5

Intuitionistic Logic

5.1 Constructive Reasoning

In the preceding chapters, we have been guided by the following, seemingly harmless extrapolation from our experience with finite sets: infinite universes can be surveyed in their totality. In particular can we in a global manner determine whether $\mathfrak{A} \models \exists x \varphi(x)$ holds, or not. To adapt Hermann Weyl's phrasing: we are used to think of infinite sets not merely as defined by a property, but as a set whose elements are so to speak spread out in front of us, so that we can run through them just as an officer in the police office goes through his file. This view of the mathematical universe is an attractive but rather unrealistic idealization. If one takes our limitations in the face of infinite totalities seriously, then one has to read a statement like "there is a prime number greater than $10^{10^{10}}$" in a stricter way than "it is impossible that the set of primes is exhausted before $10^{10^{10}}$". For we cannot inspect the set of natural numbers in a glance and detect a prime. We have to *exhibit* a prime p greater than $10^{10^{10}}$.

Similarly, one might be convinced that a certain problem (e.g. the determination of the saddle point of a zero-sum game) has a solution on the basis of an abstract theorem (such as Brouwer's fixed point theorem). Nonetheless one cannot always exhibit a solution. What one needs is a *constructive* method (proof) that determines the solution.

One more example to illustrate the restrictions of abstract methods. Consider the problem "Are there two irrational numbers a and b such that a^b is rational?" We apply the following smart reasoning: suppose $\sqrt{2}^{\sqrt{2}}$ is rational, then we have solved the problem. Should $\sqrt{2}^{\sqrt{2}}$ be irrational then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is rational. In both cases there is a solution, so the answer to the problem is: Yes. However, should somebody ask us to produce such a pair a, b, then we have to engage in some serious number theory in order to come up with the right choice between the numbers mentioned above.

Evidently, statements can be read in an inconstructive way, as we did in the preceding chapters, and in a constructive way. We will in the present chapter briefly sketch the logic one uses in constructive reasoning. In mathematics the practice of constructive procedures and reasoning has been advocated by a number of people, but the founding fathers of constructive mathematics clearly are L. Kronecker and L.E.J. Brouwer. The latter presented a complete program for the rebuilding of mathematics on a constructive basis. Brouwer's mathematics (and the accompaying logic) is called *intuitionistic*, and in this context the traditional nonconstructive mathematics (and logic) is called *classical*.

There are a number of philosophical issues connected with intuitionism, for which we refer the reader to the literature, cf. *Dummett, Troelstra-van Dalen.*

Since we can no longer base our interpretations of logic on the fiction that the mathematical universe is a predetermined totality which can be surveyed as a whole, we have to provide a heuristic interpretation of the logical connectives in intuitionistic logic. We will base our heuristics on the proof-interpretation put forward by A. Heyting. A similar semantics was proposed by A. Kolmogorov; the proof-interpretation is called the Brouwer-Heyting-Kolmogorov (BHK)-interpretation .

The point of departure is that a statement φ is considered to be true (or to hold) if we have a proof for it. By a proof we mean a mathematical construction that establishes φ, not a deduction in some formal system. For example, a proof of '$2+3=5$' consists of the successive constructions of $2, 3$ and 5, followed by a construction that adds 2 and 3, followed by a construction that compares the outcome of this addition and 5.

The primitive notion is here "a proves φ", where we understand by a proof a (for our purpose unspecified) construction. We will now indicate how proofs of composite statements depend on proofs of their parts.

($\wedge$) a proves $\varphi \wedge \psi := a$ is a pair $\langle b, c\rangle$ such that b proves φ and c proves ψ.

($\vee$) a proves $\varphi \vee \psi := a$ is a pair $\langle b, c\rangle$ such that b is a natural number and if $b=0$ then c proves φ, if $b \neq 0$ then c proves ψ.

($\rightarrow$) a proves $\varphi \rightarrow \psi := a$ is a construction that converts any proof p of φ into a proof $a(p)$ of ψ.

($\perp$) no a proves $\perp$.

In order to deal with the quantifiers we assume that some domain D of objects is given.

($\forall$) a proves $\forall x \varphi(x) := a$ is a construction such that for each $b \in D$ $a(b)$ proves $\varphi(\overline{b})$.

($\exists$) a proves $\exists x \varphi(x) := a$ is a pair (b, c) such that $b \in D$ and c proves $\varphi(\overline{b})$.

The above explanation of the connectives serves as a means of giving the reader a feeling for what is and what is not correct in intuitionistic logic. It is generally considered the intended intuitionistic meaning of the connectives.

Examples.

1. $\varphi \wedge \psi \to \varphi$ is true, for let $\langle a, b\rangle$ be a proof of $\varphi \wedge \psi$, then the construction c with $c(a,b) = a$ converts a proof of $\varphi \wedge \psi$ into a proof of φ. So c proves $(\varphi \wedge \psi \to \varphi)$.
2. $(\varphi \wedge \psi \to \sigma) \to (\varphi \to (\psi \to \sigma))$. Let a prove $\varphi \wedge \psi \to \sigma$, i.e. a converts each proof $\langle b, c\rangle$ of $\varphi \wedge \psi$ into a proof $a(b,c)$ of σ. Now the required proof p of $\varphi \to (\psi \to \sigma)$ is a construction that converts each proof b of φ into a $p(b)$ of $\psi \to \sigma$. So $p(b)$ is a construction that converts a proof c of ψ into a proof $(p(b))(c)$ of σ. Recall that we had a proof $a(b,c)$ of σ, so put $(p(b)(c) = a(b,c)$; let q be given by $q(c) = a(b,c)$, then p is defined by $p(b) = q$. Clearly, the above contains the description of a construction that converts a into a proof p of $\varphi \to (\psi \to \sigma)$. (For those familiar with the λ-notation: $p = \lambda b.\lambda c.a(b,c)$, so $\lambda a.\lambda b.\lambda c.a(b,c)$ is the proof we are looking for).
3. $\neg \exists x \varphi(x) \to \forall x \neg \varphi(x)$.
   We will now argue a bit more informal. Suppose we have a construction a that reduces a proof of $\exists x \varphi(x)$ to a proof of $\perp$. We want a construction p that produces for each $d \in D$ a proof of $\varphi(\overline{d}) \to \perp$, i.e. a construction that converts a proof of $\varphi(\overline{d})$ into a proof of $\perp$. So let b be a proof of $\varphi(\overline{d})$, then $\langle d, b\rangle$ is a proof of $\exists x \varphi(x)$, and $a(d,b)$ is a proof of $\perp$. Hence p with $(p(d))(b) = a(d,b)$ is a proof of $\forall x \neg \varphi(x)$. This provides us with a construction that converts a into p.

The reader may try to justify some statements for himself, but he should not worry if the details turn out to be too complicated. A convenient handling of these problems requires a bit more machinery than we have at hand (e.g. λ-notation). Note, by the way, that the whole procedure is not unproblematic since we assume a number of closure properties of the class of constructions.

Now that we have given a rough heuristics of the meaning of the logical connectives in intuitionistic logic, let us move on to a formalization. As it happens, the system of natural deduction is almost right. The only rule that lacks constructive content is that of Reduction ad Absurdum. As we have seen (p. 38), an application of RAA yields $\vdash \neg\neg\varphi \to \varphi$, but for $\neg\neg\varphi \to \varphi$ to hold informally we need a construction that transforms a proof of $\neg\neg\varphi$ into a proof of φ. Now a proves $\neg\neg\varphi$ if a transforms each proof b of $\neg\varphi$ into a proof of $\perp$, i.e. there cannot be a proof b of $\neg\varphi$. b itself should be a construction that transforms each proof c of φ into a proof of $\perp$. So we know that there cannot be a construction that turns a proof of φ into a proof of $\perp$, but that is a long way from the required proof of φ! (cf. ex. 1)

5.2 Intuitionistic Propositional and Predicate Logic

We adopt all the rules of natural deduction for the connectives $\vee, \wedge, \rightarrow, \perp, \exists, \forall$ with the exception of the rule RAA. In order to cover both propositional and predicate logic in one sweep we allow in the alphabet (cf. 2.3,p. 60) 0-ary predicate symbols, usually called proposition symbols.

Strictly speaking we deal with a derivability notion different from the one introduced earlier (cf. p.36), since RAA is dropped; therefore we should use a distinct notation, e.g. $\vdash_i$. However, we will continue to use $\vdash$ when no confusion arises.

We can now adopt all results of the preceding parts that did not make use of RAA.

The following list may be helpful:

Lemma 5.2.1 (1) $\vdash \varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$ (p.32)
(2) $\vdash \varphi \vee \psi \leftrightarrow \psi \vee \varphi$
(3) $\vdash (\varphi \wedge \psi) \wedge \sigma \leftrightarrow \varphi \wedge (\psi \wedge \sigma)$
(4) $\vdash (\varphi \vee \psi) \vee \sigma \leftrightarrow \varphi \vee (\psi \vee \sigma)$
(5) $\vdash \varphi \vee (\psi \wedge \sigma) \leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \sigma)$
(6) $\vdash \varphi \wedge (\psi \vee \sigma) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \sigma)$ (p.51)
(7) $\vdash \varphi \rightarrow \neg\neg\varphi$ (p.33)
(8) $\vdash (\varphi \rightarrow (\psi \rightarrow \sigma)) \leftrightarrow (\varphi \wedge \psi \rightarrow \sigma)$ (p.33)
(9) $\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$ (p.37)
(10) $\vdash \varphi \rightarrow (\neg\varphi \rightarrow \psi)$ (p.37)
(11) $\vdash \neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi$
(12) $\vdash \neg\varphi \vee \neg\psi \rightarrow \neg(\varphi \wedge \psi)$
(13) $\vdash (\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$
(14) $\vdash (\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$(p.37)
(15) $\vdash (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\varphi \rightarrow \sigma))$ (p.37)
(16) $\vdash \perp \leftrightarrow (\varphi \wedge \neg\varphi)$ (p.37)
(17) $\vdash \exists x(\varphi(x) \vee \psi(x)) \leftrightarrow \exists x\varphi(x) \vee \exists x\psi(x)$
(18) $\vdash \forall x(\varphi(x) \wedge \psi(x)) \leftrightarrow \forall x\varphi(x) \wedge \forall x\psi(x)$
(19) $\vdash \neg\exists x\varphi(x) \leftrightarrow \forall x\neg\varphi(x)$
(20) $\vdash \exists x\neg\varphi(x) \rightarrow \neg\forall x\varphi(x)$
(21) $\vdash \forall x(\varphi \rightarrow \psi(x)) \leftrightarrow (\varphi \rightarrow \forall x\psi(x))$
(22) $\vdash \exists x(\varphi \rightarrow \psi(x)) \rightarrow (\varphi \rightarrow \exists x\psi(x))$
(23) $\vdash (\varphi \vee \forall x\psi(x)) \rightarrow \forall x(\varphi \vee \psi(x))$
(24) $\vdash (\varphi \wedge \exists x\psi(x)) \leftrightarrow \exists x(\varphi \wedge \psi(x))$
(25) $\vdash \exists x(\varphi(x) \rightarrow \psi) \rightarrow (\forall x\varphi(x) \rightarrow \psi)$
(26) $\vdash \forall x(\varphi(x) \rightarrow \psi) \leftrightarrow (\exists x\varphi(x) \rightarrow \psi)$

(Observe that (19) and (20) are special cases of (26) and (25).

All of those theorems can be proved by means of straight forward application of the rules. Some well-known theorems are conspicuously absent, and in some cases there is only an implication one way; we will show later that these implications cannot, in general, be reversed.

From a constructive point of view *RAA* is used to derive strong conclusions from weak premises. E.g. in $\neg(\varphi\wedge\psi) \vdash \neg\varphi\vee\neg\psi$ the premise is weak (something has no proof) and the conclusion is strong, it asks for an effective decision. One cannot expect to get such results in intuitionistic logic. Instead there is a collection of weak results, usually involving negations and double negations.

Lemma 5.2.2 (1) $\vdash \neg\varphi \leftrightarrow \neg\neg\neg\varphi$
(2) $\vdash (\varphi \wedge \neg\psi) \to \neg(\varphi \to \psi)$
(3) $\vdash (\varphi \to \psi) \to (\neg\neg\varphi \to \neg\neg\psi)$
(4) $\vdash \neg\neg(\varphi \to \psi) \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$
(5) $\vdash \neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$
(6) $\vdash \neg\neg\forall x\varphi(x) \to \forall x\neg\neg\varphi(x)$

In order to abbreviate derivations we will use the notation $\dfrac{\Gamma}{\overline{\overline{\varphi}}}$ in a derivation when there is a derivation for $\Gamma \vdash \varphi$ (Γ has $0, 1$ or 2 elements).

Proof. (**1**) $\neg\varphi \to \neg\neg\neg\varphi$ follows from Lemma 5.2.1 (7). For the converse we again use 5.2.1(7)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1 \quad \overline{\overline{\varphi \to \neg\neg\varphi}}}{\neg\neg\varphi} \quad [\neg\neg\neg\varphi]^2}{\bot}}{\neg\varphi}\,1}{\neg\neg\neg\varphi \to \neg\varphi}\,2 \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi\wedge\neg\psi]^2}{\varphi} \quad [\varphi\to\psi]^1}{\psi} \quad \dfrac{[\varphi\wedge\neg\psi]^2}{\neg\psi}}{\bot}}{\neg(\varphi\to\psi)}\,1}{(\varphi\wedge\neg\psi)\to\neg(\varphi\to\psi)}\,2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg\varphi]^3 \quad \dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\varphi\to\psi]^4}{\psi} \quad [\neg\psi]^2}{\bot}}{\neg\varphi}\,1}{\bot}}{\neg\neg\psi}\,2}{\neg\neg\varphi\to\neg\neg\psi}\,3}{(\varphi\to\psi)\to(\neg\neg\varphi\to\neg\neg\psi)}\,4$$

Prove (**3**) also by using (14) and (15) from 5.2.1

(**4**) Apply the intuitionistic half of the contraposition (Lemma 5.2.1(14)) to (**2**):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg(\varphi\to\psi)]^4}{\neg(\varphi\wedge\neg\psi)} \qquad \dfrac{[\varphi]^1 \quad [\neg\psi]^2}{\varphi\wedge\neg\psi}}{\bot}}{\neg\varphi}\,1 \qquad [\neg\neg\varphi]^3}{\bot}}{\neg\neg\psi}\,2}{\neg\neg\varphi\to\neg\neg\psi}\,3}{\neg\neg(\varphi\to\psi)\to(\neg\neg\varphi\to\neg\neg\psi)}\,4$$

For the converse we apply some facts from 5.2.1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi\to\psi)]^1}{\neg(\neg\varphi\vee\psi)}}{\neg\neg\varphi\wedge\neg\psi}}{\neg\neg\varphi} \qquad [\neg\neg\varphi\to\neg\neg\psi]^2}{\neg\neg\psi} \qquad \dfrac{\dfrac{\dfrac{[\neg(\varphi\to\psi)]^1}{\neg(\neg\varphi\vee\psi)}}{\neg\neg\varphi\wedge\neg\psi}}{\neg\psi}}{\bot}}{\neg\neg(\varphi\to\psi)}\,1}{(\neg\neg\varphi\to\neg\neg\psi)\to\neg\neg(\varphi\to\psi)}\,2$$

(**5**) $\to$: Apply (3) to $\varphi\wedge\psi\to\varphi$ and $\varphi\wedge\psi\to\psi$. The derivation of the converse is given below.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg\varphi\wedge\neg\neg\psi]^4}{\neg\neg\psi} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi\wedge\psi)]^3 \qquad \dfrac{[\varphi]^1 \quad [\psi]^2}{\varphi\wedge\psi}}{\bot}}{\neg\varphi}\,1 \qquad \dfrac{[\neg\neg\varphi\wedge\neg\neg\psi]^4}{\neg\neg\varphi}}{\bot}}{\neg\psi}\,2}{\bot}}{\neg\neg(\varphi\wedge\psi)}\,3}{(\neg\neg\varphi\wedge\neg\neg\psi)\to\neg\neg(\varphi\wedge\psi)}\,4$$

(6) $\vdash \exists x \neg\varphi(x) \to \neg\forall x\varphi(x)$, 5.2.1(20)
so $\neg\neg\forall x\varphi(x) \to \neg\exists x\neg\varphi(x)$, 5.2.1(14)
hence $\neg\neg\forall x\varphi(x) \to \forall x\neg\neg\varphi(x)$. 5.2.1(19)

Most of the straightforward meta-theorems of propositional and predicate logic carry over to intuitionistic logic. The following theorems can be proved by a tedious but routine induction.

Theorem 5.2.3 (Substitution Theorem for Derivations) *If $\mathcal{D}$ is a derivation and \$ a propositional atom, then $\mathcal{D}[\varphi/\$]$ is a derivation if the free variables of φ do not occur bound in $\mathcal{D}$.*

Theorem 5.2.4 (Substitution Theorem for Derivability) *If $\Gamma \vdash \sigma$ and \$ is a propositional atom, then $\Gamma[\varphi/\$] \vdash \sigma[\varphi/\$]$, where the free variables of φ do not occur bound in σ or Γ.*

Theorem 5.2.5 (Substitution Theorem for Equivalence)

$\Gamma \vdash (\varphi_1 \leftrightarrow \varphi_2) \to (\psi[\varphi_1/\$] \leftrightarrow \psi[\varphi_2/\$])$,

$\Gamma \vdash \varphi_1 \leftrightarrow \varphi_2 \Rightarrow \Gamma \vdash \psi[\varphi_1/\$] \leftrightarrow \psi[\varphi_2/\$]$,

where \$ is an atomic proposition, the free variables of φ_1 and φ_2 do not occur bound in Γ or ψ and the bound variables of ψ do not occur free in Γ.

The proofs of the above theorems are left to the reader. Theorems of this kind are always suffering from unaesthetic variable-conditions. In practical applications one always renames bound variables or considers only closed hypotheses, so that there is not much to worry. For precise formulations cf. Ch. 6.

The reader will have observed from the heuristics that $\vee$ and $\exists$ carry most of the burden of constructiveness. We will demonstrate this once more in an informal argument.

There is an effective procedure to compute the decimal expansion of $\pi(3,1415927\ldots)$. Let us consider the statement $\varphi_n :=$ in the decimal expansion of π there is a sequence of n consecutive sevens.

Clearly $\varphi_{100} \to \varphi_{99}$ holds, but there is no evidence whatsoever for $\neg\varphi_{100} \vee \varphi_{99}$.

The fact that $\wedge, \to, \forall, \bot$ do not ask for the kind of decisions that $\vee$ and $\exists$ require, is more or less confirmed by the following

Theorem 5.2.6 *If φ does not contain $\vee$ or $\exists$ and all atoms but $\bot$ in φ are negated, then $\vdash \varphi \leftrightarrow \neg\neg\varphi$.*

Proof. Induction on φ.

We leave the proof to the reader. (Hint: apply 5.2.2.) ■

By definition intuitionistic predicate (propositional) logic is a subsystem of the corresponding classical systems. Gödel and Gentzen have shown, however, that by interpreting the classical disjunction and existence quantifier in a weak sense, we can embed classical logic into intuitionistic logic. For this purpose we introduce a suitable translation:

Definition 5.2.7 *The mapping* $^\circ : FORM \to FORM$ *is defined by*

(i) $\bot^\circ := \bot$ *and* $\varphi^\circ := \neg\neg\varphi$ *for atomic* φ *dinstinct from* $\bot$ *.*

(ii) $(\varphi \wedge \psi)^\circ := \varphi^\circ \wedge \psi^\circ$

(iii) $(\varphi \vee \psi)^\circ := \neg(\neg\varphi^\circ \wedge \neg\psi^\circ)$

(iv) $(\varphi \to \psi)^\circ := \varphi^\circ \to \psi^\circ$

(v) $(\forall x \varphi(x))^\circ := \forall x \varphi^\circ(x)$

(vi) $(\exists x \varphi(x))^\circ := \neg\forall x \neg\varphi^\circ(x)$

This mapping is called the *Gödel translation.*
We define $\Gamma^\circ = \{\varphi^\circ | \varphi \in \Gamma\}$. The relation between classical derivability ($\vdash_c$) and intuitionistic derivability ($\vdash_i$ is given by

Theorem 5.2.8 $\Gamma \vdash_c \varphi \Leftrightarrow \Gamma^\circ \vdash_i \varphi^\circ$*.*

Proof. It follows from the preceding chapters that $\vdash_c \varphi \leftrightarrow \varphi^\circ$, therefore $\Leftarrow$ is an immediate consequence of $\Gamma \vdash_i \varphi \Rightarrow \Gamma \vdash_c \varphi$.

For $\Rightarrow$, we use induction on the derivation $\mathcal{D}$ of φ from Γ.

1. $\varphi \in \Gamma$, then also $\varphi^\circ \in \Gamma^\circ$ and hence $\Gamma^\circ \vdash_i \varphi^\circ$.

2. The last rule of $\mathcal{D}$ is a propositional introduction or elimination rule. We consider two cases:

$\to I$
$$\dfrac{\begin{matrix}[\varphi]\\ \mathcal{D}\\ \psi\end{matrix}}{\varphi \to \psi}$$

Induction hypothesis $\Gamma^\circ, \varphi^\circ \vdash_i \psi^\circ$.
By $\to I$ $\Gamma^\circ \vdash_i \varphi^\circ \to \psi^\circ$, and so by definition $\Gamma^\circ \vdash_i (\varphi \to \psi)^\circ$.

$\vee E$
$$\dfrac{\begin{matrix} \\ \mathcal{D}\\ \varphi \vee \psi\end{matrix} \quad \begin{matrix}[\varphi]\\ \mathcal{D}_1\\ \sigma\end{matrix} \quad \begin{matrix}[\psi]\\ \mathcal{D}_2\\ \sigma\end{matrix}}{\sigma}$$

Induction hypothesis: $\Gamma^\circ \vdash_i (\varphi \vee \psi)^\circ$,
$\Gamma^\circ, \varphi^\circ \vdash_i \sigma^\circ \Gamma^\circ, \psi^\circ \vdash_i \sigma^\circ$
(where Γ contains all uncancelled hypotheses involved).

$\Gamma^\circ \vdash_i \neg(\neg\varphi^\circ \wedge \neg\psi^\circ), \Gamma^\circ \vdash_i \varphi^\circ \to \sigma^\circ, \Gamma^\circ \vdash_i \psi^\circ \to \sigma^\circ$.

The result follows from the derivation below:

$$\dfrac{\dfrac{\neg(\neg\varphi^\circ \wedge \neg\psi^\circ) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\varphi^\circ]^1 \quad \varphi^\circ \to \sigma^\circ}{\sigma^\circ} \qquad [\neg\sigma^\circ]^3}{\bot}}{\neg\varphi^\circ}\,1 \qquad \dfrac{\dfrac{\dfrac{[\psi^\circ]^2 \quad \psi^\circ \to \sigma^\circ}{\sigma^\circ} \qquad [\neg\sigma^\circ]^3}{\bot}}{\neg\psi^\circ}\,2}{\neg\varphi^\circ \wedge \neg\psi^\circ}}{\bot}}{\neg\neg\sigma^\circ}\,3$$

$$\dfrac{\neg\neg\sigma^\circ}{\sigma^\circ} \;(\text{double line})$$

The remaining rules are left to the reader.

3. The last rule of $\mathcal{D}$ is the falsum rule. This case is obvious.
4. The last rule of $\mathcal{D}$ is a quantifier introduction or elimination rule. Let us consider two cases.

$\forall I$ $$\dfrac{\begin{matrix}\mathcal{D}\\ \varphi(x)\end{matrix}}{\forall x\varphi(x)}$$

Induction hypothesis: $\Gamma^\circ \vdash_i \varphi(x)^\circ$
By$\forall I$ $\Gamma^\circ \vdash_i \forall x\varphi(x)^\circ$,so $\Gamma^\circ \vdash_i (\forall x\varphi(x))^\circ$.

$\exists E$: $$\dfrac{\begin{matrix}\mathcal{D}\\ \exists x\varphi(x)\end{matrix} \qquad \begin{matrix}[\varphi(x)]\\ \mathcal{D}_1\\ \sigma\end{matrix}}{\sigma}$$

Induction hypothesis:$\Gamma^\circ \vdash_i (\exists x\varphi(x))^\circ$,
$\Gamma^\circ, \varphi(x)^\circ \vdash_i \sigma^\circ$.
So $\Gamma^\circ \vdash_i (\neg\forall x\neg\varphi(x))^\circ$ and
$\Gamma^\circ \vdash_i \forall x(\varphi(x)^\circ \ \to \sigma^\circ)$.

$$\dfrac{\dfrac{\neg\forall x\neg\varphi(x)^\circ \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\varphi(x)^\circ]^1 \quad \dfrac{\forall x(\varphi(x)^\circ \to \sigma^\circ)}{\varphi(x)^\circ \to \sigma^\circ}}{\sigma^\circ} \qquad [\neg\sigma^\circ]^2}{\bot}}{\neg\varphi(x)^\circ}\,1}{\forall x\neg\varphi(x)^\circ}}{\bot}}{\neg\neg\sigma^\circ}\,2$$

$$\dfrac{\neg\neg\sigma^\circ}{\sigma^\circ} \;(\text{double line})$$

We now get $\Gamma^\circ \vdash_i \sigma^\circ$.

5. The last rule of $\mathcal{D}$ is RAA.

$$\begin{array}{ll} [\neg\varphi] & \text{Induction hypothesis } \Gamma^\circ, (\neg\varphi)^\circ \vdash_i \bot \,. \\ \mathcal{D} & \text{so } \Gamma^\circ \vdash_i \neg\neg\varphi^\circ\text{,and hence by Theorem 5.2.6 } \Gamma^\circ \vdash_i \varphi^\circ \\ \dfrac{\bot}{\varphi} & \end{array}$$

■

Let us call formulas in which all atoms occur negated, and which contain only the connectives $\wedge, \rightarrow, \forall, \bot$, *negative*.

The special role of $\vee$ and $\exists$ is underlined by

Corollary 5.2.9 *Classical predicate (propositional) logic is conservative over intuitionistic predicate (propositional) logic with respect to negative formulae, i.e.* $\vdash_c \varphi \Leftrightarrow \vdash_i \varphi$ *for negative* φ.

Proof. φ°, for negative φ, is obtained by replacing each atom p by $\neg\neg p$. Since all atoms occur negated we have $\vdash_i \varphi^\circ \leftrightarrow \varphi$ (apply 5.2.2(1) and 5.2.6). The result now follows from 5.2.8. ■

In some particular theories (e.g. arithmetic) the atoms are *decidable*, i.e. $\Gamma \vdash \varphi \vee \neg\varphi$ for atomic φ. For such theories one may simplify the Gödel translation by putting $\varphi^\circ := \varphi$ for atomic φ.

Observe that Corollary 5.2.9 tells us that intuitionistic logic is consistent iff classical logic is so (a not very surprising result!).

For propositional logic we have a somewhat stronger result than 5.2.8.

Theorem 5.2.10 (Glivenko's Theorem) $\vdash_c \varphi \Leftrightarrow \vdash_i \neg\neg\varphi$.

Proof. Show by induction on φ that $\vdash_i \varphi^\circ \leftrightarrow \neg\neg\varphi$ (use 5.2.2), and apply 5.2.8. ■

5.3 Kripke Semantics

There are a number of (more or less formalized) semantics for intuitionistic logic that allow for a completeness theorem. We will concentrate here on the semantics introduced by Kripke since it is convenient for applications and it is fairly simple.

Heuristic motivation. Think of an idealized mathematician (in this context traditionally called the *creative subject*), who extends both his knowledge and his universe of objects in the course of time. At each moment k he has a stock Σ_k of sentences, which he, by some means, has recognised as true and a stock A_k of objects which he has constructed (or created). Since at every moment k the idealized mathematician has various choices for his future activities (he may even stop alltogether), the stages of his activity must be thought of as being *partially ordered*, and not necessarily linearly ordered. How will the idealized mathematician interpret the logical connectives? Evidently the interpretation of a composite statement must depend on the interpretation of its parts, e.g. the idealized mathematician has established φ or (and) ψ at stage k if he has established φ at stage k or (and) ψ at stage k. The implication is more cumbersome, since $\varphi \to \psi$ may be known at stage k without φ or ψ being known. Clearly, the idealized mathematician knows $\varphi \to \psi$ at stage k if he knows that if at any future stage (including k) φ is established, also ψ is established. Similarly $\forall x\varphi(x)$ is established at stage k if at any future stage (including k) for all objects a that exist at that stage $\varphi(\overline{a})$ is established.

Evidently we must in case of the universal quantifier take the future into account since *for all elements* means more than just "for all elements that we have constructed so far"! Existence, on the other hand, is not relegated to the future. The idealized mathematician knows at stage k that $\exists x\varphi(x)$ if he has constructed an object a such that at stage k he has established $\varphi(\overline{a})$. Of course, there are many observations that could be made, for example that it is reasonable to add "in principle" to a number of clauses. This takes care of large numbers, choice sequences etc. Think of $\forall xy\exists z(z = x^y)$, does the idealized mathematician really construct 10^{10} as a succession of units? For this and similar questions the reader is referred to the literature.

We will now formalizealize the above sketched semantics.

It is for a first introduction convenient to consider a language without functions symbols. Later it will be simple to extend the language.

We consider models for some language L.

Definition 5.3.1 *A Kripke model is a quadruple $\mathcal{K} = \langle K, \Sigma, C, D\rangle$, where K is a (non-empty) partially ordered set, C a function defined on the constants of L, D a set valued function on K, Σ a function on K such that*

- $C(c) \in D(k)$ *for all* $k \in K$,
- $D(k) \neq \emptyset$ *for all* $k \in K$,
- $\Sigma(k) \subseteq At_k$ *for all* $k \in K$,

where At_k is the set of all atomic sentences of L with constants for the elements of $D(k)$. D and Σ satisfy the following conditions:

(i) $k \leq l \Rightarrow D(k) \subseteq D(l)$.
(ii) $\perp \notin \Sigma(k)$, *for all* k.
(iii) $k \leq l \Rightarrow \Sigma(k) \subseteq \Sigma(l)$.

$D(k)$ is called the *domain* of $\mathcal{K}$ at k, the elements of K are called *nodes* of $\mathcal{K}$. Instead of "φ has auxilliary constants for elements of $D(k)$" we say for short "φ has parameters in $D(k)$".

Σ assigns to each node the 'basic facts' that hold at k, the conditions (i), (ii), (iii) merely state that the collection of available objects does not decrease in time, that a falsity is never established and that a basic fact that once has been established remains true in later stages. The constants are interpreted by the same elements in all domains (they are *rigid designators*).

Note that D and Σ together determine at each node k a classical structure $\mathfrak{A}(k)$ (in the sense of 2.2.1). The universe of $\mathfrak{A}(k)$ is $D(k)$ and the relations of $\mathfrak{A}(k)$ are given by $\Sigma(k)$ as the positive diagram: $\langle \vec{a} \rangle \in R^{\mathfrak{A}(k)}$ iff $R(\vec{a}) \in \Sigma(k)$. The conditions (i) and (iii) above tell us that the universes are increasing:

$k \leq l \Rightarrow |\mathfrak{A}(k)| \subseteq |\mathfrak{A}(l)|$

and that the relations are increasing:

$k \leq l \Rightarrow R^{\mathfrak{A}(k)} \subseteq R^{\mathfrak{A}(l)}$.

Furthermore $c^{\mathfrak{A}(k)} = c^{\mathfrak{A}(l)}$ for all k and l.

In $\Sigma(k)$ there are also propositions, something we did not allow in classical predicate logic. Here it is convenient for treating propositional and predicate logic simultaneously.

The function Σ tells us which atoms are "true" in k. We now extend Σ to all sentences.

Lemma 5.3.2 *Σ has a unique extension to a function on K (also denoted by Σ) such that $\Sigma(k) \subseteq Sent_k$, the set of all sentences with parameters in $D(k)$, satisfying:*

(i) $\varphi \vee \psi \in \Sigma(k) \Leftrightarrow \varphi \in \Sigma(k)$ *or* $\psi \in \Sigma(k)$
(ii) $\varphi \wedge \psi \in \Sigma(k) \Leftrightarrow \varphi \in \Sigma(k)$ *and* $\psi \in \Sigma(k)$
(iii) $\varphi \to \psi \in \Sigma(k) \Leftrightarrow$ *for all* $l \geq k$ $(\varphi \in \Sigma(l) \Rightarrow \psi \in \Sigma(l))$
(iv) $\exists x\varphi(x) \in \Sigma(k) \Leftrightarrow$ *there is an* $a \in D(k)$ *such that* $\varphi(\overline{a}) \in \Sigma(k)$
(v) $\forall x\varphi(x) \in \Sigma(k) \Leftrightarrow$ *for all* $l \geq k$ *and for all* $a \in D(l)$ $\varphi(\overline{a}) \in \Sigma(l)$.

Proof. Immediate. We simply define $\varphi \in \Sigma(k)$ for all $k \in K$ simultaneously by induction on φ. ∎

Notation. We write $k \Vdash \varphi$ for $\varphi \in \Sigma(k)$, pronounce '*k forces φ*'.

Exercise for the reader: reformulate (i) - (v) above in terms of forcing.

Corollary 5.3.3 *(i)* $k \Vdash \neg\varphi \Leftrightarrow$ *for all* $l \geq k$ $l \nVdash \varphi$.
(ii) $k \Vdash \neg\neg\varphi \Leftrightarrow$ *for all* $l \geq k$ *there exists a* $p \geq l$ *such that* $(p \Vdash \varphi)$.

Proof. $k \Vdash \neg\varphi \Leftrightarrow k \Vdash \varphi \to \perp \Leftrightarrow$ for all $l \geq k (l \Vdash \varphi \Rightarrow l \Vdash \perp) \Leftrightarrow$ for all $l \geq k$ $l \nVdash \varphi$.
$k \Vdash \neg\neg\varphi \Leftrightarrow$ for all $l \geq k$ $l \nVdash \neg\varphi \Leftrightarrow$ for all $l \geq k$ not (for all $p \geq l$ $p \nVdash \varphi) \Leftrightarrow$ for all $l \geq k$ there is a $p \geq l$ such that $p \Vdash \varphi$. ∎

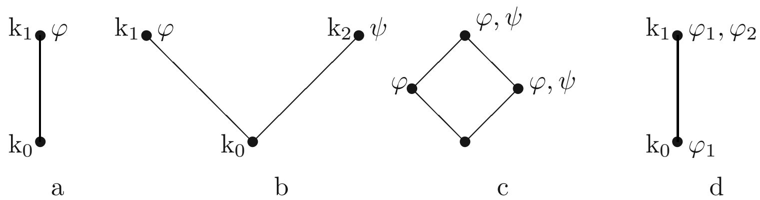
The monotonicity of Σ for atoms is carried over to arbitrary formulas.

Lemma 5.3.4 (Monotonicity of $\Vdash$) $k \le l,\ k \Vdash \varphi \Rightarrow l \Vdash \varphi$.

Proof. Induction on φ.

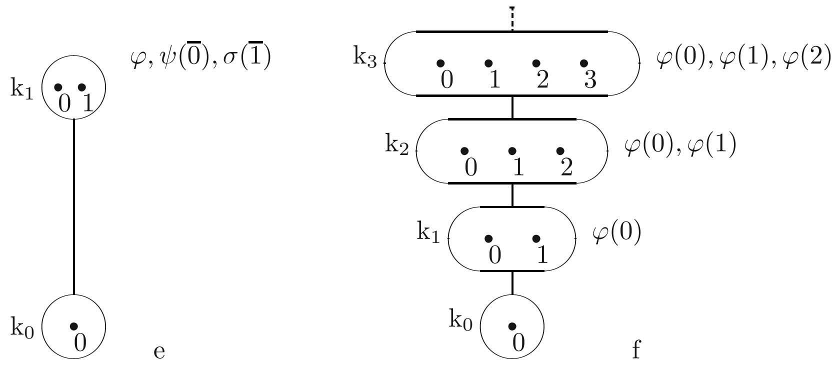
atomic φ : the lemma holds by definition 5.3.1.
$\varphi = \varphi_1 \wedge \varphi_2$: let $k \Vdash \varphi_1 \wedge \varphi_2$ and $k \le l$, then $k \Vdash \varphi_1 \wedge \varphi_2 \Leftrightarrow k \Vdash \varphi_1$ and $k \Vdash \varphi_2 \Rightarrow$ (ind. hyp.) $l \Vdash \varphi_1$ and $l \Vdash \varphi_2 \Leftrightarrow l \Vdash \varphi_1 \wedge \varphi_2$.
$\varphi = \varphi_1 \vee \varphi_2$: mimic the conjunction case.
$\varphi = \varphi_1 \to \varphi_2$ Let $k \Vdash \varphi_1 \to \varphi_2,\ l \ge k$. Suppose $p \ge l$ and $p \Vdash \varphi_1$ then, since $p \ge k, p \Vdash \varphi_2$. Hence $l \Vdash \varphi_1 \to \varphi_2$.
$\varphi = \exists x \varphi_1(x)$: immediate.
$\varphi = \forall x \varphi_1(x)$: let $k \Vdash \forall x \varphi_1(x)$ and $l \ge k$. Suppose $p \ge l$ and $a \in D(p)$, then, since $p \ge k, p \Vdash \varphi_1(\overline{a})$. Hence $l \Vdash \forall x \varphi_1(x)$. ■

We will now present some examples, which refute classically true formulas. It suffices to indicate which atoms are forced at each node. We will simplify the presentation by drawing the partially ordered set and indicating the atoms forced at each node. For propositional logic no domain function is required (equivalently, a constant one, say $D(k) = \{0\}$), so we simplify the presentation accordingly.



(a) In the bottom node no atoms are known, in the second one only φ, to be precise $k_0 \nVdash \varphi, k_1 \Vdash \varphi$. By 5.3.3 $k_0 \Vdash \neg\neg\varphi$, so $k_0 \nVdash \neg\neg\varphi \to \varphi$. Note, however, that $k_0 \nVdash \neg\varphi$, since $k_1 \Vdash \varphi$. So $k_0 \nVdash \varphi \vee \neg\varphi$.
(b) $k_i \nVdash \varphi \wedge \psi$ $(i = 0, 1, 2)$, so $k_0 \Vdash \neg(\varphi \wedge \psi)$. By definition, $k_0 \Vdash \neg\varphi \vee \neg\psi \Leftrightarrow k_0 \Vdash \neg\varphi$ or $k_0 \Vdash \neg\psi$. The first is false, since $k_1 \Vdash \varphi$, and the latter is false, since $k_2 \Vdash \psi$. Hence $k_0 \nVdash \neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$.
(c) The bottom node forces $\psi \to \varphi$, but it does not force $\neg\psi \vee \varphi$ (why?). So it does not force $(\psi \to \varphi) \to (\neg\psi \vee \varphi)$.
(d) In the bottom node the following implications are forced: $\varphi_2 \to \varphi_1, \varphi_3 \to \varphi_2, \varphi_3 \to \varphi_1$, but none of the converse implications is forced, hence $k_0 \nVdash (\varphi_1 \leftrightarrow \varphi_2) \vee (\varphi_2 \leftrightarrow \varphi_3) \vee (\varphi_3 \leftrightarrow \varphi_1)$.
We will analyse the last example a bit further. Consider a Kripke model with two nodes as in d, with some assignment Σ of atoms. We will show that for four arbitrary propositions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$
$k_0 \Vdash \bigvee\!\!\!\bigvee_{1 \le i < j \le 4} \sigma_i \leftrightarrow \sigma_j$, i.e. from any four propositions at least two are equivalent.
There are a number of cases. (1) At least two of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are forced in k_0. Then we are done. (2) Just one σ_i is forced in k_0. Then of the remaining

propositions, either two are forced in k_1, or two of them are not forced in k_1. In both cases there are σ_j and $\sigma_{j'}$, such that $k_0 \Vdash \sigma_j \leftrightarrow \sigma_{j'}$. (3) No σ_i is forced in k_0. Then we may repeat the argument under (2).



(e) (i) $k_0 \Vdash \varphi \to \exists x\sigma(x)$, for the only node that forces φ is k_1, and indeed $k_1 \Vdash \sigma(1)$, so $k_1 \Vdash \exists x\sigma(x)$.
Now suppose $k_0 \Vdash \exists x(\varphi \to \sigma(x))$, then, since $D(k_0) = \{0\}$, $k_0 \Vdash \varphi \to \sigma(0)$. But $k_1 \Vdash \varphi$ and $k_1 \nVdash \sigma(0)$.
Contradiction. Hence $k_0 \nVdash (\varphi \to \exists x\sigma(x)) \to \exists x(\varphi \to \sigma(x))$.

Remark. $(\varphi \to \exists x\sigma(x)) \to \exists x(\varphi \to \sigma(x))$ is called the *independence of premise principle*. It is not surprising that it fails in some Kripke models, for $\varphi \to \exists x\sigma(x)$ tells us that the required element a for $\sigma(\overline{a})$ may depend on the proof of φ (in our heuristic interpretation); while in $\exists x(\varphi \to \sigma(x))$, the element a must be found independently of φ. So the right hand side is stronger.

(ii) $k_0 \Vdash \neg\forall x\psi(x) \Leftrightarrow k_i \nVdash \forall x\psi(x) (i = 0, 1)$. $k_1 \nVdash \psi(\overline{1})$, so we have shown $k_0 \Vdash \neg\forall x\psi(x)$. $k_0 \Vdash \exists x\neg\psi(x) \Leftrightarrow k_0 \Vdash \neg\psi(\overline{0})$. However, $k_1 \Vdash \psi(\overline{0})$, so $k_0 \nVdash \exists x\neg\psi(x)$. Hence $k_0 \nVdash \neg\forall x\psi(x) \to \exists x\neg\psi(x)$.

(iii) A similar argument fs $k_0 \nVdash (\forall x\psi(x) \to \tau) \to \exists x(\psi(x) \to \tau)$, where τ is not forced in k_1.

(f) $D(k_i) = \{0, \ldots, i\}, \Sigma(k_i) = \{\varphi(0), \ldots, \varphi(i-1)\}$, $k_0 \Vdash \forall x\neg\neg\varphi(x) \Leftrightarrow$ for all i $k_i \Vdash \neg\neg\varphi(j), j \le i$. The latter is true since for all $p > i$ $k_p \Vdash \varphi(j)$, $j \le i$. Now $k_0 \Vdash \neg\neg\forall x\varphi(x) \Leftrightarrow$ for all i there is a $j \ge i$ such that $k_j \Vdash \forall x\varphi(x)$. But no k_j forces $\forall x\varphi(x)$. So $k_0 \nVdash \forall x\neg\neg\varphi(x) \to \neg\neg\forall x\varphi(x)$.

Remark. We have seen that $\neg\neg\forall x\varphi(x) \to \forall x\neg\neg\varphi(x)$ is derivable and it is easily seen that it holds in all Kripke models, but the converse fails in some models. The schema $\forall x\neg\neg\varphi(x) \to \neg\neg\forall x\varphi(x)$ is called the *double negation shift* (DNS).
The next thing to do is to show that Kripke semantics is sound for intuitionistic logic.

We define a few more notions for sentences:

(i) $\mathcal{K} \Vdash \varphi$ if $k \Vdash \varphi$ for all $k \in K$.
(ii) $\Vdash \varphi$ if $\mathcal{K} \Vdash \varphi$ for all $\mathcal{K}$.

For formulas containing free variables we have to be more careful. Let φ contain free variables, then we say that $k \Vdash \varphi$ iff $k \Vdash Cl(\varphi)$ (the universal closure). For a set Γ and a formula φ with free variables $x_{i_0}, x_{i_1}, x_{i_2}, \ldots$ (which we will denote by $\vec{x}$), we define $\Gamma \Vdash \varphi$ by: for all $\mathcal{K}$, $k \in K$ and for all $(\vec{a} \in D(k))$ $[k \Vdash \psi(\vec{a})$ for all $\psi \in \Gamma \Rightarrow k \Vdash \varphi(\vec{a})]$. ($\vec{a} \in D(k)$ is a convenient abuse of language).

Before we proceed we introduce an extra abuse of language which will prove extremely useful: we will freely use quantifiers in our meta-language. It will have struck the reader that the clauses in the definition of the Kripke semantics abound with expressions like "for all $l \geq k$"," for all $a \in D(k)$". It saves quite a bit of writing to use "$\forall l \geq k$", "$\forall a \in D(k)$" instead, and it increases systematic readability to boot. By now the reader is well used to the routine phrases of our semantics, so he will have no difficulty to avoid a confusion of quantifiers in the meta-language and the object-language.

By way of example we will reformulate the preceding definition:
$\Gamma \Vdash \varphi := (\forall \mathcal{K})(\forall k \in K)(\forall \vec{a} \in D(k))[\forall \psi \in \Gamma(k \Vdash \psi(\vec{a})) \Rightarrow k \Vdash \varphi(\vec{a})]$.

There is a useful reformulation of this "semantic consequence" notion.

Lemma 5.3.5 *Let Γ be finite, then $\Gamma \Vdash \varphi \Leftrightarrow \Vdash Cl(\bigwedge \Gamma \to \varphi)$ (where $Cl(X)$ is the universal closure of X).*

Proof. Left to the reader. ■

Theorem 5.3.6 (Soundness Theorem) $\Gamma \vdash \varphi \Rightarrow \Gamma \Vdash \varphi$.

Proof. Use induction on the derivation $\mathcal{D}$ of φ from Γ. We will abbreviate "$k \Vdash \psi(\vec{a})$ for all $\psi \in \Gamma$" by "$k \Vdash \Gamma(\vec{a})$". The model $\mathcal{K}$ is fixed in the proof.

(1) $\mathcal{D}$ consists of just φ, then obviously $k \Vdash \Gamma(\vec{a}) \Rightarrow k \Vdash \varphi(\vec{a})$ for all k and $(\vec{a}) \in D(k)$.
(2) $\mathcal{D}$ ends with an application of a derivation rule.

($\wedge I$) Induction hypothesis: $\forall k \forall \vec{a} \in D(k)(k \Vdash \Gamma(\vec{a}) \Rightarrow k \Vdash \varphi_i(\vec{a}))$, for $i = 1, 2$. Now choose a $k \in K$ and $\vec{a} \in D(k)$ such that $k \Vdash \Gamma(\vec{a})$, then $k \Vdash \varphi_1(\vec{a})$ and $k \Vdash \varphi_2(\vec{a})$, so $k \Vdash (\varphi_1 \wedge \varphi_2)(\vec{a})$.

Note that the choice of $\vec{a}$ did not really play a role in this proof. To simplify the presentation we will suppress reference to $\vec{a}$, when it does not play a role.

($\wedge E$)) Immediate.
($\vee I$) Immediate.

($\vee E$) Induction hypothesis: $\forall k(k \Vdash \Gamma \Rightarrow k \Vdash \varphi \vee \psi), \forall k(k \Vdash \Gamma, \varphi \Rightarrow k \Vdash \sigma)$, $\forall k(k \Vdash \Gamma, \psi \Rightarrow k \Vdash \sigma)$. Now let $k \Vdash \Gamma$, then by i.h. $k \Vdash \varphi \vee \psi$, so $k \Vdash \varphi$ or $k \Vdash \psi$. In the first case $k \Vdash \Gamma, \varphi$, so $k \Vdash \sigma$. In the second case $k \Vdash \Gamma, \psi$, so $k \Vdash \sigma$. In both cases $k \Vdash \sigma$, so we are done.

($\rightarrow I$) Induction hypothesis: $(\forall k)(\forall \vec{a} \in D(k))(k \Vdash \Gamma(\vec{a}), \varphi(\vec{a}) \Rightarrow$ $k \Vdash \psi(\vec{a}))$. Now let $k \Vdash \Gamma(\vec{a})$ for some $\vec{a} \in D(k)$. We want to show $k \Vdash (\varphi \rightarrow \psi)(\vec{a})$, so let $l \geq k$ and $l \Vdash \varphi(\vec{a})$. By monotonicity $l \Vdash \Gamma(\vec{a})$, and $\vec{a} \in D(l)$, so the ind. hyp. tells us that $l \Vdash \psi(\vec{a})$. Hence $\forall l \geq k(l \Vdash \varphi(\vec{a})$ $\Rightarrow l \Vdash \psi(\vec{a}))$, so $k \Vdash (\varphi \rightarrow \psi)(\vec{a})$.

($\rightarrow E$) Immediate.

($\perp$) Induction hypothesis $\forall k(k \Vdash \Gamma \Rightarrow k \Vdash \perp)$. Since, evidently, no k can force Γ, $\forall k(k \Vdash \Gamma \Rightarrow k \Vdash \varphi)$ is correct.

($\forall I$) The free variables in Γ are $\vec{x}$, and z does not occur in the sequence $\vec{x}$. Induction hypothesis: $(\forall k)(\forall \vec{a}, b \in D(k))(k \Vdash \Gamma(\vec{a}) \Rightarrow k \Vdash \varphi(\vec{a}, b))$. Now let $k \Vdash \Gamma(\vec{a})$ for some $\vec{a} \in D(k)$, we must show $k \Vdash \forall z \varphi(\vec{a}, z)$. So let $l \geq k$ and $b \in D(l)$. By monotonicity $l \Vdash \Gamma(\vec{a})$ and $\vec{a} \in D(l)$, so by the ind. hyp. $l \Vdash \varphi(\vec{a}, b)$. This shows $(\forall l \geq k)(\forall b \in D(l))(l \Vdash \varphi((\vec{a}, b)$, and hence $k \Vdash \forall z \varphi(\vec{a}, z)$.

($\forall E$) Immediate.

($\exists I$) Immediate.

($\exists E$) Induction hypothesis: $(\forall k)(\forall \vec{a} \in D(k)(k \Vdash \Gamma(\vec{a}) \Rightarrow k \Vdash \exists z \varphi(\vec{a}, z))$ and $(\forall k)(\forall \vec{a}, b \in D(k)(k \Vdash \varphi(\vec{a}, b), k \Vdash \Gamma(\vec{a}) \Rightarrow k \Vdash \sigma(\vec{a}))$. Here the variables in Γ and σ are $\vec{x}$, and z does not occur in the sequence $\vec{x}$. Now let $k \Vdash \Gamma(\vec{a})$, for some $\vec{a} \in D(k)$, then $k \Vdash \exists z \varphi(\vec{a}, z)$. So let $k \Vdash \varphi(\vec{a}, b)$ for some $b \in D(k)$. By the induction hypothesis $k \Vdash \sigma(\vec{a})$. ■

For the Completeness Theorem we need some notions and a few lemma's.

Definition 5.3.7 *A set of sentences Γ is a prime theory with respect to a language L if*

(i) Γ is closed under $\vdash$
(ii) $\varphi \vee \psi \in \Gamma \Rightarrow \varphi \in \Gamma$ or $\psi \in \Gamma$
(iii) $\exists x \varphi(x) \in \Gamma \Rightarrow \varphi(c) \in \Gamma$ for some constant c in L.

The following is analogue of the Henkin construction combined with a maximal consistent extension.

Lemma 5.3.8 *Let Γ and φ be closed, then if $\Gamma \nvdash \varphi$, there is a prime theory Γ' in a language L', extending Γ such that $\Gamma' \nvdash \varphi$.*

Proof. In general one has to extend the language L of Γ by a suitable set of 'witnessing' constants. So we extend the language L of Γ by a denumerable set of constants to a new language L'. The required theory Γ' is obtained by series of extensions $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \ldots$. We put $\Gamma_0 := \Gamma$.

Let Γ_k be given such that $\Gamma_k \nvdash \varphi$ and Γ_k contains only finitely many new constants. We consider two cases.

k is even. Look for the first existential sentence $\exists x\psi(x)$ in L' that has not yet been treated, such that $\Gamma_k \vdash \exists x\psi(x)$. Let c be the first new constant not in Γ_k. Now put $\Gamma_{k+1} := \Gamma_k \cup \{\psi(c)\}$.

k is odd. Look for the first disjunctive sentence $\psi_1 \vee \psi_2$ with $\Gamma_k \vdash \psi_1 \vee \psi_2$ that has not yet been treated. Note that not both $\Gamma_k, \psi_1 \vdash \varphi$ and $\Gamma_k, \psi_2 \vdash \varphi$ for then by $\vee\exists$ $\Gamma_k \vdash \varphi$.
Now we put: $\Gamma_{k+1} := \begin{cases} \Gamma_k \cup \{\psi_1\} \text{ if } \Gamma_k, \psi_1 \nvdash \varphi \\ \Gamma_k \cup \{\psi_2\} \text{ otherwise.} \end{cases}$

Finally: $\Gamma' := \bigcup_{k \geq 0} \Gamma_k$.

There are a few things to be shown:

1. $\Gamma' \nvdash \varphi$. We first show $\Gamma_i \nvdash \varphi$ by induction on i. For $i = 0$, $\Gamma_0 \nvdash \varphi$ holds by assumption. The induction step is obvious for i odd. For i even we suppose $\Gamma_{i+1} \vdash \varphi$. Then $\Gamma_i, \psi(c) \vdash \varphi$. Since $\Gamma_i \vdash \exists x\psi(x)$, we get $\Gamma_i \vdash \varphi$ by $\exists E$, which contradicts the induction hypothesis. Hence $\Gamma_{i+1} \nvdash \varphi$, and therefore by complete induction $\Gamma_i \nvdash \varphi$ for all i.
   Now, if $\Gamma' \vdash \varphi$ then $\Gamma_i \vdash \varphi$ for some i. Contradiction.

2. Γ' is a prime theory.
   (a) Let $\psi_1 \vee \psi_2 \in \Gamma'$ and let k be the least number such that $\Gamma_k \vdash \psi_1 \vee \psi_2$. Clearly $\psi_1 \vee \psi_2$ has not been treated before stage k, and $\Gamma_h \vdash \psi_1 \vee \psi_2$ for $h \geq k$. Eventually $\psi_1 \vee \psi_2$ has to be treated at some stage $h \geq k$, so then $\psi_1 \in \Gamma_{h+1}$ or $\psi_2 \in \Gamma_{h+1}$, and hence $\psi_1 \in \Gamma'$ or $\psi_2 \in \Gamma'$.
   (b) Let $\exists x\psi(x) \in \Gamma'$, and let k be the least number such that $\Gamma_k \vdash \exists x\psi(x)$. For some $h \geq k$ $\exists x\psi(x)$ is treated, and hence $\psi(c) \in \Gamma_{h+1} \subseteq \Gamma'$ for some c.
   (c) Γ' is closed under $\vdash$. If $\Gamma' \vdash \psi$, then $\Gamma' \vdash \psi \vee \psi$, and hence by (a) $\psi \in \Gamma'$.

Conclusion: Γ' is a prime theory containing Γ, such that $\Gamma' \nvdash \varphi$. ■

The next step is to construct for closed Γ and φ with $\Gamma \nvdash \varphi$, a Kripke model, with $\mathcal{K} \Vdash \Gamma$ and $k \nVdash \varphi$ for some $k \in K$.

Lemma 5.3.9 (Model Existence Lemma) *If $\Gamma \nvdash \varphi$ then there is a Kripke model $\mathcal{K}$ with a bottom node k_0 such that $k_0 \Vdash \Gamma$ and $k_0 \nVdash \varphi$.*

Proof: We first extend Γ to a suitable prime theory Γ' such that $\Gamma' \nvdash \varphi$. Γ' has the language L' with set of constants C'. Consider a set of distinct constants $\{c_m^i | i \geq 0, m \geq 0\}$ disjoint with C'. A denumerable family of denumerable sets of constants is given by $C^i = \{c_m^i | m \geq 0\}$. We will construct a Kripke model over the poset of all finite sequences of natural numbers, including the empty sequence $\langle\rangle$, with their natural ordering, "initial segment of".

Define $C(\langle\rangle) := C'$ and $C(\vec{n}) = C(\langle\rangle) \cup C^0 \cup \ldots \cup C^{k-1}$ for $\vec{n}$ of positive length k. $L(\vec{n})$ is the extension of L by $C(\vec{n})$, with set of atoms $At(\vec{n})$. Now put $D(\vec{n}) := C(\vec{n})$. We define $\Sigma(\vec{n})$ by induction on the length of $\vec{n}$.

$\Sigma(\langle\rangle) := \Gamma' \cap At(\langle\rangle)$. Suppose $\Sigma(\vec{n})$ has already been defined. Consider an enumeration $\langle\sigma_0, \tau_0\rangle, \langle\sigma_1, \tau_1\rangle, \ldots$ of all pairs of sentences in $L(\vec{n})$ such that $\Gamma(\vec{n}), \sigma_i \nvdash \tau_i$. Apply Lemma 5.3.8 to $\Gamma(\vec{n}) \cup \{\sigma_i\}$ and τ_i for each i. This yields a prime theory $\Gamma(\vec{n}, i)$ and $L(\vec{n}, i)$ such that $\sigma_i \in \Gamma(\vec{n}, i)$ and $\Gamma(\vec{n}, i) \nvdash \tau_i$.

Now put $\Sigma(\vec{n}, i) := \Gamma(\vec{n}, i) \cap At(\vec{n}, i)$. We observe that all conditions for a Kripke model are met. The model reflects (like the model of 3.1.11) very much the nature of the prime theories involved.

Claim: $\vec{n} \Vdash \psi \Leftrightarrow \Gamma(\vec{n}) \vdash \psi$.

We prove the claim by induction on ψ.

- For atomic ψ the equivalence holds by definition.
- $\psi = \psi_1 \wedge \psi_2$ – immediate
- $\psi = \psi_1 \vee \psi_2$.
  (a) $\vec{n} \Vdash \psi_1 \vee \psi_2 \Leftrightarrow \vec{n} \Vdash \psi_1$ or $\vec{n} \Vdash \psi_2 \Rightarrow$ (ind. hyp.) $\Gamma(\vec{n}) \vdash \psi_1$ or $\Gamma(\vec{n}) \vdash \psi_2 \Rightarrow \Gamma(\vec{n}) \vdash \psi_1 \vee \psi_2$.
  (b) $\Gamma(\vec{n}) \vdash \psi_1 \vee \psi_2 \Rightarrow \Gamma(\vec{n}) \vdash \psi_1$ or $\Gamma(\vec{n}) \vdash \psi_2$, since $\Gamma(\vec{n})$ is a prime theory (in the right language $L(\vec{n})$). So, by induction hypothesis, $\vec{n} \Vdash \psi_1$ or $\vec{n} \Vdash \psi_2$, and hence $\vec{n} \Vdash \psi_1 \vee \psi_2$.
- $\psi = \psi_1 \to \psi_2$.
  (*a*) $\vec{n} \Vdash \psi_1 \to \psi_2$. Suppose $\Gamma(\vec{n}) \nvdash \psi_1 \to \psi_2$, then $\Gamma(\vec{n}), \psi_1 \nvdash \psi_2$. By the definition of the model there is an extension $\vec{m} = \langle n_0, \ldots, n_{k-1}, i\rangle$ of $\vec{n}$ such that $\Gamma(\vec{n}) \cup \{\psi_1\} \subseteq \Gamma(\vec{m})$ and $\Gamma(\vec{m}) \nvdash \psi_2$. By induction hypothesis $\vec{m} \Vdash \psi_1$ and by $\vec{m} \geq \vec{n}$ and $\vec{n} \Vdash \psi_1 \to \psi_2$, $\vec{m} \Vdash \psi_2$. Applying the induction hypothesis once more we get $\Gamma(\vec{m}) \vdash \psi_2$. Contradiction. Hence $\Gamma(\vec{n}) \vdash \psi_1 \to \psi_2$.
  (b) The converse is simple; left to the reader.
- $\psi = \forall x \psi(x)$.
  (a) Let $\vec{n} \Vdash \forall x \varphi(x)$, then we get $\forall \vec{m} \geq \vec{n}\, \forall c \in C(\vec{m})(\vec{m} \Vdash \varphi(c))$. Assume $\Gamma(\vec{n}) \nvdash \forall x \varphi(x)$, then for a suitable i $\Gamma(\vec{n}, i) \nvdash \forall x \varphi(x)$ (take $\top$ for σ_i in

the above construction). Let c be a constant in $L(\vec{n}, i)$ not in $\Gamma(\vec{n}, i)$, then $\Gamma(\vec{n}, i) \nvdash \varphi(c)$, and by induction hypothesis $(\vec{n}, i) \nVdash \varphi(c)$. Contradiction.
(b)$\Gamma(\vec{n}) \vdash \forall x \varphi(x)$. Suppose $\vec{n} \nVdash \forall x \varphi(x)$, then $\vec{m} \nVdash \varphi(c)$ for some $\vec{m} \geq \vec{n}$ and for some $c \in L(\vec{m})$, hence $\Gamma(\vec{m}) \nvdash \varphi(c)$ and therefore $\Gamma(\vec{m}) \nvdash \forall x \varphi(x)$. Contradiction.

– $\psi = \exists x \varphi(x)$.
The implication from left to right is obvious. For the converse we use the fact that $\Gamma(\vec{n})$ is a prime theory. The details are left to the reader.

We now can finish our proof. The bottom node forces Γ and φ is not forced. ■

We can get some extra information from the proof of the Model Existence Lemma: (i) the underlying partially ordered set is a *tree*, (ii) all sets $D(\vec{m})$ are denumerable.

From the Model Existence Lemma we easily derive the following

Theorem 5.3.10 (Completeness Theorem – Kripke) $\Gamma \vdash_i \varphi \Leftrightarrow \Gamma \Vdash \varphi$ *(Γ and φ closed).*

Proof. We have already shown $\Rightarrow$. For the converse we assume $\Gamma_i \nvdash \varphi$ and apply 5.3.9, which yields a contradiction. ■

Actually we have proved the following refinement: intuitionistic logic is complete for countable models over trees.

The above results are completely general (safe for the cardinality restriction on L), so we may as well assume that Γ contains the identity axioms $I_1, \ldots, I_4$ (2.6). May we also assume that the identity predicate is interpreted by the real equality in each world? The answer is no, this assumption constitutes a real restriction, as the following theorem shows.

Theorem 5.3.11 *If for all $k \in K$ $k \Vdash \overline{a} = \overline{b} \Rightarrow a = b$ for $a, b \in D(k)$ then $\mathcal{K} \Vdash \forall xy(x = y \vee x \neq y)$.*

Proof. Let $a, b \in D(k)$ and $k \nVdash \overline{a} = \overline{b}$, then $a \neq b$, not only in $D(k)$, but in all $D(l)$ for $l \geq k$, hence for all $l \geq k, l \nVdash \overline{a} = \overline{b}$, so $k \Vdash \overline{a} \neq \overline{b}$. ■

For a kind of converse, cf. Exercise 18.

The fact that the relation $a \sim_k b$ in $\mathfrak{A}(k)$, given by $k \Vdash \overline{a} = \overline{b}$, is not the identity relation is definitely embarrassing for a language with function symbols. So let us see what we can do about it. We assume that a function symbol F is interpreted in each k by a function F_k. We require $k \leq l \Rightarrow F_k \subseteq F_l$. F has to obey $I_4 : \forall \vec{x} \vec{y} \ (\vec{x} = \vec{y} \rightarrow F(\vec{x}) = F(\vec{y}))$. For more about functions see Exercise 34.

Lemma 5.3.12 *The relation $\sim_k$ is a congruence relation on $\mathfrak{A}(k)$, for each k.*

Proof. Straightforward, by interpreting $I_1 - I_4$ ■

We may drop the index k, this means that we consider a relation $\sim$ on the whole model, which is interpreted node-wise by the local $\sim_k$'s.
We now define new structures by taking equivalence classes: $\mathfrak{A}^\star(k){:=}\mathfrak{A}(k)/{\sim_k}$, i.e. the elements of $|\mathfrak{A}^\star(k)|$ are equivalence classes $a/{\sim_k}$ of elements $a \in D(k)$, and the relations are canonically determined by

$R_k^\star(a/\sim, \ldots) \Leftrightarrow R_k(a, \ldots)$, similarly for the functions $F_k^\star(a/\sim, \ldots) = F_k(a, \ldots)/\sim$.

The inclusion $\mathfrak{A}(k) \subseteq \mathfrak{A}(l)$, for $k \leq l$, is now replaced by a map $f_{kl} : \mathfrak{A}^\star(k) \to \mathfrak{A}^\star(l)$, where f_{kl} is defined by $f_{kl}(a) = a^{\mathfrak{A}(l)}$ for $a \in |\mathfrak{A}^\star(k)|$. To be precise:

$a/{\sim_k} \longmapsto a/{\sim_l}$, so we have to show $a \sim_k a' \Rightarrow a \sim_l a'$ to ensure the well-definedness of f_{kl}. This, however, is obvious, since $k \Vdash \overline{a} = \overline{a'} \Rightarrow l \Vdash \overline{a} = \overline{a'}$.

Claim 5.3.13 *f_{kl} is a homomorphism.*

Proof. Let us look at a binary relation. $R_k^\star(a/\sim, b/\sim) \Leftrightarrow R_k(a,b) \Leftrightarrow k \Vdash R(a,b) \Rightarrow l \Vdash R(a,b) \Leftrightarrow R_l(a,b) \Leftrightarrow R_l^\star(a/{\sim}, b/{\sim})$.

The case of an operation is left to the reader. ■

The upshot is that we can define a modified notion of Kripke model.

Definition 5.3.14 *A modified Kripke model for a language L is a triple $\mathcal{K} = \langle K, \mathfrak{A}, f\rangle$ such that K is a partially ordered set, $\mathfrak{A}$ and f are mappings such that for $k \in K, \mathfrak{A}(k)$ is a structure for L and for $k, l \in K$ with $k \leq l$ $f(k,l)$ is a homomorphism from $\mathfrak{A}(k)$ to $\mathfrak{A}(l)$ and $f(l,m) \circ f(k,l) = f(k,m)$,*
$f(k,k) = id$.

Notation. We write f_{kl} for $f(k,l)$, and $k \Vdash^\star \varphi$ for $\mathfrak{A}(k) \models \varphi$, for atomic φ.

Now one may mimic the development presented for the original notion of Kripke semantics.

In particular the connection between the two notions is given by

Lemma 5.3.15 *Let $\mathcal{K}^\star$ be the modified Kripke model obtained from $\mathcal{K}$ by dividing out $\sim$. Then $k \Vdash \varphi(\overrightarrow{a}) \Leftrightarrow k \Vdash^\star \varphi(\overrightarrow{a}/{\sim})$ for all $k \in K$.*

Proof. Left to the reader. ■

Corollary 5.3.16 *Intuitionistic logic (with identity) is complete with respect to modified Kripke semantics.*

Proof. Apply 5.3.10 and 5.3.15. ■

We will usually work with ordinary Kripke models, but for convenience we will often replace inclusions of structures $\mathfrak{A}(k) \subseteq \mathfrak{A}(l)$ by inclusion mappings $\mathfrak{A}(k) \hookrightarrow \mathfrak{A}(l)$.

5.4 Some Model Theory

We will give some simple applications of Kripke's semantics. The first ones concern the so-called *disjunction* and *existence properties.*

Definition 5.4.1 *A set of sentences Γ has the*

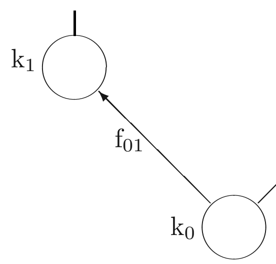
(i) disjunction property (DP) *if* $\Gamma \vdash \varphi \vee \psi \Rightarrow \Gamma \vdash \varphi$ *or* $\Gamma \vdash \psi$.
(ii) existence property (EP) if $\Gamma \vdash \exists x\varphi(x) \Rightarrow \Gamma \vdash \varphi(t)$ *for some closed term* t *(where* $\varphi \vee \psi$ *and* $\exists x\varphi(x)$ *are closed).*

In a sense DP and EP reflect the constructive character of the theory Γ (in the frame of intuitionistic logic), since it makes explicit the clause 'if we have a proof of $\exists x\varphi(x)$, then we have a proof of a particular instance', similarly for disjunction.

Classical logic does not have DP or EP, for consider in propositional logic $p_0 \vee \neg p_0$. Clearly $\vdash_c p_0 \vee \neg p_0$, but neither $\vdash_c p_0$ nor $\vdash_c \neg p_0$!

Theorem 5.4.2 *Intuitionistic propositional and predicate logic without functions symbols have* DP.

Proof. Let $\vdash \varphi \vee \psi$, and suppose $\nvdash \varphi$ and $\nvdash \psi$, then there are Kripke models $\mathcal{K}_1$ and $\mathcal{K}_2$ with bottom nodes k_1 and k_2 such that $k_1 \nVdash \varphi$ and $k_2 \nVdash \psi$.



It is no restriction to suppose that the partially ordered sets K_1, K_2 of $\mathcal{K}_1$ and $\mathcal{K}_2$ are disjoint.

We define a new Kripke model with $K = K_1 \cup K_2 \cup \{k_0\}$ where $k_0 \notin K_1 \cup K_2$, see picture for the ordering.

We define $\mathfrak{A}(k) = \begin{cases} \mathfrak{A}_1(k) \text{ for } k \in K_1 \\ \mathfrak{A}_2(k) \text{ for } k \in K_2 \\ |\mathfrak{A}| \text{ for } k = k_0. \end{cases}$

where $|\mathfrak{A}|$ consists of all the constants of L, if there are any, otherwise $|\mathfrak{A}|$ contains only one element a. The inclusion mapping for $\mathfrak{A}(k_0) \hookrightarrow \mathfrak{A}(k_i)(i = 1, 2)$ is defined by $c \mapsto c^{\mathfrak{A}(k_i)}$ if there are constants, if not we pick $a_i \in \mathfrak{A}(k_i)$ arbitrarily and define $f_{01}(a) = a_1, f_{02}(a) = a_2$. $\mathfrak{A}$ satisfies the definition of a Kripke model.
The models $\mathcal{K}_1$ and $\mathcal{K}_2$ are 'submodels' of the new model in the sense that the forcing induced on $\mathcal{K}_i$ by that of $\mathcal{K}$ is exactly its old forcing, cf. Exercise 13. By the Completeness Theorem $k_0 \vdash \varphi \vee \psi$, so $k_0 \Vdash \varphi$ or $k_0 \Vdash \psi$. If $k_0 \Vdash \varphi$, then $k_1 \Vdash \varphi$. Contradiction. If $k_0 \vdash \psi$, then $k_2 \vdash \psi$. Contradiction. So $\nvdash \varphi$ and $\nvdash \psi$ is not true, hence $\vdash \varphi$ or $\vdash \psi$. ■

Observe that this proof can be considerably simplified for propositional logic, all we have to do is place an extra node under k_1 and k_2 in which no atom is forced (cf. Exercise 19).

Theorem 5.4.3 *Let the language of intuitionistic predicate logic contain at least one constant and no function symbols, then EP holds.*

Proof. Let $\vdash \exists x \varphi(x)$ and $\nvdash \varphi(c)$ for all constants c. Then for each c there is a Kripke model $\mathcal{K}_c$ with bottom node k_c such that $k_c \nVdash \varphi(c)$. Now mimic the argument of 5.4.2 above, by taking the disjoint union of the $\mathcal{K}_c$'s and adding a bottom node k_0. Use the fact that $k_0 \Vdash \exists x \varphi(x)$. ■

The reader will have observed that we reason about our intuitionistic logic and model theory in a classical meta-theory. In particular we use the principle of the excluded third in our meta-language. This indeed detracts from the constructive nature of our considerations. For the present we will not bother to make our arguments constructive, it may suffice to remark that classical arguments can often be circumvented, cf. Chapter 6.

In constructive mathematics one often needs stronger notions than the classical ones. A paradigm is the notion of *inequality*. E.g. in the case of the real numbers it does not suffice to know that a number is unequal (i.e. not equal) to 0 in order to invert it. The procedure that constructs the inverse for a given Cauchy sequence requires that there exists a number n such that the distance of the given number to zero is greater than 2^{-n}. Instead of a negative notion we need a positive one, this was introduced by Brouwer, and formalized by Heyting.

Definition 5.4.4 *A binary relation # is called an apartness relation if*
(i) $\forall xy(x = y \leftrightarrow \neg x\#y)$
(ii) $\forall xy(x\#y \leftrightarrow y\#x)$
(iii) $\forall xyz(x\#y \to x\#z \vee y\#z)$

Examples.

1. For rational numbers the inequality is an apartness relation.
2. If the equality relation on a set is decidable (i.e. $\forall xy(x = y \vee x \neq y)$), then $\neq$ is an apartness relation (Exercise 22).
3. For real numbers the relation $|a - b| > 0$ is an apartness relation (cf. *Troelstra-van Dalen*, 2.7, 2.8.).

We call the theory with axioms (i), (ii), (iii) of 5.4.4 **AP**, the theory of apartness (obviously, the identity axiom $x_1 = x_2 \wedge y_1 = y_2 \wedge x_1\#y_1 \to x_2\#y_2$ is included).

Theorem 5.4.5 **AP** $\vdash \forall xy(\neg\neg x = y \to x = y)$.

Proof. Observe that $\neg\neg x = y \leftrightarrow \neg\neg\neg x\#y \leftrightarrow \neg x\#y \leftrightarrow x = y$. ∎

We call an equality relation that satisfies the condition $\forall xy(\neg\neg x = y \to x = y)$ *stable*. Note that *stable* is essentially weaker than *decidable* (Exercise 23).

In the passage from intuitionistic theories to classical ones by adding the principle of the excluded third usually a lot of notions are collapsed, e.g. $\neg\neg x = y$ and $x = y$. Or conversely, when passing from classical theories to intuitionistic ones (by deleting the principle of the excluded third) there is a choice of the right notions. Usually (but not always) the strongest notions fare best. An example is the notion of *linear order*.

The theory of linear order, **LO**, has the following axioms:
(i) $\forall xyz(x < y \wedge y < z \to x < z)$
(ii) $\forall xyz(x < y \to z < y \vee x < z)$
(iii) $\forall xyz(x = y \leftrightarrow \neg x < y \wedge \neg y < x)$.

One might wonder why we did not choose the axiom $\forall xyz(x < y \vee x = y \vee y < x)$ instead of (ii), it certainly would be stronger! There is a simple reason: the axiom is *too* strong, it does not hold, e.g., for the reals.

We will next investigate the relation between linear order and apartness.

Theorem 5.4.6 *The relation* $x < y \vee y < x$ *is an apartness relation.*

Proof. An exercise in logic. ∎

Conversely, Smoryǹski has shown how to introduce an order relation in a Kripke model of **AP**: Let $\mathcal{K} \Vdash$ **AP**, then in each $D(k)$ the following is an equivalence relation: $k \nVdash a\#b$.

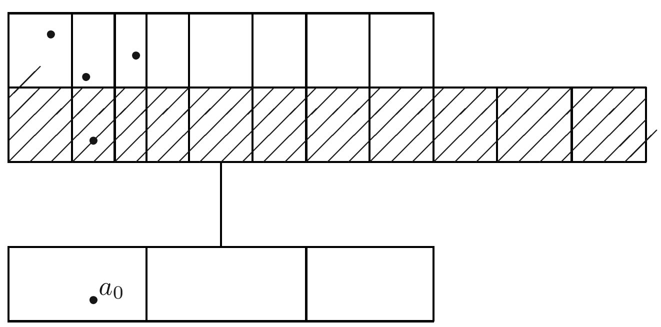
(a) $k \Vdash a = a \leftrightarrow \neg a\#a$, since $k \Vdash a = a$ we get $k \Vdash \neg a\#a$ and hence $k \nVdash a\#a$.

(b) $k \Vdash a\#b \leftrightarrow b\#a$, so obviously $k \nVdash a\#b \Leftrightarrow k \nVdash b\#a$.
(c) let $k \nVdash a\#b, k \nVdash b\#c$ and suppose $k \Vdash a\#c$, then by axiom (iii) $k \Vdash a\#b$ or $k \Vdash c\#b$ which contradicts the assumptions. So $k \nVdash a\#c$.

Observe that this equivalence relation contains the one induced by the identity; $k \Vdash a = b \Rightarrow k \nVdash a\#b$. The domains $D(k)$ are thus split up in equivalence classes, which can be linearly ordered in the classical sense. Since we want to end up with a Kripke model, we have to be a bit careful. Observe that equivalence classes may be split by passing to a higher node, e.g. if $k < l$ and $k \nVdash a\#b$ then $l \Vdash a\#b$ is very well possible, but $l \nVdash a\#b \Rightarrow k \nVdash a\#b$. We take an arbitrary ordering of the equivalence classes of the bottom node (using the axiom of choice in our meta-theory if necessary). Next we indicate how to order the equivalence classes in an immediate successor l of k.
The 'new' elements of $D(l)$ are indicated by the shaded part.

(i) Consider an equivalence class $[a_0]_k$ in $D(k)$, and look at the corresponding set $\hat{a}_0 := \bigcup\{[a]_l | a \in [a_0]_k\}$.
This set splits in a number of classes; we order those linearly. Denote the equivalence classes of $\hat{a}_0$ by $a_0 b$ (where b is a representative). Now the classes belonging to the $b's$ are ordered, and we order all the classes on $\bigcup \hat{a}_0 | a_0 \in D(k)\}$ lexicographically according to the representation $a_0 b$.
(ii) Finally we consider the new equivalence classes, i.e. of those that are not equivalent to any b in $\bigcup\{\hat{a}_0 | a_0 \in D(k)\}$. We order those classes and put them in that order behind the classes of case (i).

Under this procedure we order all equivalence classes in all nodes.
We now define a relation R_k for each k: $R_k(a, b) := [a]_k < [b]_k$, where $<$ is the ordering defined above. By our definition $k < l$ and $R_k(a, b) \Rightarrow R_l(a, b)$. We leave it to the reader to show that I_4 is valid, i.e. in particular $k \Vdash \forall xyz(x = x' \wedge x < y \rightarrow x' < y)$, where $<$ is interpreted by R_k.



Observe that in this model the following holds:

$$(\#) \qquad \forall xy(x\#y \leftrightarrow x < y \vee y < x),$$

for in all nodes $k, k \Vdash a\#b \leftrightarrow k \Vdash a < b$ or $k \Vdash b < a$.
Now we must check the axioms of linear order.

(i) *transitivity.* $k_0 \Vdash \forall xyz(x < y \wedge y < z \to x < z) \Leftrightarrow$ for all $k \geq k_0$, for all $a, b, c \in D(k) k \Vdash a < b \wedge b < c \to a < c \Leftrightarrow$ for all $k \geq k_0$, for all $a, b, c \in D(k)$ and for all $l \geq k$ $l \Vdash a < b$ and $l \Vdash b < c \Rightarrow l \Vdash a < c$.
So we have to show $R_l(a, b)$ and $R_l(b, c) \Rightarrow R_l(a, c)$, but that is indeed the case by the linear ordering of the equivalence classes.

(ii) (*weak*)*linearity.* We must show $k_0 \Vdash \forall xyz(x < y \to z < y \vee x < z)$. Since in our model $\forall xy(x\#y \leftrightarrow x < y \vee y < x)$ holds the problem is reduced to pure logic: show:
$\mathbf{AP} + \forall xyz(x < y \wedge y < z \to x < z) + \forall xy(x\#y \leftrightarrow x < y \vee y < x) \vdash \forall xyz(x < y \to z < y \vee x < z)$.
We leave the proof to the reader.

(iii) *anti-symmetry.* We must show $k_0 \Vdash \forall xy(x = y \leftrightarrow \neg x < y \wedge \neg y < x)$. As before the problem is reduced to logic. Show:
$\mathbf{AP} + \forall xy(x\#y \leftrightarrow x < y \vee y < x) \vdash \forall xy(x = y \leftrightarrow \neg x < y \wedge \neg y < x)$.

Now we have finished the job – we have put a linear order on a model with an apartness relation. We can now draw some conclusions.

Theorem 5.4.7 $\mathbf{AP} + \mathbf{LO} + (\#)$ *is conservative over* $\mathbf{LO}$.

Proof. Immediate, by Theorem 5.4.6. ■

Theorem 5.4.8 (van Dalen-Statman) $\mathbf{AP} + \mathbf{LO} + (\#)$ *is conservative over* $\mathbf{AP}$.

Proof. Suppose $\mathbf{AP} \nvdash \varphi$, then by the Model Existence Lemma there is a tree model $\mathcal{K}$ of $\mathbf{AP}$ such that the bottom node k_0 does not force φ.
We now carry out the construction of a linear order on K, the resulting model $\mathcal{K}^*$ is a model of $\mathbf{AP} + \mathbf{LO} + (\#)$, and, since φ does not contain $<, k_0 \nVdash \varphi$. Hence $\mathbf{AP} + \mathbf{LO} + (\#) \nvdash \varphi$. This shows the conservative extension result:
$\mathbf{AP} + \mathbf{LO} + (\#) \vdash \varphi \Rightarrow \mathbf{AP} \vdash \varphi$, for φ in the language of $\mathbf{AP}$. ■

There is a convenient tool for establishing elementary equivalence between Kripke models:

Definition 5.4.9 *(i) A bisimulation between two posets A and B is a relation $R \subseteq A \times B$ such that for each a, a', b with $a \leq a', aRb$ there is a b' with $a'Rb'$ and for each a, b, b' with $aRb, b \leq b'$ there is an a' such that $a'Rb'$.*
(ii) R is a bisimulation between propositional Kripke models $\mathcal{A}$ and $\mathcal{B}$ if it is a bisimulation between the underlying posets and if $aRb \Rightarrow \Sigma(a) = \Sigma(b)$ (i.e. a and b force the same atoms).

Bisimulations are useful to establish elementary equivalence node-wise.

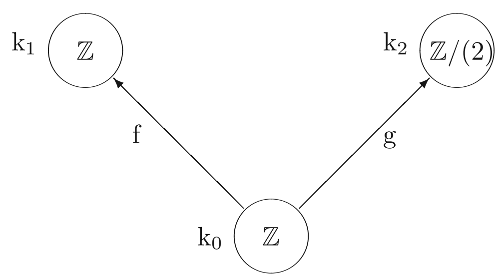
Lemma 5.4.10 *Let R be a bisimulation between $\mathcal{A}$ and $\mathcal{B}$ then for all $a, b, \varphi, \quad aRb \Rightarrow (a \Vdash \varphi \Leftrightarrow b \Vdash \varphi)$.*

Proof. Induction on φ. For atoms and conjunctions and disjunctions the result is obvious.
Consider $\varphi = \varphi_1 \to \varphi_2$.
Let aRb and $a \Vdash \varphi_1 \to \varphi_2$. Suppose $b \nVdash \varphi_1 \to \varphi_2$, then for some $b' \geq b \quad b' \Vdash \varphi_1$ and $b' \nVdash \varphi_2$. By definition, there is an $a' \geq a$ such that $a'Rb'$. By induction hypothesis $a' \Vdash \varphi_1$ and $a' \nVdash \varphi_2$. Contradiction.
The converse is completely similar. ■

Corollary 5.4.11 *If R is a total bisimulation between $\mathcal{A}$ and $\mathcal{B}$, i.e. $domR = A, ranR = B$, then $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent ($\mathcal{A} \Vdash \varphi \Leftrightarrow \mathcal{B} \Vdash \varphi$).*

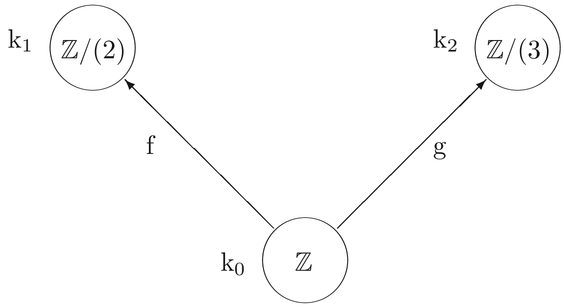
We end this chapter by giving some examples of models with unexpected properties.

1.



f is the identity and g is the canonical ring homomorphism $\mathbb{Z} \to \mathbb{Z}/(2)$. $\mathcal{K}$ is a model of the ring axioms (p. 86).
Note that $k_0 \Vdash 3 \neq 0, k_0 \nVdash 2 = 0, k_0 \nVdash 2 \neq 0$ and $k_0 \nVdash \forall x(x \neq 0 \to \exists y(xy = 1))$, but also $k_0 \nVdash \exists x(x \neq 0 \wedge \forall y(xy \neq 1))$. We se that $\mathcal{K}$ is a commutative ring in which not all non-zero elements are invertible, but in which it is impossible to exhibit a non-invertible, non-zero element.

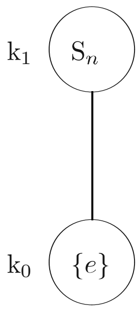
2.



Again f and g are the canonical homomorphisms. $\mathcal{K}$ is an intuitionistic, commutative ring, as one easily verifies. medskip

$\mathcal{K}$ has no zero-divisors: $k_0 \Vdash \neg\exists xy(x \neq 0 \wedge y \neq 0 \wedge xy = 0) \Leftrightarrow$ for all i $k_i \nVdash \exists xy(x \neq 0 \wedge y \neq 0 \wedge xy = 0)$. (1)
For $i = 1, 2$ this is obvious, so let us consider $i = 0$. $k_0 \Vdash \exists xy(x \neq 0 \wedge y \neq 0 \wedge xy = 0) \Leftrightarrow k_0 \Vdash m \neq 0 \wedge n \neq 0 \wedge mn = 0$ for some m, n. So $m \neq 0, n \neq 0, mn = 0$. Contradiction. This proves (1).
The cardinality of the model is rather undetermined. We know $k_0 \Vdash \exists xy(x \neq y)$ - take 0 and 1, and $k_0 \Vdash \neg\exists x_1x_2x_3 \bigwedge\limits_{1 \leq i < j \leq 4} x_i \neq x_j$. But note that $k_0 \nVdash \exists x_1x_2x_3 \bigwedge\limits_{1 \leq i < j \leq 3} x_1 \neq x_j, k_0 \nVdash \forall x_1x_2x_3x_4 \bigvee\limits_{1 \leq i < j \leq 4} x_i = x_j$ and $k_0 \nVdash \neg\exists x_1x_2x_3 \bigwedge\limits_{1 \leq i < j \leq 3} x_1 \neq x_j$.
Observe that the equality relation in $\mathcal{K}$ is not stable: $k_0 \Vdash \neg\neg 0 = 6$, but $k_0 \nVdash 0 = 6$.

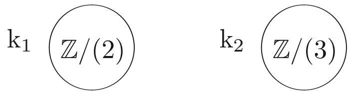
3.



S_n is the (classical) symmetric group on n elements. Choose $n \geq 3$. k_0 forces the group axioms (p. 85). $k_0 \Vdash \neg\forall xy(xy = yx)$, but $k_0 \nVdash \exists xy(xy \neq$

yx), and $k_0 \nVdash \forall xy(xy = yx)$. So this group is not commutative, but one cannot indicate non-commuting elements.

4.



Define an apartness relation by $k_1 \Vdash a\#b \Leftrightarrow a \neq b$ in $\mathbb{Z}/(2)$, idem for k_2. Then $\mathcal{K} \Vdash \forall x(x\#0 \to \exists y(xy = 1))$.
This model is an intuitionistic field, but we cannot determine its characteristic. $k_1 \vdash \forall x(x + x = 0), k_2 \Vdash \forall x(x + x + x = 0)$. All we know is $\mathcal{K} \Vdash \forall x(6.x = 0)$.

In the short introduction to intuitionistic logic that we have presented we have only been able to scratch the surface. We have intentionally simplified the issues so that a reader can get a rough impression of the problems and methods without going into the finer foundational details. In particular we have treated intuitionistic logic in a classical meta-mathematics, e.g. we have freely applied proof by contradiction (cf. 5.3.10). Obviously this does not do justice to constructive mathematics as an alternative mathematics in its own right. For this and related issues the reader is referred to the literature. A more constructive appraoch is presented in the next chapter.

Exercises

1. (informal mathematics). Let $\varphi(n)$ be a decidable property of natural numbers such that neither $\exists n\varphi(n)$, nor $\forall n\neg\varphi(n)$ has been established (e.g. "n is the largest number such that n and $n+2$ are prime"). Define a real number a by the cauchy sequence:

$$a_n := \begin{cases} \sum_{i=1}^{n} 2^{-i} \text{ if } \forall k < n \neg\varphi(k) \\ \\ \sum_{i=1}^{k} 2^{-i} \text{ if } k < n \text{ and } \varphi(k) \text{ and } \neg\varphi(i) \text{for } i < k. \end{cases}$$

Show that (a_n) is a cauchy sequence and that "$\neg\neg a$ is rational", but there is no evidence for "a is rational".

2. Prove
$\vdash \neg\neg(\varphi \to \psi) \to (\varphi \to \neg\neg\psi)$, $\vdash \neg\neg(\varphi \vee \neg\varphi)$,
$\vdash \neg(\varphi \wedge \neg\varphi)$, $\vdash \neg\neg(\neg\neg\varphi \to \varphi)$,
$\neg\neg\varphi, \neg\neg(\varphi \to \psi) \vdash \neg\neg\psi$, $\vdash \neg\neg(\varphi \to \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)$,
$\vdash \neg(\varphi \vee \psi) \leftrightarrow \neg(\neg\varphi \to \psi)$.

3. (a) $\varphi \vee \neg\varphi, \psi \vee \neg\psi \vdash (\varphi \blacksquare \psi) \vee \neg(\varphi \blacksquare \psi)$, where $\blacksquare \in \{\wedge, \vee, \rightarrow\}$.
   (b) Let the proposition φ have atoms $p_0, \ldots, p_n$, show
   $\bigwedge (p_i \vee \neg p_i) \vdash \varphi \vee \neg\varphi$.

4. Define the double negation translation $\varphi^{\neg\neg}$ of φ by placing $\neg\neg$ in front of each subformula. Show $\vdash_i \varphi^\circ \leftrightarrow \varphi^{\neg\neg}$ and $\vdash_c \varphi \Leftrightarrow \vdash_i \varphi^{\neg\neg}$.

5. Show that for propositional logic $\vdash_i \neg\varphi \Leftrightarrow \vdash_c \neg\varphi$.

6. Intuitionistic arithmetic **HA** (Heyting's arithmetic) is the first-order intuitionistic theory with the axioms of page 87 as mathematical axioms. Show $\mathbf{HA} \vdash \forall xy(x = y \vee x \neq y)$ (use the principle of induction). Show that the Gödel translation works for arithmetic, i.e. $\mathbf{PA} \vdash \varphi \Leftrightarrow \mathbf{HA} \vdash \varphi^\circ$ (where **PA** is Peano's (classical) arithmetic). Note that we need not doubly negate the atoms.

7. Show that **PA** is conservative over **HA** with respect to formula's not containing $\vee$ and $\exists$.

8. Show that $\mathbf{HA} \vdash \varphi \vee \psi \leftrightarrow \exists x((x = 0 \rightarrow \varphi) \wedge (x \neq 0 \rightarrow \psi))$.

9. (a) Show $\nvdash (\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$; $\nvdash (\neg\neg\varphi \rightarrow \varphi) \rightarrow (\varphi \vee \neg\varphi)$;
   $\nvdash \neg\varphi \vee \neg\neg\varphi$; $\nvdash (\neg\varphi \rightarrow \psi \vee \sigma) \rightarrow [(\neg\varphi \rightarrow \psi) \vee (\neg\varphi \rightarrow \sigma)]$;
   $\nvdash \neg\varphi \vee \neg\neg\varphi$; $\nvdash \bigvee_{1 \leq i < j \leq n} (\varphi_i \leftrightarrow \varphi_j)$, for all $n > 2$.
   (b) Use the completeness theorem to establish the following theorems:
   (i) $\varphi \rightarrow (\psi \rightarrow \varphi)$
   (ii) $(\varphi \vee \varphi) \rightarrow \varphi$
   (iii) $\forall xy\varphi(x, y) \rightarrow \forall yx\varphi(x, y)$
   (iv) $\exists x\forall y\varphi(x, y) \rightarrow \forall y\exists x\varphi(x, y)$
   (c) Show $k \Vdash \forall xy\varphi(xy) \Leftrightarrow \forall l \geq k \forall a, b \in D(l)\ l \Vdash \varphi(\overline{a}, \overline{b})$.
   $k \nVdash \varphi \rightarrow \psi \Leftrightarrow \exists l \geq k(l \Vdash \varphi \text{ and } l \nVdash \psi)$.

10. Give the simplified definition of a Kripke model for (the language of) propositional logic by considering the special case of def. 5.3.1 with $\Sigma(k)$ consisting of propositional atoms only, and $D(k) = \{0\}$ for all k.

11. Give an alternative definition of Kripke model based on the "structure-map" $k \mapsto \mathfrak{A}(k)$ and show the equivalence with definition 5.3.1 (without propositional atoms).

12. Prove the soundness theorem using lemma 5.3.5.

13. A subset K' of a partially ordered set K is closed (under $\leq$) if $k \in K'$, $k \leq l \Rightarrow l \in K'$. If K' is a closed subset of the underlying partially ordered set K of a Kripke model $\mathcal{K}$, then K' determines a Kripke model $\mathcal{K}'$ over K' with $D'(k) = D(k)$ and $k \Vdash' \varphi \Leftrightarrow k \Vdash \varphi$ for $k \in K'$ and φ atomic. Show $k \Vdash' \varphi \Leftrightarrow k \Vdash \varphi$ for all φ with parameters in $D(k)$, for $k \in K'$ (i.e. it is the future that matters, not the past).

14. Give a modified proof of the model existence lemma by taking as nodes of the partially ordered set prime theories that extend Γ and that have a language with constants in some set $C^0 \cup C^1 \cup \ldots \cup C^{k-1}$ (cf. proof of 5.3.9) (note that the resulting partially ordered set need not (and, as a matter of fact, is not) a tree, so we lose something. Compare however exercise 16).

15. Consider a propositional Kripke model $\mathcal{K}$, where the Σ function assigns only subsets of a finite set Γ of the propositions, which is closed under subformulas. We may consider the sets of propositions forced at a node instead of the node: define $[k] = \{\varphi \in \Gamma | k \Vdash \varphi\}$. The set $\{[k] | k \in K\}$ is partially ordered by inclusion define $\Sigma_\Gamma([k]) := \Sigma(k) \cap At$, show that the conditions of a Kripke model are satisfied; call this model $\mathcal{K}_\Gamma$, and denote the forcing by $\Vdash_\Gamma$. We say that $\mathcal{K}_\Gamma$ is obtained by *filtration* from $\mathcal{K}$.
    (a) Show $[k] \Vdash_\Gamma \varphi \Leftrightarrow k \Vdash \varphi$, for $\varphi \in \Gamma$.
    (b) Show that $\mathcal{K}_\Gamma$ has an underlying finite partially ordered set.
    (c) Show that $\vdash \varphi \Leftrightarrow \varphi$ holds in all finite Kripke models.
    (d) Show that intuitionistic propositional logic is *decidable* (i.e. there is a decision method for $\vdash \varphi$), apply 3.3.17.

16. Each Kripke model with bottom node k_0 can be turned into a model over a tree as follows: K_{tr} consists of all finite increasing sequences $\langle k_0, k_1, \ldots, k_n \rangle, k_i < k_{i+1} (0 \leq i < n)$, and $\mathfrak{A}_{tr}(\langle k_0, \ldots, k_n \rangle) := \mathfrak{A}(k_n)$. Show $\langle k_0, \ldots, k_n \rangle, \Vdash_{tr} \varphi \Leftrightarrow k_n \Vdash \varphi$, where $\Vdash_{tr}$ is the forcing relation in the tree model.

17. (a) Show that $(\varphi \to \psi) \vee (\psi \to \varphi)$ holds in all linearly ordered Kripke models for propositional logic.
    (b) Show that $\mathbf{LC} \nvdash \sigma \Rightarrow$ there is a linear Kripke model of $\mathbf{LC}$ in which σ fails, where $\mathbf{LC}$ is the propositional theory axiomatized by the schema $(\varphi \to \psi) \vee (\psi \to \varphi)$ (Hint: apply Exercise 15). Hence $\mathbf{LC}$ is complete for linear Kripke models (Dummett).

18. Consider a Kripke model $\mathcal{K}$ for decidable equality (i.e. $\forall xy(x = y \vee x \neq y)$). For each k the relation $k \Vdash \overline{a} = \overline{b}$ is an equivalence relation. Define a new model $\mathcal{K}'$ with the same partially ordered set as $\mathcal{K}$, and $D'(k) = \{[a]_k | a \in D(k)\}$, where $[a]$ is the equivalence class of a. Replace

the inclusion of $D(k)$ in $D(l)$, for $k < l$, by the corresponding canonical embedding $[a]_k \mapsto [a]_l$. Define for atomic φ $k \Vdash' \varphi := k \Vdash \varphi$ and show $k \Vdash' \varphi \Leftrightarrow k \Vdash \varphi$ for all φ.

19. Prove DP for propositional logic directly by simplifying the proof of 5.4.2.

20. Show that **HA** has DP and EP, the latter in the form: $\mathbf{HA} \vdash \exists x\varphi(x)) \Rightarrow \mathbf{HA} \vdash \varphi(\overline{n})$ for some $n \in N$. (Hint, show that the model, constructed in 5.4.2 and in 5.4.3, is a model of **HA**).

21. Consider predicate logic in a language without function symbols and constants. Show $\vdash \exists x\varphi(x) \Rightarrow \vdash \forall x\varphi(x)$, where $FV(\varphi) \subseteq \{x\}$. (Hint: add an auxiliary constant c, apply 5.4.3, and replace it by a suitable variable).

22. Show $\forall xy(x = y \vee x \neq y) \vdash \bigwedge \mathbf{AP}$, where **AP** consists of the three axioms of the apartness relation, with $x\#y$ replaced by $\neq$.

23. Show $\forall xy(\neg\neg x = y \to x = y) \nvdash \forall xy(x = y \vee x \neq y)$.

24. Show that $k \Vdash \varphi \vee \neg\varphi$ for maximal nodes k of a Kripke model, so $\Sigma(k) = Th(\mathfrak{A}(k))$ (in the classical sense). That is, "the logic in maximal node is classical."

25. Give an alternative proof of Glivenko's theorem using Exercises 15 and 24.

26. Consider a Kripke model with two nodes $k_0, k_1; k_0 < k_1$ and $\mathfrak{A}(k_0) = \mathbb{R}$, $\mathfrak{A}(k_1) = \mathbb{C}$. Show $k_0 \nVdash \neg\forall x(x^2 + 1 \neq 0) \to \exists x(x^2 + 1 = 0)$.

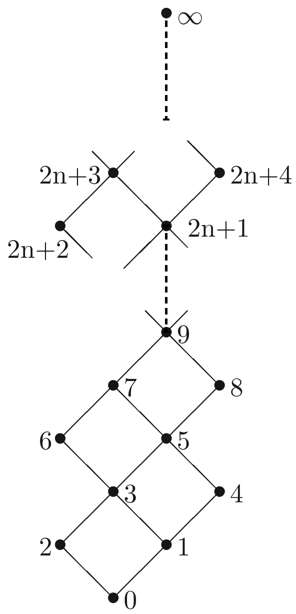
27. Let $\mathbb{D} = \mathbb{R}[X]/X^2$ be the ring of dual numbers. $\mathbb{D}$ has a unique maximal ideal, generated by X. Consider a Kripke model with two nodes $k_0, k_1; k_0 < k_1$ and $\mathfrak{A}(k_0) = \mathbb{D}$, $\mathfrak{A}(k_1) = \mathbb{R}$, with $f : \mathbb{D} \to \mathbb{R}$ the canonical map $f(a + bX) = a$. Show that the model is an intuitionistic field, define the apartness relation.

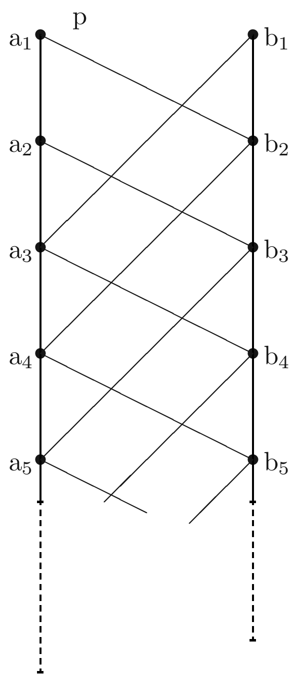
28. Show that $\forall x(\varphi \vee \psi(x)) \to (\varphi \vee \forall x\psi(x))(x \notin FV(\varphi))$ holds in all Kripke models with constant domain function (i.e. $\forall kl(D(k) = D(l))$.

29. This exercise will establish the undefinability of propositional connectives in terms of other connectives. To be precise the connective $\blacksquare_1$ is not definable in (or 'by') $\blacksquare_2, \ldots, \blacksquare_n$ if there is no formula φ, containing only the connectives $\blacksquare_2, \ldots, \blacksquare_n$ and the atoms p_0, p_1, such that $\vdash p_0 \blacksquare_1 p_1 \leftrightarrow \varphi$.

(i) $\vee$ is not definable in $\rightarrow, \wedge, \bot$. Hint: suppose φ defines $\vee$, apply the Gödel translation.
(ii) $\wedge$ is not definable in $\rightarrow, \vee, \bot$. Consider the Kripke model with three nodes k_1, k_2, k_3 and $k_1 < k_3, k_2 < k_3, k_1 \Vdash p, k_2 \Vdash q, k_3 \Vdash p, q$. Show that all $\wedge$-free formulas are either equivalent to $\bot$ or are forced in k_1 or k_2.
(iii) $\rightarrow$ is not definable in $\wedge, \vee, \neg, \bot$. Consider the Kripke model with three nodes k_1, k_2, k_3 and $k_1 < k_3, k_2 < k_3, k_1 \Vdash p, k_3 \Vdash p, q$. Show for all $\rightarrow$ −free formulas $k_2 \Vdash \varphi \Rightarrow k_1 \Vdash \varphi$.

30. In this exercise we consider now only propositions with a single atom p. Define a sequence of formulas by $\varphi_0 := \bot, \varphi_1 := p, \varphi_2 := \neg p, \varphi_{2n+3} := \varphi_{2n+1} \vee \varphi_{2n+2}, \varphi_{2n+4} := \varphi_{2n+2} \rightarrow \varphi_{2n+1}$ and an extra formula $\varphi_\infty := \top$. There is a specific set of implications among the φ_i, indicated in the diagram on the left.





(i) Show that the following implications hold:
$\vdash \varphi_{2n+1} \rightarrow \varphi_{2n+3}$, $\vdash \varphi_{2n+1} \rightarrow \varphi_{2n+4}$, $\vdash \varphi_{2n+2} \rightarrow \varphi_{2n+3}$,
$\vdash \varphi_0 \rightarrow \varphi_n$, $\vdash \varphi_n \rightarrow \varphi$.
(ii) Show that the following 'identities' hold:
$\vdash (\varphi_{2n+1} \rightarrow \varphi_{2n+2}) \leftrightarrow \varphi_{2n+2}$, $\vdash (\varphi_{2n+2} \rightarrow \varphi_{2n+4}) \leftrightarrow \varphi_{2n+4}$,
$\vdash (\varphi_{2n+3} \rightarrow \varphi_{2n+1}) \leftrightarrow \varphi_{2n+4}$, $\vdash (\varphi_{2n+4} \rightarrow \varphi_{2n+1}) \leftrightarrow \varphi_{2n+6}$,
$\vdash (\varphi_{2n+5} \rightarrow \varphi_{2n+1}) \leftrightarrow \varphi_{2n+1}$, $\vdash (\varphi_{2n+6} \rightarrow \varphi_{2n+1}) \leftrightarrow \varphi_{2n+4}$,

$\vdash (\varphi_k \to \varphi_{2n+1}) \leftrightarrow \varphi_{2n+1}$ for $k \geq 2n+7$,
$\vdash (\varphi_k \to \varphi_{2n+2}) \leftrightarrow \varphi_{2n+2}$ for $k \geq 2n+3$.
Determine identities for the implications not covered above.

(iii) Determine all possible identities for conjunctions and disjunctions of φ_i's (look at the diagram).

(iv) Show that each formula in p is equivalent to some φ_i.

(v) In order to show that there are no other implications than those indicated in the diagram (and the compositions of course) it suffices to show that no φ_n is derivable. Why?

(vi) Consider the Kripke model indicated in the diagram on the right. $a_1 \Vdash p$ and no other node forces p. Show: $\forall a_n \exists \varphi_i \forall k (k \Vdash \varphi_i \Leftrightarrow k \geq a_n)$, $\forall b_n \exists \varphi_j \forall k (k \Vdash \varphi_j \Leftrightarrow k \geq b_n)$
Clearly the $\varphi_i(\varphi_j)$ is uniquely determined, call it $\varphi(a_n)$, resp. $\varphi(b_n)$. Show $\varphi(a_1) = \varphi_1,\ \varphi(b_1) = \varphi_2,\ \varphi(a_2) = \varphi_4,\ \varphi(b_2) = \varphi_6,\ \varphi(a_{n+2}) = [(\varphi(a_{n+1}) \vee \varphi(b_n)) \to (\varphi(a_n) \vee \varphi(b_n))] \to (\varphi(a_{n+1}) \vee \varphi(b_n)), \varphi(b_{n+2}) = [(\varphi(a_{n+1}) \vee \varphi(b_{n+1})) \to (\varphi(a_{n+1}) \vee \varphi(b_n))] \to (\varphi(a_{n+1}) \vee \varphi(b_{n+1})$.

(vii) Show that the diagram on the left contains all provable implications.

Remark. The diagram of the implications is called the *Rieger-Nishimura lattice* (it actually is the free Heyting algebra with one generator).

31. Consider intuitionistic predicate logic without function symbols. Prove the following extension of the existence property: $\vdash \exists y \varphi(x_1, \ldots, x_n, y) \Leftrightarrow \vdash \varphi(x_1, \ldots, x_n, t)$, where t is a constant or one of the variables $x_1, \ldots, x_n$. (Hint: replace $x_1, \ldots, x_n$ by new constants $a_1, \ldots, a_n$).

32. Let $Q_1 x_1 \ldots Q_n x_n \varphi(\vec{x}, \vec{y})$ be a prenex formula (without function symbols), then we can find a suitable substitution instance φ' of φ obtained by replacing the existentially quantified variables by certain universally quantified variables or by constants, such that $\vdash Q_1 x_1 \ldots Q_n x_n \varphi(\vec{x}, \vec{y}) \Leftrightarrow \vdash \varphi'$ (use Exercise 31).

33. Show that $\vdash \varphi$ is decidable for prenex φ. (use 3.3.17 and Exercise 32).
Remark. Combined with the fact that intuitionistic predicate logic is undecidable, this shows that not every formula is equivalent to one in prenex normal form.

34. Consider a language with identity and function symbols, and interpret an n-ary symbol F by a function $F_k : D(k)^n \to D(k)$ for each k in a given Kripke model $\mathcal{K}$. We require *monotonicity*: $k \leq l \Rightarrow F_k \subseteq F_l$, and *preservation of equality*: $\vec{a} \sim_k \vec{b} \Rightarrow F_k(\vec{a}) \sim_k F_k(\vec{b})$, where $a \sim_k b \Leftrightarrow k \Vdash \overline{a} = \overline{b}$.
(i) Show $K \Vdash \forall \vec{x} \exists! y (F(\vec{x}) = y)$
(ii) Show $\mathcal{K} \Vdash I_4$.

(iii) Let $\mathcal{K} \Vdash \forall \vec{x} \exists ! y \varphi(\vec{x}, y)$, show that we can define for each k and F_k satisfying the above requirements such that $\mathcal{K} \Vdash \forall(\vec{x} \varphi(\vec{x}, F(\vec{x}))$.

(iv) Show that one can conservatively add definable Skolem functions.

Note that we have shown how to introduce functions in Kripke models, when they are given by "functional" relations. So, strictly speaking, Kripke models with just relations are good enough.