Peano Arithmetic and Finite Zermelo Fraenkel Set Theory

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Outline

- Natural numbers and Peano arithmetic.
- Notions of finiteness.
- Theories of finite sets and axioms.
- Ackermann interpretation of $\mathrm{ZF}_{\mathrm{fin}}$ in PA.
- (time permitting) Theories of finite sets and classes.

Source material

Main source:

Richard Kaye, Tin Lok Wong: On Interpretations of Arithmetic and Set Theory. *Notre Dame J. Formal Logic* 48, 2007.

Supplementary sources:

S. Baratella, R. Ferro: A theory of Sets with the Negation of the Axiom of Infinity. *Math. Logic Quarterly* 39, 1993, 338–52.

P. Vopěnka: *Mathematics in the Alternative Set Theory.* Teubner Verlag, 1979.

A. Sochor: Metamathematics of the Alternative Set Theory I;II;III. *Commentationes Mathematicae Universitatis Carolinae* 20(4);23(1);24(1), 1979;1982;1983,

L. Běhounek: *Nezávislost axiomů ve dvou axiomatikách teorie konečných množin.* Bachelor thesis, Dpt. of Logic, Faculty of Arts, Charles University in Prague, 1998.

Other points of interest:

Models of Peano Arithmetic ("MOPA") seminar:

`https://nylogic.github.io/MOPA.html`

First-order logic with equality

Fix a first-order language $\mathcal{L}$ (such as, e.g., $\{\in\}$).
This yields the notion of well-formed $\mathcal{L}$-formula.

Propositional axioms:

- $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
- $\alpha \to (\beta \to \alpha)$
- $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$

Axioms for quantifiers:

- $\forall x\, \varphi \to \varphi(x/t)$ t any $\mathcal{L}$-term substitutable in φ
- $\forall x\, (\alpha \to \beta) \to (\alpha \to \forall x\, \beta)$ x not free in α

Rules:

- (mp) α, $\alpha \to \beta$ / β
- (gen) α / $\forall x\, \alpha$

Throughout, α, β, γ are wff's of $\mathcal{L}$.

First-order logic with equality (cont'd)

Axioms of equality:

- $=$ is an equivalence;
- $\bigwedge_{i \leq n}(x_i = y_i) \to F(\bar{x}) = F(\bar{y})$
  for each $n \in \mathbb{N}$ and each n-ary function symbol F of $\mathcal{L}$
- $\bigwedge_{i \leq n}(x_i = y_i) \to (P(\bar{x}) \to P(\bar{y}))$
  for each $n \in \mathbb{N}$ and each n-ary predicate symbol P of $\mathcal{L}$.

A theory in $\mathcal{L}$ is a set of $\mathcal{L}$-sentences.
E.g., ZF is a theory in the language $\{\in\}$.

Convention on bounded quantifiers:
- $(\forall y \in x)\, \alpha$ means $(\forall y)(y \in x \to \alpha)$
- $(\exists y \in x)\, \alpha$ means $(\exists y)(y \in x \,\&\, \alpha)$

Natural numbers

The term “arithmos”, used in Euclid’s *Elements*, always denotes a positive integer.
Number theory is treated in Books VII – IX.

`http://aleph0.clarku.edu/ djoyce/java/elements/elements.html`

The *Elements* were a blueprint of the axiomatic method,
to be surpassed only in 19th century
(starting with the works of Frege and Hilbert).

Elements clearly demarcate the discrete quantities and the continuous ones.
The investigation of the natural number series and the continuum have remained the agenda for the ongoing developmnent of mathematics.

Develops and employs set-theoretic methods, starts with (or works his way down to) investigating natural numbers.

Set S and an $N \subseteq S$.
N is called **simply infinite** if there exists a function f on S and an element 1 of N such that:

- $f : N \mapsto N$;
- N is the chain (minimal closure) of $\{1\}$ in S under f;
- 1 is not in the image of N under f;
- f is 1-1.

Simply infinite systems are isomorphic.

Peano Arithmetic (PA)

First-order theory in logic with equality.

Language: $\{0, S, +, \cdot\}$.

Axioms:

(Q1) $S(x) \neq \bar{0}$

(Q2) $S(x) = S(y) \rightarrow x = y$

(Q3) $x \neq \bar{0} \rightarrow (\exists y)(x = S(y))$

(Q4) $x + \bar{0} = x$

(Q5) $x + S(y) = S(x + y)$

(Q6) $x \cdot \bar{0} = \bar{0}$

(Q7) $x \cdot S(y) = (x \cdot y) + x$

(Q8) $x \leq y \leftrightarrow (\exists v)(v + x = y)$

(Ind) $\varphi(\bar{0}) \mathbin{\&} (\forall x)(\varphi(x) \rightarrow \varphi(S(x))) \rightarrow (\forall x)\varphi(x)$

[R. Kaye: *Models of Peano Arithmetic.* Clarendon Press, Oxford, 1991.]
[P. Hájek, P. Pudlák. *Metamathematics of First-Order Arithmetic.* Springer, 1993.]
[C. Smoryński. *Lectures on Nonstandard Models of Arithmetic.* Proc. LC 1982.]

(Infinite) sets provide enough structure to define (embed, interpret) mathematical theories and structures usually under consideration.

This includes the "arithmetization" and axiomatic rendering of geometry.

Some structural properties of finite sets (and natural numbers) were transferred also to infinite sets and infinite ordinals, by design.

The set theoretic universe is "uniform" in this sense.
(This is called "Cantorian finitism" by J. P. Mayberry.)

We shall investigate a connection between hereditarily finite sets and finite ordinals.

Some axioms for sets

(extensionality) $\forall x, y\, (\forall z\, (z \in x \leftrightarrow z \in y) \to x = y)$

(empty set) $\exists x\, \forall y\, \neg(y \in x)$ Introduce $\emptyset$.

(pair) $\forall x, y\, \exists z\, \forall u\, (u \in z \leftrightarrow u = x \vee u = y)$

(union) $\forall x\, \exists y\, \forall z\, (z \in y \leftrightarrow \exists u\, (z \in u \wedge u \in x))$

(power set) $\forall x\, \exists y\, \forall z (z \in y \leftrightarrow \forall u\, (u \in z \to u \in x))$

(separation) $\forall x\, \exists y\, \forall z (z \in y \leftrightarrow z \in x \,\&\, \varphi(z))$
(φ any formula, y not free in φ)

(replacement)
$\forall x\, [\forall u \in x\, \exists! v\, \varphi(u, v) \to \exists y\, \forall v\, (v \in y \leftrightarrow \exists u \in x\, \varphi(u, v))]$
(φ any formula, y not free in φ)

(regularity) $\forall x\, (x \neq \emptyset \to \exists y\, (y \in x \wedge y \cap x = \emptyset))$
a.k.a. (foundation)

NB: all these axioms are exactly as in ZF.

Finiteness

In the theory ZF, the axiom

(infinity) $\exists x\,(\emptyset \in x \,\&\, \forall y \in x\,(y \cup \{y\} \in x))$

can be viewed as confirming **actual infinity**: any process, incl. "infinite processes" (such as successively adding a distinct element) s has been completed.

Theory of finite sets formalizes the opposite view.

"We want to study a set theory in which the cantorian axiom of infinity is explicitly negated, precisely because we do not want to admit the possibility of considering a procedure going on forever as completed, as one element."

[Baratella, Ferro: A Theory of Sets with the Negation of the Axiom of Infinity]

$\mathrm{ZF}_{\mathbf{fin}}$ has the following axioms:
(extensionality), (empty set), (pair), (union), (power set), (separation), (replacement), (regularity), ¬(infinity)

(Called "ZF–inf" in [Kaye–Wong]).

Finiteness (2)

(induction) $\varphi(\emptyset) \,\&\, \forall x, y\, [\varphi(x) \rightarrow \varphi(x \cup \{y\})] \rightarrow \forall x\, \varphi(x)$
(φ any formula, y not free in φ)

Claim: $\mathrm{ZF}_{\mathrm{fin}}$ proves induction for any formula φ.

Proof: let φ be given.

For contradiction, assume there is a set x s.t.
$\varphi(\emptyset)$ and $\forall y, q\, [\varphi(y) \rightarrow \varphi(y \cup \{q\})]$ but $\neg\varphi(x)$.

Let $z = \{y \mid y \subseteq x \,\&\, \varphi(y)\}$. Separation from $P(x)$.

By assumption $x \notin z$.

We have $\emptyset \in z$, since by assumption $\varphi(\emptyset)$; thus z is nonempty.

Let $y \in z$. This means $y \subseteq x$ and $\varphi(y)$, and since $\neg\varphi(x)$, we have $x \setminus y \neq \emptyset$.
Let $q \in x \setminus y$. Then by assumption $\varphi(y \cup \{q\})$ and $y \cup \{q\} \subseteq x$, so
$y \cup \{q\} \in z$.

This contradicts $\neg$(infinity). QED

[Sochor: Meta – AST II, p. 60]

Finiteness (3)

Write
– $x \approx y$ if there is a bijection between x and y;
– $x \preceq y$ if there is a 1-1 function f mapping x into y;
– $x \prec y$ if $x \preceq y$ but $x \not\approx y$.

A set x is Dedekind finite ($\mathrm{Fin}^{\mathrm{D}}$) provided that $\forall y (y \subsetneq x \to y \prec x)$.

Claim [$\mathrm{ZF}_{\mathbf{fin}}$]**:** every set is Dedekind finite.

Proof: we prove $\forall x \, \forall u \subset x \; u \not\approx x$ by induction.

It is enough to verify the induction step.
Let $u \subset x \cup \{y\}$ for $y \notin x$.

Suppose first $u \subset x$ and let f be a bijection of $x \cup \{y\}$ onto u. Then $f \setminus \{\langle y, f(y) \rangle\}$ is a bijection of x onto $u \setminus f(y) \subseteq u \subset x$. This contradicts the assumption $w \subset x \to w \not\approx x$.

Now suppose $y \in u$. Again let f be a bijection of $x \cup \{y\}$ onto u. We have $f(y) = w \in u$ and also $f(v) = y$ for some $v \in x$. Then $f \setminus \{\langle y, w \rangle\} \cup \{\langle v, w \rangle\}$ is a bijection of x onto u, again this contradicts assumption. QED

[Vopěnka: Mathematics in the AST, p.24]

Finiteness (4)

A set x is Tarski finite ($\mathrm{Fin}^{\mathrm{T}}$) provided that every subset of $P(x)$ has a maximal element w.r.t. inclusion.

Claim [$\mathrm{ZF}_{\mathbf{fin}}$]**:** every set has a minimal/maximal element w.r.t. inclusion.

Proof by induction, cf. [Vopěnka. p.24]

Vopěnka's book uses a stronger version of regularity (cf. p. 25) than ours. (More on that below.)
However the proofs on this and previous slide do not use regularity.

NB. Instead of ¬(infinity), one can postulate that every set is Tarski finite.

See [Běhounek] for the nuances.

Ordinal numbers in $\mathrm{ZF}_{\mathrm{fin}}$

A set is an ordinal number iff is transitive and totally ordered by $\in$.

Let On denote the class of ordinals.
Then On is transitive and totally ordered by $\in$.

As usual, $\emptyset \in \mathrm{On}$ and $\emptyset$ has no predecessor;
$\alpha \in \mathrm{On}$ implies $\alpha \cup \{\alpha\} \in \mathrm{On}$ and the latter is immediate successor of α within On.

Claim: For $\alpha \neq \emptyset$ there is a β s.t. $\alpha = \beta \cup \{\beta\}$.

Proof by induction. [Vopěnka, Mathematics AST, p. 59]

Observe On is not a set.

Claim: for each set x there is a unique $\alpha \in \mathrm{On}$ s.t. $x \approx \alpha$.

Proof of existence by induction, unicity from Dedekind finiteness.

NB. Existence of bijection of x to a finite ordinal is yet another way of defining finiteness of x — $\mathrm{Fin}^{\mathrm{f}}(x)$ in [Baratella–Ferro, p. 5].

Equip On with ordinal addition and multiplication (defined as usual).
This yiels a structure that satisfies all axioms of PA.

(WO) Every set can be well-ordered.
a.k.a. "Zermelo's theorem" within context of ZF.

$\leq$ is well order on x iff $\leq$ is a total order on x and
each non-empty subset of x has a least element w.r.t. $\leq$.

[Vopěnka, Math. AST, p.32] proves the following: if x totally ordered by $\leq$ and $x' \subseteq x$ is nonempty, then x' has a least and a greatest element w.r.t. $\leq$. Proved by induction.

This means that each $\alpha \in \mathrm{On}$ is well ordered by "$x \in y \vee x = y$".
Then use the above claim $\forall x\, \exists! \alpha \in \mathrm{On}\, x \approx \alpha$.

[Baratella, Ferro, p. 7] formulate the axiom as follows:

(WO) $\forall x\, \exists \alpha \in \mathrm{On}\, (x \approx \alpha)$

(having established well-order on each ordinal). The axiom is formulated in a context beyond $\mathrm{ZF}_{\mathrm{fin}}$ and eventually proved as a theorem in $\mathrm{ZF}_{\mathrm{fin}}$.

More axioms

(regularity') $\exists x\, \varphi(x) \to \exists x(\varphi(x) \,\&\, \forall y \in x\, \neg\varphi(y))$
(φ any formula)
[Vopěnka] and [Sochor] call this axiom simply (regularity).
In particular, Sochor mentions a "regularity in a form which is strong enough" in the context of finite sets [Sochor, Meta-AST I, p. 699]. More on this below.

($\in$-induction) $\forall x(\forall y \in x\, \varphi(y) \to \varphi(x)) \to \forall x\, \varphi(x)$
(φ any formula)

(TC) $\forall x\, \exists y\, (x \subseteq y \,\&\, \mathrm{Trans}(\mathrm{y}))$ "transitive closure"

NB. The axiom states that each set is contained in a transitive set.
This is equivalent to having $\mathrm{TC}(x)$ as a (provably total in $\mathrm{ZF_{fin}}$) function where $y = \mathrm{TC}(x)$ is defined as

$$x \subseteq y \,\&\, \mathrm{Trans}(y) \,\&\, \forall y'(x \subseteq y' \,\&\, \mathrm{Trans}(y') \to y \subseteq y')$$

over a weak fragment of $\mathrm{ZF_{fin}}$. Cf. [Kaye, Wong, p. 501] for details.

A theorem on TCs

Claim [ZFC]**:** $\mathrm{ZF}_{\mathrm{fin}} \cup \{\neg(\mathrm{TC})\}$ is consistent.

Proof sketch:

A. Let f be a (definable) bijection on V_ω.
Define a new membership relation $\in^f$ on V_ω as follows:

$$x \in^f y \ \text{ iff } \ x \in f(y).$$

Claim 1: $\langle V_\omega, \in^f \rangle$ is a model of $\mathrm{ZF}_{\mathrm{fin}} \setminus \{(\text{regularity})\}$.

B. Define f as follows.
Let $\omega^\star$ be the set $\{\{n+1\} \mid n \in \omega\} = \{\{x \cup \{x\}\} \mid x \in \omega\}$.

ω is $\{0, 1, 2, 3, 4, \dots\}$
$\omega^\star$ is $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\} \dots\}$. Clearly $\omega \cap \omega^\star = \emptyset$.

Let $f(n) = \{n+1\}$ and $f(\{n+1\}) = n$ and $f(a) = a$ for $a \notin \omega \cup \omega^\star$.
Clearly f is a bijection.

Claim 2: $\langle V_\omega, \in^f \rangle \models (\text{regularity})$. (!)

We have $n + 1 \in^f n$ since $n + 1 \in \{n+1\} = f(n)$.
In particular, the set $\mathrm{TC}^f(\emptyset)$ is clearly not in V_ω.

[A. Mancini, D. Zambella: A Note on Recursive Models of Set Theories. *Notre Dame J. Formal Logic* 42(2), 2001, p. 112]. See also [Baratella, Ferro, Thm. 5.5].

T. f. a. e.:

$\exists x\, \varphi(x) \to \exists x(\varphi(x) \,\&\, \forall y \in x\, \neg\varphi(y))$

$\neg\exists x(\varphi(x) \,\&\, \forall y \in x\, \neg\varphi(y)) \to \neg\exists x\, \varphi(x)$

$\forall x\, \neg(\varphi(x) \,\&\, \forall y \in x\, \neg\varphi(y)) \to \forall x\, \neg\varphi(x)$

$\forall x(\varphi(x) \to \neg\forall y \in x\, \neg\varphi(y)) \to \forall x\, \neg\varphi(x)$

$\forall x(\forall y \in x\, \neg\varphi(y) \to \neg\varphi(x)) \to \forall x\, \neg\varphi(x)$, which is $\in$-induction for $\neg\varphi$.

NB. No properties of $\in$ have been used.

(regularity) and (TC) as instances of (regularity')

We work in $\mathrm{ZF}_{\mathrm{fin}} \setminus \{(\mathrm{reg.})\}$.

A. Assume (regularity') , i.e., $\exists x\, \varphi(x) \to \exists x\, (\varphi(x)\, \&\, \forall y \in x\, \neg\varphi(y))$.

In particular, for a given z, we have
$\exists x\, (x \in z) \to \exists x\, (x \in z\, \&\, \forall y \in x\, \neg(y \in z))$.

B. Assume ($\in$-induction) $\forall x(\forall y \in x\, \varphi(y) \to \varphi(x)) \to \forall x\, \varphi(x)$.
(Already established to be equivalent to (regularity').)

Let x be given.
Assume $\forall y \in x\, \exists w\, (w = \mathrm{TC}(y))$.
Clearly the set $\bigcup\{\mathrm{TC}(y) \mid y \in x\}$ satisfies the requirements for $\mathrm{TC}(x)$,
i.e., $\exists w\, (w = \mathrm{TC}(x))$.

(regularity) and (TC) yield (regularity')

Let $\varphi(x)$ be arbitrary formula.

Let v be s.t. $\varphi(v)$ holds. Apply (TC) and get $\exists y\,(v \in y \,\&\, \mathrm{Trans}(y))$ (take $v \cup \mathrm{TC}(v)$).

Let $x = \{u \in y \mid \varphi(u)\}$, so in particular, $v \in x$.

Since $x \neq \emptyset$, we have $\exists q \in x\,(q \cap x = \emptyset)$ by (reg.)

Notice $\varphi(q)$ since $q \in x$.
But no element of q satisfies φ, since $q \cap x = \emptyset$.

By trans. of y, $q \subseteq y$, but $q \cap x = \emptyset$, therefore all elements of q are in $y \setminus x$.

The AST is a theory of sets and classes. Here we refer to its set fragment.

AST axioms for sets:

- (extensionality) $\forall xy(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)$;
- (existence of sets) $\exists x \forall y(y \notin x)$ and $\forall xy\, \exists z\, (z = x \cup \{y\})$;
- (induction) $\varphi(\emptyset)\, \&\, \forall xy(\varphi(x) \to \varphi(x \cup \{y\})) \to \forall x \varphi(x)$;
- (regularity') $\exists x\, \varphi(x) \to \exists x(\varphi(x)\, \&\, \forall y \in x \neg\varphi(y))$.

(φ a set formula).

The set fragment of the AST is equivalent to $\mathrm{ZF_{fin}} \cup \{\mathrm{TC}\}$.

In fact, (ZFC proves that) AST is conservative over its set fragment.

Stocktaking — on finite sets

Caution is advised whenever removing axioms.
— What does "$\mathrm{ZF}_{\mathrm{fin}}$" mean?

- Replacing (inf.) with $\neg$(inf.) impacts other axioms — such as (reg.);
- different versions of *finity* may engender different properties.

A first-order "theory of hereditarily finite sets" can, of course, have nonstandard models.