski. Der Aussagenkalkül und die Topologie. *Fund. Math.*, 31:103–134.

[Troelstra 1973] A.S. Troelstra. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis.* Springer Verlag, Berlin.

[Troelstra 1977] ——— *Choice Sequences. A chapter of intuitionistic mathematics.* Clarendon Press, Oxford.

[Troelstra 1978] ——— Heyting, "Die formale Regeln der intuitionistischen Logik" (1930). Commentary. In "*Two decades of mathematics in the Netherlands*", Eds. Bertin, E.M.J., H.J.M. Bos, A.W. Grootendorst, 1. Mathematical Centr. Amsterdam. 163–175.

[Troelstra–van Dalen 1988] Troelstra, A.S., D. van Dalen. *Constructivism in Mathematics, I, II.* North-Holland Publ. Co., Amsterdam.

[Veldman 1976] W. Veldman. An intuitionistic completeness theorem for intuitionistic predicate calculus. *J. Symb. Logic*, 41:159–166.