Peano Arithmetic and Finite Zermelo Fraenkel Set Theory III: Vopěnka's Alternative Set Theory

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Outline

- Natural numbers and Peano arithmetic.
- Notions of finiteness.
- Theories of finite sets and axioms.
- Ackermann interpretation of $\mathrm{ZF}_{\mathrm{fin}}$ in PA.
- A theory of finite sets and classes.

Main source:

[P. Vopěnka: *Mathematics in the Alternative Set Theory.* Teubner Verlag, 1979]

Set theory is primarily a theory of infinity.

Cantor's set theory develops infinity according to Cantor's ideas.

- **Actually infinite sets:** iterative constructions are completed.

Recall the short quote from [Baratella – Ferro]:

"We want to study a set theory in which the cantorian axiom of infinity is explicitly negated, precisely because we do not want to admit the possibility of considering a procedure going on forever as completed, as one element."

- **Cantorian finitism:** universal applicability of set theoretical constructions. (extrapolation)

Cantor made a distinction between the **transfinite** (e.g., each infinite ordinal) and the **absolute** infinite.

The term "Cantorian finitism" is often attributed to Michael Hallett.
[Hallett: Cantorian set theory and the limitations of size. Clarendon Press, 1986]

It seems to have been coined by John Mayberry.

[Mayberry: The Consistency Problem for Set Theory (II). British J. Phil. Science 28(2), 1977]

Vopěnka's views on set theory – cont'd

"It is commonly noted that set theory produces far more superstructure than is needed to support classical mathematics."

[Holmes: Alternative axiomatic set theories. The Stanford Encyclopedia of Philosophy (SEP), Winter 2021 Edition), E. N. Zalta (ed.)]

Cantor's set theory depends on its formal means:

". . . we are unable to give evidence of any actually infinite set in the real world. Thus we deal here with a construction extending the real world and surpassing qualitatively the limits of the space of possibilities of our observation. Assertions about infinite sets thus lose their phenomenal content."

Eventually, theories such as ZFC tend to be self-absorbed.

"Mathematics based on Cantor set theory changed to mathematics of Cantor set theory."

[Vopěnka: Mathematics in the AST, Introduction]

Vopěnka's views on set theory – cont'd

Vopěnka and his research group used the term
"Cantor's set theory" for a broad range of axiomatic set theories.

These include

- **Quine's New Foundations:**
  "Very soon, theories (consistent, we hope) based on Cantor's ideas were constructed — now we have e. g. the Zermelo–Fraenkel, Gödel–Bernays, Morse and New Foundations set theories. We shall speak about all these theories as Cantor's set theories."
- **the Theory of Semisets**, also previously investigated by Vopěnka, Hájek, Sochor and other members of the group.
  "AST is similar to the theory of semisets in the sense that both admit classes which are subclasses of sets and which are not sets. [...] But the main difference is again in **what** we want to do in AST: from this point of view, the theory of semisets is very near to Cantor's theory."

[Sochor: The Alternative Set Theory. In *Set Theory and Hierarchy Theory: A Memorial Tribute to Andrzej Mostowski*, vol. 537 LNM, 259–271, 1976]

What is the Alternative Set Theory?

Alternative Set Theory (AST) is an axiomatic theory of sets and classes in classical first-order logic.

Aims to be a new mathematical ontology (and a competitor to ZFC).
Not mimicking existing constructions of classical set theory.

The axiomatization was intended as an open system.

Vopěnka's notion of infinity is a natural infinity,
aimed at bringing experience and perception back to mathematics.

Infinity manifests itself as an absence of easy survey.
"Our infinity is a phenomenon occurring when we observe large sets."

Motivating example

"Suppose that we have a large library. Of course, it is natural to assume that we are able to make a list of all books in our library and hence the collection of all books in our library will be treated as a set. However, can we make a list of all interesting books in the library? Some books are definitely interesting and others completely dull, but usually we have also books about which it is difficult to decide whether they are really interesting or not. Thus it would be very difficult to write a list of all interesting books in our library though such a list would not be so extensive as the list of all books in the library. We assume that, in principle, we are able about each book to decide whether it is interesting or not, but such a decision might be very time consuming. [...] In the real situation we shall discard books as long as we have enough place; for lack of time we would approximate the collection of all uninteresting books its subcollection which can be better described. Hence we shall never be sure that just those books which are interesting remain in the library. On the other hand let us realize that it is very convenient to speak about the collection of all interesting books in our library - and we often do it. Our idealization holds this collection as already created although its extension is not precisely described."

[Sochor. The alternative set theory and its approach to Cantor's set theory. In *Aspects of Vagueness*, D. Reidel Publ. Company, Trillas, Skala, Termini (eds.), 1984]

Sets in the AST

Axioms for sets:

- (extensionality for sets) $\forall xy(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)$;
- (existence of sets) $\forall xy\,\exists z\,(z = x \cup \{y\})$;
- (induction for set formulas) $\varphi(\emptyset)\,\&\,\forall xy(\varphi(x) \to \varphi(x \cup \{y\})) \to \forall x\varphi(x)$;
- (regularity') $\exists x\,\varphi(x) \to \exists x\,(\varphi(x)\,\&\,\forall y \in x\,\neg\varphi(y))$,

where φ is a set formula.

These axioms prove all axioms of the theory $\mathrm{ZF}_{\mathrm{fin}} + \mathrm{TC}$;
and vice versa.

(It can be proved in ZF that) AST is conservative over its set fragment.

All **sets in AST are finite** "in the classical sense".

Sets in the AST – cont'd

Let x and y be sets.

- x is set-subvalent to y ($x \mathrel{\hat{\preceq}} y$) iff there is a set function f s.t. $\mathrm{Dom}(f) = x$ and $\mathrm{Rng}(f) \subseteq y$
- x is set-equivalent to y ($x \mathrel{\hat{\approx}} y$) iff there is a set bijection of x onto y.

Then a proper subset of x is strictly set-subvalent to x.
(Dedekind finiteness of x.)

Let x be a set and r a linear order on x. Let y be a nonempty subset of x. Then z has a first and a last element in the order r.

Natural numbers in the AST

α is a natural number provided α is transitive and linearly ordered by $\in$.

For each x there is a unique natural number α s.t. $x \mathrel{\hat{\approx}} \alpha$.

Class $\mathbb{N}$ of natural numbers. ($\mathbb{N}$ is not a set.)

Arithmetic operations:

- $S(\alpha) = \alpha \cup \{\alpha\}$;
- $\alpha + \beta = \gamma$ iff $\gamma \mathrel{\hat{\approx}} \alpha \cup (\{\beta\} \times \beta)$;
- $\alpha \cdot \beta = \gamma$ iff $\gamma \mathrel{\hat{\approx}} \alpha \times \beta$;
- $\alpha^\beta = \gamma$ iff $\gamma \mathrel{\hat{\approx}} \{f \mid \mathrm{Dom}(f) = \beta \text{ and } \mathrm{Rng}(f) \subseteq \alpha\}$.

$\mathbb{N}$ (with successors, addition, and multiplication) interprets PA.

NB. Some elements of $\mathbb{N}$ will turn out infinite “in the sense of the AST”.

Classes represent **properties** (of sets).
(As per example,) they are possibly not clearly perceived.

Universal sort of objects (similar to, e.g., NBG): $\forall x \exists X (x = X)$.
$\mathrm{Set}(X)$ iff $\exists Y (X \in Y)$. Then $\exists x (x = X)$.

Examples: V is the universal class, $\mathbb{N}$ is the class of natural numbers.

(extensionality for classes) $\forall X, Y (X = Y \leftrightarrow \forall z (z \in X \leftrightarrow z \in Y))$.
(class comprehension) $\exists Y \forall x \, (x \in Y \leftrightarrow \Phi(x))$ for $\Phi(x)$ not containing Y.
NB. Φ is **arbitrary** formula in language of sets and classes.

A class X is finite iff all subclasses are sets:

$$\mathrm{Fin}(X) \;\; \text{iff} \;\; \forall Y \subseteq X (\mathrm{Set}(Y)).$$

In particular, each finite class is a set.

Infinity in the AST

A semiset is a subclass of a set.

Axiom: there exists a proper semiset.
(proper semiset) $\exists x \exists Y (Y \subseteq x \,\&\, \neg \mathrm{Set}(Y))$

It follows that infinite sets exist: e.g., x **as above is infinite**.
Here "infinite" is used in Vopěnka's sense. Classically all sets are finite.

"Infinity is brought into our theory by means of semisets. But this kind of infinity is different from the actual infinity in Cantor's sense. Our infinity is a phenomenon occurring when we observe large sets. It manifests itself as absence of an easy survey, as our inability to grasp the set in its totality."
[Vopěnka, Mathematics in the AST, pp. 34–35]

There is a natural number α such that $x \mathrel{\hat{\approx}} \alpha$ (via f).

Lemma: α (above) is an infinite natural number.

Proof: f maps Y to some $Z \subseteq \alpha$.
Take $F = \{\langle a, b\rangle \in f \mid a \in Z\}$.
If Z is a set, then so is F (separation).
If F is a set, then so is Y; contradiction.
Therefore, α subsumes the proper semiset Z, and is infinite.

$X \approx Y$ iff there is a bijection F that maps X onto Y.

Wang's paradox

- 0 is small;
- if n is small, so is $n+1$;
- every natural number is small.

"It might be urged that it is not a paradox, since, on the ordinary understanding of 'small', the conclusion is true. A small elephant is an elephant that is smaller than most elephants; and, since every natural number is larger than only finitely many natural numbers, and smaller than infinitely many, every natural number is small, i.e., smaller than most natural numbers.
But it is a paradox, since we can evidently find interpretations of 'small' under which the conclusion is patently false and the premisses apparently true. It is, in fact, a version of the ancient Greek paradox of the heap."

[Dummett: Wang's paradox. *Synthese* 30, 1975]

Finite and infinite natural numbers in the AST

Recall: $\mathbb{N}$ is the proper class of natural numbers.

$\mathbb{FN} = \{\alpha \in \mathbb{N} : \mathrm{Fin}(\alpha)\}$.
Finite natural numbers.

$\mathbb{FN}$ is a proper initial segment of $\mathbb{N}$, as well as of any infinite $\alpha \in \mathbb{N}$.
$\mathbb{FN}$ is also closed under successors: a cut in $\mathbb{N}$.

Lemma: $\mathbb{FN}$ is not a set.

Since $\mathbb{FN} \subset \alpha$ for each infinite $\alpha \in \mathbb{N}$, $\mathbb{FN}$ is a proper semiset.

$\mathbb{FN}$ closed under closed under addition, multiplication, and (even) exponentiation.

Induction in $\mathbb{FN}$: for any formula Φ,
$\Phi(0)$ and $\forall n \in \mathbb{FN}\,(\Phi(n) \to \Phi(n \cup \{n\})$ implies $\forall n \in \mathbb{FN}\,\Phi(n)$.

$\mathbb{FN}$ interprets PA.

Prolongation axiom

Recalling $\mathbb{FN}$ is a prototypical semiset in the AST:

Figure: https://texashillcountry.com/3-texas-train-tour-examples-day-trip/train-tracks-to-the-horizon-landscape/

Prolongation axiom: any class function on FN is a part of a set function.

"The prolongation axiom is a hypothesis which serves as a base for exact knowledge exceeding evidence."

NB. Not all AST axioms have been introduced here.
Cf. [Vopěnka: Mathematics in the AST.]

Hilbert's hotel in the AST

Figure: https://www.ias.edu/ideas/2016/pires-hilbert-hotel[Ana Pires: How big is infinity?]

Theorem: Let x be infinite and $y \notin x$. Then $x \approx x \cup \{y\}$.

Rational numbers in AST

The integers are built up from natural numbers:
$\mathbb{N}^* = \mathbb{N} \cup \{\langle 0, \alpha \rangle | \alpha \neq 0\}$; analogously for $\mathbb{FN}^*$.

Then $\mathbb{RN}$ is defined as the quotient field of $\mathbb{N}^*$, and analogously for $\mathbb{FRN}$.

Rationals x, y are infinitely near, $x \doteq y$ provided that
– $|x - y| < 1/n$ for each nonzero $n \in \mathbb{FN}$, or
– $n < x$ and $n < y$ for each $n \in \mathbb{FN}$, or
– $x < -n$ and $y < -n$ for each $n \in \mathbb{FN}$.

A rational x is infinitely small iff $x \doteq 0$.

Real numbers are the quotient of rational numbers by $\doteq$.

Nonstandard universes

Skolem 1934: nonstandard model of PA.
"Strong" model — elementarily equivalent to the standard one.

Vopěnka learned about Skolem's construction from L. S. Rieger.
1962: Vopěnka provided nonstandard models of NBG;
(i.e., inner model of NBG in itself, with nonstandard natural numbers).

However, (in connection with AST) Vopěnka never speaks about "nonstandard models" or indeed refers to models of AST in ZFC.

Instead, AST provides an ontology for mathematics:

- in the universe of sets,
  elements of $\mathbb{N}$ that play the role of natural numbers;
- in a limit universe,
  the role of standard natural numbers is played by $\mathbb{FN}$..

[Robinson: Non-standard analysis. Proc. Royal Acad. Sci., 1961]
[Robinson: Non-standard Analysis. North Holland, 1966]

Robinson showed infinitesimal calculus can be consistently modelled using NSA. Infinitesimals are reciprocals of infinite (i.e., nonstandard) natural numbers.

Recall: NSA has transfer principle (elementary equivalence).

(Not! available in AST.)

Vopěnka buidls his AST axiomatically.
This is independent of those structures in ZFC that NSA describes.

It appears that NSA alerted Vopěnka to the possibility of modelling infinitesimal calculus using nonstandard structures.

Vopěnka's Prague school of set theory (ca. 1963 – 1968)

Czech mathematical logic was pioneered by L. S. Rieger (1916–1963).
Rieger worked at Czech Technical University in Prague
and at the Math Institute of the Czech Academy of Sciences.
He lectured at the Faculty of Mathematics and Physics.

Vopěnka attended Rieger's lectures. After Rieger's death in 1963,
he started his own seminar at the Faculty of Mathematics and Physics.

With B. Balcar, L. Bukovský, P. Hájek, K. Hrbáček, K. Příkrý, T. Jech,
A. Sochor, P. Štěpánek, ...

At that time, **Vopěnka was 28**.

Tarski (as reported by Sochor): "I do not know if there is at this point another place in the world, having as numerous and cooperative a group of young and talented researchers in foundations of mathematics."

Boolean-valued models and independence proofs.

[Vopěnka: General theory of ∇-models. *Comment. Math. Univ. Carolinae* 8, 1967]

Vopěnka biography (up to 2001):

[Sochor: Petr Vopěnka (∗ 16.5.1935). *Annals Pure Appl. Logic* 109, 2001]

1968: point of discontinuity

"In 1966 P. Vopěnka was appointed an associate dean [Faculty of Mathematics and Physics, Charles University in Prague] and used his new position to help found a Department of Mathematical Logic and also to create new curricula.
Because of these changes, students could major in areas such as algebra, geometry, topology and most notably in theoretical cybernetics.
[. . .]
Feeling unacceptably constrained by the Czechoslovak communist regime, a number of participants in P. Vopěnka seminar decided to leave the country. After the invasion of Czechoslovakia in 1968, the exodus accelerated [. . .] the original enthusiastic group had been decimated. In 1971, the Department of Mathematical Logic was abolished and P. Vopěnka lost his influence on the theoretical cybernetics major.

[Sochor: Petr Vopěnka (∗ 16.5.1935). Annals Pure Appl. Logic, 2001]

Theory of semisets

[Vopěnka, Hájek: The theory of semisets. North Holland, 1972]
[Hájek: Why Semisets? Comment. Mathematicae Universitatis Carolinae, 1973]

The theory of semisets arose from the study of certain universes, e.g., generic extensions or nonstandardness.

It is conservative over ZFC.

Sets are comprehensible collections. E.g.,

- Power set: If x is a set, then $P(x)$ is the set of all subsets of x.
- Comprehension: each definable subcollection of a set is a set.

Semisets are arbitrary subcollections of sets.

Do not assume that "each subcollection of a set is a set".

The existence of proper semisets is not guaranteed.

Early records of AST

During early 1970's Vopěnka was prevented from publishing his own work, travelling abroad, taking doctoral students, etc., for political reasons.

No records on AST by Vopěnka from this period.

Chudáček (Vopěnka student 1972–76) reports that there was no published material on AST from Vopěnka, which made the students' contributions awkward to present.

1973 Vopěnka circulated a manuscript on AST locally.

1976 Sochor presented a paper The Alternative Set Theory
in LNM volume 537 "Set Theory and Hierarchy Theory: A Memorial Tribute to Andrzej Mostowski".

1979 Vopěnka publishes Mathematics in the Alternative Set Theory with Teubner. (translated into English by Hájek)

In the 1970's, Vopěnka ran a seminar on the AST at the Faculty of Mathematics and Physics.

The theory exists in its developed form of today through the efforts of Antonín Sochor, Karel Čuda, Jaroslav Chudáček, Josef Mlček, Michal Resl, Jiří Sgall, Kateřina Trlifajová, Alena Vencovská, Blanka Vojtášková, Jiří Witzany, and the Bratislava group: Jaroslav Guričan, Martin Kalina, and Pavol Zlatoš.

Many works are accessible through the Czech Digital Mathematical Library:

www.dml.cz

Some materials exists in Czech (Slovak) only. Notably, the (useful) book [Vopěnka: Úvod do matematiky v alternatívnej teórii množín. *Alfa*, Bratislava, 1989] is in Slovak.

Models of the AST in ZF

In ZF, take the set V_ω of hereditarily finite sets.

Let $(V_\omega^*, \in^*)$ be an ultrapower (over non-trivial ultrafilter on ω).
Add each $X \subseteq V^*$ unless there is $x \in V_\omega^*$ s.t. $X = \{y \mid (V_\omega^*, \in^*) \models y \in x\}$
(Assume CH to cater for the AST-axiom of cardinalities)

This yields a model of the AST.
[Pudlák, Sochor: Models of the Alternative Set Theory. *The Journal of Symbolic Logic* 49(2), 570–585, 1984]

The intended (ZF-) interpretation of $\mathbb{FN}$ was the standard natural numbers.

Timeline

1934 Skolem's nonstandard model of PA
1961 Robinson's paper on NSA
1962 Vopěnka's nonstandard model of NBG
1963 – 1968 Vopěnka Prague school of set theory
1972 The Theory of Semisets published by North Holland
1970's and 1980's seminar on AST at Faculty of Mathematics and Physics
1973 Vopěnka circulates notes on AST
1975 (onwards) Sochor publishes papers on AST in English
1979 Mathematics in the Alternative Set Theory published by Teubner
1968 – 1980 Polívka employed at Faculty of Mathematics and Physics
1989 Introduction to mathematics in the Alternative Set Theory published by Alfa (in Slovak)
1989 1st Symposium on the Alternative Set Theory held in Stará Lesná
1990 Vopěnka appointed full professor of mathematics at Faculty of Mathematics and Physics. Shortly afterwards he becomes Minister of Education.