Chapter V
APPLICATIONS OF KRIPKE MODELS

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§ 1. Kripke models.

5.1.1. Discussion. In Kripke 1965, S. Kripke introduced a set-theoretic semantics for the intuitionistic predicate calculus. In this Chapter, we study this set-theoretic machinery and apply it to the investigation of Heyting's Arithmetic. Since the set-theoretic approach may seem out of place in a study of intuitionistic systems, we remark in Section 5.1.26 on how intuitionistic proofs of some of the results can be recovered.

Kripke's model theory bears no resemblance to intuitionistic reasoning despite various attempts to make it a plausible interpretation of intuitionistic reasoning. (The reader who disagrees will certainly change his mind by the time he finishes this chapter.) Formally, however, the same logical laws are valid in the Kripke models and in the intuitionistic predicate calculus. This fact, combined with the ease in handling the Kripke models, makes them an extremely useful tool in the metamathematical investigation of Heyting's Arithmetic.

Before defining the Kripke models, let us consider one of these interpretations in order to motivate somewhat the formal definition of a Kripke model. The interpretation we consider is that of intuitionistic logic as a logic of "positivistic research". We have various "states of knowledge", which form themselves into a partial order. At each state of knowledge there is a collection of objects we have mentally constructed. A larger state of knowledge may require us to mentally construct new objects. Also, an atomic relation, e.g. an equation, may or may not be seen to hold on the basis of a given state of knowledge. Obviously, if it is seen to be true on the basis of a given state of knowledge, it must be seen to be true on the basis of any extension of the given state of knowledge. Further, this should hold for more complicated properties than atomic relations. The problem, then, is to find an interpretation of the logical connectives and quantifiers which preserve this property. Conjunction, disjunction, and existential quantification are straightforward - e.g. we see $\exists x\, Ax$ to be true on the basis of some state of knowledge iff we have some mentally constructed object a such that Aa is seen to be true on the basis of this state of knowledge.

The other connectives and quantifier are problematical and it is here that the interpretation loses its plausibility. Consider, e.g., the implication $A \rightarrow B$. If $A \rightarrow B$ is adjudged true on the basis of a state of knowledge, then $A \to B$ is also true in any extension of this state of knowledge and, if A is true in such an extension, so is B. The converse, that, if, for every extension of our knowledge, once we know A to be true we also know B to be true, then we know $A \to B$ to be true, is not at all obvious; but a usable condition to define the connective $\to$ is needed and we accept it. Negation and the universal quantifier are treated similarly.

The interpretations of &, $\vee$, and $\exists$ seem natural enough, but those of the more negative connectives and quantifiers are a little forced. The net result is that, to show that we cannot assert the truth of a statement on the basis of a given state of knowledge, we appeal not to the lack of positive knowledge - but to the fact that some extension of our knowledge contains false assertions. Modifying the treatment of the negative connectives might make the interpretation more palatable. Such a task, however, lies beyond the scope of this Chapter and we turn now to the formal definition of Kripke's models.

5.1.2. Definition. By a Kripke model (Kripke 1965) we shall mean a quadruple $\underline{K} = (K, \leq, D, \Vdash)$, where $(K, \leq)$ is a non-empty partially ordered set, D is a non-descreasing function associating elements of K with non-empty sets, and $\Vdash$ is a relation between elements of K and formulae with no free variables (but which may possess constants denoting elements of the $D\alpha$'s) which satisfies the following (where small greek letters denote elements of K):

i) for $A(x_1,\ldots,x_n)$ atomic, $\beta \geq \alpha$, $a_1,\ldots,a_n \in D\alpha$, if $\alpha \Vdash A(a_1,\ldots,a_n)$, then $\beta \Vdash A(a_1,\ldots,a_n)$;
ii) $\alpha \Vdash A \,\&\, B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$;
iii) $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$;
iv) $\alpha \Vdash A \to B$ iff $\forall \beta \geq \alpha (\beta \Vdash A \Rightarrow \beta \Vdash B)$;
v) $\alpha \Vdash \neg A$ iff $\forall \beta \geq \alpha (\beta \nVdash A)$;
vi) $\alpha \Vdash \exists x Ax$ iff $\exists a \in D\alpha (\alpha \Vdash Aa)$;
vii) $\alpha \Vdash \forall x Ax$ iff $\forall \beta \geq \alpha\ \forall b \in D\beta (\beta \Vdash Ab)$.

The relation " $\alpha \Vdash A$ " may be read " A is true at α " or, for those familiar with set theory, " α forces A ". The elements of K will be denoted by small greek letters and will be called nodes in order to avoid confusion with the elements of the domains of the nodes - i.e. elements of the sets $D\alpha$. The triple $(K, \leq, D)$ is often called a quantificational model structure (or qms). If we restrict our attention to the propositional calculus, a propositional model structure (pms) is just a partially ordered set $(K, \leq)$ and a propositional model is a triple $(K, \leq, \Vdash)$, where $\Vdash$ satisfies (i) - (v).

As in classical model theory, one may define a notion of validity: A will be called valid in the model $\underline{K}$ iff $\alpha \Vdash A$ for all $\alpha \in K$. A will be called valid (universally valid) if A is valid in every model $\underline{K}$. More generally, if Γ is a set of formulae, we say Γ entails A, written $\Gamma \models A$, iff A is valid in every model in which every formula of Γ is valid. We shall prove later on:

$$\Gamma \models A \quad \text{iff} \quad \Gamma \vdash A .$$

5.1.3. Some basic properties of Kripke models.

Before giving some examples of Kripke models, let us remark on some of their basic properties. The first is that conditions (ii) - (vii) on the forcing relation $\Vdash$ constitute the recursion clauses for an inductive definition of a forcing relation on a qms. In particular, if we specify which atomic formulae are forced at which nodes of the qms (in such a manner that (i) holds), then the relation extends uniquely (by using clauses (ii) - (vii)) to a forcing relation on that qms.

A second remark is that the first condition on atomic formulae specifies a property that holds for all formulae. I.e. if $\alpha \Vdash A$ and $\alpha \leq \beta$, then $\beta \Vdash A$. The proof of this is by induction on the length of a formula. For atomic formulae, the result is immediate. Let A be a conjunction, say $A = B \& C$. Then

$$\begin{aligned} \alpha \Vdash A &\Rightarrow \alpha \Vdash B \text{ and } \alpha \Vdash C \\ &\Rightarrow \beta \Vdash B \text{ and } \beta \Vdash C, \text{ by induction hypothesis,} \\ &\Rightarrow \beta \Vdash B \& C . \end{aligned}$$

Disjunction and existential quantification are handled similarly. For implication, negation, and universal quantification, we use the fact that we have required our condition defining $\alpha \Vdash A$ to hold for all $\beta \geq \alpha$. For example, let $A = B \rightarrow C$ and $\beta \geq \alpha$.

$$\begin{aligned} \alpha \Vdash B \rightarrow C &\Rightarrow \forall \gamma \geq \alpha (\gamma \Vdash B \Rightarrow \gamma \Vdash C) \\ &\Rightarrow \forall \gamma \geq \beta (\gamma \Vdash B \Rightarrow \gamma \Vdash C), \text{ since } \beta \geq \alpha \\ &\Rightarrow \beta \Vdash B \rightarrow C . \end{aligned}$$

Negation and universal quantification are treated similarly.

A final remark is that the truth of $\alpha \Vdash A$ depends only on those β which are $\geq \alpha$ - each clause in the definition of the forcing relation refers only to those $\beta \geq \alpha$. Let $\underline{K}$ be a model and define $\underline{K}_\alpha = (K_\alpha, \leq_\alpha, D_\alpha, \Vdash_\alpha)$ for $\alpha \in K$ by:

$$K_\alpha = \{\beta \in K : \beta \geq \alpha\},$$

$\leq_\alpha$ and D_α are the restrictions of $\leq$ and D to K_α, and $\Vdash_\alpha$ is defined by letting $\beta \Vdash_\alpha A$ iff $\beta \Vdash A$, for A atomic and $\beta \in K_\alpha$. We should expect that, for any A, $\alpha \Vdash A$ in $\underline{K}$ iff $\alpha \Vdash_\alpha A$ in $\underline{K}_\alpha$. Indeed, a simple induction on the length of A shows that $\beta \Vdash A$ in $\underline{K}$ iff $\beta \Vdash_\alpha A$ in $\underline{K}_\alpha$ for all $\beta \in K_\alpha$. Thus, to verify that $\alpha \Vdash A$, we need only look at those $\beta \geq \alpha$ (i.e. we may restrict ourselves to the model $\underline{K}_\alpha$).

5.1.4. Examples. Let us first consider examples of models for the propositional calculus. We indicate the model by drawing a graph, the vertices of which determine nodes of the model. A node α precedes a node β in the ordering if the vertex corresponding to α is connected by a series of ascending lines to the vertex corresponding to β. E.g. $\alpha \leq \beta$ in the pms:

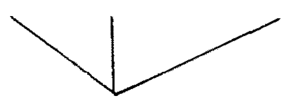
β |
α | .

We indicate the forcing relation by writing atomic formulae next to the nodes forcing them. E.g. using the pms just given, we obtain a model by letting $\alpha \Vdash A$, $\beta \Vdash A, B$:

β | A, B
α | A .

Observe that, in the model just given, (i) $\alpha \nVdash B \vee \neg B$; (ii) $\alpha \Vdash \neg\neg B$, but $\alpha \nVdash B$, whence $\alpha \nVdash \neg\neg B \rightarrow B$; (iii) β forces any tautology; and (iv) $\alpha \Vdash (C \rightarrow D) \vee (D \rightarrow C)$ for any formulae C, D.

One can get more complicated models by allowing the graphs to branch:



For the quantificational theory, we must add domains. Just as it is hard to draw models for classical theories, it will be hard to do this for intuitionistic theories. For simple cases, however, we may indicate the domains by listing their elements at each vertex of the graph. E.g.:

{a,b}
|
{a} .

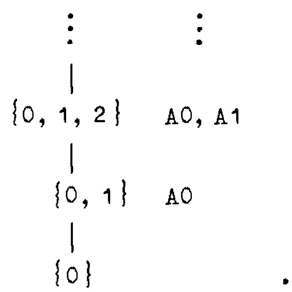
We may use this qms to construct a model:

β {a,b} A,Ba
|
α { a } Ba .

Here A is a propositional sentence. Observe that $\alpha \Vdash \forall x(A \vee Bx)$, but $\alpha \nVdash A \vee \forall x Bx$.

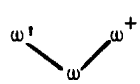
As one may easily verify, the formula $\forall x(A \vee Bx) \rightarrow A \vee \forall x Bx$ is valid in all models with constant domains (i.e. models in which D is a constant function), where we again assume that x does not occur free in A. (It is known that this class of models is <u>complete</u> for intuitionistic logic with this scheme added. Cf. <u>Gabbay</u> 1969 A or <u>Görnemann</u> 1971.)

Another interesting classically valid sentence which is not intuitionistically valid is $\neg\neg\forall x(Ax \vee \neg Ax)$. Consider the model:



I.e. we have a sequence $\alpha_0 < \alpha_1 < \ldots$ of nodes with $D\alpha_n = \{0, \ldots, n\}$ and $\alpha_m \Vdash An$ iff $m > n$. Suppose $\alpha_0 \Vdash \neg\neg\forall x(Ax \vee \neg Ax)$. Then $\forall \beta \geq \alpha_0$ $\beta \nVdash \neg\forall x(Ax \vee \neg Ax)$. In particular, $\alpha_0 \nVdash \neg\forall x(Ax \vee \neg Ax)$. But then $\exists \beta \geq \alpha_0$ $\beta \Vdash \forall x(Ax \vee \neg Ax)$. Let $\beta = \alpha_n$. Letting $x = n$, $\alpha_n \Vdash An \vee \neg An$, i.e. $\alpha_n \Vdash An$ or $\alpha_n \Vdash \neg An$. But $\alpha_n \nVdash An$ by definition and $\alpha_n \nVdash \neg An$ since $\alpha_{n+1} \Vdash An$. It not only follows that $\alpha_0 \nVdash \neg\neg\forall x(Ax \vee \neg Ax)$, but, in fact, that $\alpha_0 \Vdash \neg\forall x(Ax \vee \neg Ax)$.

When we have a classical model, e.g. the standard model, ω, of arithmetic, instead of listing the domain and the atomic formulae to be forced, if we wish to force those atomic formulae true in the model, we simply place an ω at the vertex. E.g. if ω^+ and ω' are non-standard models of arithmetic, we will write



for the intended Kripke model.

We could continue to give several further examples of Kripke models, but feel it would be more instructive for the reader to construct some of his own. E.g. he may wish to construct countermodels to $\neg A \vee \neg\neg A$, $((A \rightarrow B) \rightarrow A) \rightarrow A$, $(A \rightarrow B) \vee (B \rightarrow A)$. We should like to stress that he should pay close attention to the geometry of his countermodels. The geometry of the Kripke models is the basic tool used in this Chapter.

5.1.5 - 5.1.11. The completeness theorem.

5.1.5. So far we have constructed a model theory for the intuitionistic predicate calculus and used this model theory to demonstrate the failure of certain basic laws of classical logic which are not intuitionistically valid. It is now our job to demonstrate how closely the model theory fits intuitionistic reasoning. Formally, the fit is exact:

5.1.6. Theorem. (The completeness theorem.) $\Gamma \vdash A$ iff $\Gamma \models A$.

The proof of soundness, $\Gamma \vdash A$ implies $\Gamma \models A$, is long but easy. One merely has to show that each axiom is valid and that the rules of inference preserve truth. E.g. consider the rule PL2: $A, A \to B \vdash B$. If $\underline{K} = (K, \leq, D, \Vdash)$ is given and $\alpha \in K$ is such that $\alpha \Vdash A$, $\alpha \Vdash A \to B$, then, by the definition of $\alpha \Vdash A \to B$, it follows that $\alpha \Vdash B$. Hence, this rule is sound.

The more ambitious reader may prove the soundness theorem for any of the formulations of the intuitionistic predicate calculus given in Chapter I.

We now turn to proving the completeness theorem. The weak form, $\vdash A$ iff $\models A$, is due to Kripke 1965. The form we shall prove, often called a strong completeness theorem, is due independently to Aczel 1968, Fitting 1969, and Thomason 1968. For the sake of subsection 5.1.26, we shall follow Thomason's treatment. These proofs are modelled on Henkin's proof for classical logic.

Let M be a first-order language containing

i) a denumerable set V_M of individual variables;

ii) a denumerable set C_M of individual constants, and

iii) for each $j \geq 0$, a denumerable set F_M^j of j-ary predicate letters.

Formulae are to be built up from atomic formulae by using $\&$, $\vee$, $\to$, $\neg$, $\exists$, and $\forall$. Fm_M will denote the set of such formulae. Note that Fm_M is denumerable. Sn_M will denote the set of sentences - i.e. the formulae with no free variables.

5.1.7. Definition. A set $\Gamma \subseteq Sn_M$ is called M-saturated if

i) Γ is consistent;

ii) $A \in Sn_M$ and $\Gamma \vdash A \Rightarrow A \in \Gamma$;

iii) $A, B \in Sn_M$ & $A \vee B \in \Gamma \Rightarrow A \in \Gamma$ or $B \in \Gamma$; and

iv) if $Ax \in Fm_M$, x is the only free variable in A and $\exists x Ax \in \Gamma$, then, for some $c \in C_M$, $Ac \in \Gamma$.

Those familiar with the algebraic representation theorems may consider a saturated set Γ to be a sort of counterpart to a prime filter in a distributive lattice. Basically, these prime filters will yield nodes of a model and their inclusion relations will yield an ordering. Matters are slightly complicated by the necessity of introducing new constants to successively enlarge the domains.

5.1.8. Lemma. Let $\Gamma \cup \{A\} \subseteq Sn_M$ and suppose $\Gamma \not\vdash A$. Let $\{c_1, c_2, \ldots\}$ be a denumerably infinite set of symbols disjoint from C_M and let M' be obtained by adding $\{c_1, c_2, \ldots\}$ to the constants C_M of M. Then there is an M'-saturated superset Γ_ω of Γ such that $A \notin \Gamma_\omega$.

Proof. Set $\Gamma_o = \Gamma$ and define Γ_{k+1} inductively as follows:

Case 1. k is even. Let $\exists xB$ be the first existential sentence of M' not already treated such that $\Gamma_k \vdash \exists xB$ and let c be the first constant in $\{c_1, c_2, \ldots\}$ not occurring in Γ_k. Then set $\Gamma_{k+1} = \Gamma_k \cup \{Bc\}$.

Case 2. k is odd. Let $B \vee B'$ be the first disjunctive formula of M' not already treated such that $\Gamma_k \vdash B \vee B'$. If $\Gamma_k \cup \{B\} \not\vdash A$, put $\Gamma_{k+1} = \Gamma_k \cup \{B\}$. Otherwise $\Gamma_{k+1} = \Gamma_k \cup \{B'\}$.

Finally, set $\Gamma_\omega = \bigcup_{k=o}^{\infty} \Gamma_k$. We must show that Γ_ω satisfies conditions (i) - (iv) of 5.1.7 above.

(i). We show by induction that $\Gamma_k \not\vdash A$. Let $\Gamma_{2n+1} \vdash A$. Then $\Gamma_{2n+1} = \Gamma_{2n} \cup \{Bc\}$ for some B, c where c does not occur in any formula of Γ_{2n}. Thus $\Gamma_{2n}, Bc \vdash A$, whence $\Gamma_{2n} \vdash Bc \rightarrow A$ and, by Q4, $\Gamma_{2n} \vdash \exists xB \rightarrow A$. But $\Gamma_{2n} \vdash \exists xB$, whence $\Gamma_{2n} \vdash A$, a contradiction.

Similarly, PL5 allows us to conclude that, if $\Gamma_{2n+2} \vdash A$, then $\Gamma_{2n+1} \vdash A$.

Hence, for all k $\Gamma_k \not\vdash A$. But $\Gamma_\omega \vdash A$ iff $\Gamma_k \vdash A$ for some k, from which it follows that $\Gamma_\omega \not\vdash A$.

(iii), (iv). If $B \vee C \in \Gamma_\omega$, then $\Gamma_i \vdash B \vee C$ for some i. Hence, for some odd $k \geq i$, $B \vee C$ is the first disjunction not treated. Thus $\Gamma_{k+1} = \Gamma_k \cup \{B\}$ or $\Gamma_k \cup \{C\}$, i.e. $B \in \Gamma_\omega$ or $C \in \Gamma_\omega$. Similarly, if $\exists xB \in \Gamma_\omega$, then $Bc \in \Gamma_\omega$ for some c.

(ii). If $\Gamma_\omega \vdash A$, then $\Gamma_\omega \vdash A \vee A$ and, by (iii), $A \in \Gamma_\omega$. Q. E. D.

5.1.9. Theorem. If $\underline{\Gamma}$ is M-saturated, then for some Kripke model $\underline{K} = (K, \leq, D, \Vdash)$, and for some $\alpha \in K$,

$$\underline{\Gamma} = \{A : \alpha \Vdash A\}.$$

In fact, α may be assumed to be a minimum element of $\underline{K}$.

Proof. Let $M_o = M$ and let M_{i+1} be obtained from M_i by adding the set $S_i = \{c_1^{i+1}, \ldots, c_n^{i+n}, \ldots\}$ to C_{M_i}, where $S_i \cap C_{M_i} = \emptyset$. Set $K = \{\Delta : \underline{\Gamma} \subseteq \Delta$ and Δ is M_i-saturated for some $i\}$. We define $\Delta \leq \Delta'$ iff $\Delta \subseteq \Delta'$, $D\Delta = C_{M_i}$, where Δ is M_i-saturated. Finally, for atomic formulae $A(c_1, \ldots, c_n)$ with $c_1, \ldots, c_n \in C_{M_i}$, let

$$\Delta \Vdash A(c_1, \ldots, c_n) \text{ iff } A(c_1, \ldots, c_n) \in \Delta.$$

We wish to show that this last equivalence holds for all applicable formulae (i.e. formulae with no free variables and whose parameters are from the proper language). For this, we need the following

5.1.10. Lemma. Let $\Delta \in K$.

(a) $B \to C \in \Delta$ iff $\forall \Delta' \supseteq \Delta\ (B \in \Delta' \Rightarrow C \in \Delta')$;

(b) $\neg B \in \Delta$ iff $\forall \Delta' \supseteq \Delta\ B \notin \Delta'$;

(c) $\forall x Bx \in \Delta$ iff $\forall \Delta' \supseteq \Delta\ \forall c \in D\Delta'\ Bc \in \Delta'$.

Proof. (a) If $B \to C \in \Delta$, $B \in \Delta'$, and $\Delta' \supseteq \Delta$, then $\Delta' \vdash C$ and, by saturation, $C \in \Delta'$. Conversely, suppose $B \to C \notin \Delta$. Then $\Delta \cup \{B\} \nvdash C$ and, for Δ M_i - saturated, there is, by lemma 5.1.8, an M_{i+1} - saturated $\Delta' \supseteq \Delta \cup \{B\}$ such that $C \notin \Delta'$. This contradicts the assumption $B \in \Delta' \Rightarrow C \in \Delta'$.

(b) Similar to (a).

(c) Again, one direction is trivial. Suppose, conversely, that $\forall \Delta' \supseteq \Delta\ \forall c \in D\Delta'\ Bc \in \Delta'$ and $\forall x Bx \notin \Delta$. Let Δ be M_i - saturated. Since $\Delta \nvdash \forall x Bx$, we conclude, by Q 1, that $\Delta \nvdash Bc$ for $c \in C_{M_{i+1}} - C_{M_i}$. Hence, there is an M_{i+1} - saturated $\Delta' \supseteq \Delta$ such that $\Delta' \nvdash Bc$, i.e. $Bc \notin \Delta'$.
Q. E. D.

We may now complete the proof of theorem 5.1.9 by proving by induction on the length of A that

$$\Delta \Vdash A(c_1, \ldots, c_n) \text{ iff } A(c_1, \ldots, c_n) \in \Delta,$$

for $c_1, \ldots, c_n \in C_{M_i}$, where Δ is M_i - saturated. The case A is atomic follows by definition. The case $A = B \& C$ is trivial. Let $A = B \vee C$:

$$\begin{aligned} \Delta \Vdash B \vee C &\text{ iff } \Delta \Vdash B \text{ or } \Delta \Vdash C \\ &\text{ iff } B \in \Delta \text{ or } C \in \Delta, \quad \text{by induction hypothesis} \\ &\text{ iff } B \vee C \in \Delta, \quad \text{by saturation.} \end{aligned}$$

Let $A = B \to C$:

$$\begin{aligned} \Delta \Vdash B \to C &\text{ iff } \forall \Delta' \supseteq \Delta (\Delta' \Vdash B \to \Delta' \Vdash C) \\ &\text{ iff } \forall \Delta' \supseteq \Delta (B \in \Delta' \to C \in \Delta'), \text{ by induction hypothesis} \\ &\text{ iff } B \to C \in \Delta, \text{ by lemma 5.1.10.} \end{aligned}$$

The cases $A = \neg B$ and $\forall x B$ are similar.

Let $A = \exists x Bx$:

$$\begin{aligned} \Delta \Vdash \exists x Bx &\text{ iff } \exists c \in D\Delta\ \Delta \Vdash Bc \\ &\text{ iff } \exists c \in D\Delta\ Bc \in \Delta, \quad \text{by induction hypothesis} \\ &\text{ iff } \exists x Bx \in \Delta, \quad \text{by saturation.} \end{aligned}$$

This completes the proof. Q. E. D.

We may now complete the proof of the completeness theorem.

5.1.11. Proof of theorem 5.1.6. We have yet to prove $\Gamma \models A$ implies $\Gamma \vdash A$. Let $\Gamma \nvdash A$ and find a saturated $\underline{\Gamma} \supseteq \Gamma$ such that $\underline{\Gamma} \nvdash A$. By theorem 5.1.9, there is a model $\underline{K} = (K, \leq, D, \Vdash)$ and $\alpha \in K$ such that for all B,

$$\alpha \Vdash B \text{ iff } B \in \underline{\Gamma}.$$

In particular, $\alpha \Vdash B$ for $B \in \Gamma$ and $\alpha \nVdash A$. Hence $\Gamma \nvDash A$. Q. E. D.

5.1.12 - 5.1.18. <u>The Aczel slash</u>.

5.1.12. By theorem 5.1.9, for any M - saturated set Γ, there is a Kripke model $\underline{K}$ and a node α such that

$$\Gamma = \{A : \alpha \Vdash A\}.$$

The converse, that every such set is M' - saturated, where M' is obtained from M by extending C_M to include names for all elements of $D\alpha$, is an easy verification which we leave to the reader. As observed in <u>Aczel</u> 1968, we can obtain more information on M - saturation from the proof of theorem 5.1.9 than just this.

Observe that $\Gamma = \{A : \alpha \Vdash A\}$ for some α implies that α is a minimum element in the pms constructed. Thus, let us start with the model $\underline{K}$ constructed and add a new node α_0 such that $\alpha_0 \leq \alpha$ for all $\alpha \in K$, let $D\alpha_0 = C_M$, and extend the forcing relation by defining, for A atomic,

$$\alpha_0 \Vdash A \text{ iff } \Gamma \vdash A.$$

The <u>Aczel slash</u> is defined by

$$\Gamma \mid A \quad \text{iff} \quad \alpha_0 \Vdash A.$$

Also, define $|(\Gamma) = \{A : \Gamma \mid A\}$.

5.1.13. <u>Lemma</u>. $|(\Gamma)$ is M - saturated and $|(\Gamma) \subseteq \{A : \Gamma \vdash A\}$.
<u>Proof</u>. Clear.

5.1.14. <u>Theorem</u>. $|(\Gamma)$ is a maximal M - saturated subtheory of Γ.
<u>Proof</u>. Let $|(\Gamma) \subseteq \Delta \subseteq \{A : \Gamma \vdash A\}$, Δ M - saturated. We show that $\Delta \vdash A$ implies $A \in |(\Gamma)$.
(i). If A is atomic,

$$\Delta \vdash A \Rightarrow \Gamma \vdash A \Rightarrow A \in |(\Gamma).$$

(ii), (iii). $A = B \& C$, $B \vee C$. These cases are trivial.
(iv). Let $A = B \to C$:

$$\Delta \vdash B \to C \Rightarrow \Gamma \vdash B \to C.$$

a) $|(\Gamma) \vdash B$. Then $\Delta \vdash B$ and so $\Delta \vdash C$. Thus $|(\Gamma) \vdash C$.
b) $|(\Gamma) \nvdash B$. $B \to C \notin |(\Gamma)$ implies $\exists \beta \geq \alpha (\beta \Vdash B \text{ and } \nVdash C)$.

But $\beta = \Delta' \supset \Gamma$ and so $B \in \Delta' \Rightarrow C \in \Delta'$, a contradiction.
(v) $A = \neg B$. Similar to (iv).
(vi). Let $A = \exists x Bx$:

$$\Delta \vdash \exists x Bx \Rightarrow \exists a \in C_M \; \Delta \vdash Ba, \text{ by } M\text{ - saturation}$$

$$\Delta \vdash \exists x Bx \Rightarrow Ba \in |(\Gamma)$$
$$\Rightarrow \exists x Bx \in |(\Gamma) .$$

(vii) $A = \forall x Bx$. Similar to (iv). Q. E. D.

5.1.15. Corollary. Let Γ be closed under deducibility. Then

$$\Gamma \text{ is } M\text{-saturated iff } \Gamma = |(\Gamma)$$
$$\text{iff } \Gamma \subseteq |(\Gamma) .$$

5.1.16. Corollary. The intuitionistic predicate calculus is saturated.

5.1.17. Corollary. $A|A$ in the sense of Aczel iff $A|A$ in the sense of Kleene (cf. § 3.1).

5.1.18. Theorem (Characterization of the Aczel slash by an inductive definition). The relation $\Gamma|A$ is inductively defined by the following:

(i) For atomic A

$$\Gamma|A \text{ iff } \Gamma \vdash A ;$$

(ii) $\Gamma | B \& C$ iff $\Gamma|B$ and $\Gamma|C$;

(iii) $\Gamma | B \vee C$ iff $\Gamma|B$ or $\Gamma|C$;

(iv) $\Gamma | B \to C$ iff $\Gamma \vdash B \to C$ and $(\Gamma|B \Rightarrow \Gamma|C)$;

(v) $\Gamma | \neg B$ iff $\Gamma \vdash \neg B$ and $\Gamma \not| B$;

(vi) $\Gamma | \exists x Ax$ iff $\Gamma | Aa$ for some $a \in C_M$;

(vii) $\Gamma | \forall x Ax$ iff $\Gamma \vdash \forall x Ax$ and $\Gamma|Aa$ for all $a \in C_M$.

Proof. (i) by definition ; (ii), (iii), and (vi) are obvious.

(iv) Let $\Gamma|B \to C$, i.e. $\alpha_0 \Vdash B \to C$. Then $\alpha_0 \Vdash B \Rightarrow \alpha_0 \Vdash C$, i.e. $\Gamma|B \Rightarrow \Gamma|C$. Since $|(\Gamma) \subseteq \Gamma$, $\Gamma \vdash B \to C$.

Conversely, if $\Gamma \not| B \to C$, i.e. $\alpha_0 \not\Vdash B \to C$, then either $\alpha_0 \Vdash B$ and $\alpha_0 \not\Vdash C$ or $\Delta \vdash B$, $\Delta \not\vdash C$ for some saturated $\Delta \supseteq \Gamma$. The latter can only be true if $\Gamma \not\vdash B \to C$; the former if $\Gamma|B$, $\Gamma \not| C$.

(v) and (vii) are similar. Q. E. D.

5.1.19 - 5.1.21. The operation $(\ ) \to (\Sigma \ )'$.

5.1.19. The Aczel slash, like the Kleene slash, may be used to prove saturation results (often called explicit definability results). In Aczel 1968, Aczel used the inductive characterization (theorem 5.1.18) to give a version of Kleene's slash - theoretic proof of the ED - property for HA. However, our interest in this chapter is primarily in the model theory and in model-theoretic proofs. Thus, let us ignore theorem 5.1.18 and reconsider what we did in proving theorem 5.1.14.

The proof of the completeness theorem involved our constructing a model of a theory Γ. We observed (i) that Γ is saturated iff it is the set of formulae forced by a minimum node of that model, and (ii) that, if we added a minimum node, we got a maximal saturated subtheory of Γ. We shall generalize the model-theoretic construction of (ii).

Let $\underline{K}$ be a Kripke model and let a language M with a non-empty set C_M of constants be given. We will let $\underline{K}'$ denote any model $(K', \leq', D', \Vdash)$ obtained by adding a new node α_0 to K such that

(i) $\alpha_0 \leq' \alpha$ for all $\alpha \in K$;

(ii) $D'\alpha_0 = C_M$;

(iii) if A is atomic, $\alpha_0 \Vdash' A \Rightarrow \alpha \Vdash A$ for all $\alpha \in K$.

Then, for $\alpha \in K$ and any formula A, $\alpha \Vdash' A$ iff $\alpha \Vdash A$. Of special interest is the case in which the implication in (iii) is replaced by an equivalence. This is the case we most often encounter.

By theorem 5.1.14, if the class of models of a theory Γ is closed under the operation $\underline{K} \rightarrow \underline{K}'$, then Γ is M-saturated. We shall give another proof of this shortly. First we must introduce another operation on models.

Let $\underline{F} = \{\underline{K}_\mu : \mu \in N\}$ be a family of Kripke models. The disjoint sum, $\Sigma\underline{F}$, of the model $\underline{K} = (K, \leq, D, \Vdash)$ defined by

(i) $K = \bigcup_{\mu \in N} K_\mu \times \{\mu\}$;

(ii) $(\alpha, \mu) \leq (\beta, \nu)$ iff $\mu = \nu$ and $\alpha \leq_\mu \beta$;

(iii) $D(\alpha, \mu) = D_\mu \alpha$;

(iv) for atomic A, $(\alpha, \mu) \Vdash A$ iff $\alpha \Vdash_\mu A$.

E.g. suppose $\underline{F}$ is the family consisting of the following models (where $C_M = \{a\}$):

$$\underline{K}_1 : \begin{array}{ccc} \beta & \{a,b\} & Pa \\ & | & \\ \alpha & \{a\} & \end{array} \qquad \underline{K}_2 : \begin{array}{ccc} \gamma & \{a,b\} & Pb \\ & | & \\ \alpha & \{a,b\} & \end{array} .$$

Then $\Sigma\underline{F}$ is the model :

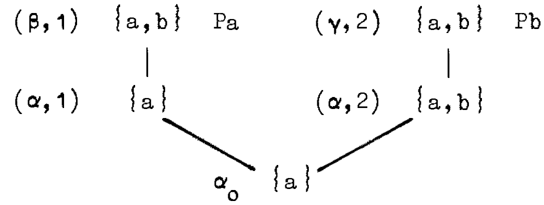
$$\underline{K}_1 + \underline{K}_2 : \begin{array}{ccc} (\beta,1) & \{a,b\} & Pa \\ & | & \\ (\alpha,1) & \{a\} & \end{array} \qquad \begin{array}{ccc} (\gamma,2) & \{a,b\} & Pb \\ & | & \\ (\alpha,2) & \{a,b\} & \end{array} .$$

The relation (iv) in the definition of $\Sigma\underline{F}$ may be shown by induction to hold for all A:

$$(\alpha, \mu) \Vdash A \quad \text{iff} \quad \alpha \Vdash_\mu A .$$

Remark. Alternatively, we may use the final remark of subsection 5.1.3 to prove this without another induction.

If $\underline{F}$ is a family of Kripke models, we can apply the two operations successively: $F \to \Sigma\underline{F} \to (\Sigma\underline{F})'$. E.g. for the family $\underline{F}$ given, $(\Sigma\underline{F})'$ is:



5.1.20. Theorem. Let the class of models of the theory Γ be closed under the operation $\underline{F} \to (\Sigma\underline{F})'$. Then Γ is M-saturated.

Proof. Let Ax contain only x free and let, for each $a \in C_M$, $\Gamma \not\vdash Aa$. Then, for each $a \in C_M$, we can find a model $\underline{K}_a$ such that $\underline{K}_a$ has a least node, say α_a, and $\alpha_a \not\Vdash Aa$.

Let $\underline{F} = \{\underline{K}_a : a \in C_M\}$ and let α_o be the least node of $(\Sigma F)'$. Suppose $\alpha_o \Vdash \exists x Ax$. Then, for some $a \in C_M = D\alpha_o$, $\alpha_o \Vdash Aa$. But $\alpha_a \geq \alpha_o$ and so $\alpha_a \Vdash Aa$, a contradiction.

(Recall that forcing at α_a in $\underline{K}_a$ is the same as that in $(\Sigma\underline{F})'$.)

Disjunction being handled similarly, we have the required result. Q. E. D.

Observing that the class of models of Γ is closed under the operation $\underline{F} \to \Sigma\underline{F}$, we have the immediate

5.1.21. Corollary. Let the class of models of the theory Γ be closed under the operation $\underline{K} \to \underline{K}'$. Then Γ is M-saturated.

Remark. The difference between using theorem 5.1.14 and theorem 5.1.20 to prove that Γ is saturated is that, to apply theorem 5.1.14, one has to show that a particular model $\underline{K}$ of Γ yields a model $\underline{K}'$ of Γ, while theorem 5.1.20 requires one to show that, for any model $\underline{K}$ of Γ, $\underline{K}'$ is a model of Γ. Model-theoretically, both tasks should be equally difficult. Theorem 5.1.18 makes the first task easier - but, it is the second approach that we will find more useful.

5.1.22 - 5.1.23. Models with equality.

5.1.22. In working with Kripke models, one may treat equality as a binary relation satisfying certain axioms. One doesn't always have the option one had in classical model theory to assume that equality is interpreted by actual identity - if equality is interpreted by identity, then $\forall xy(x = y \vee \neg x = y)$ is forced - but there are intuitionistic equality relations which are not decidable (e.g. the equality of the reals).

When equality is decidable, however, it suffices to consider the class of

<u>normal</u> <u>models</u> - i.e. models in which the equality predicate is interpreted as actual identity.

5.1.23. <u>Theorem</u>. Let Γ have a decidable equality. Then Γ is strongly complete for the class of models in which the equality of two constants is forced iff they denote the same object.

<u>Proof</u>. Let $\underline{K} = (K, \leq, D, \Vdash)$ be a model of Γ. We shall define a corres sponding normal model $\underline{K}^n$ by using the following equalence relation $\bigcup_{\alpha \in K} D\alpha$:

$$x \approx y \quad \text{iff} \quad \exists\alpha(\alpha \Vdash x = y).$$

Let $[x] = \{y : x \approx y\}$ be the equivalence class of x under $\approx$. Define $\underline{K}^n$ by

(i) $K^n = K$;

(ii) $\leq^n = \leq$;

(iii) $D^n\alpha = \{[x] : x \in D\alpha\}$; and

(iv) for atomic A, $\alpha \Vdash^n A([a_1], \ldots, [a_n])$ if $\alpha \Vdash A(a_1', \ldots, a_n')$, where $a_i' \approx a_i$ and $a_i' \in D\alpha$.

By the standard induction on the length of A, the equivalence (iv) is seen to hold for all A. Q. E. D.

Since Heyting's arithmetic has a decidable equality, we shall, in the sequel, only consider normal models.

5.1.24. <u>Function symbols</u>. Another device we could use is function symbols. While we can show proof - theoretically that function symbols are eliminable, we cannot conclude from this that the theories determined by the classes of models with and without functions coincide. (To do this, we would have to prove completeness of the theories possessing function symbols with respect to their models possessing functions.) We shall, therefore, indicate the model - theoretic proof of the eliminability of function symbols for the special case of a theory with decidable equality.

Let Γ be a theory with the language M and let M possess function symbols. An interpretation of the symbol f in a model $\underline{K}$ is given by choosing a family of functions $\{f_\alpha : \alpha \in K\}$ such that (if f is n-ary) $f_\alpha : (D\alpha)^n \to D\alpha$ and, if $\alpha \leq \beta$, $f_\beta \upharpoonright D\alpha = f_\alpha$. The interpretation of atomic formulae involving terms constructed by the use of such function symbols is handled as in classical model theory.

Suppose we now replace M by a language M' in which every n-ary function symbol is replaced by an $n+1$-ary relation symbol, as discussed in § 1.2. If Γ' is obtained by translating the axioms of Γ into M' and adding the function axioms, then, just as in classical model theory, there is a natural correspondence between models of Γ and models of Γ'.

This is proven by mimicking the classical proof. Thus we may restrict our attention to models with functions replacing certain relations (namely their graphs). The details are left to the reader.

5.1.25. Conventions. Let us finally make, in addition to our convention concerning models of HA that they be normal, a convention that they do not possess functions and the simplifying convention that they all possess minimum (or least) nodes, which we shall call origins. An origin of K will usually be denoted by α_0 and has the defining property that $\alpha_0 \leq \alpha$ for all $\alpha \in K$. (Observe that such models are not closed under $F \rightarrow \Sigma F$ and, hence, we shall have to apply theorem 5.1.20 rather than its corollary.)

5.1.26. Intuitionism?

What, one may ask, does all of this set-theoretic machinery have to do with intuitionism? We shall not attempt to answer this question - instead we merely outline how certain proofs obtained by the use of this machinery can be transformed into intuitionistically meaningful proofs. (See e.g. Mints 1969.)

The key to this transformation lies in the Hilbert - Bernays completeness theorem (cf. e.g. Kleene 1952), by which certain outwardly set-theoretic constructions may be replaced by arithmetical ones. Specifically, by arithmetizing the completeness theorem for classical logic, one can show that, for any r.e. theory T, if Con(T) is added to classical arithmetic, then a provably arithmetical model exists - i.e. there is a model with an arithmetically definable domain and arithmetically definable relations such that the translations of the axioms of T are all provable.

The same is true of the completeness theorem given above (especially in the treatment by Thomason). Thus, if we use the completeness theorem to prove (say) an independence result, we can prove the result in classical arithmetic augmented by some consistency statements. This is true of all the results of this chapter. If, in addition, the result is Π^0_2 (e.g. as in the case of an independence result), we know from a previous chapter that the proof in the classical system can be transformed into a proof in the corresponding intuitionistic system.

We shall not prove this result here, however, since most of the results we give can be obtained constructively by less devious means and since the only results which we need for our classical proofs are (i) the existence of arithmetically definable models for any r.e. theory (intuitionistic or classical) and (ii) the fact that the models are provably arithmetical if we add the statement of consistency of the theory to classical arithmetic. For classical theories, this is the Hilbert - Bernays completeness theorem.

For intuitionistic theories this almost reduces to the Hilbert - Bernays completeness theorem as follows: Observe that a Kripke model is a classical model when viewed as a structure in its own right. That is, given $\underline{K}$, Γ, and M, let M' be obtained by replacing each atomic formula $A(x_1,\ldots,x_n)$ by a new formula $A(\alpha, x_1,\ldots,x_n)$ and adding new atomic formulae $D(\alpha,x)$, $K(\alpha)$, and $\alpha \leq \beta$. (Let us assume for simplicity that there are no function symbols.) The relation $A(\alpha,x_1,\ldots,x_n)$ is to be interpreted by $\alpha \Vdash A(x_1,\ldots,x_n)$. We then translate all statements about $\underline{K}$ into M' as follows:

(i) for A atomic, $(\alpha \Vdash A(x_1,\ldots,x_n))^T = A(\alpha,x_1,\ldots,x_n)$;

(ii) $(\alpha \Vdash A \& B)^T = (\alpha \Vdash A)^T \& (\alpha \Vdash B)^T$;

(iii) $(\alpha \Vdash A \vee B)^T = (\alpha \Vdash A)^T \vee (\alpha \Vdash B)^T$;

(iv) $(\alpha \Vdash A \rightarrow B)^T = \forall\beta \geq \alpha((\beta \Vdash A)^T \rightarrow (\beta \Vdash B)^T)$;

(v) $(\alpha \Vdash \neg A)^T = \forall\beta \geq \alpha \neg(\beta \Vdash A)^T$;

(vi) $(\alpha \Vdash \exists x A x)^T = \exists x(D(\alpha,x) \& (\alpha \Vdash Ax)^T)$;

(vii) $(\alpha \Vdash \forall x A x)^T = \forall\beta \geq \alpha\ \forall x[D(\beta,x) \rightarrow (\beta \Vdash Ax)^T]$.

We define Γ' by taking, in addition to axioms asserting that we have a Kripke model (e.g. $(\alpha \Vdash A)^T \& \alpha \leq \beta \rightarrow (\beta \Vdash A)^T$), the axioms $(\alpha \Vdash A)^T$ for axioms A of Γ. Then Γ is r.e. iff Γ' is r.e. and we obtain an arithmetical model of Γ from one of Γ'. The only problem at this stage is that the provable arithmeticity of the models depends here on the consistency statement for Γ' rather than for Γ. However, this loss of precision will cause us no trouble.

§ 2. The treatment of Heyting's arithmetic

5.2.1 - 5.2.4. The operation ( ) → (Σ)'.

5.2.1. So far, aside from specializations of the form of the models used (to being normal, to not having functions, and to having origins), the only results which we have proven concern saturation or explicit definability. The result we wish to apply first to Heyting's arithmetic is theorem 5.1.20 which implies that, if we show the class of models of HA to be closed under the operation ( ) → (Σ)', then we may conclude the following

5.2.2. Theorem (Explicit definability). If Ax has only x free and $HA \vdash \exists x Ax$, then $HA \vdash An$ for some n.

5.2.3. Theorem (Disjunction property). Let A,B be closed. If $HA \vdash A \vee B$, then $HA \vdash A$ or $HA \vdash B$.

To prove this, we shall have to choose a formulation of HA. The simplest one for our purposes is the one with constants 0, 1, ... for each natural number, relations $S(x,y)$, $A(x,y,z)$, and $M(x,y,z)$ defining the functions of successor, addition, and multiplication.

Typographically, we find it convenient to reserve in this chapter the letters n, m (possibly indexed) to denote numerals (in contrast to the other chapters, where n, m usually stood for numerical variables, and numerals were written with a bar: $\bar{n}$, $\bar{m}$, $\bar{x}$, $\bar{y}$, ... etc.).

The axioms of HA are, in addition to the axioms of the predicate calculus with equality:

(i) $\neg S(x,0)$,
$\neg x = 0 \rightarrow \exists y\, S(y,x)$,
$S(x,y) \,\&\, S(x,z) \rightarrow y = z$,
$S(y,x) \,\&\, S(z,x) \rightarrow y = z$,
$\exists y\, S(x,y)$;

(ii) $A(x,y,z) \,\&\, A(x,y,w) \rightarrow z = w$,
$\exists z\, A(x,y,z)$,
$A(x,0,x)$,
$A(x,y,z) \,\&\, S(y,w) \,\&\, S(z,v) \rightarrow A(x,w,v)$;

(iii) $M(x,y,z) \,\&\, M(x,y,w) \rightarrow z = w$,
$\exists z\, M(x,y,z)$,
$M(x,0,0)$,
$M(x,y,z) \,\&\, S(y,w) \,\&\, A(z,x,v) \rightarrow M(x,w,v)$;

(iv) $S(n, n+1)$, for each constant n;

and the scheme, for any formula A whose free variables include x and do not include y:

(v) $A0 \& \forall xy(Ax \& S(x,y) \rightarrow Ay) \rightarrow \forall x Ax$.

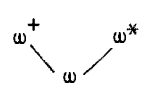
Aesthetically, it is more pleasing to use a formulation with function symbols and, as shown in 5.1.24, we may do so. However, that would require a little more care in defining various structures and a little more work in proving results about them. We shall, occasionally, however, freely use the fact that there is a natural correspondence between models of our official system above and the system with function symbols (or, if one prefers, we shall abuse notation by using function symbols).

Our first step is to prove the following

5.2.4. Theorem. The class of models of HA is closed under the operation $(\ ) \rightarrow (\Sigma\ )'$.

Recall that, in the definition of $K \rightarrow K'$, we left open the problem of deciding which atomic formulae to force at α_0 , stating that we usually have $\alpha_0 \Vdash A$ iff $\alpha \Vdash A$ for all $\alpha \in K$. (Recall also that the proof of theorem 5.1.20 merely required us to have some model of the form K' .) For HA, there is no ambiguity - closed atomic formulae are decided by the theory and, if K' is to be a model of HA, we must have $\alpha_0 \Vdash A$ iff A is true in the standard model.

Thus our operation $F \rightarrow (\Sigma F)'$ is given by tacking on a new node α_0 below all nodes of ΣF, setting $D\alpha_0 = \{0, 1, \ldots\}$, and letting $\alpha_0 \Vdash' A$ iff A is true, for any atomic A. E.g. if ω^+ and ω^* are non-standard models of classical arithmetic, then (using the graphic representation of subsection 5.1.4) $(\omega^+ + \omega^*)'$ is



Proof of theorem 5.2.4. The assertion that F is a model of HA means that every axiom of HA is valid in every member of F (i.e. forced at each node of each model in F). For $(\Sigma F)'$ not to be a model of HA, some node α of $(\Sigma F)'$ must fail to force some axiom of HA. Obviously, we cannot have $\alpha > \alpha_0$, since then $\alpha \in K$ for some $K \in F$ (making the obvious identification - i.e. ignoring the operation used to make members of F disjoint). Thus, to prove that $(\Sigma F)'$ is a model of HA, it suffices to show that $\alpha_0 \Vdash A$ for each axiom A of HA.

The only non-trivial case to consider is the induction axiom. For simplicity, we assume that Ax has only the variable x free. The general case is left to the reader (i.e. we let the reader verify the validity of the universal closure of the scheme with free variables).

Let $\alpha_o \Vdash\!\!\!\!/\ A0 \,\&\, \forall xy(Ax \,\&\, S(x,y) \to Ay) \to \forall xAx$. Then, for some $\beta \geq \alpha_o$, $\beta \Vdash A0 \,\&\, \forall xy(Ax \,\&\, S(x,y) \to Ay)$, but $\beta \Vdash\!\!\!\!/\ \forall xAx$. Now we cannot have $\beta > \alpha_o$, since then $\beta \in K$ for some $\underline{K} \in \underline{F}$ and β forces all axioms of $\underset{\sim}{HA}$. Hence $\alpha_o \Vdash A0 \,\&\, \forall xy(Ax \,\&\, S(x,y) \to Ay)$, but $\alpha_o \Vdash\!\!\!\!/\ \forall xAx$. Since $\alpha_o \Vdash\!\!\!\!/\ \forall xAx$, there is some $\beta \geq \alpha_o$ and some $b \in D\beta$ such that $\beta \Vdash\!\!\!\!/\ Ab$. Again $\beta = \alpha_o$ and b is some natural number. Let m be the smallest such number. Since $\alpha_o \Vdash A0$, m is a successor, say $n+1$, and, since m is the smallest number b such that $\alpha_o \Vdash\!\!\!\!/\ Ab$, $\alpha_o \Vdash An$. But $\alpha_o \Vdash \forall xy(Ax \,\&\, S(x,y) \to Ay)$, whence $\alpha_o \Vdash An \,\&\, S(n, n+1) \to A(n+1)$. Thus $\alpha_o \Vdash A(n+1)$, i.e. $\alpha_o \Vdash Am$, a contradiction. Q. E. D.

Theorems 5.2.4 and 5.1.20 immediately yield theorems 5.2.2 and 5.2.3 as corollaries.

5.2.5 - 5.2.7. Applications of the operation $(\ ) \to (\Sigma\ )'$.

5.2.5. The closure of the class of models of $\underset{\sim}{HA}$ under $(\ ) \to (\Sigma\ )'$ is one of the basic tools of the Kripke model approach to studying $\underset{\sim}{HA}$. E.g. we have already used this to prove ED, the explicit definability property. Its use here is simply that it allows us to take countermodels to $A0, A1, \ldots$ and put them together to construct a countermodel to $\exists xAx$. It is in this construction of models that this operation is so useful. Consider, e.g., the old result of Kreisel's (Kreisel 1958):

5.2.6. Theorem. Let Ax have only x free and suppose $\vdash \forall x(Ax \vee \neg Ax)$, $\vdash$ denoting derivability in $\underset{\sim}{HA}$. Then

$$\vdash \forall xAx \vee \exists x \neg Ax \quad \text{iff} \quad \vdash \neg \forall xAx \to \exists x \neg Ax$$
$$\text{iff} \quad \vdash \exists y[\neg \forall xAx \to \neg Ay].$$

Proof. (Cf. also 3.8.5). We shall show that $\vdash \neg \forall xAx \to \exists x \neg Ax$ implies $\vdash \exists y[\neg \forall xAx \to \neg Ay]$ and leave the rest to the reader. Suppose $\vdash \neg \forall xAx \to \exists x \neg Ax$ and $\not\vdash \exists y[\neg \forall xAx \to \neg Ay]$. Then, $\not\vdash \neg \forall xAx \to An$ for each n. But $\vdash \neg An \to (\neg \forall xAx \to \neg An)$, whence $\not\vdash \neg An$. By the decidability of A and the DP, $\vdash An$.

On the other hand, $\vdash \forall xAx \to [\neg \forall xAx \to \neg A0]$, and so $\not\vdash \forall xAx$.

Let $\underline{K}$ be a model of $\underset{\sim}{HA}$ with $\alpha \in K$ such that $\alpha \Vdash\!\!\!\!/\ \forall xAx$. Then $\exists \beta \geq \alpha \, \exists b \in D\beta \ \ \beta \Vdash\!\!\!\!/\ Ab$. By decidability , $\beta \Vdash \neg Ab$ and hence $\beta \Vdash \neg \forall xAx$.

Now consider $\underline{K}_\beta$ (recall the definition from subsection 5.1.3) and especially $(\underline{K}_\beta)'$:

$$\begin{array}{c} \underline{K}_\beta \\ | \\ \omega \end{array} .$$

Observe that $\gamma \geq \alpha_o$ implies $\gamma = \alpha_o$ or $\gamma \geq \beta$ and that $\gamma \geq \beta$ implies $\gamma \Vdash \neg \forall x Ax$. Also, $\alpha_o \not\Vdash \forall x Ax$ since, if $\alpha_o \Vdash \forall x Ax$, then $\beta \Vdash \forall x Ax$ and one has a contradiction. Thus $\gamma \not\Vdash \forall x Ax$ for all $\gamma \geq \alpha_o$ and $\alpha_o \Vdash \neg \forall x Ax$. We now use the fact that $\vdash \neg \forall x Ax \rightarrow \exists x \neg Ax$ to conclude $\alpha_o \Vdash \exists x \neg Ax$. But $D\alpha_o = \{0, 1, \ldots\}$ and, for some n, $\alpha_o \Vdash \neg An$, a contradiction. Q. E. D.

5.2.7. We have not really used the basic operation $(\ ) \rightarrow (\Sigma\ )'$ in the direct construction of models. We turn our attention now to this task.

Let $\underline{T} = (T, \leq)$ be a finite tree. By a <u>terminal node</u> of the tree we shall mean a maximal node of the tree - i.e. a node with no successors. We shall let Ter denote the set of terminal nodes of T. For any node $\alpha \in T - \text{Ter}$, $S(\alpha)$ will denote the set of successors of α.

Let us assume that we have assigned models of classical arithmetic to each of the terminal nodes - say ω_α is assigned to $\alpha \in \text{Ter}$. We now associate with each $\alpha \in T$ a Kripke model $\underline{K}(\alpha)$ as follows:

(i) if $\alpha \in \text{Ter}$, $\underline{K}(\alpha) = \omega_\alpha$ (viewed as a one-node Kripke model);

(ii) if $\alpha \notin \text{Ter}$, $\underline{K}(\alpha) = (\sum_{\beta \in S(\alpha)} \underline{K}(\beta))'$.

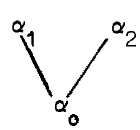
Finally, define $\underline{K}_{\underline{T}} = \underline{K}(\alpha_o)$, where α_o is the origin of $\underline{T}$.

5.2.8. <u>Theorem</u>. $\underline{K}_{\underline{T}}$ is a model of $\underset{\sim}{HA}$.

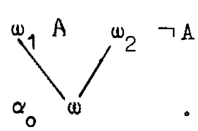
<u>Proof</u>. We show by bar induction that $\underline{K}(\alpha)$ is a model of $\underset{\sim}{HA}$. The theorem is trivial for terminal nodes. If α is not terminal, apply theorem 5.2.4. It follows that $\underline{K}(\alpha)$ is a model of $\underset{\sim}{HA}$ for all $\alpha \in K$. Letting $\alpha = \alpha_o$ completes the proof. Q. E. D.

<u>Note</u>. Obviously, we may replace the finiteness restriction on $\underline{T}$ by the well-foundedness restriction.

As an example, we know by Gödel's theorem that there is an independent sentence A of classical arithmetic. Thus there are models ω_1 and ω_2 of A and $\neg A$, respectively. Associating these models with the terminal nodes of the tree,

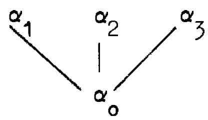
$\underline{T}$:  ,

we have the model:

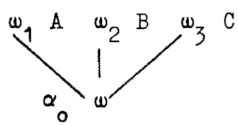
$\underline{K}_{\underline{T}}$:  .

Observe e.g. that $\alpha_o \not\Vdash A \vee \neg A$.

A stronger version of Gödel's theorem allows us, for any n, to find Σ^0_1 sentences $A_1, \dots, A_n$ which are mutually independent over classical arithmetic. In particular, we can find models $\omega_1, \dots, \omega_n$ such that A_j is true in ω_i iff $i = j$. (We shall discuss this further in section 3.) Letting $n = 3$ and relabelling A_1, A_2, A_3 as A, B, and C, let ω_1, ω_2, and ω_3 be associated with the terminal nodes of the tree

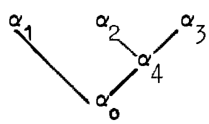
$\underline{T}$: 

Then $\underline{K}_{\underline{T}}$ is

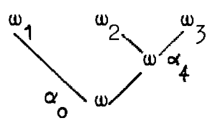


Observe that $\alpha_0 \not\Vdash (\neg A \to B \vee C) \to ((\neg A \to B) \vee (\neg A \to C))$. (See chapter III, section 2.26 for an application.)

Let ω_1, ω_2, and ω_3 be as in the preceding example and let $\underline{T}$ be:



Associating ω_1, ω_2, and ω_3 with α_1, α_2, and α_3, we have

$\underline{K}_{\underline{T}}$: 

Observe that, although α_0 and α_4 both have copies of ω associated with them, they do not behave alike, e.g. $\alpha_4 \Vdash \neg A, \neg\neg(B \vee C)$, but $\alpha_0 \not\Vdash \neg A, \neg\neg(B \vee C)$.

5.2.9 - 5.2.12. <u>Formulae preserved under</u> $(\ ) \to (\Sigma\ )'$.

5.2.9. If Γ is a set of sentences, we may ask whether or not various metamathematical properties of $\underset{\sim}{HA}$ also hold for $\underset{\sim}{HA} + \Gamma$. For instance, one may ask whether or not the explicit definability theorem holds for $\underset{\sim}{HA} + \Gamma$ or whether or not $\underset{\sim}{HA} + \Gamma$ is closed under the derived rules given by theorem 5.2.6. Since the only property used in deriving these properties of $\underset{\sim}{HA}$ is the closure of the class of models of $\underset{\sim}{HA}$ under the operation $(\ ) \to (\Sigma\ )'$, to prove these results for $\underset{\sim}{HA} + \Gamma$, we need only show that the class of models of $\underset{\sim}{HA} + \Gamma$ is closed under this basic operation

Of course, to prove explicit definability, one could use the Aczel slash - its inductive definition makes it fairly usable. The operation $(\ ) \to (\Sigma\ )'$

has the advantage that, if Γ and Δ are preserved by it, then $\Gamma + \Delta$ is preserved - i.e. if the validity of $\underset{\sim}{HA} + \Gamma$ is preserved by the operation $\underline{F} \to (\Sigma\underline{F})'$ and if the same holds of $\underset{\sim}{HA} + \Delta$, then $\underset{\sim}{HA} + \Gamma + \Delta$ is also preserved by this operation. Thus, the class of sets of formulae preserved by this operation exhibit better closure properties than the class of sets, Γ, of formulae which yield saturated extensions, $\underset{\sim}{HA} + \Gamma$, of $\underset{\sim}{HA}$.

5.2.10. Lemma. Let the sentence A have no strictly positive $\vee$ or $\exists$ (i.e. A is a Harrop sentence, see 1.10.5). Then A is preserved under the operation $(\ ) \to (\Sigma\ )'$.

Proof. We shall prove this by induction on the length of A. To carry out the induction step corresponding to (v), we must make a convention involving free variables. Let A have $x_1, \dots, x_n$ as free variables - we shall prove that $A(m_1, \dots, m_n)$ is preserved for all numbers $m_1, \dots, m_n$. The result then follows trivially for sentences.

(i) The preservation of atomic formulae follows by the decidability of atomic formulae in $\underset{\sim}{HA}$.

(ii) Let $A(m_1, \dots, m_n) \& B(m_1, \dots, m_n)$ be valid in $\underline{F}$. Then $A(m_1, \dots, m_n)$ and $B(m_1, \dots, m_n)$ are valid in $\underline{F}$. But each of these is preserved under $\underline{F} \to (\Sigma\underline{F})'$, whence $A \& B$ is valid in $(\Sigma\underline{F})'$.

(iii) Let $A(m_1, \dots, m_n) \to B(m_1, \dots, m_n)$ be valid in $\underline{F}$. For this implication to fail to be valid in $(\Sigma\underline{F})'$, we must have $\alpha_0 \Vdash A(m_1, \dots, m_n)$, $\alpha_0 \not\Vdash B(m_1, \dots, m_n)$. But then $A(m_1, \dots, m_n)$ is valid in $\underline{F}$, whence $B(m_1, \dots, m_n)$ is valid in $\underline{F}$. Again $B(m_1, \dots, m_n)$ is preserved, whence $\alpha_0 \Vdash B(m_1, \dots, m_n)$, a contradiction.

(iv) Similar to (iii).

(v) Let $A(x, m_1, \dots, m_n)$ be given, $\forall x A(x, m_1, \dots, m_n)$ valid in $\underline{F}$. For $\forall x A(x, m_1, \dots, m_n)$ to fail to be valid in $(\Sigma\underline{F})'$, we must have $\alpha_0 \not\Vdash A(m, m_1, \dots, m_n)$ for some $m \in D\alpha_0 = \{0, 1, \dots\}$. But $A(m, m_1, \dots, m_n)$ is valid in $\underline{F}$ and is preserved, leading to a contradiction. Q.E.D.

5.2.11. Theorem. The class $\mathfrak{P}$ of sets, Γ, such that the validity of $\underset{\sim}{HA} + \Gamma$ is preserved by the operation $(\ ) \to (\Sigma\ )'$ has the following closure properties:

(i) $\mathfrak{P}$ is closed under arbitrary union;

(ii) if $\Gamma \in \mathfrak{P}$ and A is a Harrop-sentence, then $\Gamma \cup \{A\} \in \mathfrak{P}$;

(iii) if $\Gamma \in \mathfrak{P}$, A has only the variable x free, and $\underset{\sim}{HA} + \Gamma \vdash An$ for each numeral n, then $\Gamma \cup \{\forall x A x\} \in \mathfrak{P}$.

Proof. The only case we haven't proven already is (iii). The proof of this is basically the same as that of case (v) in the preceding proof.

5.2.12. Corollary. (Friedman A) Let $\Gamma \in \mathfrak{P}$. Then ED and DP hold for $\underset{\sim}{HA} + \Gamma$.

5.2.13 - 5.2.23. Examples. Reflection principles and transfinite induction.

5.2.13. Condition (iii) in the definition of $\mathfrak{P}$ was introduced in Friedman A for the purpose of proving results like corollary 5.2.12. By it, if we have an axiom scheme for which we wish to prove a preservation theorem, we need only prove the theorem for the scheme without free variables. For induction,

$$A0 \,\&\, \forall xy(Ax \,\&\, S(x,y) \rightarrow Ay) \rightarrow Ax\ ,$$

we need only prove the preservation result for each instance,

$$A0 \,\&\, \forall xy(Ax \,\&\, S(x,y) \rightarrow Ay) \rightarrow An\ .$$

If we examine the proof we gave, we notice that we reduced the problem to proving the preservation of this last sentence. We shall now consider some further schemata and apply condition (iii) to prove preservation theorems for them.

Let $<$ be a primitive recursive (or even provably decidable - i.e. $\Vdash\!\!\!\!/\ \vdash x<y \vee \neg x<y$) well-ordering of the natural numbers. By the scheme, $TI(<)$, of transfinite induction on $<$ is meant the following:

$$A0 \,\&\, \forall x[\forall y<x Ay \rightarrow Ax] \rightarrow \forall x Ax\ ,$$

where, for convenience, 0 is taken to be the first element of the ordering.

5.2.14. Lemma. Let Γ be the subscheme of $TI(<)$ determined by the restriction that Ax have only x free. Then the preservation theorem holds for $\underset{\sim}{HA} + \Gamma$.

Proof. Let $\underline{F}$ be a family of models of $\underset{\sim}{HA} + \Gamma$ and observe that $<$ is a genuine well-ordering on ω. Thus, if Γ is not valid in $(\Sigma\underline{F})'$, we have

$$\alpha_0 \Vdash A0 \,\&\, \forall x[\forall y<x Ay \rightarrow Ax]\ ,$$
$$\alpha_0 \not\Vdash An\ .$$

Letting n_0 be the least such n, $\alpha_0 \Vdash Am$ for all $m<n_0$ and $\beta \Vdash \forall x Ax$ for all $\beta > \alpha_0$, whence $\alpha_0 \Vdash \forall y<n_0 Ay$, whence $\alpha_0 \Vdash An_0$, a contradiction.

Q. E. D.

5.2.15. Theorem. The scheme $TI(<)$ is preserved.

Proof. Let Γ be as in lemma 5.2.14 and let $B(x_1,\ldots,x_n)$ denote the instance,

$$A(0,x_1,\ldots,x_n) \,\&\, \forall x[\forall y<x A(y,x_1,\ldots,x_n) \rightarrow A(x,x_1,\ldots,x_n)] \rightarrow \forall x A(x,x_1,\ldots,x_n)\ ,$$

if $TI(<)$. For each choice $m_1,\ldots,m_n$ of numerals to replace

$x_1,\ldots,x_n$, $B(m_1,\ldots,m_n) \in \Gamma$, whence $\underset{\sim}{HA} + \Gamma \vdash B(m_1,\ldots,m_n)$. It follows by condition (iii) that $\underset{\sim}{HA} + \Gamma + B \in \mathfrak{P}$. Thus $\underset{\sim}{HA} + TI(<) = \bigcup \{\underset{\sim}{HA} + \Gamma + B \mid B \in TI(<)\} \in \mathfrak{P}$. Q.E.D.

5.2.16. Corollary. Let $\underset{\sim}{T}$ extend $\underset{\sim}{HA}$ by the addition of some schemata of transfinite induction on primitive recursive well-orderings. Then $\underset{\sim}{T}$ satisfies DP and ED.

5.2.17. To discuss the next set of schemata, let, for an r.e. extension $\underset{\sim}{T}$ of $\underset{\sim}{HA}$, $Proof_T(x,y)$ be the canonical proof predicate. The properties of $Proof_T(x,y)$ which we use are that

(i) $Proof_T$ is decidable, and

(ii) $\exists n\ \underset{\sim}{HA} \vdash Proof_T(n, \ulcorner A \urcorner)$ iff $\underset{\sim}{T} \vdash A$,

where $\ulcorner A \urcorner$ is the gödel number of A. If A contains the free variable y, we let $\ulcorner A\bar{y} \urcorner$ denote $s(y, \ulcorner A \urcorner)$, where s is a primitive recursive function such that

$$s(n, \ulcorner A \urcorner) = \ulcorner [y/n]A \urcorner \text{ is the gödel number of the sentence}$$

obtained by replacing the variable y in A by the numeral n.

We may use this notation to list the following schemata:

Local reflection for $\underset{\sim}{T}$, $RF(\underset{\sim}{T})$:

$RF(T)$ $\quad \exists x\ Proof_T(x, \ulcorner A \urcorner) \to A$, for sentences A.

Uniform reflection for $\underset{\sim}{T}$, $RFN(\underset{\sim}{T})$:

$RFN(T)$ $\quad \forall y[\exists x\ Proof_T(x, \ulcorner A\bar{y} \urcorner) \to Ay]$, for A containing only y free.

Uniform' reflection for $\underset{\sim}{T}$, $RFN'(\underset{\sim}{T})$:

$RFN'(T)$ $\quad \forall y\ \exists x\ Proof_T(x, \ulcorner A\bar{y} \urcorner) \to \forall y Ay$, for A containing only y free.

Consistency of $\underset{\sim}{T}$, $CON(T)$:

$CON(T)$ $\quad \neg \exists x\ Proof_T(x, \ulcorner 0=1 \urcorner)$.

ω - Consistency of $\underset{\sim}{T}$, ω - C $(\underset{\sim}{T})$:

ω-$CON(T)$ $\quad \exists x\ Proof_T(x, \ulcorner \neg \forall y Ay \urcorner) \to \neg \forall y\ \exists x\ Proof_T(x, \ulcorner A\bar{y} \urcorner)$, for A containing only y free.

Feferman 1962, theorem 2.19 gives an intuitionistic proof of the following:

5.2.18. Lemma. The schemata $RFN(\underset{\sim}{T})$ and $RFN'(\underset{\sim}{T})$ are equivalent.

Thus, we need not consider $RFN'(\underset{\sim}{T})$. For the relative strengths of these reflection principles, see Feferman 1962 and Kreisel - Levy 1968.

5.2.19. Theorem. $CON(\underset{\sim}{T})$ and ω - $CON(\underset{\sim}{T})$ are preserved by the operation $(\ ) \to (\Sigma\ )'$.

Proof. $CON(\underset{\sim}{T})$ and ω - $CON(\underset{\sim}{T})$ have no strictly positive $\vee$ or $\exists$. Q.E.D.

5.2.20. Lemma. Let A be a sentence. If $\underset{\sim}{T} \vdash A$, then $\underset{\sim}{HA} + RF(\underset{\sim}{T}) \vdash A$.
Proof. Observe $\underset{\sim}{T} \vdash A$ implies $\underset{\sim}{HA} \vdash \exists x\, Proof_T(x, \ulcorner A \urcorner)$. $RF(\underset{\sim}{T})$ yields $\underset{\sim}{HA} + RF(\underset{\sim}{T}) \vdash A$. Q. E. D.

5.2.21. Theorem. If T is preserved by the operation $(\ ) \to (\Sigma\ )'$, then so is $\underset{\sim}{HA} + RF(\underset{\sim}{T})$.
Proof. Let $\underline{F}$ be a family of models of $\underset{\sim}{HA} + RF(\underset{\sim}{T})$ and let $(\Sigma\underline{F})'$ fail to be a model of $\underset{\sim}{HA} + RF(\underset{\sim}{T})$. Then

$$\alpha_0 \Vdash \exists x\ Proof_T(x, \ulcorner A \urcorner),\quad \alpha_0 \nVdash A,$$

for some sentence A. Thus, for some n, $\alpha_0 \Vdash Proof_T(n, \ulcorner A \urcorner)$.
But $Proof_T$ is decidable, whence $\underset{\sim}{HA} \vdash Proof_T(n, \ulcorner A \urcorner)$ and $\underset{\sim}{T} \vdash A$.
By lemma 2.4.6, $\underset{\sim}{T}$ is valid in $\underline{F}$, whence, by hypothesis, $\underset{\sim}{T}$ is valid in $(\Sigma\underline{F})'$. Thus $\alpha_0 \Vdash A$, a contradiction. Q. E. D.

5.2.22. Corollary. If $\underset{\sim}{T}$ is preserved by the operation $(\ ) \to (\Sigma\ )'$, then so is $\underset{\sim}{HA} + RFN(\underset{\sim}{T})$.

5.2.23. Corollary. If T is preserved by the operation $(\ ) \to (\Sigma\ )'$, then so are $\underset{\sim}{T} + RF(\underset{\sim}{T})$, $\underset{\sim}{T} + RFN(\underset{\sim}{T})$, $\underset{\sim}{T} + RF(\underset{\sim}{T} + RF(\underset{\sim}{T}))$, etc.

§ 3. Additional information from $(\ )\to(\Sigma\ )'$: de Jongh's theorem.

5.3.1. Statement of de Jongh's theorem.

In addition to its use in proving explicit definability results and the validity of an occasional derived rule, we observed in 5.2.4 that we could use the operation $(\ )\to(\Sigma\ )'$ to construct Kripke models of $\underset{\sim}{HA}$ out of models of classical arithmetic. This last application has, as a corollary, a simple proof of the propositional case of an interesting theorem of de Jongh. In the sequel, Pp denotes intuitionistic propositional logic.

Let $A(p_1,\dots,p_n)$ be a propositional formula constructed from the propositional variables $p_1,\dots,p_n$. In an as yet unpublished paper (de Jongh A, see de Jongh 1970), D.H.J. de Jongh proved the following

5.3.2. Theorem. If $\mathrm{Pp} \not\vdash A(p_1,\dots,p_n)$, then $\underset{\sim}{HA} \not\vdash A(B_1,\dots,B_n)$, for some sentences $B_1,\dots,B_n$ of arithmetic.

According to this theorem, if $A(p_1,\dots,p_n)$ is not an intuitionistic tautology, there are arithmetical substitution instances resulting in a sentence underivable in $\underset{\sim}{HA}$. Alternatively, we can view this as a completeness result if we define the validity of a formula $A(p_1,\dots,p_n)$ in $\underset{\sim}{HA}$ to be the validity of the scheme $A(B_1,\dots,B_n)$ determined by $A(p_1,\dots p_n)$.

Actually, de Jongh proved a stronger result: The choice of substitution instances $B_1,\dots,B_n$ of $p_1,\dots,p_n$ can be made uniformly in all $A(p_1,\dots,p_n)$. A proof of this by means of Kripke models is more difficult and will be given in section 6.

Another result of de Jongh's is a completeness theorem for the predicate calculus. To date, the only proof of this result is de Jongh's original proof, which combines the use of Kripke models and realizability.

5.3.3 - 5.3.8. Preliminaries on the propositional calculus.

5.3.3. The proof of the completeness theorem given in 5.1.6 - 5.1.11 specializes easily to the propositional calculus. Kripke's original proof (Kripke 1965) also yields the completeness (but not strong completeness) of the intuitionistic propositional calculus, Pp, for the class of models whose underlying pms is a finite tree. Our first task is to retrieve this result. We do this by starting with a countermodel to a formula A and pluck out finitely many nodes needed to falsify A, splitting and ordering them into a tree in the process.

We will let σ, τ denote finite sequences. $\langle\ \rangle$ denotes the empty sequence. $\langle a\rangle$ denote the sequence whose only element is a. $\sigma * \tau$ will denote the concatenation of σ,τ - i.e. if $\sigma = \langle s_1,\dots,s_m\rangle$, $\tau = \langle t_1,\dots,t_n\rangle$, then $\sigma * \tau = \langle s_1,\dots,s_m,t_1,\dots,t_n\rangle$. In particular, $\sigma * \langle a\rangle = \langle s_1,\dots,s_m,a\rangle$.

5.3.4. Theorem. (Finite tree theorem.) Let $(K,\leq,\Vdash)$ be a model with origin α_0 such that $\alpha_0 \nVdash A$. Then there is a finite tree model $(K^*,\leq^*,\Vdash^*)$ such that $\langle\ \rangle \nVdash^* A$.

Proof. Let S be the set of subformulae of A, and, for $\beta\in K$, let $S(\beta) = \{B\in S : \beta \Vdash B\}$.

Set $\beta_{\langle\,\rangle} = \alpha_0$.

Given β_σ, let $\beta_{\sigma*\langle 1\rangle},\dots,\beta_{\sigma*\langle k\rangle}$ be a maximal set of $\gamma_1,\dots,\gamma_k$ such that

(i) $\beta_\sigma \leq \gamma_i$ for all i,

(ii) $S(\beta_\sigma) \neq S(\gamma_i)$ for all i,

(iii) if $\beta_\sigma \leq \gamma \leq \gamma_i$, then $S(\gamma) = S(\beta_\sigma)$ or $S(\gamma) = S(\gamma_i)$, and

(iv) $S(\gamma_i) \neq S(\gamma_j)$ for $i \neq j$.

Now let $K^* = \{\sigma : \beta_\sigma$ has been defined$\}$, and let $\leq^*$ be the usual tree ordering. For atomic B, define $\sigma \Vdash^* B$ iff $\beta_\sigma \Vdash B$.

We prove by induction on the length of B, for $B\in S$, that $\sigma \Vdash^* B$ iff $\beta_\sigma \Vdash B$.

(i) The atomic case follows by definition.

(ii)-(iii). Let B be $C \& D$ or $C \vee D$. The proofs are trivial.

(iv) Let B be $C\to D$. Let $\sigma \nVdash^* C\to D$.

Then there is a $\tau \geq \sigma$ such that $\tau \Vdash^* C$, $\tau \nVdash^* D$. But then $\beta_\tau \Vdash C$, $\beta_\tau \nVdash D$ and, since $\beta_\sigma \leq \beta_\tau$, $\beta_\sigma \nVdash C\to D$.

Conversely, let $\beta_\sigma \nVdash C\to D$.

Case 1. $\beta_\sigma \Vdash C$. Then $\beta_\sigma \nVdash D$ and $\sigma \Vdash^* C$, $\sigma \nVdash^* D$. Then $\sigma \nVdash^* C\to D$.

Case 2. $\beta_\sigma \nVdash C$. Then there is a γ such that $\beta_\sigma \leq \gamma$, $\gamma \Vdash C$, $\gamma \nVdash D$. But, by construction, there is a $\tau \geq^* \sigma$ such that $S(\beta_\tau) = S(\gamma)$ and so $C\in S(\beta_\tau)$, $D \notin S(\beta_\tau)$. Thus $\tau \Vdash^* C$, $\tau \nVdash^* D$ and $\sigma \nVdash^* C\to D$.

(v) Let B be $\neg C$. The proof is similar to (iv). Q. E. D.

5.3.5. Corollary (Kripke). Pp is complete for the class of finite tree models, i.e. $\mathrm{Pp} \nvdash A$ iff A has a countermodel in a finite tree.

We shall find it convenient to work with a special class of trees. To prove completeness for them, we prove the following result (which generalizes a result of Gabbay 1969 B).

5.3.6. Theorem (Extension theorem). Let $(K_0,\leq_0)$ be a finite subtree of the finite tree $(K_1,\leq_1)$. Let $\Vdash_0$ be a forcing relation defined on $(K_0,\leq_0)$. Then there is a forcing relation $\Vdash_1$ on $(K_1,\leq_1)$ such that, for all $\alpha\in K$ and all formulae A,

$$\alpha \Vdash_0 A \text{ iff } \alpha \Vdash_1 A .$$

<u>Note</u>. By "subtree" we do not merely mean "tree which is a subordering of" - the result is false in this case. The successors of a node $\alpha \in K_o$ must be successors in the tree $(K_1, \leq_1)$. For convenience, we also require the origins of the two trees to coincide.

<u>Proof</u>. For $\alpha \in K_o$ and atomic A, define $\alpha \Vdash_1 A$ iff $\alpha \Vdash_o A$. For each $\beta \in K_o$, choose a terminal node $t_\beta \geq \beta$ in the tree $(K_o, \leq_o)$. Let $\alpha \in K_1 - K_o$. Then there is a maximum $\beta \in K_o$ such that $\alpha \geq_1 \beta$. Define, for atomic A, $\alpha \Vdash_1 A$ iff $t_\beta \Vdash_o A$.

We now show, for all A,

(i) if $\alpha \in K_o$, , $\alpha \Vdash_1 A$ iff $\alpha \Vdash_o A$,

(ii) if $\alpha \in K_1 - K_o$, $\alpha \Vdash_1 A$ iff $t_\beta \Vdash_o A$,

where t_β is defined as above.

(i) For atomic A, the result follows by definition.

(ii) - (iii) The cases $A = B \& C$, $B \vee C$ are trivial

(iv) Let $A = B \to C$.

(a) Let $\alpha \Vdash_o B \to C$, $\beta \geq_1 \alpha$ such that $\beta \Vdash_1 B$. If $\beta \in K_o$, $\beta \Vdash_o B$ by induction hypothesis and so $\beta \Vdash_o C$. Thus $\beta \Vdash_1 C$. If $\beta \in K_1 - K_o$, we have, for some $\gamma \leq_1 \beta$, $\beta \Vdash_1 B$ iff $t_\gamma \Vdash_o B$. Now $t_\gamma \geq \gamma \geq \alpha$ (since the predecessors of β are linearly ordered) and so $t_\gamma \Vdash_o C$, whence the induction hypothesis yields $\beta \Vdash_1 C$. Hence $\beta \geq_1 \alpha$ implies that, if $\beta \Vdash_1 B$, then $\beta \Vdash_1 C$ and we have $\alpha \Vdash_1 B \to C$.

(b) Let $\alpha \nVdash_o B \to C$. Then there is a $\beta \in K_o$, $\beta \geq_o \alpha$ such that $\beta \Vdash_o B$, $\beta \nVdash_o C$. Then $\beta \Vdash_1 B$, $\beta \nVdash_1 C$ and $\alpha \nVdash_1 B \to C$.

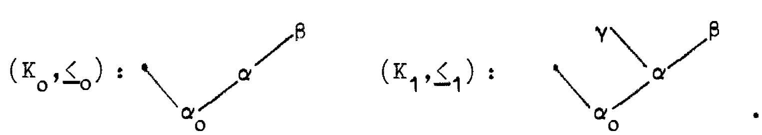
(c) Let $\alpha \in K_1 - K_o$. Let $\beta \geq_1 \alpha$ and let $\gamma \in K_o$ be maximal such that $\gamma \leq_1 \alpha$. Then γ is maximal in K_o such that $\gamma \leq_1 \beta$. By induction hypothesis, $\alpha, \beta \Vdash_1 B$ iff $t_\gamma \Vdash_o B$ and $\alpha, \beta \Vdash_1 C$ iff $t_\gamma \Vdash_o C$.

$$\begin{aligned} \alpha \Vdash_1 B \to C \quad & \text{iff} \quad \forall \beta \geq_1 \alpha (\beta \Vdash_1 B \Rightarrow \beta \Vdash_1 C) \\ & \text{iff} \quad \alpha \Vdash_1 B \Rightarrow \alpha \Vdash_1 C \\ & \text{iff} \quad t_\gamma \Vdash_o B \Rightarrow t_\gamma \Vdash_o C \\ & \text{iff} \quad t_\gamma \Vdash_o B \to C, \end{aligned}$$

since, for terminal t_γ, the forcing semantics is the same as in classical logic.

(v) The proof for negation is similar to that for case (iv). Q. E. D.

E.g. Consider the trees $(K_o, \leq_o)$ and $(K_1, \leq_1)$:



If we embed $(K_o, \leq_o)$ in $(K_1, \leq_1)$ in the obvious manner, and if we have a forcing relation $\Vdash_0$ on $(K_0, \leq_0)$, in order to extend to a relation $\Vdash_1$, we must decide how γ is to behave. We cannot necessarily let γ behave like α, because α has an extra node beyond it which may affect α's behavior. However, we can make γ behave like any terminal node beyond α - which in this case is β. Now, α cannot distinguish γ from β and hence α behaves the same in $(K_0, \leq_0, \Vdash_0)$ and $(K_1, \leq_1, \Vdash_1)$.

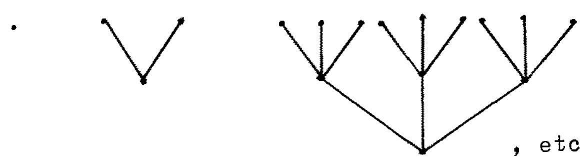
Note. In this proof, we need only assume $(K_0, \leq_0)$ is finite. The theorem holds when both $(K_0, \leq_0)$ and $(K_1, \leq_1)$ are infinite. In this case, the terminal nodes are replaced by complete sequences (in the sense of Cohen 1966).

Theorem 5.3.6 has the immediate corollary:

5.3.7. Corollary. Let $\{(K_n, \leq_n)\}_n$ be a sequence of finite trees with the property that every finite tree $(K, \leq)$ can be embedded as a subtree of some $(K_n, \leq_n)$. Then $P\rho$ is complete for the sequence $(K_n, \leq_n)$.

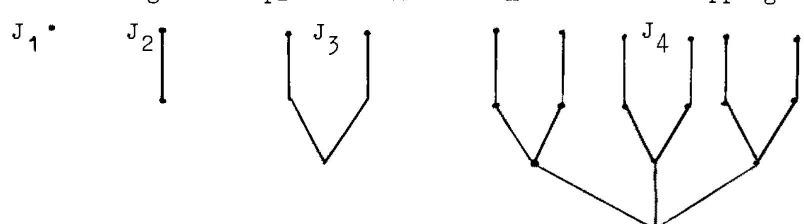
5.3.8. Examples.

A) The diagonal sequence. This is the sequence whose n-th element $(n \geq 1)$ is n-ary and of height n:



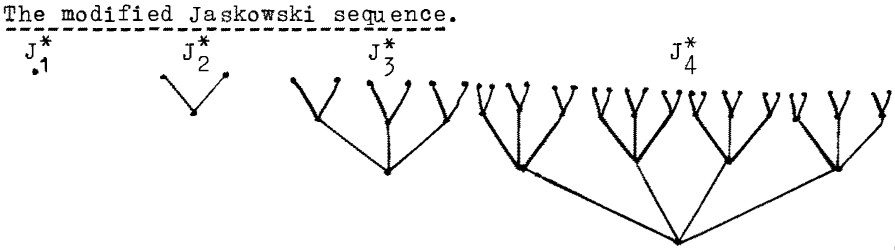
, etc.

B) The Jaskowski sequence. This economical sequence, due to Jaskowski (Jaskowski 1936) (see also Rose 1953, Gal, Rosser, and Scott 1958, Scott 1957, Gabbay 1969 , and Mostowski 1966. Gabbay 1969 gives a treatment similar to ours for this sequence.), is obtained by letting the $n+1$-st tree be the result of taking n copies of the n-th tree and dropping a node below them:



, etc.

C) The modified Jaskowski sequence.



, etc.

Let us finish this subsection by remarking on a useful property of the modified Jaskowski trees (a property shared, incidentally by the diagonal trees): Every node of J_n^* is determined by the terminal nodes lying beyond it. From this, we can prove the following lemma:

5.3.9. <u>Lemma</u>. Let $\alpha_1,\dots,\alpha_{n!}$ be the terminal nodes of J_n^*. Suppose we have a forcing relation, $\Vdash$, defined on J_n^* such that, for each i, there is a formula A_i such that

$$\alpha_j \Vdash A_i \quad \text{iff} \quad j = i .$$

Then, if S is a set of nodes of J_n^* such that $\alpha \in S$ and $\alpha \leq \beta$ imply $\beta \in S$, there is a formula A constructed from the A_i's, for which $S = \{\alpha : \alpha \Vdash A\}$. In particular, for any α, there is an A_α such that $\{\beta : \beta \geq \alpha\} = \{\beta : \beta \Vdash A_\alpha\}$.

<u>Proof</u>. Since $\alpha \in J_n^*$ is determined by those $\alpha_i \geq \alpha$, it is also determined by those $\alpha_i \not\geq \alpha$. Let $A_\alpha = \bigwedge\!\!\!\bigwedge_{\alpha_i \not\geq \alpha} \neg A_i$ for *) $\alpha \neq \alpha_0$ and let $A_{\alpha_0} = A_1 \to A_1$.

We must show that $\beta \Vdash A_\alpha$ iff $\beta \geq \alpha$. For α_0 this is trivial. Let $\alpha \neq \alpha_0$. If $\beta \Vdash A_\alpha$, the set of terminal nodes <u>not</u> beyond β includes those <u>not</u> beyond α (otherwise $\beta \nVdash \neg A_i$ for some i). Hence the set of terminal nodes of β (i.e. beyond β) is included in the set of terminal nodes of α. Let α_i be one of these. Since the set of predecessors of α_i is linearly ordered, either $\alpha \leq \beta$ or $\beta < \alpha$. But $\beta < \alpha$ would imply that α and β have the same terminal nodes beyond them, a contradiction. Thus $\beta \geq \alpha$. The converse is easy: $\alpha_i \not\geq \alpha$ implies $\alpha_i \not\geq \beta$ and $\beta \Vdash \neg A_i$ (since no extension of β forces A_i). Hence $\beta \Vdash A_\alpha$.

Finally, let S have the property stated in the lemma and let $A = \bigvee\!\!\!\bigvee_{\alpha \in S} A_\alpha$. Q. E. D.

Let us comment briefly on the content of this lemma. We know that $\mathrm{P}\mathfrak{p}$ is complete with respect to the modified Jaskowski sequence. Thus, if $\mathrm{P}\mathfrak{p} \nvdash A(p_1,\dots,p_n)$, there is a J_n^* and a forcing relation, $\Vdash$, on J_n^* such that $\alpha_0 \nVdash A(p_1,\dots,p_n)$. With each p_i, we can associate the set S_i of nodes β such that $\beta \Vdash p_i$. The lemma gives a simple sufficient condition on a forcing relation on J_n^* that there exists a formula behaving like p_i. Recall that, if we have associated non-standard models $\omega_1,\dots,\omega_{n!}$ of arithmetic with the nodes $\alpha_1,\dots,\alpha_{n!}$, we can define a model $\underline{K}_{J_n^*}$ of $\underset{\sim}{\mathrm{HA}}$. As long as the sentences A_i have no constants

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*) $\bigwedge\!\!\!\bigwedge$, $\bigvee\!\!\!\bigvee$ are used for repeated conjunctions and disjunctions respectively.

denoting non-standard numbers, the proof of the lemma carries through for $\underline{K}_{J_n^*}$. This is the key to our simple proof of de Jongh's theorem.

5.3.10-12 The Gödel - Rosser - Mostowski - Kripke - Myhill theorem.

A straightforward iteration of the Gödel - Rosser theorem will give us independent sentences $A_1,\dots,A_m$ for any m. For the sake of obtaining the simplest possible substitution instances in theorem 5.3.2, we want the independent sentences $A_1,\dots,A_m$ to be Σ_1^0. This result has been proven by Mostowski 1961, Kripke 1963, and Myhill 1972.

We shall present Myhill's proof of the following

5.3.11. Theorem. Let $T_0, T_1, \dots$ be an r.e. sequence of consistent r.e. extensions of classical (Peano) arithmetic, HA^c. Then we can find a Σ_1^0 sentence A such that A is independent over each theory T_i.

Proof. Let X,Y be recursively inseparable sets and let X,Y be represented by formulae $\exists yR(x,y)$, $\exists yS(x,y)$, respectively, in such a way that:

$$n \in X \text{ iff } \exists yR(n,y) \text{ iff } HA \vdash \exists yR(n,y),$$
$$n \in Y \text{ iff } \exists yS(n,y) \text{ iff } HA \vdash \exists yS(n,y),$$

and $$HA \vdash \neg(\exists yR(n,y) \,\&\, \exists yS(n,y)).$$

Let $$X_i = \{x : T_i \vdash \exists yR(x,y)\},$$
$$X_i' = \{x : T_i \vdash \neg\exists yR(x,y)\},$$

and consider $X^* = \bigcup_i X_i$, $X' = \bigcup_i X_i'$.

By the reduction theorem in recursion theory (see e.g. Rogers 1967, p. 72), there are r.e. sets U, V such that

$$U \cup V = X^* \cup X',$$
$$U \cap V = \emptyset,$$
$$U \subseteq X^*, \text{ and } V \subseteq X'.$$

Clearly $X \subseteq U$. For, if $n \in X \cap V$, $n \in X'$ and, for some T_i, $T_i \vdash \exists yR(n,y)$ and $T_i \vdash \neg\exists yR(n,y)$, contradicting the consistency of T_i.

Also, $Y \subseteq V$, since $n \in Y$ implies $T_i \vdash \exists yS(n,y)$, whence $T_i \vdash \neg\exists yR(n,y)$. But $n \in U$ implies $T_i \vdash \exists yR(n,y)$ for some i, again contradicting consistency.

Now, U and V separate X and Y, whence there is an $n_0 \notin U \cup V$. Then, if we let $A = \exists yR(n_0,y)$, we see that A is independent over each T_i.

Q. E. D.

5.3.12. Corollary. For any m, we can find m Σ_1^0 sentences independent over HA^c.

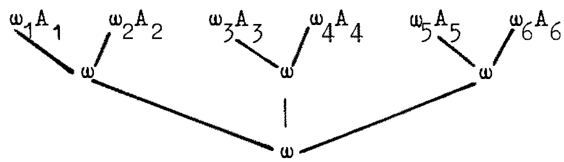
Proof. Let A_1 be independent over $\underset{\sim}{HA}^c$; A_2 independent over $\underset{\sim}{HA}^c + A_1$, $\underset{\sim}{HA}^c + \neg A_1$; A_3 independent over $\underset{\sim}{HA}^c + A_1 + A_2$, $\underset{\sim}{HA}^c + A_1 + \neg A_2$, $\underset{\sim}{HA}^c + \neg A_1 + A_2$, $\underset{\sim}{HA}^c + \neg A_1 + \neg A_2$; etc. Q. E. D.

5.3.13 - 5.3.15. de Jongh's theorem. Let us now combine the results of 5.3.3 - 5.3.12 to prove theorem 5.3.2, which we restate here:

5.3.13. Theorem. If $\mathbf{P_P} \not\vdash A(p_1,\ldots,p_n)$, then $\underset{\sim}{HA} \not\vdash A(B_1,\ldots,B_n)$, for some sentence $B_1,\ldots,B_n$ of arithmetic.

Proof. Let $\alpha_0 \Vdash\!\!\!\!/\, A(p_1,\ldots,p_n)$ for some forcing relation on J_k^* and let $A_1,\ldots,A_{n!}$ be independent over $\underset{\sim}{HA}^c$ and find, for each i, a model ω_i of $A_i + \bigwedge_{j \neq i} \neg A_j$. Associate these models with the terminal nodes of J_k^* and look at $\underline{K}_{J_k^*}$.

E.g. $\underline{K}_{J_3^*}$ is



Let, for each p_i, S_i be the set of nodes forcing p_i. By lemma 5.3.9, we can find a sentence B_i of arithmetic built up from the A_j's such that $S_i = \{\beta : \beta \Vdash B_i\}$.

Now, a simple induction can be used to show that, for any formula $C(p_1,\ldots,p_n)$ and any node β,

$$\beta \Vdash C(p_1,\ldots,p_n) \quad \text{iff} \quad \beta \Vdash C(B_1,\ldots,B_n),$$

under the two forcing relations. In particular,

$$\alpha_0 \Vdash\!\!\!\!/\, A(B_1,\ldots,B_n). \qquad \text{Q. E. D.}$$

The sentence corresponding to S is (except in the trivial case $\alpha_0 \in S$) of the form,

$$\bigvee_i \bigwedge_j \neg A_{ij},$$

whence we have the following corollary due to Myhill:

5.3.14. Corollary. The substitution instances $B_1, \ldots, B_n$ in theorem 5.3.13 may be taken to be disjunctions of Π_1^0 sentences.

Observe that one cannot use Π_1^0 sentences, because, if B is Π_1^0, $\underset{\sim}{HA} \vdash \neg\neg B \to B$.

Starting with $A_1,\ldots,A_m$ Π_1^0 and independent, we have the following

5.3.15. Corollary. The substitution instances $B_1,\ldots,B_{n_0}$ in theorem 5.3.13 may be taken to be disjunctions of double negations of Σ_1^0 sentences.

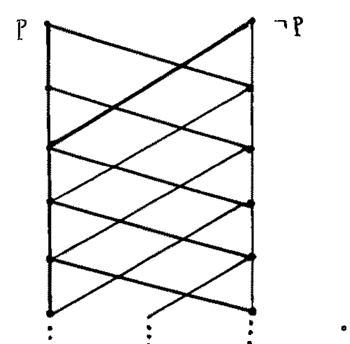
Since the only properties of HA used in proving de Jongh's theorem were the closure of the class of models of HA under the operation $(\ )\to(\Sigma\ )'$, the consistency of HA with classical logic (so that $\omega_1,\ldots,\omega_m$ could be chosen), and the incompleteness of HA^c, we can conclude that de Jongh's theorem also holds for $HA+\Gamma$ for any r.e. $\Gamma\in\mathfrak{P}$ (as in section 5.2.11) which is consistent with classical logic. In particular, de Jongh's theorem holds for $HA+TI(<)$, $HA+RF(HA)$, $HA+RFN(HA)$, etc. If $HA+\Gamma$ is not consistent with classical logic, the independence of A is replaced by the independence of $\neg A$, so that models of A and $\neg A$ exist. Then the models $\omega_1,\ldots,\omega_m$ are replaced by Kripke models.

5.3.16. de Jongh's theorem for one propositional variable.

In Nishimura 1960, Iwao Nishimura characterized the lattice of formulae in one propositional variable in the intuitionistic propositional calculus. It happens that there are close relations between these lattices and pms's. From Nishimura's work, it is not hard to prove the following

5.3.17. Theorem. Let $(K,\leq,\Vdash)$ be the model shown below and let $A(p)$ be a formula in the variable p such that $\mathbf{P}p\not\vdash A(p)$. Then for some $\alpha\in K$, $\alpha\not\Vdash A(p)$.

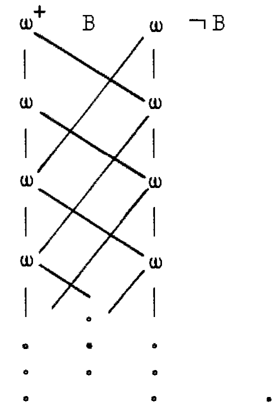
$(K,\leq,\Vdash)$:



The proof of this lies beyond the scope of these notes. A proof avoiding the use of lattices may be found in de Jongh B.

Let B be Σ_1^0, independent over HA^c. Then there is a model ω^+ in which B is true. Consider the model

$\underline{K}$:



This allows us to prove the following result, due independently to de Jongh and ourselves.

5.3.18. <u>Theorem</u>. Let B be Σ_1^0, independent over $\mathbf{HA}^c$, and let $\mathfrak{P}p \nvdash A(p)$. Then $\mathbf{HA} \nvdash A(B)$.

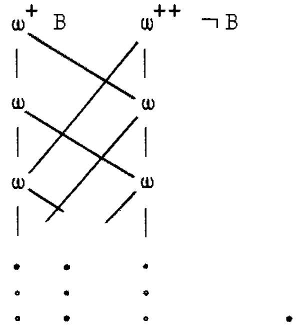
<u>Proof</u>. Prove by induction on the length of $C(p)$ that, for $\alpha \in K$, $\alpha \Vdash C(p)$ in the first model iff $\alpha \Vdash C(B)$ in the second model. Then apply theorem 5.3.17. Q. E. D.

Recall theorem 5.2.6, by which e.g. we showed $\mathbf{HA} \vdash \forall x Ax \vee \exists x \neg Ax$ iff $\mathbf{HA} \vdash \neg \forall x Ax \to \exists x \neg Ax$, when $\mathbf{HA} \vdash \forall x(Ax \vee \neg Ax)$.

We may restate this equivalence as $\mathbf{HA} \vdash \exists x Ax \vee \neg \exists x Ax$ iff $\mathbf{HA} \vdash \neg\neg \exists x Ax \to \exists x Ax$, for decidable A. But, for this formulation, theorem 5.3.18 readily applies. In $\mathbf{HA} \nvdash \exists x Ax \vee \neg \exists x Ax$, then $\mathbf{HA} \nvdash \exists x Ax$ and $\mathbf{HA} \nvdash \neg \exists x Ax$, whence $\mathbf{HA}^c \nvdash \exists x Ax$, $\mathbf{HA}^c \nvdash \neg \exists x Ax$. The decidability of A implies that $\neg \exists x Ax$ is true in ω and there is a non-standard model ω^+ of $\exists x Ax$. Now, applying the proof of theorem 5.3.18 to the sentence $\exists x Ax$ and the propositional formula $\neg\neg p \to p$, we have the result.
The case $\mathbf{HA} \vdash \exists x Ax \vee \neg \exists x Ax$ is trivial. (In particular, we have an independence proof for Markov's schema $\neg\neg \exists x Ax \to \exists x Ax$, Ax primitive recursive. - Markov's schema is studied in section 4, below.)

Another point worth stressing is that, for one propositional variable, we have a best possible result: uniform Σ_1^0 substitution instances.

Finally, observe that we do not have the result for all $\mathbf{HA} + \Gamma$, $\Gamma \in \mathfrak{P}$, since we must have Γ valid in ω. One can get around this slightly by considering models :



We leave the investigation of such results to the reader, taking time only to mention the following result of de Jongh's:

5.3.19. Theorem. Let B be a sentence such that $\underset{\sim}{HA} \nvdash \neg\neg B$, $\underset{\sim}{HA} \nvdash \neg\neg B \neg B$. Then, if $P_P \nvdash A(p)$, we have $\underset{\sim}{HA} \nvdash A(B)$.

Remark. We may use theorem 5.3.6 (the extension theorem) to prove the following: If $P_P \nvdash A(p)$, then $\alpha_0 \Vdash\!\!\!\!/ A(p)$ for some finite tree model $\underline{K}$ in which p is forced only at terminal nodes (if at all). This will give a simple proof of theorem 5.3.18 without using the Nishimura pms.

5.3.20. Another theorem of de Jongh (digression).

The Nishimura pms was used by de Jongh to solve a problem of Kreisel. In the last paragraph of Kreisel - Levy 1968, Kreisel and Levy mention that, when one wants a truth definition for formulae of bounded complexity, one must include the number of nested implications and negations occurring in a formula as well as the number of nested alternating quantifiers in one's measure of complexity. The infinitude of the Nishimura lattice tells us that there are infinitely many propositional formulae in one variable which are non-equivalent over P_P. Kreisel asked for a proof that there is no truth definition within $\underset{\sim}{HA}$ for the substitution instances - i.e. for any formula Tx and some sentence B, not all of the following equivalences are derivable:

$$T(\ulcorner A(B) \urcorner) \leftrightarrow A(B),$$

where A(p) ranges over all propositional formulae in one variable and $\ulcorner A(B) \urcorner$ denotes the gödel number of A(B).

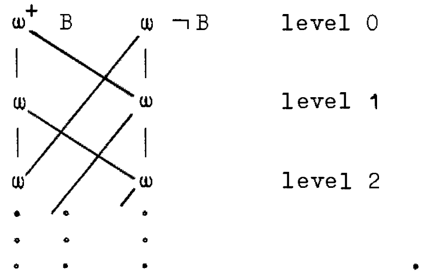
We present de Jongh's proof of this result here:

5.3.21. Theorem. Let B be Σ_1^0, independent of $\underset{\sim}{HA}^c$, and let Tx have only x free. Then, for some propositional formula A(p),

$$\underset{\sim}{HA} \nvdash T(\ulcorner A(B) \urcorner) \longleftrightarrow A(B).$$

In other words, for any independent Σ_1^0 sentence B, there is no truth definition for the set of propositional formulae in B.

Proof. Since B is Σ_1^0 and independent, B is false in the standard model. Let ω^+ be a non-standard model of $\underset{\sim}{HA}^c + B$, and assign levels to the nodes of the Nishimura model as follows:

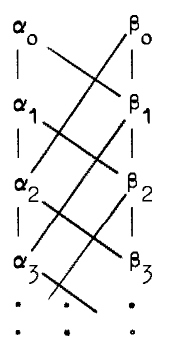


Let $C(x_1, \ldots x_n)$ be a formula with free variables as indicated. We shall prove by induction on the length of C that there is a level $n_C \geq 1$ such that, for any $m_1, \ldots, m_n$, if $C(m_1, \ldots, m_n)$ is forced by a node of level n_C, then $C(m_1, \ldots, m_n)$ is forced by all nodes.

(i) Let C be atomic. Then $n_C = 1$.

(ii) - (iii) If C is $D \& E$ or $D \vee E$, $n_C = \max(n_D, n_E)$ will do the trick.

(iv) Let C be $D \rightarrow E$. Then take $n_C = \max(n_D, n_E) + 1$. To see this, label the nodes as follows:



Let $n = \max(n_D, n_E)$ and let $\alpha_{n+1} \Vdash D \rightarrow E$. First, observe that $\alpha_n \Vdash D \rightarrow E$ and, if $\alpha_n \Vdash D$, $\alpha_n \Vdash E$ and all nodes force D and E, whence they force $D \rightarrow E$. If $\beta_n \Vdash D$, all nodes force D, whence $\alpha_n \Vdash D$, whence all nodes force $D \rightarrow E$.

If, for some γ, $\gamma \nVdash D \rightarrow E$, then there is a $\delta \geq \gamma$ such that $\delta \Vdash D$, $\delta \nVdash E$. If the level of δ is $\geq n$, then α_n or β_n is $\geq \delta$. But, if this is the case, α_n or β_n forces D and all nodes force the implication. On the other hand, there is no node of level $< n$ which is not $\geq \alpha_{n+1}$. Hence all nodes force $D \rightarrow E$.

The case in which $\beta_{n+1} \Vdash D \rightarrow E$ is similar.

(v) C is $\neg D$. Similar to (iv).

(vi) Let $C(x_1,\dots,x_n)$ be $\exists x\, D(x,x_1,\dots,x_n)$. Then $n_C = n_D$. Since $n_D \geq 1$, the domain at level n_D is $\{0, 1, \dots\}$. Let e.g. $\alpha_{n_D} \Vdash \exists x\, D(x,m_1,\dots,m_n)$. Then, for some m, $\alpha_{n_D} \Vdash D(m,m_1,\dots,m_n)$ and all nodes force $D(m,m_1,\dots,m_n)$, whence they force $\exists x\, D(x,m_1,\dots,m_n)$. β_{n_D} is handled similarly.

(vii) $C(x_1,\dots,x_n)$ is $\forall x\, D(x,x_1,\dots,x_n)$. Then $n_C = n_D$ and the proof is similar to that in case (vi).

Thus, any formula Tx will have a level n_T associated with it in such a way that, for all $\ulcorner A(B) \urcorner$, if $\alpha_{n_T} \Vdash T(\ulcorner A(B) \urcorner)$ or $\beta_{n_T} \Vdash T(\ulcorner A(B) \urcorner)$, then $\alpha \Vdash T(\ulcorner A(B) \urcorner)$ for all nodes α in the model. The proof is completed by the following lemma.

5.3.22. Lemma. For each level n, there is a sentence $A(B)$ which is forced at a node of level n, but at no nodes of level $n+1$ or higher.

The proof of this is not difficult, but is rather long and we omit it. The reader is referred to de Jongh B, for the proof. Alternatively, if he is willing to accept theorem 5.3.17 and the infinitude of the set of inequivalent formulae in one propositional variable, lemma 5.3.22 for arbitrarily large n follows by a simple cardinality argument - with only finitely many nodes of level $\leq n$ to distinguish these formulae, we can only find finitely many inequivalent formulae. By either approach, the proof of theorem 5.3.21 is completed. Q. E. D.

5.3.23. Further results on de Jongh's theorem.

In theorem 5.3.13 (theorem 5.3.2), we proved that, for any underivable $A(p_1,\dots,p_n)$, arithmetical $B_1,\dots,B_n$ can be found such that $A(B_1,\dots,B_n)$ is underivable in $\underset{\sim}{HA}$. de Jongh's original proof (de Jongh A) gave $B_1,\dots,B_n$ uniformly in all $A(p_1,\dots,p_n)$. Friedman A improved this by showing that, where uniformity is desired, any collection $B_1,\dots,B_n$ of Π^0_2 sentences independent over $\underset{\sim}{HA}^c$ augmented by all true Π^0_1 sentences will work. Friedman's proof made use of his generalization of the Kleene slash. We shall present a (Kripke) model-theoretic proof of his result in section 6, below.

In terms of the simplicity of the substitution instances, corollary 5.3.15 shows that $B_1,\dots,B_n$ (in the non-uniform version) may be taken to be disjunctions of double negations of Σ^0_1 sentences, which, classically, would be Σ^0_1. We obtain Σ^0_1 substitution instances in section 6.

When restricting one's attention to a particular number of variables, we outlined a proof of the existence of uniform Σ^0_1 counterexamples in 5.3.16 - 5.3.19 above. Using an alternate proof involving the arithmetization of the Kleene slash, de Jongh 1971 and B reproved his theorem 5.3.19 (cf. also 3.1.16).

§ 4. Markov's schema

5.4.1 - 5.4.3 . Markov's schema.

5.4.1. We have already encountered Markov's schema in our discussion of the Kripke models. In section 2, we presented a model-theoretic proof of Kreisel's version of the independence of Markov's schema (theorem 5.2.6) and in section 3, we observed that this result also came as a corollary to a special version of de Jongh's theorem for the special case of one propositional variable. Both proofs are off-shoots of the simple fact that Markov's schema is <u>not</u> preserved by the operation $(\ ) \to (\Sigma\ )'$.

Before discussing this last fact, let us consider several formulations of Markov's schema:

(i) $\forall x(Ax \vee \neg Ax) \,\&\, \neg\neg \exists x Ax \to \exists x Ax$,

(ii) $\forall xy(Axy \vee \neg Axy) \,\&\, \forall x\, \neg\neg \exists y Axy \to \forall x\, \exists y\, Axy$,

(iii) $\forall xy(Axy \vee \neg Axy) \to \forall x[\neg\neg \exists y Axy \to \exists y\, Axy]$,

(iv) $\forall x[\forall y(Axy \vee \neg Axy) \,\&\, \neg\neg\, \exists y Axy \to \exists y\, Axy]$,

(v) $\forall z[\forall xy(Axyz \vee \neg Axyz) \,\&\, \forall x\, \neg\neg \exists y Axyz \to \forall x\, \exists y\, Axyz]$,

where A contains only the free variables shown. Observe that the schemata obtained by replacing $x, y,$ or z by finite sequence of variables reduce to the present schemata via a pairing function. Thus, there is no loss of generality in considering only (i) - (v). For the treatment of Markov's schema by other methods, see also 1.11.5 and § 3.8.

5.4.2. <u>Lemma</u>. (v) $\leftrightarrow$ (iv) $\to$ (iii) $\to$ (ii) $\to$ (i).

<u>Proof</u>. (v) $\to$ (iv). Trivial.

(iv) $\to$ (v). Let $\underline{K}$ be a model of (iv) and let $\alpha \in K$ be such that $\alpha \not\Vdash \forall z[\forall xy(Axyz \vee \neg Axyz) \,\&\, \forall x \neg\neg \exists y\, Axyz \to \forall x\, \exists y\, Axyz]$. Then, for some $\beta \geq \alpha$ and $b \in D\beta$, we have $\beta \Vdash \forall xy(Axyb \vee \neg Axyb)$, $\beta \Vdash \forall x \neg\neg \exists y\, Axyb$, and $\beta \not\Vdash \forall x\, \exists y\, Axyb$.

But then there are $\gamma \geq \beta$ and $c \in D\gamma$ such that $\gamma \not\Vdash \exists y\, Acyb$. We also have $\gamma \Vdash \forall y(Acyb \vee \neg Acyb) \,\&\, \neg\neg \exists y\, Acyb$. Let $d = j(c,b)$, where j is the standard primitive recursive pairing function with inverses j_1, j_2. Then $\gamma \Vdash \forall y(A(j_1 d, y, j_2 d) \vee \neg A(j_1 d, y, j_2 d))$ and $\gamma \Vdash \neg\neg \exists y A(j_1 d, y, j_2 d)$, whence, applying (iv) to $A'xy : A(j_1 x, y, j_2 x)$, we have $\gamma \Vdash \exists y\, A(j_1 d, y, j_2 d)$, a contradiction.

(iv) $\to$ (iii). Let $\underline{K}$ be a model of (iv), $\alpha \Vdash \forall xy(Axy \vee \neg Axy)$ and $\alpha \not\Vdash \forall x[\neg\neg \exists y\, Axy \to \exists y\, Axy]$. Then there are $\beta \geq \alpha$ and $b \in D\beta$ such that $\beta \Vdash \neg\neg \exists y\, Aby$, $\beta \not\Vdash \exists y\, Aby$. Now $\beta \Vdash \forall y(Aby \vee \neg Aby)$ and, by (iv), $\beta \Vdash \forall y(Aby \vee \neg Aby) \,\&\, \neg\neg \exists y\, Aby \to \exists y\, Aby$, whence $\beta \Vdash \exists y\, Aby$, a contradiction.

(iii) → (ii). Let $\underset{\sim}{K}$ be a model of (iii), $\alpha \Vdash \forall xy(Axy \vee \neg Axy)$, and $\alpha \not\Vdash \forall x \neg\neg \exists y Axy \rightarrow \forall x \exists y Axy$. Then there are $\beta \geq \alpha$ and $b \in D\beta$ such that $\beta \Vdash \neg\neg \exists y Aby$ and $\beta \not\Vdash \exists y Aby$. But, by (iii), $\beta \Vdash \neg\neg \exists y Aby \rightarrow \exists y Aby$, a contradiction.

(ii) → (i). Trivial. Q. E. D.

Unfortunately, we cannot settle any of the converse implications (simple model-theoretic independence proofs are ruled out - when we replace the operation $(\ ) \rightarrow (\Sigma \ )'$ by one which preserves Markov's schema, we will see that all five schemata are preserved.). However, we can prove the following:

5.4.3. Theorem. The scheme (iv) is derivable in $\underset{\sim}{HA} + RFN((i))$.

Proof. The proof is based on a remark of Kreisel's that the uniform reflection principle allows one to add free variables. We show

$$\underset{\sim}{HA} \vdash \forall x \exists z \ Proof_{HA+(i)}(z, \ulcorner \forall y(A\bar{x}y \vee \neg A\bar{x}y) \ \& \ \neg\neg \exists y A\bar{x}y \rightarrow \exists y A\bar{x}y \urcorner).$$

Let $B_0(x), \ldots$ be a primitive recursive enumeration of all instances of $\forall y(Axy \vee \neg Axy) \ \& \neg\neg \exists y Axy \rightarrow \exists y Axy$.

a) $\underset{\sim}{HA} \vdash \forall w \ Proof_{HA+(i)}(\ulcorner B_w(0) \urcorner, \ulcorner B_w(0) \urcorner)$, i.e. every axiom is its own proof.

b) Let $\forall w \exists z \ Proof_{HA+(i)}(z, \ulcorner B_w(\bar{x}) \urcorner)$. Also, let f be primitive recursive such that

$$\underset{\sim}{HA} \vdash B_w(x+1) \longleftrightarrow B_{fw}(x).$$

By well-known properties of Proof,

$$\underset{\sim}{HA} \vdash \exists z \ Proof_{HA+(i)}(z, \ulcorner B_w(\overline{x+1}) \urcorner) \longleftrightarrow \exists z \ Proof_{HA+(i)}(z, \ulcorner B_{fw}(x) \urcorner).$$

But $\exists x Proof_{HA+(i)}(z, \ulcorner B_{fw}(\bar{x}) \urcorner)$ and so $\exists z \ Proof_{HA+(i)}(z, \ulcorner B_w(\overline{x+1}) \urcorner)$.

c) Thus

$$\underset{\sim}{HA} \vdash \forall w \exists z \ Proof_{HA+(i)}(z, \ulcorner B_w(\bar{x}) \urcorner) \rightarrow \forall w \exists z \ Proof_{HA+(i)}(z, \ulcorner B_w(\overline{x+1}) \urcorner).$$

This and (a) yields

$$\underset{\sim}{HA} \vdash \forall w \ \forall x \ \exists z \ Proof_{HA+(i)}(z, \ulcorner B_w(\bar{x}) \urcorner).$$

$RFN'(\underset{\sim}{HA} + (i))$ (which is equivalent to $RFN(\underset{\sim}{HA} + (i))$ by lemma 5.2.18 - the implication $RFN \rightarrow RFN'$, however, is trivial) yields, for w the index of $\forall y(Axy \vee \neg Axy) \ \& \ \neg\neg \exists y Axy \rightarrow \exists y Axy$,

$$\underset{\sim}{HA} + RFN(\underset{\sim}{HA} + (i)) \vdash \forall x[\ \forall y(Axy \vee \neg Axy) \ \& \neg\neg \exists y Axy \rightarrow \exists y Axy].$$ Q. E. D.

Thus, if schemata (i) - (iv) are not formally equivalent, they are almost equivalent. Combining this with our model-theoretic inability to distinguish these schemata, we shall allow ourselves to be sloppy and let MP denote any of the schemata (i) - (v) (for the present chapter).

We note that MP may be formulated as a rule of inference (see 3.8.1).

5.4.4 - 5.4.6. The independence of MP.

As remarked above, we have already proven the independence of MP twice. This time, however, we shall be more direct.

5.4.4. Theorem. Let $\mathbf{HA} + \Gamma$ be r.e., $\Gamma \in \mathfrak{P}$ (as defined in section 5.2.11). Then, some instance of MP is not derivable in $\mathbf{HA} + \Gamma$.

In fact, let $\exists x Ax$ be independent with Ax primitive recursive (so $\mathbf{HA} \vdash \forall x(Ax \vee \neg Ax))$. Then

$$\mathbf{HA} + \Gamma \nvdash \neg\neg \exists x Ax \rightarrow \exists x Ax .$$

Proof. Let $\exists x Ax$ be independent of $\mathbf{HA} + \Gamma$ and let $\underline{K}$ be a model of $\mathbf{HA} + \Gamma$ with a node β such that $\beta \Vdash \exists x Ax$ and consider $(\underline{K}_\beta)'$:

$$\begin{array}{c} \underline{K}_\beta \\ | \\ \alpha_0 \;\; \omega \end{array} .$$

We will show $\alpha_0 \nVdash \neg\neg \exists x Ax \rightarrow \exists x Ax$. Since $\gamma \geq \alpha_0$ implies $\gamma = \alpha_0$ or $\gamma \geq \beta$, $\beta \Vdash \exists x Ax$ implies $\alpha_0 \Vdash \neg\neg \exists x Ax$. But, if $\alpha_0 \Vdash \exists x Ax$, then $\alpha_0 \Vdash A\bar{n}$ for some n. As usual, this means $\mathbf{HA} \vdash A\bar{n}$ and so $\mathbf{HA} \vdash \exists x Ax$, contradicting independence.

But $\Gamma \in \mathfrak{P}$ and so $\mathbf{HA} + \Gamma$ is preserved by the step from $\underline{K}_\beta$ to $(\underline{K}_\beta)'$. Q.E.D.

For example, MP is independent of $\mathbf{HA} + RFN(\mathbf{HA})$, $\mathbf{HA} + TI(<)$, $\mathbf{HA} + CON(\mathbf{HA} + MP)$, etc.

In addition to outright independence results, one can obtain results on the form of the axiomatization of MP as follows. First, let us define a measure, m, of the complexity of a formula of number theory. We do this inductively as follows,

(i) if A is atomic, $m(A) = 1$,

(ii) - (iv) $m(A \& B) = m(A \vee B) = m(A \rightarrow B) = \max(m(A), m(B))$,

(v) $m(\neg A) = m(A)$,

(vi) $m(\exists x Ax) = \begin{cases} m(Ax), & Ax = \exists y Bxy \text{ for some } B \\ m(Ax) + 1, & \text{otherwise,} \end{cases}$

(vii) $m(\forall x Ax) = \begin{cases} m(Ax), & Ax = \forall y Bxy \text{ for some } B \\ m(Ax) + 1, & \text{otherwise.} \end{cases}$

Observe that, for classical arithmetic, $\mathbf{HA}^c$, this is a reasonable measure in the sense that a truth definition for formulae of bounded complexity can be given. We have observed in 5.3.20 that no such definition can be given in $\mathbf{HA}$. (To obtain one, redefine $m(A \rightarrow B) = \max(m(A), m(B)) + 1$ and $m(\neg A) = m(A) + 1$.)

5.4.5. <u>Theorem</u>. $\underset{\sim}{HA} + MP$ is not axiomatized by any restriction of the schema to formulae A for which $m(A) \leq n_o$ for any n_o. I.e. no set of instances of MP of bounded complexity can axiomatize MP (over $\underset{\sim}{HA}$).

<u>Proof</u>. We know from recursion theory that all formulae whose complexity is of measure $\leq n_o$ lie in a certain level of the arithmetical hierarchy. We also know from the hierarchy theorem and the completeness theorem (for classical logic) that there is some non-standard model ω^+ of $\underset{\sim}{HA}^c$ in which the truth of a sentence at or below the given level of the hierarchy agrees with truth in the standard model, but truth at higher levels does not. Thus consider $(\omega^+)'$:

$$\begin{array}{ll} \alpha & \omega^+ \\ & | \\ \alpha_o & \omega \quad . \end{array}$$

We first show by induction on $m(A)$ and on the length of A, that, if $m(A) \leq n_o$, $\alpha, \alpha_o \Vdash A$ iff A is true in ω (written $\omega \models A$). For this, we use the facts that $\alpha \Vdash A$ iff $\omega^+ \models A$

iff $\omega \models A$.

Atomic A are decidable and there is no problem. The connectives & and $\vee$ also offer no difficulty. Consider $B \to C$. If $\alpha_o \Vdash B \to C$, then $\alpha \Vdash B \to C$, whence $\omega^+ \models A$. Since $m(B \to C) \leq n_o$, $\omega \models A$. If $\alpha \nVdash B \to C$, then $\alpha_o \nVdash B \to C$. So suppose $\alpha_o \nVdash B \to C$, whence α or $\alpha_o \Vdash B$, $\nVdash C$. $\alpha \Vdash B$ and the length of B is less than that of $B \to C$, whence $\omega \models B$, $\alpha_o \Vdash B$. But $\alpha_o \nVdash B \to C$, whence $\alpha_o \nVdash C$. Again $\omega \not\models C$, whence $\omega \not\models B \to C$. Also $\alpha \nVdash B \to C$.

$\neg B$ is handled similarly.

Now consider the quantified formulae. For convenience, we only exhibit one quantifier, although there may actually be a block of like quantifiers. Thus, consider $\exists x Bx$.

$$\begin{aligned} \alpha_o \Vdash \exists x Bx &\Rightarrow \alpha \Vdash \exists x Bx \\ &\Rightarrow \omega^+ \models \exists x Bx \\ &\Rightarrow \omega \models \exists x Bx, \text{ by choice of } \omega^+. \end{aligned}$$

Conversely,

$$\begin{aligned} \omega \models \exists x Bx &\Rightarrow \underset{\sim}{\exists} n \ \omega \models Bn \\ &\Rightarrow \alpha, \alpha_o \Vdash Bn, \text{ by induction hypothesis.} \end{aligned}$$

For $\forall x Bx$,

$$\begin{aligned} \alpha_o \Vdash \forall x Bx &\Rightarrow \alpha \Vdash \forall x Bx \\ &\Rightarrow \omega^+ \models \forall x Bx \\ &\Rightarrow \omega \models \forall x Bx, \text{ by choice of } \omega^+. \end{aligned}$$

If $\omega \models \forall x Bx$, then $\omega^+ \models \forall x Bx$ and $\alpha \Vdash \forall x Bx$. To conclude $\alpha_0 \Vdash \forall x Bx$, it suffices to show $\alpha_0 \Vdash Bn$ for all numerals n. But

$$\begin{aligned} \omega \models \forall x Bx \;&\Rightarrow\; \underset{\sim}{\forall} n\ \omega \models Bn \\ &\Rightarrow\; \underset{\sim}{\forall} n\ \alpha_0 \Vdash Bn, \text{ by induction hypothesis.} \end{aligned}$$

Thus, we see that, for $m(A) \leq n_0$,

$$\begin{aligned} \alpha_0 \Vdash A \quad &\text{iff} \quad \alpha \Vdash A \\ &\text{iff} \quad \omega \models A. \end{aligned}$$

But ω^+ is not an elementary extension of ω and, for some prenex sentence A, $\omega^+ \models A$, $\omega \not\models A$. Let A be such a sentence for which $m(A)$ is minimal.

Then A is of the form $\exists x_1 \ldots x_n B$, where $m(B) < m(A)$. To see this, observe that, by minimality of $m(A)$, the above argument holds for all prenex C for which $m(C) < m(A)$. In particular,

$$\begin{aligned} \alpha_0 \Vdash C \quad &\text{iff} \quad \alpha \Vdash C \\ &\text{iff} \quad \omega \models C, \end{aligned}$$

and C is decidable (since $\alpha_0 \Vdash C \leftrightarrow \alpha \Vdash C$ and $\alpha_0 \Vdash \neg C \leftrightarrow \alpha \Vdash \neg C$).

Suppose A is of the form $\forall x_1 \ldots x_n B(x_1, \ldots, x_n)$, $m(B) < m(A)$. Then

$$\begin{aligned} \omega^+ \models \forall x_1 \ldots x_n B x_1 \ldots x_n \;&\Rightarrow\; \alpha \Vdash \forall x_1 \ldots x_n B \\ &\Rightarrow\; \underset{\sim}{\forall} a_1 \ldots a_n \in D\alpha\, (\alpha \Vdash B(a_1, \ldots, a_n)) \\ &\Rightarrow\; \underset{\sim}{\forall} m_1 \ldots m_n \in \omega\ (\alpha \Vdash B(m_1, \ldots, m_n)) \\ &\Rightarrow\; \underset{\sim}{\forall} m_1 \ldots m_n \in \omega\, (\alpha_0 \Vdash B(m_1, \ldots, m_n)), \end{aligned}$$

since $m(B) < m(A)$. This last fact, together with the fact that $\alpha \Vdash \forall x_1 \ldots x_n B$ implies $\alpha_0 \Vdash \forall x_1 \ldots x_n B$. But, more importantly, since $m(B) < m(A)$,

$$\alpha_0 \Vdash B(m_1, \ldots, m_n) \Rightarrow \omega \models B(m_1, \ldots, m_n),$$

whence $\omega \models \forall x_1 \ldots x_n B$ and A cannot be of the form suggested.

Hence we have $\omega^+ \models \exists x_1 \ldots x_n B$, $\omega \models \neg \exists x_1 \ldots x_n B$, and $\alpha_0 \Vdash \forall x_1 \ldots x_n (B(x_1, \ldots, x_n) \vee \neg B(x_1, \ldots, x_n))$ (by the fact that $m(B) < m(A)$). But $\alpha \Vdash \exists x_1 \ldots x_n B$, and so $\alpha_0 \Vdash \neg\neg \exists x_1 \ldots x_n B$. But we cannot have $\alpha_0 \Vdash \exists x_1 \ldots x_n B$. Contracting quantifiers, we have an instance,

$$\forall x (B'x \vee \neg B'x) \wedge \neg\neg \exists x B'x \rightarrow \exists x B'x,$$

of (i) which is not forced at α_0.

Finally, for $m(C) < n_0 - 1$, we have $m(\exists x\ C) < n_0$, whence $\exists x C$ is decidable in the model ($m(\exists x C \vee \neg \exists x C) = m(\exists x C) < n_0$). But, whether $\alpha_0 \Vdash \exists x C$ or $\alpha_0 \Vdash \neg \exists x C$, we have

$$\alpha_0 \Vdash \forall x (Cx \vee \neg Cx) \wedge \neg\neg \exists x Cx \rightarrow \exists x Cx.$$

Hence, MP is true in the model for instances of low complexity, but not for high complexity. The fact that we can make the "low complexity" large enough to include n_o yields the theorem. Q.E.D.

5.4.6. Corollary. MP is not derivable from the subscheme,

$$\forall x \neg\neg \exists y Axy \rightarrow \forall x \exists y Axy ,$$

where A is primitive recursive.

The above proof was rather long, but the idea is simple. If we start with a model ω^+ of HA^c which agrees with ω in the truth of formulae of low complexity, but not for formulae of high complexity, then formulae of low complexity are decidable in the model:

$$\begin{array}{cc} & \omega^+ \\ & | \\ \alpha_o & \omega \end{array} ,$$

whence MP holds for sentences of low complexity. But, sentences of high complexity are not decidable and, in particular, there is some sentence $\exists xBx$ which is not decidable at α_o, but for which. B yields a decidable property. Hence MP fails in some instance of high complexity.

We might also comment on the measure of complexity used. One might object that we should consider an intuitionistically meaningful measure - i.e. one for which the bounded truth definition can be given in HA as well as in HA^c. As observed above, such a measure m' is obtained by defining $m'(A \rightarrow B) = \max(m'(A), m'(B)) + 1$ and $m'(\neg A) = m'(A) + 1$. But then, for any formula A, $m'(A) \geq m(A)$, whence a bound on $m'(A)$ yields one on $m(A)$ and the result follows from theorem 5.4.5. Further, concerning the specific choice of a measure m, theorem 5.4.5 can be shown to hold for any measure for which truth definitions for formulae of bounded complexity can be given in HA^c (and hence for those measures whose bounded truth definitions can be given in HA).

The above theorem 5.4.5 and corollary 5.4.6 easily generalize to any r.e. $HA + \Gamma$, where $\Gamma \in \mathfrak{P}$ and Γ is true in the standard model, e.g. $HA + RFN(HA)$.

5.4.7 - 5.4.9. A comment on proof-theoretic closure properties.

5.4.7. We have used the failure of MP to be preserved by the operation $(\ ) \rightarrow (\Sigma\ )'$ to prove its independence and to prove that it cannot be replaced by a bounded set of instances of itself. In 5.4.10-14, we will replace the operation $(\ ) \rightarrow (\Sigma\ )'$ by one which will allow us to extend many standard results for HA to HA + MP.

We have also given a derived rule (whose proof was based on this failure to be preserved):

5.4.8. <u>Theorem</u>. Let A contain only x free and let $\underset{\sim}{HA} \vdash \forall x(Ax \vee \neg Ax)$. Then the following are equivalent:

(i) $\underset{\sim}{HA} \vdash \exists x Ax \vee \neg \exists x Ax$,

(ii) $\underset{\sim}{HA} \vdash \neg\neg \exists x Ax \rightarrow \exists x Ax$,

(iii) $\underset{\sim}{HA} \vdash \exists y[\neg\neg \exists x Ax \rightarrow Ay]$.

(This is theorem 5.2.6.)

An almost trivial derived rule is

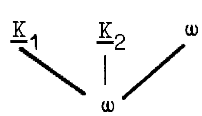
5.4.9. <u>Theorem</u>. Let A contain only x free and let $\underset{\sim}{HA} \vdash \forall x(Ax \vee \neg Ax)$ and $\underset{\sim}{HA} \vdash \neg\neg \exists x Ax$. Then $\underset{\sim}{HA} \vdash \exists x Ax$.

<u>Proof</u>. Observe that $\underset{\sim}{HA} \vdash \neg\neg \exists x Ax$ implies that $\exists x Ax$ is true in the standard model and hence An is true for some n. But $\underset{\sim}{HA} \vdash An \vee \neg An$ and the disjunction property yields $\underset{\sim}{HA} \vdash An$ or $\underset{\sim}{HA} \vdash \neg An$. Hence $\underset{\sim}{HA} \vdash An$, i.e. $\underset{\sim}{HA} \vdash \exists x Ax$. Q. E. D.

From an earlier chapter, we know that this last result holds when A is allowed to have other free variables. The present proof admits no easy extension to this generalization. Suppose then that we decide to attempt to prove this directly. That is, let $\underline{K}$ be a model with node α such that $\alpha \not\Vdash \forall x \exists y Axy$. Then, for some $\beta \geq \alpha$, $b \in D\beta$, $\beta \not\Vdash \exists y Aby$. Since $\underset{\sim}{HA} \vdash \forall x(Axy \vee \neg Axy)$, we see $\beta \Vdash \neg Abc$ for all $c \in D\beta$. Also, since $\underset{\sim}{HA} \vdash \forall x \neg\neg \exists y Axy$, we can find some $\gamma > \beta$ and $c \in D\gamma$ such that $\gamma \Vdash Abc$. But $c \in D\gamma - D\beta$ and we obtain no contradiction. We cannot pull $D\beta$ out of the model $\underline{K}$ and define a nonstandard model of arithmetic on it, because it does not follow from the fact that an axiom is forced at a given node that it will be true in the classical model determined by the domain and atomic formulae forced at that node. (Note: We have not here done anything that we would not do to show $\underset{\sim}{HA} \vdash \forall xy(Axy \vee \neg Axy) \,\&\, \forall y \neg\neg \exists x Axy \rightarrow \forall y \exists x Axy$. Thus we should not expect to get anywhere. Also, unfortunately, there seems to be no place at which to apply the trick used in proving theorem 5.2.6.)

5.4.10 - 5.4.14. $(\ ) \rightarrow (\Sigma \ + \omega)'$.

5.4.10. The failure of MP to be preserved under $(\ ) \rightarrow (\Sigma \ )'$ has its applications. But applications such as the ED - theorem required preservation under $(\ ) \rightarrow (\Sigma \ )'$ and, to obtain such results, we must give a similar such operation under which MP is preserved. Fortunately, the solution to this problem is simple: If $\underline{F}$ is a family of models of MP, define an operation on $\underline{F}$ by $\underline{F} \rightarrow (\Sigma\underline{F} + \omega)'$. E.g. let $\underline{F} = \{\underline{K}_1, \underline{K}_2\}$. Then $(\Sigma\underline{F} + \omega)'$ is:



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5.4.11. Theorem. If $\underset{\sim}{HA} + MP$ is valid in $\underline{F}$, then it is valid in $(\Sigma\underline{F} + \omega)'$ - i.e. the validity of $\underset{\sim}{HA} + MP$ is preserved by the operation $(\ ) \to (\Sigma\ + \omega)'$.
Proof. We consider the schema (i). It being valid in $\Sigma\underline{F} + \omega$, we need only look at α_0. Let $\alpha_0 \Vdash \forall x(Ax \vee \neg Ax) \& \neg\neg \exists x Ax$. Let $\alpha > \alpha_0$ be the node corresponding to ω. Then $\alpha \Vdash \neg\neg \exists x Ax$, whence $\alpha \Vdash \exists x Ax$. $D\alpha = \{0, 1, \ldots\}$ and so, for some n, $\alpha \Vdash An$. A is decidable and so $\underset{\sim}{HA} \vdash An$, whence $\alpha_0 \Vdash An$, i.e. $\alpha_0 \Vdash \exists x Ax$. Thus

$$\alpha_0 \Vdash \forall x(Ax \vee \neg Ax) \& \neg\neg \exists x Ax \to \exists x Ax. \qquad \text{Q. E. D.}$$

5.4.12. Corollary. $\underset{\sim}{HA} + MP$ possesses ED, DP.
Proof. We can't quite quote theorem 5.1.20, but we can observe that the proof carries over easily, the additional summand being used only to guarantee the validity of MP. Q. E. D.

Regarding closure properties of the class $\mathfrak{P}^\omega$ of sets Γ such that the validity of $\underset{\sim}{HA} + \Gamma$ is preserved by the operation $(\ ) \to (\Sigma\ + \omega)'$, we get almost exactly the properties corresponding to $(\ ) \to (\Sigma\ )'$ (theorem 5.2.11). The difference is that we must also insist that Γ be true in the standard model.

5.4.13. Theorem. The class $\mathfrak{P}^\omega$ of sets, Γ, such that the validity of $\underset{\sim}{HA} + \Gamma$ is preserved by the operation $(\ ) \to (\Sigma\ + \omega)'$ has the following closure properties:

(i) $\mathfrak{P}^\omega$ is closed under arbitrary union;

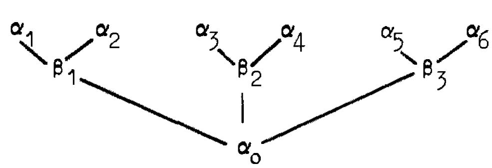
(ii) if $\Gamma \in \mathfrak{P}^\omega$ and A is a Harrop - sentence and $\omega \models A$, then $\Gamma \cup \{A\} \in \mathfrak{P}^\omega$;

(iii) if $\Gamma \in \mathfrak{P}^\omega$, A has only the variable x free, and $\underset{\sim}{HA} + \Gamma \vdash An$ for each numeral n, then $\Gamma \cup \{\forall x Ax\} \in \mathfrak{P}^\omega$.

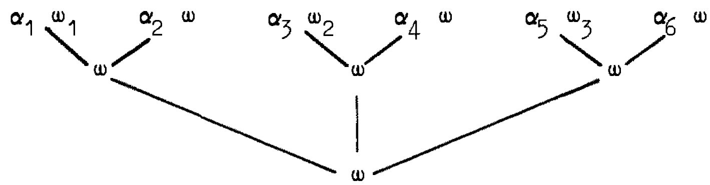
(Note that, in (iii), the fact $\omega \models \forall x Ax$ follows from the facts that $\underset{\sim}{HA} + \Gamma \vdash An$ for all n and $\omega \models \Gamma$.)
Proof. The proof of theorem 5.4.13 is identical to that of theorem 5.2.11 and we omit it.

The results of 5.2.13 - 5.2.23 carry over easily. We leave the verification to the reader.

5.4.14. Remark. The proof of de Jongh's theorem does not carry over: Consider J_3^*:



If we wish β_1 to force MP when we turn this into a model of $\underset{\sim}{HA}$, we must associate the standard model with either α_1 or α_2. Similarly, it must be associated with one of α_3 and α_4 and with one of α_5 and α_6. Thus, we end up with something of the form:



But now the proof of de Jongh's theorem does not go through: Lemma 5.3.9 does not apply since α_2, α_4 and α_6 all behave identically. A sophistication of our technique in section 6 will allow us to get around this problem. Also, it will yield a method of generalizing theorem 5.4.13 to cases where Γ need not be true in the standard model.

§ 5. The schema IP_0^c

5.5.1. In addition to MP, the schema

$$IP_0^c: \forall x(Ax \vee \neg Ax) \& (\forall xAx \to \exists yBy) \to \exists y(\forall xAx \to By),$$

where A, B have only the free variables indicated, is valid under Gödel's interpretation (see 3.5.10). As in section 4, where we considered variants of MP, we may consider variants of this schema. IP_0, however, is simply not as susceptible to study by means of the Kripke models as MP, and, thus, we shall only consider the schema as presented (i.e. with no free variables). The reader may consider variants as he pleases (in particular, IP).

5.5.2 - 5.5.3. Proof theoretic closure results.

Let $\mathfrak{P}$ and $\mathfrak{P}^\omega$ be as defined in sections 5.2.11 and 5.4.13, respectively.

5.5.2. Theorem. Let A have only the variable x free, B only the variable y free. Let $\Gamma \in \mathfrak{P}$. If $\underset{\sim}{HA} + \Gamma \vdash \forall x(Ax \vee \neg Ax)$ and $\underset{\sim}{HA} + \Gamma \vdash \forall xAx \to \exists yBy$, then

$$\underset{\sim}{HA} + \Gamma \vdash \exists y(\forall xAx \to By).$$

Proof. Suppose $\underset{\sim}{HA} + \Gamma \nvdash \exists y(\forall xAx \to By)$. Then, for each n there is a model $\underline{K}_n$ with origin β_n such that $\beta_n \Vdash \forall xAx$, $\beta_n \nVdash Bn$. Let α_0 be the origin of $(\Sigma\underline{K}_n)'$. By the decidability of A and and the fact that $\beta_n \Vdash \forall xAx$ for all n, $\alpha_0 \Vdash \forall xAx$. Hence $\alpha_0 \Vdash \exists yBy$ and, for some n, $\alpha_0 \Vdash Bn$. But $\beta_n > \alpha_0$ and so $\beta_n \Vdash Bn$, a contradiction. Q. E. D.

5.5.3. Theorem. Let A, B be as in theorem 5.5.2 and let $\Gamma \in \mathfrak{P}^\omega$. If $\underset{\sim}{HA} + \Gamma \vdash \forall x(Ax \vee \neg Ax)$ and $\underset{\sim}{HA} + \Gamma \vdash \forall xAx \to \exists yBy$, then

$$HA + \Gamma \vdash \exists y(\forall xAx \to By).$$

Proof. Replace $(\Sigma\underline{K}_n)'$ by $(\Sigma\underline{K}_n + \omega)'$. Q. E. D.

For instance, we may let Γ = MP, TI(<), RFN($\underset{\sim}{HA}$), RFN($\underset{\sim}{HA}$ + MP), etc.

5.5.4 - 5.5.7. Mutual independence of MP and IP_0^c.

There is one useful property that IP_0^c has: It is not preserved under $(\ ) \to (\Sigma\ )'$ or $(\ ) \to (\Sigma\ + \omega)'$. Because of this, we may prove the following

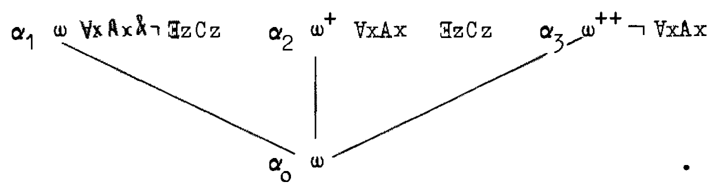
5.5.4. Theorem. Let $\Gamma \in \mathfrak{P}^\omega$ be r.e. Then there is a primitive recursive A Ax and a formula By (each with only the free variables indicated) such that

$$\underset{\sim}{HA} + \Gamma \nvdash (\forall xAx \to \exists yBy) \to \exists y(\forall xAx \to By).$$

<u>Proof</u>. Let $\exists z Cz$ and $\forall x Ax$ be formally undecidable in $\underset{\sim}{HA}^c + \Gamma$, Ax and Cz primitive recursive. Also, let $\forall x Ax \& \exists z Cz$ be consistent (e.g. use corollary 5.3.12). Define

$$By \equiv (y = 0 \& \exists z Cz) \vee (y = 1 \& \neg \exists z Cz).$$

Let ω^+, ω^{++} be classical models of $\forall x Ax \& \exists z Cz$ and $\neg \forall x Ax$, respectively. Also, observe that ω is a model of $\forall x Ax \& \neg \exists z Cz$. Consider $(\omega + \omega^+ + \omega^{++})'$:



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Now $\alpha_0 \Vdash \Gamma$ by theorem 5.4.13.and $\alpha_0 \Vdash \forall x(Ax \vee \neg Ax)$ by the primitive recursiveness of Ax. Also, $\alpha_0 \Vdash \forall x Ax \rightarrow \exists y By$, since only $\alpha_1, \alpha_2 \Vdash \forall x Ax$ and $\alpha_1 \Vdash B1$, $\alpha_2 \Vdash B0$.

But $\alpha_0 \not\Vdash \exists y(\forall x Ax \rightarrow By)$, since then $\alpha_0 \Vdash \forall x Ax \rightarrow B0$ or $\alpha_0 \Vdash \forall x Ax \rightarrow B1$. In the first case, it follows that $\alpha_1 \Vdash \forall x Ax \rightarrow B0$ and, in the second case, that $\alpha_2 \Vdash \forall x Ax \rightarrow B1$, both implications leading to contradictions. Q. E. D.

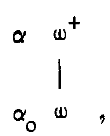
It is worth singling out the case $\Gamma = MP$:

5.5.5. <u>Corollary</u>. IP_0^c is not derivable from MP.

However, IP_0^c is preserved under a special case of $(\ )\rightarrow(\Sigma\ )'$, under which MP is not preserved:

5.5.6. <u>Theorem</u>. Let ω^+ be a model of $\underset{\sim}{HA}^c$. Then $\underset{\sim}{HA} + IP_0^c$ is valid in $(\omega^+)'$.

<u>Proof</u>. Suppose that, in the model $(\omega^+)'$,



,

α_0 does not force an instance of IP_0^c:

$$\alpha_0 \not\Vdash \forall x(Ax \vee \neg Ax) \& (\forall x Ax \rightarrow \exists y By) \rightarrow \exists y(\forall x Ax \rightarrow By).$$

IP_0^c being valid at α, we must have

$$\alpha_0 \Vdash \forall x(Ax \vee \neg Ax) \& (\forall x Ax \rightarrow \exists y By),$$
$$\alpha_0 \not\Vdash \exists y(\forall x Ax \rightarrow By).$$

By this last statement, $\alpha_0 \not\Vdash \forall x Ax \rightarrow B0$ and, for some $\beta \geq \alpha_0$, $\beta \Vdash \forall x Ax$, $\beta \not\Vdash B0$. In particular, $\beta \Vdash \forall x Ax$ and, A being decidable, $\alpha_0 \Vdash An$ for all n. It follows that $\alpha_0 \Vdash \forall x Ax$. But $\alpha_0 \Vdash \forall x Ax \rightarrow \exists y By$, whence

$\alpha_o \Vdash \exists y By$ and, for some n, $\alpha_o \Vdash Bn$. Thus $\alpha_o \Vdash \forall x Ax \to Bn$ and so $\alpha_o \Vdash \exists y(\forall x Ax \to By)$, a contradiction. Q. E. D.

The immediate corollary is

5.5.7. Corollary. Let $\underset{\sim}{HA} + \Gamma$ be r.e., $\Gamma \in \mathfrak{P}$, $\underset{\sim}{HA}^c + \Gamma$ consistent. Then some instance of MP is not derivable in $\underset{\sim}{HA} + \Gamma + IP_o^c$.

Proof. $\underset{\sim}{HA} + \Gamma + IP_o$ is valid in $(\omega^+)'$ for any model ω^+ of $\underset{\sim}{HA}^c + \Gamma$. But, as shown in the proof of theorem 5.4.5, MP is not valid in $(\omega^+)'$ unless ω^+ is an elementary extension of ω. Q. E. D.

5.5.8. Final comments on IP_o^c. The non-preservation of IP_o^c under $(\ ) \to (\Sigma\ )'$ allows us to prove for IP_o^c an analogue to theorem 5.4.5. We may also generalize theorem 5.4.5 by using the fact that IP_o^c is preserved under $\omega^+ \to (\omega^+)'$. Also, aside from such results, and formulations of such corollaries as the independence of IP_o^c from $\underset{\sim}{HA} + MP + RFN(\underset{\sim}{HA} + MP) + CON(\underset{\sim}{HA} + IP_o^c) + TI(<)$, etc., we may observe that we can obtain subtler results such as the following:

5.5.9. Theorem. There is a formula By and a primitive recursive Ax (each with only the indicated free variables) such that

(i) $\underset{\sim}{HA} + MP \nvdash (\forall x Ax \to \exists y By) \to \exists y(\forall x Ax \to By)$,

(ii) $\underset{\sim}{HA} + IP \nvdash \neg\neg \exists x Ax \to \exists x Ax$.

(In other words, the same formula A works in both independence proofs.)

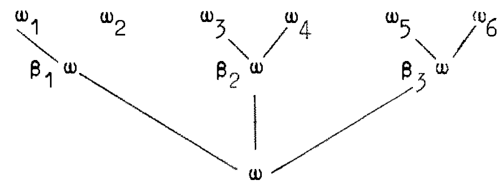
Proof. Observe that A may be taken in both independence proofs above to be arbitrary up to the requirement that $\exists x Ax$ be independent of $\underset{\sim}{HA}^c$. Q. E. D.

Beyond this, there is little we can do model-theoretically since (i) the only models of IP_o^c we have so far are (except for those given by the completeness theorem) models ω^+ of $\underset{\sim}{HA}^c$ and models of the form $(\omega^+)'$, and (ii) the only models to which we know we can apply the operation $(\ ) \to (\ )'$ and preserve IP_o^c are the models of $\underset{\sim}{HA}^c$.

§ 6. Definability of models of $\underset{\sim}{HA}^c$: applications

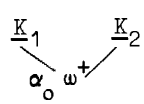
5.6.1. The operation $(\ ) \to (\Sigma\ )^*$.

The operation $(\ ) \to (\Sigma\ )'$, while extremely useful, is too restrictive for certain purposes. In proving de Jongh's theorem, for instance, we had models of the form



This cannot possibly give us Σ_1^o substitution instances since β_1, β_2 and β_3 must all force the same Σ_1^o sentences (and similar results for models on the other modified Jaskowski trees). The observant reader will also have noticed that, to apply $(\ ) \to (\Sigma\ + \omega)'$ in proving (say) the ED - property for $\underset{\sim}{HA} + MP + \Gamma$, we had to assume that Γ was true in the standard model.

Let $\underline{F}$ be a family of models and let ω^+ be a non-standard model of $\underset{\sim}{HA}^c$ such that (i) the domain of ω^+ is contained in $D\alpha$ for all $\alpha \in K$, $\underline{K} \in \underline{F}$, and (ii) atomic formulae whose constants name elements in ω^+ are forced at any node α exactly when they are true in ω^+. Then, we can define a model $(\Sigma\underline{F})^*$ in the manner in which we defined $(\Sigma\underline{F})'$. E.g. let $\underline{F} = \{\underline{K}_1, \underline{K}_2\}$. Then $(\Sigma\underline{F})^*$ is



Unfortunately, we don't know if $(\Sigma\underline{F})^*$ will always be a model of HA. For, what have we got to guarantee that the induction schema will be forced at α_o? Truth in ω^+ is not convincing - the law of the excluded middle is true in ω^+, but not forced at α_o. In $(\ ) \to (\Sigma\ )'$, we did not merely have a model of $\underset{\sim}{HA}^c$ placed at α_o - we had the natural numbers themselves.

Let us consider how we proved the induction axiom to be valid in $(\Sigma\underline{F})'$. Our "second proof" consisted in observing that, by theorem 5.3.11 (iii), to conclude that induction was valid, we had not to look at the schema without free variables other than x in Ax, but we only had to look at all instances

$$A0 \,\&\, \forall xy(Ax \,\&\, S(x,y) \to Ay) \to An .$$

This is obviously valid in $(\Sigma\underline{F})^*$ - but, since the domain at α_o has non-standard integers, we cannot conclude from the fact that $\alpha_o \Vdash A0 \,\&\, \forall xy(Ax \,\&\, S(x,y) \to Ay)$ that $\alpha_o \Vdash Aa$ for all $a \in D\alpha_o$, but only that $\alpha_o \Vdash A0, A1, \ldots$.

The actual proof we gave in proving theorem 5.2.4 was based on the following reasoning: If $\alpha_0 \Vdash A0 \,\&\, \forall xy(Ax \,\&\, S(x,y) \to Ay)$ and $\alpha_0 \not\Vdash \forall x Ax$, there is a <u>least</u> n such that $\alpha_0 \not\Vdash An$. There are two cases in which we can guarantee the existence of a least element in ω^+ of which A is not forced: (i) $\omega^+ = \omega$, and (ii) the condition " $\alpha_0 \Vdash A$ " is expressible in the language of ω^+ - in other words, if the truth (or, forcing) definition for a formula in the Kripke model can be given within the classical model ω^+.

5.6.2 - 5.6.7. <u>Definability</u>.

5.6.2. In this subsection, we formally define what we mean by the definability of a Kripke model in a model of $\underset{\sim}{HA}^c$. Suppose $\underline{K}$ is a Kripke model, in which $\underset{\sim}{HA}$ is valid, ω^+ a nonstandard model of arithmetic. Let, for each $\alpha \in K$, $a_\alpha \in D\alpha$, $\bar{a}_\alpha$ denote a number in ω^+ indexing a_α. We assume that we have formulae as follows, with the free variables as indicated:

$K(\alpha)$,
$D(\alpha, x)$,
$\alpha \leq \beta$,
$S(\alpha, x, y)$,
$A(\alpha, x, y, z)$,
$M(\alpha, x, y, z)$.

Also, $\bar{0}$, $\bar{1}$, ... will denote indices of 0, 1, ...

Let us suppose that there is a one - to - one correspondence between elements $\alpha \in K$ and elements a of ω^+ for which $\omega^+ \models Ka$, in such a way that, if a, b are associated with α, β, respectively, then

$$\alpha \leq \beta \quad \text{iff} \quad \omega^+ \models a \leq b ,$$

and

$$\omega^+ \models \forall xy(x \leq y \to Kx \,\&\, Ky) .$$

Then, obviously, we may identify elements of K with a definable subset of the domain of ω^+ and $\leq$ with the definable partial ordering on this subset. We also assume that

$$\omega^+ \models D(\alpha, \bar{a}_\alpha) ,$$

and

$$\omega^+ \models D(\alpha, a) \Rightarrow a \text{ is an index } \bar{a}_\alpha \text{ for some } a_\alpha \in D\alpha .$$

We may assume either that the set of constants $\{\bar{a}_\alpha : a_\alpha \in D\alpha\}$ is contained in $\{\bar{b}_\beta : b_\beta \in D\beta\}$ or that there is a definable function $f(x,y,z)$ such that

$$f(\alpha, \beta, \bar{a}_\alpha) \text{ is an index of } a_\alpha \in D\beta \; (\alpha \leq \beta).$$

In what follows, however, we shall ignore this minor distinction.

Finally, if, in addition to all of this, we have

$$\omega^+ \models B(\alpha, \bar{a}_{1\alpha}, \ldots, \bar{a}_{n\alpha}) \text{ iff } \alpha \Vdash B(a_{1\alpha}, \ldots, a_{n\alpha}),$$

for all atomic B, nodes α, and elements $a_{1\alpha}, \ldots, a_{n\alpha} \in D\alpha$, then we say that the model $\underline{K}$ is definable in ω^+.

The definability of a classical model, ω^{++}, in ω^+ can be taken as the definability of the one node Kripke model, or can be taken in the obvious manner.

Three rather obvious lemmas are

5.6.3. Lemma. Let $\underline{K}$ be definable in ω^+. Then, for any formula $A(x_1, \ldots, x_n)$ with free variables as shown, there is a formula $A^*(x, x_1, \ldots, x_n)$ with free variables as shown and parameters from ω^+ such that, for all $\alpha \in K$, $a_{1\alpha}, \ldots, a_{n\alpha} \in D\alpha$,

$$\alpha \Vdash A(a_{1\alpha}, \ldots, a_{n\alpha}) \text{ iff } \omega^+ \models A^*(\alpha, \bar{a}_{1\alpha}, \ldots, \bar{a}_{n\alpha}).$$

5.6.4. Lemma. Let $\underline{F} = \{\underline{K}_1, \ldots, \underline{K}_n\}$ be such that each $\underline{K}_i$ is definable in ω^+, then $\Sigma\underline{F}$ is definable in ω^+.

5.6.5. Lemma. Let $\underline{K}$ be definable in ω^{++} and let ω^{++} be definable in ω^+. Then $\underline{K}$ is definable in ω^+.

These lemmas will be applied shortly in the construction of models. For this, we will need to prove that $(\Sigma\underline{F})^*$ is a model of $\underline{HA}$ when each $\underline{K} \in \underline{F}$ is definable in ω^+ ($\underline{F}$ finite). But, before we can do this, we need the following:

5.6.6. Lemma. Let $\underline{K}$ be a model of $\underline{HA}$ definable in ω^+ and let $\underline{K}$ have a least node α_0. There is a canonical embedding of ω^+ into the domain of α_0 - i.e. a map of the domain of ω^+ into $D\alpha_0$ which is one-to-one and preserves the atomic formulae.

Proof. Obviously one can match up the 0 of ω^+ with the 0 of $D\alpha_0$, the 1 of ω^+ with the 1 of $D\alpha_0$, etc. But, for non-standard elements, we must observe that, in $\underline{HA}$, the relation "y is the result of the x-fold application of the successor function of $\underline{K}$ to 0" (i.e. $y = S^x 0$) is expressible. By the closure of $D\alpha_0$ (in $\underline{K}$) under the successor function, and by induction in ω^+, for all x in ω^+ there is an element in $D\alpha_0$ which is the x-fold application of successor to 0. The truth of atomic formulae $S(a,b)$ is obviously preserved under the map which associates x with the object $S^x 0$ in $D\alpha_0$. The preservation of the truth of other atomic formulae follows from the validity of the recursion equations in $\underline{K}$ and induction in ω^+.

Q.E.D.

(Note: It is in steps like this that the embedding functions $f(\alpha, \beta, x)$ are introduced in the model. We shall, however, suppress further mention of these functions in favor of a more informal approach to the proofs, in much the same way that algebraists avoid mentioning such minor difficulties.)

We may now prove the following

5.6.7. Theorem. Let $\underline{F} = \{\underline{K}_1, \ldots, \underline{K}_n\}$ be definable in ω^+. Then $(\Sigma\underline{F})^*$,

,

is a model of $\underline{\underline{HA}}$. If, in addition, ω^+ is definable in ω^{++}, $(\Sigma\underline{F})^*$ is definable in ω^{++}.

Proof. By lemma 5.6.4, $\Sigma\underline{F}$ is definable in ω^+. Obviously, ω^+ is definable in ω^+. To define $(\Sigma\underline{F})^*$ in ω^+, first recall that, formally, $\Sigma\underline{F} = \underline{K}$ where

$$K = K_1 \times \{1\} \cup K_2 \times \{2\} \cup \ldots \cup K_n \times \{n\}, \text{ etc.}$$

Thus $\alpha_o = (0,0) \notin K$ and we may let α_o be the node for ω^+ and define

$$\alpha \leq^* \beta \text{ iff } \alpha = \alpha_o \vee (\alpha \neq \alpha_o \,\&\, \alpha \leq \beta).$$

Thus K, $\leq$ are definable. Let

$$D^*(\alpha,x) \leftrightarrow (\alpha = \alpha_o \,\&\, x \in \omega^+) \vee (\alpha > \alpha_o \quad D(\alpha,x)).$$

(To explain "$x \in \omega^+$", recall that the elements of ω^+ index elements in all domains and so we must choose special indices to denote elements of ω^+ - that is we have a formula singling out the indices for ω^+. If $a \in \omega^+$, $\bar{a}$ will denote its index. (Recall that ω^+ is definable in ω^+ and consider what we mean by this.).)

To complete the proof that $(\Sigma\underline{F})^*$ is definable in ω^+, we need only show how to define the atomic formulae. Let us assume that ω^+ is defined in ω^+ as a Kripke model (say with node α_o). Let B be atomic: B^Σ denotes its definition in $\Sigma\underline{F}$; B^+ its definition in ω^+. Then

$$B^*(\alpha,x_1,\ldots,x_n) \leftrightarrow (\alpha = \alpha_o \,\&\, B^+(\alpha_o,x_1,\ldots,x_n)) \vee (\alpha > \alpha_o \,\&\, B^\Sigma(\alpha,x_1,\ldots,x_n)).$$

Now, by lemma 5.6.3, for any formula A, there is a formula $A^*(x,x_1,\ldots,x_n)$ such that, for any node α, and elements $a_1,\ldots,a_n \in D\alpha$,

$$\alpha \Vdash A(a_1,\ldots,a_n) \text{ iff } \omega^+ \models A^*(\alpha,\bar{a}_1,\ldots,\bar{a}_n).$$

Suppose $(\Sigma\underline{F})^*$ is not a model of $\underline{\underline{HA}}$. Then, for some $A(x,x_1,\ldots,x_n)$ with only $x,x_1,\ldots,x_n$ free,

$$\alpha_o \not\Vdash \forall x_1 \ldots x_n[A(0,x_1,\ldots,x_n) \,\&\, \forall xy(A(x,x_1,\ldots,x_n) \,\&\, S(x,y) \rightarrow A(y,x_1,\ldots,x_n)) \rightarrow \rightarrow \forall x A(x,x_1,\ldots,x_n)].$$

Then, for some $a_1,\dots,a_n \in \omega^+$,

$$\alpha_0 \Vdash\!\!\!/ \ A(0,a_1,\dots,a_n) \ \& \ \forall xy(A(x,a_1,\dots,a_n) \ \& \ S(x,y) \to A(y,a_1,\dots,a_n)) \to \to \forall x A(x,a_1,\dots,a_n) .$$

Then

$$\alpha_0 \Vdash A(0,a_1,\dots,a_n), \ \forall xy(A(x,a_1,\dots,a_n) \ \& \ S(x,y) \to A(y,a_1,\dots,a_n)),$$
$$\alpha_0 \Vdash\!\!\!/ \ A(a,a_1,\dots,a_n)$$

for some $a \in \omega^+$. Letting a be such that $\alpha_0 \Vdash\!\!\!/ \ A(a,a_1,\dots,a_n)$, we see, for A^* defining $\alpha_0 \Vdash A(x,x_1,\dots,x_n)$,

$$\omega^+ \models A^*(\alpha_0, \bar{0}, \bar{a}_1,\dots,\bar{a}_n),$$

(1) $$\omega^+ \models \forall xy[A^*(\alpha_0,\bar{x},\bar{a}_1,\dots,\bar{a}_n) \ \& \ S^*(\alpha_0,\bar{x},\bar{y}) \to A^*(\alpha_0,\bar{y},\bar{a}_1,\dots,\bar{a}_n)],$$
$$\omega^+ \not\models A^*(\alpha_0, \bar{a}, \bar{a}_1,\dots,\bar{a}_n).$$

Now $S^*(\alpha_0,\bar{x},\bar{y}) \longleftrightarrow S(x,y)$, whence (1) becomes

(2) $$\omega^+ \models \forall xy[A^*(\alpha_0,\bar{x},\bar{a}_1,\dots,\bar{a}_n) \ \& \ S(x,y) \to A^*(\alpha_0,\bar{y},\bar{a}_1,\dots,\bar{a}_n)].$$

But the map $a \to \bar{a}$ is definable in ω^+, whence there is a least a_0 such that

$$\omega^+ \not\models A^*(\alpha_0, \bar{a}_0, \bar{a}_1,\dots,\bar{a}_n).$$

$a_0 \neq 0$, whence $a_0 = b+1$ for some b. By minimality,

$$\omega^+ \models A^*(\alpha_0, \bar{b}, \bar{a}_1,\dots,\bar{a}_n).$$

By this last statement and (2),

$$\omega^+ \models A^*(\alpha_0, \bar{a}_0, \bar{a}_1,\dots,\bar{a}_n),$$

a contradiction. Thus $(\Sigma\underline{F})^*$ is a model of $\underset{\sim}{HA}$.

The final comment, that $(\Sigma\underline{F})^*$ is definable in ω^{++} if ω^+ is definable in ω^{++} follows from the definability of $(\Sigma\underline{F})^*$ in ω^+ and lemma 5.6.5.

Q. E. D.

5.6.8 - 5.6.9. <u>The Hilbert - Bernays completeness theorem.</u>

5.6.8. In 5.6.2 - 5.6.7, we proved two important results: (i) If $\underline{K}_1,\dots,\underline{K}_n$ are definable in ω^+, then $(\underline{K}_1 + \dots + \underline{K}_n)^*$ is a model of $\underset{\sim}{HA}$: and (ii) if ω^+, as above, is definable in ω^{++}, then $(\underline{K}_1 + \dots + \underline{K}_n)^*$ is also definable in ω^{++}. But, to be able to apply these results, we need a stock of definable Kripke models and definable non-standard models of $\underset{\sim}{HA}^c$. This is obtained by appeal to the Hilbert - Bernays completeness theorem.

5.6.9. <u>Theorem</u> (Hilbert - Bernays completeness theorem). Let $\underset{\sim}{T}$ be a consistent r.e. extension of $\underset{\sim}{HA}^c$. Then, for any model ω^+ of $\underset{\sim}{HA}^c + CON(\underset{\sim}{T})$, there is a non-standard model ω^{++} of $\underset{\sim}{T}$ which is definable in ω^{+}.

For a proof, see Kleene 1952, XIV, Thms 36 - 40, or Feferman 1960, theorem 6.2.

5.6.10 - 5.6.12. The Gödel - Rosser - Mostowski - Kripke - Myhill theorem revisited.

5.6.10. Our first two applications of the above results will be a proof of the existence of Σ_1^o substitution instances and uniform Π_2^o substitution instances in de Jongh's theorem. For these results, we need two refinements of theorem 3.3.1 and its corollary. For Σ_1^o substitution, we need, for r.e. $\underset{\sim}{T}$, $A_1,\ldots,A_m$ such that $\underset{\sim}{T}+A_i$ is consistent and such that $\underset{\sim}{HA}^c \vdash A_i \rightarrow \neg A_j$ for $i \neq j$. We present Kripke's proof:

5.6.11. Theorem. Let $\underset{\sim}{T}$ be a consistent r.e. extension of $\underset{\sim}{HA}^c$. There is an r.e. relation $P(y)$ such that, for every natural number n, $\underset{\sim}{T} + Pn + \exists! x Px$ is consistent.

Proof. (Kripke 1963.) Let $R(e,x,y)$ numeralwise represent the relation $\{e\}(x) = y$. Define a partial recursive function as follows:

$$\varphi(x) = y \quad \text{if} \quad \underset{\sim}{T} \vdash \neg(R(x,x,y) \ \& \ \exists! z R(x,x,z)),$$

(choosing the first theorem of this form if there are more than one). Then φ has an index, e. Let Px be $R(e,e,x)$. We show that, for all n,

$$Pn \ \& \ \exists! x Px$$

is consistent with $\underset{\sim}{T}$.

First, observe that $\varphi(e)$ is undefined. If not, $\varphi(e) = n_o$ for some n_o. Then

$$\underset{\sim}{T} \vdash \neg(R(e,e,n_o) \ \& \ \exists! z R(e,e,z)).$$

But clearly, if $\varphi(e) = n_o$,

$$\underset{\sim}{T} \vdash R(e,e,n_o) \ \& \ \exists! z R(e,e,z),$$

a contradiction. Hence $\varphi(e)$ is undefined and for no n do we have

$$\underset{\sim}{T} \vdash \neg(R(e,e,n) \ \& \ \exists! x R(e,e,x)).$$

Hence, for all n, $\underset{\sim}{T} + R(e,e,n) + \exists! x R(e,e,x)$ is consistent. Q. E. D.

Letting A_n be Pn, we have the desired result. One might mention that we have $\underset{\sim}{HA} \vdash A_i \rightarrow \neg A_j$ for $i \neq j$ as well as $\underset{\sim}{HA}^c \vdash A_i \rightarrow \neg A_j$.

For the Π_2^o substitution, we need the following

5.6.12. Theorem. Let $\underset{\sim}{H}$ denote $\underset{\sim}{HA}^c$ augmented by all true Π_1^o sentences of arithmetic. If $\underset{\sim}{T} \supseteq \underset{\sim}{H}$ is consistent and has a Σ_2^o enumeration, then there is an infinite family, $\{A_1,\ldots,A_n,\ldots\}$ of Π_2^o sentences independent over

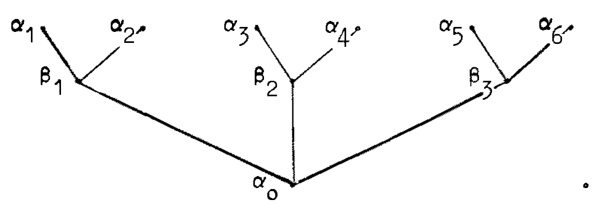
$\underset{\sim}{T}$ (in the sense of 5.3.10 that we may choose any subset of them to be true and the rest to be false).

If we observe that the proof predicate is Σ_2^0 and that the Σ_2^0 relations are precisely those numeralwise representable in $\underset{\sim}{T}$, we can mimic the proofs of theorem 5.3.11 and corollary 5.3.12 to obtain an infinite set of independent Σ_2^0 sentences. Replacing these sentences by their negations yields the theorem.

5.6.13 - 5.6.16. Σ_1^0 Substitution instances in de Jongh's theorem.

5.6.13. Recall that the reason we used the modified Jaskowski trees in proving de Jongh's theorem was that every node was determined by the set of terminal nodes not lying beyond it. Thus, if each terminal node was the unique node satisfying a particular sentence, it followed that every node was the least node satisfying a conjunction of negations of sentences corresponding to terminal nodes. Then, any set which could be the set of nodes forcing a propositional variable under a propositional forcing relation was now the set of nodes forcing a disjunction of such conjunctions of negations. In proving the existence of Σ_1^0 substitution instances, we will assign to each node of a tree a Σ_1^0 sentence which is forced only at and above that node. The substitution instances will be disjunctions of these sentences (and will thus be Σ_1^0).

Note that we no longer need to use the special property of the modified Jaskowski trees that every node is determined by a set of terminal nodes. Nonetheless, it will still be convenient to work with them. Consider, e.g., J_3^* :



Starting at the terminal nodes and working our way down the tree, we shall assign theories to the nodes. Let $A_1,\ldots,A_6$ be Σ_1^0 mutually independent over $\underset{\sim}{HA}^c$ (or, let them be obtained by theorem 5.6.11). Assign to α_i the theory

$$\underset{\sim}{T}_i = \underset{\sim}{HA}^c + A_i + \bigwedge_{j\neq i} \neg A_j .$$

By the independence of the family $\{A_1,\ldots,A_6\}$, $\underset{\sim}{T}_i$ is consistent. Now choose B_1, B_2, B_3 individually independent over $\underset{\sim}{HA}^c + CON(\underset{\sim}{T}_1) + \ldots + CON(\underset{\sim}{T}_6) + \neg A_1 + \ldots + \neg A_6$ (which is true in ω and hence consistent) such that $\underset{\sim}{HA}^c \vdash B_i \to \neg B_j$ for $i \neq j$. Assign to β_i the theory

$$\mathbf{T}_i' = \mathbf{HA}^c + B_i + \bigwedge_{i=1}^{6} CON(\mathbf{T}_i) + \bigwedge_{i=1}^{6} \neg A_i \, .$$

Again, $\mathbf{T}_i'$ is consistent. Finally, assign to α_0 the theory

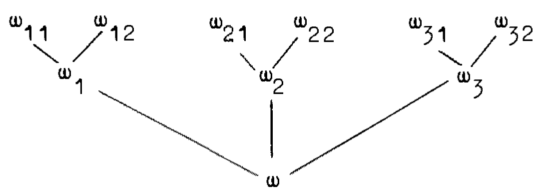
$$\mathbf{HA}^c + CON(\mathbf{T}_1') + CON(\mathbf{T}_2') + CON(\mathbf{T}_3') + \neg B_1 + \neg B_2 + \neg B_3 \, .$$

Having assigned such theories, we now assign models of $\mathbf{HA}^c$ to the nodes. Place ω at α_0. Now $\omega \models CON(\mathbf{T}_1') + CON(\mathbf{T}_2') + CON(\mathbf{T}_3')$, whence there are models ω_1, ω_2, and ω_3 of $\mathbf{T}_1'$, $\mathbf{T}_2'$, and $\mathbf{T}_3'$, respectively, such that each ω_i is definable in ω. Now,

$$\omega_i \models T_i' = \mathbf{HA}^c + B_i + \bigwedge_{j=1}^{6} CON(\mathbf{T}_j) + \bigwedge_{j=1}^{6} \neg A_j \, .$$

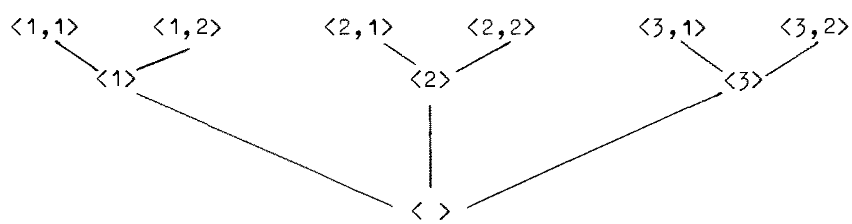
Thus, in ω_i there are definable models of $\mathbf{T}_1, \ldots, \mathbf{T}_6$. Let ω_{11}, ω_{12} be models of $\mathbf{T}_1$, $\mathbf{T}_2$, respectively, definable in ω_1; ω_{21}, ω_{22} models of $\mathbf{T}_3$, $\mathbf{T}_4$ definable in ω_2; and ω_{31}, ω_{32} models of $\mathbf{T}_5$, $\mathbf{T}_6$ definable in ω_3.

Thus, we have



Now, successively apply the lemmas 5.6.3 - 5.6.6 to conclude that the resulting structure is a model of $\mathbf{HA}$. Further, as we shall prove below, A_1 is forced only at the node corresponding to ω_{11}, A_2 at ω_{12}, A_3 at ω_{21}, ... B_1 is forced only at ω_1 and above, B_2 at ω_2 and above, and B_3 at ω_3 and above. Any provable Σ_1^0 sentence is forced at ω. Hence each node is characterized by a Σ_1^0 sentence and we may proceed from here.

For ease in assigning theories and models to nodes in the general case, and for ease in giving the proof, let us use the notation for trees of finite sequences as described in 5.3.3. J_3^*, e.g., will be represented by



Let J_n^* be given and let $\sigma_1, \ldots, \sigma_{n!}$ be its terminal nodes. Choose $A_{\sigma_1}, \ldots, A_{\sigma_{n!}}$ such that $\mathbf{HA}^c + A_{\sigma_i} + \bigwedge_{j \neq i} \neg A_{\sigma_j}$ is consistent and let $\mathbf{T}_\sigma = \mathbf{HA}^c + A_{\sigma_i} + \bigwedge_{j \neq i} \neg A_{\sigma_j}$. Let $\sigma_1, \ldots, \sigma_k$ be the non-terminal nodes of length m and let $\sigma_i*\langle 1\rangle, \ldots, \sigma_i*\langle l\rangle$ $(i=1,\ldots,k)$ be the nodes of length m+1. Let $A_{\sigma_1}, \ldots, A_{\sigma_k}$ be chosen such that $\underset{\sim}{HA}^c \vdash A_{\sigma_i} \to \neg A_{\sigma_j}$ for $i \neq j$ and such that each A_{σ_i} is consistent with

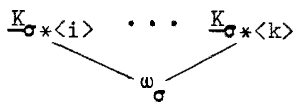
$$\underset{\sim}{T}_{m+1} = \underset{\sim}{HA}^c + \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{l} CON(\underset{\sim}{T}_{\sigma_1*\langle j\rangle}) + \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{l} \neg A_{\sigma_i*\langle j\rangle} .$$

(Observe that ω is a model of this theory and, thus, it is consistent.) Let $\underset{\sim}{T}_{\sigma_i} = \underset{\sim}{T}_{m+1} + A_{\sigma_i}$.

Thus, every node σ gets a theory $\underset{\sim}{T}_\sigma$ assigned to it. Further, if $\sigma*\langle j\rangle$ is a successor of σ, then $CON(\underset{\sim}{T}_{\sigma*\langle j\rangle})$ is provable in $\underset{\sim}{T}_\sigma$. Thus every model ω_σ of $\underset{\sim}{T}_\sigma$ has a definable model of $\underset{\sim}{T}_{\sigma*\langle j\rangle}$. Let $\omega_{\langle\,\rangle}$ be ω and, for each ω_σ and successor $\sigma*\langle j\rangle$, let $\omega_{\sigma*\langle j\rangle}$ be a model of $\underset{\sim}{T}_{\sigma*\langle j\rangle}$ definable in ω_σ. Having defined these models, assign Kripke models $\underset{\sim}{K}_\sigma$ to the nodes as follows:

(i) $\underset{\sim}{K}_\sigma = \omega_\sigma$ for terminal σ,

(ii) let $\sigma*\langle 1\rangle, \ldots, \sigma*\langle k\rangle$ be the successors of σ, and let $\underset{\sim}{K}_\sigma = (\Sigma \underset{\sim}{K}_{\sigma*\langle i\rangle})^*$:



.

By the lemmas 5.6.3 - 5.6.6, each $\underset{\sim}{K}_\sigma$ is a model of $\underset{\sim}{HA}$.

5.6.14. <u>Lemma</u>. Let A be a Σ_1^0 sentence. $\sigma \Vdash A$ iff $\omega_\sigma \models A$.

If $\sigma \Vdash \exists x Bx$, say B primitive recursive, we know that $\sigma \Vdash Bs$ for some $s \in D\sigma$. Let $\tau \geq \sigma$ be terminal. Then $\omega_\tau \models Bs$. Now, it does not follow trivially that $\omega_\sigma \models Bs$ - model-theoretically, the well-known characterization of recursiveness is preservation under extension and restriction for <u>end</u> extensions (i.e. extensions in which all of the new elements are larger than the old ones. Of course, <u>Matiyasevich</u> 1970 now gives us the result for arbitrary extensions - but this is far from trivial.).

<u>Proof of lemma 5.6.14</u>. There are three tricks we can use here:

(i) Expand the language so that primitive recursive relations are atomic. The lemma follows trivially.

(ii) Observe that the definable extension ω^{++} of ω^+ is an end extension. The lemma now follows, because, for $s \in \omega^+$, $\forall x < s$ means the same in both models.

(iii) Apply the theorem of Matiyasevich 1970 by which, if A is Σ_1^0, $\underset{\sim}{HA} \vdash A \leftrightarrow \exists x_1 \ldots x_m D(x_1 \ldots x_m)$, where D is quantifier free. The lemma then follows trivially. Q.E.D.

5.6.15. Lemma. $\omega_\sigma \models A_\tau$ iff $\sigma \geq \tau$.

Proof. Clearly $\omega_\tau \models A_\tau$. Then $\tau \Vdash A_\tau$, whence, if $\sigma \geq \tau$, $\sigma \Vdash A_\tau$, whence $\omega_\sigma \models A_\tau$.

Conversely, consider A_τ. First, assume τ is terminal. Then $\omega_\sigma \models A_\tau$ implies that $\omega_{\tau'} \models A_\tau$ for all terminal $\tau' \geq \sigma$. But, the only terminal $\omega_{\tau'} \models A_\tau$ is τ. Hence $\sigma = \tau$, by the property of the modified Jaskowski trees. (To avoid using this property, let $\sigma < \tau$. Then, for some $\sigma < \sigma'$, $\tau = \sigma' * \langle i \rangle$, but $\omega_\sigma \models A_\tau \Rightarrow \sigma \Vdash A_\tau \Rightarrow \sigma' \Vdash A_\tau \Rightarrow \omega_{\sigma'} \models A_\tau$, contradicting the fact that $\omega_{\sigma'} \models \underset{\sim}{T}_{\sigma'} = \underset{\sim}{HA}^c + \bigwedge_{\tau'} \neg A_{\tau'} + \bigwedge_{\tau'} CON(A_{\tau'} + \bigwedge_{\tau'' \neq \tau'} \neg A_{\tau''})$, where τ', τ'' range over terminal nodes.)

Let τ not be terminal and let $\omega_\sigma \models A_\tau$. Assume $\sigma \not\geq \tau$. σ cannot be of length less than τ, because, by choice,

$$\omega_\sigma \models \neg A_\rho,$$

for any ρ of length greater than that of σ. Thus, the length of σ is at least that of τ and $\sigma \geq \sigma'$ for some σ' of length the same as τ. Now, $\omega_{\sigma'} \models A_{\sigma'}$, whence $\sigma' \Vdash A_{\sigma'}$, whence $\sigma \Vdash A_{\sigma'}$, whence $\omega_\sigma \models A_{\sigma'}$. But $\underset{\sim}{HA}^c \vdash A_{\sigma'} \rightarrow \neg A_\tau$, by definition, a contradiction. Q. E. D.

Note. The above proof could have been simplified by unifying the cases - which could have been done by stipulating that theorem 5.6.11 be used in treating the terminal nodes. The non-terminal nodes must be treated by using theorem 5.6.11. - Unless we know $\underset{\sim}{HA}^c \vdash A_{\sigma'} \rightarrow \neg A_{\tau'}$ we have no guarantee that A_τ will be false in extensions σ of σ'. (This observation is due to de Jongh, who found and corrected the corresponding error in our original attempt at proving the existence of Σ_1^0 substitution instances.)

We may now prove

5.6.16. Theorem. Let $Pp \not\vdash A(p_1,\ldots,p_n)$. Then there are Σ_1^0 sentences $B_1,\ldots,B_n$ such that $\underset{\sim}{HA} \not\vdash A(B_1,\ldots,B_n)$.

Proof. Let J_n^* be given with a forcing relation on it such that $\langle\ \rangle \not\Vdash A(p_1,\ldots,p_n)$. Let, for each p_i,

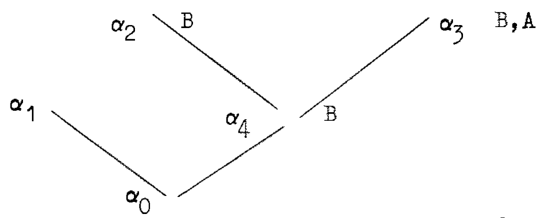
$$B_i \longleftrightarrow \bigvee_{\sigma \Vdash p_i} A_\sigma .$$

(If no σ forces p_i, let B_i be any refutable Σ_1^0 sentence.) Then B_i is Σ_1^0, $\sigma \Vdash B_i$ iff $\sigma \Vdash p_i$, and a simple induction on the length of $C(p_1,\ldots,p_n)$ shows

$$\sigma \Vdash C(p_1,\ldots,p_n) \text{ iff } \sigma \Vdash C(B_1,\ldots,B_n). \qquad \text{Q. E. D.}$$

5.6.17 - 5.6.19. Uniform Π^0_2 substitutions in de Jongh's theorem.

5.6.17. The problem with the Σ^0_1 substitutions is that we could not choose the nodes at which we wanted a particular Σ^0_1 sentence to be forced. E.g. consider the tree



Suppose we want A, B forced as indicated, $A, B \in \Sigma^0_1$. Then, at α_4 we must have a model ω_{α_4} of

$$\underset{\sim}{HA}^c + B + CON(\underset{\sim}{HA}^c + B + \neg A) + CON(\underset{\sim}{HA}^c + B + A) .$$

Since B is false in the standard model,

$$\underset{\sim}{HA}^c + \neg B + CON(\underset{\sim}{HA}^c + B + \neg A) + CON(\underset{\sim}{HA}^c + B + A)$$

is also consistent and B must be independent over $\underset{\sim}{HA}^c + CON(\underset{\sim}{HA}^c + B + \neg A) + CON(\underset{\sim}{HA}^c + B + A)$. Larger trees will require larger nested consistency statements and we just don't know if any such Σ^0_1 sentences exist. (Observe that we cannot have as much independence from consistency statements as with Π^0_2 - sentences, since, if $B \in \Sigma^0_1$, then $B + CON(\underset{\sim}{HA}^c + \neg B)$ is inconsistent.)

If, however, $B_1, \ldots, B_n$ are Π^0_2 and mutually independent over $\underset{\sim}{HA}^c$ when augmented by all true Π^0_1 sentences of arithmetic, we can assign nodes to the formulae as desired. This is the basis of the following model-theoretic proof of a result of Friedman A:

5.6.18. Theorem. (Friedman). Let $B_1, \ldots, B_n$ be Π^0_2, independent over $\underset{\sim}{HA}^c$ augmented by all true Π^0_1 sentences, and let $Pp \not\vdash A(p_1, \ldots, p_n)$. Then

$$\underset{\sim}{HA} \not\vdash A(B_1, \ldots, B_n) .$$

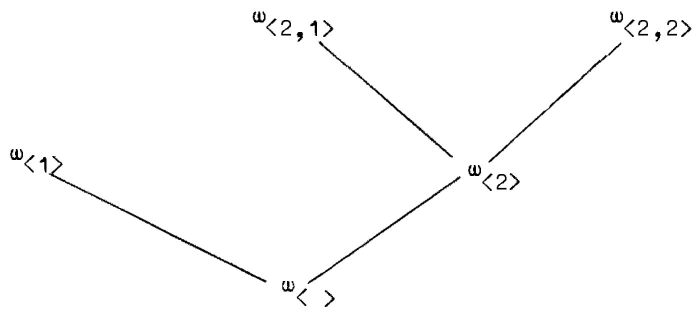
Proof. Let $(K, \leq, \Vdash)$ be an arbitrary tree model of P and let $\langle\, \rangle \not\Vdash A(p_1, \ldots, p_n)$. Let $B_1, \ldots, B_n$ satisfy the hypothesis of the theorem and assign theories to nodes as follows:

If τ is terminal, $\underset{\sim}{T}_\tau = \underset{\sim}{HA}^c + \bigwedge_{\tau \Vdash p_i} B_i + \bigwedge_{\tau \not\Vdash p_i} \neg B_i$.

If σ has successors $\sigma * \langle 1 \rangle, \ldots, \sigma * \langle k \rangle$,

$$\underset{\sim}{T}_\sigma = \underset{\sim}{HA}^c + \bigwedge_{\sigma \Vdash p_i} B_i + \bigwedge_{\sigma \not\Vdash p_i} \neg B_i + \bigwedge_{j=1}^{k} CON(\underset{\sim}{T}_{\sigma * \langle j \rangle}) .$$

Each $\underset{\sim}{T}_\sigma$ is obviously consistent and we may define models ω_σ as usual, starting at $\langle\ \rangle$ with an arbitrary model of $\underset{\sim}{T}_{\langle\ \rangle}$. Then $\underline{K}_\sigma$ is defined and, finally, we have a model of $\underset{\sim}{HA}$. E.g. with the tree featured above, we have



where $\omega_{\langle 2,1\rangle}, \omega_{\langle 2,2\rangle}, \omega_{\langle 2\rangle} \models B$, $\omega_{\langle 2,2\rangle} \models A$.

5.6.19. Lemma. Let A be Π^0_2. $\sigma \Vdash A$ iff $\underset{\sim}{\forall}\tau \geq \sigma\ \ \omega_\tau \models A$.

Proof. Probably the simplest thing to do is to appeal to Matiyasevich 1970 or add new predicate symbols so that A is of the form,

$$\forall x_1 \ldots x_n\ \exists y_1 \ldots y_m C(x_1, \ldots, x_n, y_1, \ldots, y_m),$$

where C is quantifier-free and decidable. Then

$$\begin{aligned}\sigma \Vdash A &\Rightarrow \underset{\sim}{\forall}\tau \geq \sigma\ \ \tau \Vdash A\\ &\Rightarrow \underset{\sim}{\forall}\tau \geq \sigma\ \ \underset{\sim}{\forall}s_1 \ldots s_n \in D\tau\ \ \underset{\sim}{\exists}t_1 \ldots t_m \in D\tau(\tau \Vdash C(s_1, \ldots, s_n, t_1, \ldots, t_m))\\ &\Rightarrow \underset{\sim}{\forall}\tau \geq \sigma\ \underset{\sim}{\forall}s_1 \ldots s_n \in D\tau\ \ \underset{\sim}{\exists}t_1 \ldots t_n\ \ \omega_\tau \models C(s_1, \ldots, s_n, t_1, \ldots, t_m)\\ &\Rightarrow \underset{\sim}{\forall}\tau \geq \sigma\ \omega_\tau \models \forall x_1 \ldots x_n \exists y_1 \ldots y_m C, \text{ i.e. } \omega_\tau \models A.\end{aligned}$$

The converse is just the definition of forcing for an $\forall\exists$ combination.

Q. E. D.

To finish the proof of the theorem, observe that

$$\begin{aligned}\sigma \Vdash B_i &\text{ iff } \underset{\sim}{\forall}\tau \geq \sigma\ \ \omega_\tau \models B_i\\ &\text{ iff } \underset{\sim}{\forall}\tau \geq \sigma\ \ \tau \Vdash p_i\\ &\text{ iff } \sigma \Vdash p_i.\end{aligned}$$

The rest is just the usual induction. Q. E. D.

5.6.20 - 5.6.22. De Jongh's theorem for MP .

5.6.20. Just as MP was not preserved by $(\ ) \to (\Sigma\ )'$, it is not in general preserved by $(\ ) \to (\Sigma\ )^*$. If, however, each element of $\underline{F} = \{\underline{K}_1, \ldots, \underline{K}_n\}$ is definable in ω^+, and if MP is valid in $\underline{F}$, then MP is valid in $(\Sigma\underline{F} + \omega^+)^*$. This is just a variation of the result we will need. A direct verification of this variant is left to the reader. The general lemma we will need is the following:

5.6.21. Lemma. Let $\underset{\sim}{K}$ be a tree model of $\underset{\sim}{HA}$ obtained by the process of placing non-standard models of arithmetic at the nodes. Suppose that, for every node σ, there is a terminal node $\tau \geq \sigma$ such that $\omega_\sigma = \omega_\tau$. Then MP is valid in $\underset{\sim}{K}$.

Proof. Assume MP is not valid in $\underset{\sim}{K}$ - take MP in the form (iv) of section 4:

$$\langle\ \rangle \not\Vdash \forall x[\forall y(Axy \vee \neg Axy) \ \&\ \neg\neg\exists y Axy \rightarrow \exists y Axy].$$

Then, for some $\sigma \geq \langle\ \rangle$, $s \in D\sigma$,

$$\alpha \not\Vdash \forall y(Asy \vee \neg Asy) \ \&\ \neg\neg\exists y Asy \rightarrow \exists y Asy.$$

Thus, for some $\rho \geq \sigma$, whence $t \in D_p$. Also,

$$\rho \Vdash \forall y(Asy \vee \neg Asy), \quad \rho \Vdash \neg\neg\exists y Asy, \quad \rho \not\Vdash \exists y Asy.$$

Let $\tau \geq \rho$ be terminal with $\omega_\tau = \omega_\rho$. Then $\tau \Vdash \exists y Asy$, say $\tau \Vdash Ast$, $t \in D\tau$. But $D\tau = D\rho$, whence $t \in D\rho$. Also,

$$\rho \Vdash Ast \vee \neg Ast,$$

whence $\rho \Vdash Ast$, i.e. $\rho \Vdash \exists y Asy$, a contradiction. Q. E. D.

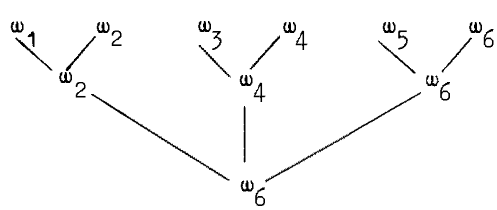
5.6.22. Theorem. Let $Pp \not\vdash A(p_1,\ldots,p_n)$. Then there are sentences $B_1,\ldots,B_n$ such that

$$\underset{\sim}{HA} + MP \not\vdash A(B_1,\ldots,B_n).$$

Proof. Let J_n^* be given and define theories as follows: $\underset{\sim}{T}_1 = \underset{\sim}{HA}^c + A_1$, where A_1 is Σ_1^0, independent of $\underset{\sim}{HA}^c$.

$$\underset{\sim}{T}_{m+1} = \underset{\sim}{HA}^c + CON(\underset{\sim}{T}_m) + \neg A_m + A_{m+1},$$

where A_{m+1} is independent of $\underset{\sim}{HA}^c + CON(\underset{\sim}{T}_m) + \neg A_m$. Let $\alpha_1,\ldots,\alpha_{n!}$ be the terminal nodes of J_n^*. Let $\omega_{n!}$ be a model of $\underset{\sim}{T}_{n!}$. Given a model ω_{m+1} of $\underset{\sim}{T}_{m+1}$, let ω_m be a model of $\underset{\sim}{T}_m$ definable in ω_{m+1}. Assign ω_m to the terminal node α_m. In going down the tree, assign to σ the classical model assigned to the right-most successor of σ. J_3^*, e.g., looks like



This gives us a Kripke model of $\underset{\sim}{HA}$. By the lemma, it is also a model of MP. Finally, each terminal node is the unique node forcing a particular sentence . The proof of de Jongh's theorem in section 3 now goes through easily. Q.E.D.

Note that Σ_1^0 substitutions are impossible: If A is Σ_1^0, $\underset{\sim}{HA} + MP \vdash \neg\neg A \to A$.

5.6.23 - 5.6.25. Other applications.

We first present a lemma.

5.6.23. Lemma. Let $<$ be primitive recursive, $\underline{K}_1, \ldots, \underline{K}_n$ models of $\underset{\sim}{HA} + TI(<)$, each $\underline{K}_i$ definable in ω^+, and $\omega^+ \models TI(<)$. Then $TI(<)$ is valid in $(\Sigma\underline{F})^*$.

Proof. Use $TI(<)$ in ω^+ applied to $A^*(\alpha_0, \bar{x}, \bar{x}_1, \ldots, \bar{x}_n)$ to verify that TI applied to $A(x, x_1, \ldots, x_n)$ is forced at α_0. Q.E.D.

By insisting that each theory $\underset{\sim}{T}_\sigma$ used in 5.6.13 - 5.6.22 also contain $TI(<)$, every model $\underline{K}_\sigma$ encountered is a model of $TI(<)$. Thus, the Σ_1^0 substitution and uniform Π_2^0 substitution results hold for $\underset{\sim}{HA} + TI(<)$ (where independence over $\underset{\sim}{HA}^c$ ($\underset{\sim}{HA}^c$ + true Π_1^0) is replaced by independence over $\underset{\sim}{HA} + TI(<)$ ($\underset{\sim}{HA}^c + TI(<)$ + true Π_1^0)). Further, the unrefined version of de Jongh's theorem holds for $\underset{\sim}{HA} + MP + TI(<)$.

A similar proof does not work for $RF(\underset{\sim}{T})$ or $RFN(\underset{\sim}{T})$. Recall that, to prove $RF(\underset{\sim}{T})$ was preserved if $\underset{\sim}{T}$ was, when we assumed $\exists x\, Proof_T(x, \ulcorner A \urcorner)$ was forced by α_0 in $(\Sigma\underline{F})'$, it followed that $Proof_T(n, \ulcorner A \urcorner)$ was forced for some natural number n. From this it followed that A was indeed provable. We can no longer reason in this manner for $(\Sigma\underline{F})^*$. We can, however, appeal to the following result of Kreisel - Levy 1968; (Theorem 12, p. 125):

5.6.24. Theorem. For small ordinals α, $\underset{\sim}{HA}$ together with the scheme $TI(<_{\epsilon_\alpha})$ (transfinite induction on the canonical well-ordering of type ϵ_α) is equivalent to the system obtained from $\underset{\sim}{HA}$ by iterating the process $\underset{\sim}{T} \to \underset{\sim}{T} + RFN(\underset{\sim}{T})$ $1+\alpha$ times; e.g. $\underset{\sim}{HA} + RFN(\underset{\sim}{HA}) = \underset{\sim}{HA} + TI(<_{\epsilon_0})$.

5.6.25. It follows from theorem 5.6.24 and the above remark that the Σ_1^0 and Π_2^0 results hold for $\underset{\sim}{HA} + RFN(\underset{\sim}{HA})$, $\underset{\sim}{HA} + RFN(\underset{\sim}{HA} + RFN(\underset{\sim}{HA}))$, etc. The results for $\underset{\sim}{HA} + RF(\underset{\sim}{HA})$ follow trivially from the results for the extension $\underset{\sim}{HA} + RFN(\underset{\sim}{HA})$.

It also follows that we get the unrefined version of de Jongh's theorem for $\underset{\sim}{HA} + MP + TI(<)$, $\underset{\sim}{HA} + MP + RFN(\underset{\sim}{HA})$, $\underset{\sim}{HA} + MP + RF(\underset{\sim}{HA})$ - we do not have the result for $\underset{\sim}{HA} + MP + RFN(\underset{\sim}{HA} + MP)$ because it is not known if

$$\underset{\sim}{HA} + MP + RFN(\underset{\sim}{HA} + MP) = \underset{\sim}{HA} + MP + TI(<_{\epsilon_0}).$$

In discussing the operations $(\ )\to(\Sigma\ )'$, $(\ )\to(\Sigma\ +\omega)'$, we gave some closure properties of the classes $\mathfrak{P}$, $\mathfrak{P}^{\omega}$, respectively, of sets Γ preserved by these operations. In disussing the operation $(\ )\to(\Sigma\ )^*$ and the class of models described in the statement of lemma 5.6.21, we should comment on the classes $\mathfrak{P}^*$ of sets of sentences valid in the Kripke model of the lemma provided they are valid in all of the non-standard models of which the Kripke model is composed. We both lose and gain some closure conditions.

In both cases, we lose Friedman's condition (iii) (theorems 5.2.11 and 5.4.13 above) which we restate here for convenience:

Condition (iii) If $\Gamma\in\mathfrak{P}(\mathfrak{P}^{\omega})$, A has only x free, and $\underset{\sim}{HA}+\Gamma\vdash An$ for all n, then $\Gamma\cup\{\forall xAx\}\in\mathfrak{P}$ (resp., $\mathfrak{P}^{\omega}$).
Recall that, if $\Gamma+\forall xAx$ was valid in $\underline{F}$, $\forall xAx$ could only fail to be valid in $(\Sigma\underline{F})'$ or $(\Sigma\underline{F}+\omega)'$ when some instance An was not forced at the node α_0 - which is ruled out by the hypothesis. Obviously, this is no longer valid reasoning in the present situation where the new origins have non-standard integers in their domains.

For $(\ )\to(\Sigma\ )^*$, the use of definability does not give us an alternative to condition (iii). For applications to MP, however, we do have a slight rebate. Rather than to try to state an intelligible analogue to theorem 5.4.13, let us consider an example:

5.6.26. Theorem. Let $\mathfrak{P}_1$ be the class obtained by the following:

(i) $\underset{\sim}{HA}+MP\in\mathfrak{P}_1$;

(ii) $\underset{\sim}{HA}+TI(<)\in\mathfrak{P}_1$;

(iii) the union of any r.e. sequence of elements of $\mathfrak{P}_1$ is in $\mathfrak{P}_1$;

(iv) if $\Gamma\in\mathfrak{P}_1$, A is a Harrop-sentence and A is consistent with $\underset{\sim}{HA}^c$ + all true Π^0_1 sentences, then $\Gamma\cup\{A\}\in\mathfrak{P}_1$.

Then, for any $\Gamma\in\mathfrak{P}_1$, $\underset{\sim}{HA}+\Gamma$ has DP.

Proof. First, by (iii) all $\Gamma\in\mathfrak{P}_1$ are consistent with $\underset{\sim}{HA}^c$ + all true Π^0_1 sentences, which includes the consistency statements needed to define models. Also, if $\Gamma\in\mathfrak{P}_1$, Γ is r.e. and, if $\underset{\sim}{HA}+\Gamma\not\vdash A$, $\underset{\sim}{HA}+\Gamma\not\vdash B$, we can find Kripke models $\underline{K}_1$, $\underline{K}_2$, definable in some model ω^+ of $\underset{\sim}{HA}+\Gamma$, in which A, resp. B, fails to be forced. (To see that these are definable Kripke models, use the reduction of the problem to the Hilbert-Bernays completeness theorem outlined in 5.1.26.) Then $(\underline{K}_1+\underline{K}_2+\omega^+)^*$ is a model of $\underset{\sim}{HA}+\Gamma$ in which $A\vee B$ is false. Q.E.D.

We cannot generalize this to obtain ED. First, the domain at the origin would have non-standard integers and $\alpha_0\Vdash\exists xAx$ would not imply $\alpha_0\Vdash An$ for some n. Second, to have the definability of forcing for $(\Sigma\underline{F}+\omega^+)^*$ in ω^+, for an infinite family $\underline{F}$, we must have a uniform forcing definition for all elements in the class. But, the minute we have this, we have models

$\underline{K}_a$ for non-standard a occurring in our description of $\underline{F}$ - i.e. we are not defining the model we want to define.

Such esoteric results as theorem 5.6.26 are of little interest in themselves. They do, however, illustrate the differences in our ability to treat $\underset{\sim}{HA}$ and $\underset{\sim}{HA} + MP$ (as do such negative results as our inability to prove ED for $\Gamma \in \mathfrak{P}_1$).

§ 7. Other systems

5.7.1 - 5.7.2. Subsystems of Heyting's arithmetic.

5.7.1. In van Dalen - Gordon 1971, van Dalen and Gordon apply Kripke models to settle some independence questions regarding subsystems of $\underset{\sim}{HA}$. For example, consider the system $\underset{\sim}{T}$ with the constant 0, function symbols $'$, $+$, $\cdot$, and axioms (in addition to axioms of the intuitionistic predicate calculus with equality):

$$\begin{array}{ll} x' = y' \rightarrow x = y & \neg x' = 0 \\ x + 0 = x & x + y' = (x+y)' \\ x \cdot 0 = 0 & x \cdot y' = x \cdot y + x \\ x + y = y + x & x \cdot y = y \cdot x \\ (x+y) + z = x + (y+z) & (x \cdot y) \cdot z = x \cdot (y \cdot z) \end{array}$$

$$\neg x = 0 \rightarrow \exists y (y' = x) .$$

Then, van Dalen and Gordon proved:

5.7.2. Theorem. $\underset{\sim}{T} \nvdash \forall xy(x=y \vee \neg x=y)$.

Proof. (van Dalen - Gordon 1971). Let ${}^*\mathbb{R}$ be a non-standard extension of $\mathbb{R}$, the field of real numbers. Let ${}^*N$ denote the set of elements of ${}^*\mathbb{R}$ which are infinitesimally close to some natural number - i.e.

$${}^*N = \{x \in {}^*\mathbb{R} : \underset{\sim}{\exists} n \in \omega \ \underset{\sim}{\exists} \text{infinitesimal } \delta (x = n \pm \delta)\} .$$

Now, consider the model,

$$\begin{array}{ll} \alpha_1 & {}^*N_2 \\ & \ \ | \\ \alpha_0 & {}^*N_1 \end{array} ,$$

where $D\alpha_1 = D\alpha_0 = {}^*N$, the operations $'$, $+$, $\cdot$ on ${}^*N_1$, ${}^*N_2$ are those inherited from ${}^*\mathbb{R}$, $\alpha_0 \Vdash a=b$ iff a, b actually denote the same element of ${}^*\mathbb{R}$, and $\alpha_1 \Vdash a=b$ iff a and b are infinitesimally close (i.e. iff a, b are close to the same natural number). The axioms of $\underset{\sim}{T}$ are obviously forced at α_0, but the decidability of equality is not forced, since, if δ is infinitesimal, $\alpha_0 \nVdash n=n+\delta$ and, since $\alpha_1 \Vdash n=n+\delta$, $\alpha_0 \nVdash \neg n=n+\delta$. Thus $\alpha_0 \nVdash n=n+\delta \vee \neg n=n+\delta$. Q. E. D.

Observe that, in the model given in the proof of the theorem, the model at α_1 is, basically, the standard model ω. Thus, every instance of induction is forced at α_1 and the double negation of every instance of induction is forced at α_0. Hence, although equality is provably decidable by induction, it is not provably decidable by the double negation of induction - or by induction on Harrop - formulae, since, as the reader may easily verify, if such a formula is forced at α_1, it is forced at α_0. Thus, as one might expect, one cannot use a negative form of induction to prove the positive result that equality is decidable.

This raises the question:
How much induction is needed to prove the decidability of equality?
By induction on quantifier-free formulae, one can prove, for each n, $\forall x(n=x \vee \neg n=x)$. (Thus, quantifier-free induction fails to hold in the above model and quantifier-free induction is not derivable from the negative formulations of induction mentioned.) Can one use induction on quantifier-free formulae to derive the decidability of equality - $\forall xy(x=y \vee \neg x=y)$?

5.7.3. Extensions of HA: Theory of species.

Aside from some comments on free choice sequences in Kripke's original paper Kripke 1965, the only discussion of Kripke models and higher systems published to date is Prawitz 1970 in which Prawitz proves the completeness of the cut-free rules of the second-order intuitionistic predicate calculus with respect to second-order Kripke models (and also with respect to a proper subclass of these models, namely, the second-order Beth models). Since the induction scheme is given by a single axiom of this second-order language, this yields a completeness theorem for second-order arithmetic plus comprehension with respect to these second-order models.

The simplest way to describe the second-order Kripke models is to say that the domain function D splits into two functions D_1 and D_2, each satisfying the monotonicity condition, and such that there is a binary membership relation, $\in$, between elements of $D_1\alpha$ and $D_2\alpha$ (for given α). Further, when discussing comprehension, one assumes that for every node α and every formula $A(x)$ with parameters from $D_1\alpha$ and $D_2\alpha$, x the only free variable in A, there is an element X of $D_2\alpha$ such that

$$\alpha \Vdash \forall x[Ax \longleftrightarrow x \in X].$$

(In the presence of a little arithmetic, we can restrict ourselves to unary relations in $D_2\alpha$.)

In attempting to construct models of second-order arithmetic, the obvious approach is to mimic our procedure in constructing models of first-order arithmetic - but also insisting that the classical models being used be models of the second-order theory with comprehension. We have not considered this possibility thoroughly enough to say whether or not it will lead anywhere.

Let us consider a simple example. Suppose (ω^{++}, B) is definable in (ω^{+}, A), where B and A are the classes of sets of numbers in the two models. We would define a model:

$$\begin{array}{c} (\omega^{++}, B) \\ | \\ \alpha_0\ (\omega^{+}, A^{*})\ . \end{array}$$

The choice of B is obvious; but what do we choose for A^*? We cannot simply choose A since, e.g., $\{x \mid C(x)$ is true in $(\omega^+, A)\}$ need not be $\{x \mid C(x)$ is forced at $\alpha_0\}$. (E.g. as long as ω^{++} is not an elementary extension of ω^+, there will be arithmetical $C(x)$ for which these sets will have to differ.) The obvious approach is to start with species X_C for arithmetical C such that

$$\alpha_0 \Vdash \forall x[x \in X_C \leftrightarrow C(x)] .$$

We cannot do this for C with species variables since we don't know yet what A^* is. Adding a species at a time, one can handle comprehension for formulae

$$\exists X_1 \ldots X_n\ C(x, X_1, \ldots, X_n) ,$$

where C has no bound species variables; but one cannot automatically handle more complex formulae - each new species added changes the domain of species and hence the nature of universal quantification.

Another possibility is the use of ω-models - i.e. models in which the individuals are precisely the natural numbers. The induction scheme, even applied to second-order formulae, will obviously be forced and the only problematic scheme is that of comprehension. A simple way to guarantee comprehension is to guarantee that all possible species are in the domains. Let $\underline{K}$ be a model, with partial order $(K, \leq)$, and let Ax be a formula with only the variable x free. Let $A_\alpha = \{a \in D_1\alpha : \alpha \Vdash Aa\}$. To guarantee comprehension, we need at α_0 a set $X \in D_2\alpha_0$ such that for any α and any $a \in D_1\alpha$, $\alpha \Vdash a \in X$ iff $\alpha \Vdash a \in A_\alpha$. To guarantee this, we simply let $D_2\alpha = D_2\alpha_0$ be the set of all partially ordered systems, $\underline{S} = \{S_\alpha\}_{\alpha \in K}$, of sets of natural numbers indexed by K satisfying $\alpha \leq \beta \Rightarrow S_\alpha \subseteq S_\beta$, and let $\underline{S}$ behave like S_α at node α. For any formula A, the system $\underline{A} = \{A_\alpha\}_{\alpha \in K}$ automatically represents A and comprehension is valid.

This latter type of model can be used to obtain certain formal results, e.g. it is easy to construct a model (the full binary tree) in which

$$\forall x \neg \forall X_1 \ldots \forall X_n A(x \in X_1, \ldots, x \in X_n)$$

is valid for any propositional formula $A(p_1, \ldots, p_n)$ which is not derivable in the intuitionistic propositional calculus. As a second-order counterexample, $A(x \in X_1, \ldots, x \in X_n)$ is as simple as they come - one might hope for an arithmetic counterexample, or at least a version of de Jongh's theorem; but these models will not yield such results, because all true arithmetic formulae are valid in them. Similarly, they cannot be used to prove the explicit definability or disjunction theorems.

5.7.4. Other set-theoretic approaches.

The Kripke models only form one of several classical modellings of intuitionistic systems. Others include the Beth models, interpretations in lattices, and topological interpretations. Their applications to the propositional calculus and the first-order predicate calculus are well-known. For higher systems, they have barely been applied. Prawitz 1970 applies the Beth bodels (and also Kripke models) to the theory of species; Scott 1968 and Scott 1970 apply the topological interpretation to the theory of the order of the continuum; and Moschovakis A applies the topological interpretation to second-order arithmetic - i.e. arithmetic with quantification over functions.

In Scott 1970, Scott proved the validity of Kripke's schema in his model. Moschovakis, in Moschovakis A, showed the consistency of this schema with a system of second-order intuitionistic arithmetic. Kripke's schema,

$$\exists\alpha[\exists x(\alpha x \neq 0 \leftrightarrow A(x))],$$

is important in that it contradicts many theorems of classical analysis (see e.g. Hull 1967).