1.4 Pseudospektra matice a pole hodnot

(B)

idea: studujeme $\sigma(A+E)$ pro všechna E, $\|E\| \le \varepsilon$
pro dané ε ... jak se liší od $\sigma(A)$?
$\rightarrow$ pro $\varepsilon \to 0$ lze určit také studium citlivosti individuálních vl. č.

$\hat\lambda$ A-norm. ... $\varepsilon = \|E\|$ – poloměr

def: $A, E \in \mathbb{C}^{N\times N}$, $\hat A = A+E$, $\varepsilon \ge 0$. Pak
$\sigma_\varepsilon(A) \overset{(*)}{:=} \{\hat\lambda \in \mathbb{C} : \hat\lambda \in \sigma(A+E), \|E\| \le \varepsilon\}$ ... (kde spekt. norma)
... ε-pseudospektrum matice A.

pozn: lze def. pro lib. generovanou operátorovou normu

Věta: $A, E \in \mathbb{C}^{N\times N}$, $\hat A = A+E$, $\varepsilon \ge 0$. Pak násl. def. jsou ekvivalentní:

a) $(*)$

b) $\sigma_\varepsilon(A) := \{\hat\lambda \in \mathbb{C} : \exists x \in \mathbb{C}^N : \|x\| = 1, \|(A-\hat\lambda I)x\| \le \varepsilon\}$ (rezolventa)

c) $\sigma_\varepsilon(A) := \{\hat\lambda \in \mathbb{C} : \|(A-\hat\lambda I)^{-1}\| \ge \varepsilon^{-1}\}$, kde
pro $\hat\lambda \in \sigma(A)$: $\|(A-\hat\lambda I)^{-1}\| \overset{\text{def}}{=} +\infty$

d) $\sigma_\varepsilon(A) := \{\hat\lambda \in \mathbb{C} : \sigma_{min}(A-\hat\lambda I) \le \varepsilon\}$. (sing. číslo)

D: a$\Rightarrow$b: $(A+E)x = \hat\lambda x$, $\|x\| = 1 \Rightarrow$
(cvič) $\Rightarrow (A-\hat\lambda I)x = -Ex \Rightarrow \|(A-\hat\lambda I)x\| \le \|Ex\| \le \varepsilon$

b$\Rightarrow$c: BÚNO: $\hat\lambda \notin \sigma(A) \Rightarrow \exists (A-\hat\lambda I)^{-1}$

$(A-\hat\lambda I)x = w$, $\|x\| = 1$, $\|w\| \overset{b)}{\le} \varepsilon \Rightarrow$

$\Rightarrow (A-\hat\lambda I)^{-1} \cdot \frac{w}{\|w\|} = \frac{x}{\|w\|} \Rightarrow$

$\Rightarrow \|(A-\hat\lambda I)^{-1}\| \ge \frac{1}{\varepsilon}$

c$\Rightarrow$d: $\|(A-\hat\lambda I)^{-1}\| = \frac{1}{\sigma_{min}(A-\hat\lambda I)} \overset{c}{\ge} \frac{1}{\varepsilon}$

d$\Rightarrow$a: $(A-\hat\lambda I) - \sigma_{min}, u_{min}, v_{min}$ ... sing. triplet $(A-\hat\lambda I)$ ... sing. tr.

$\Rightarrow \exists x : \|x\| = 1 : (A - \tilde{\lambda} I - \underbrace{\sigma_{min} u_{min} v_{min}^T}_{=:E}) x = 0$ 2

chci najít E tak, aby $\|E\| \leq \varepsilon$ a $\tilde{\lambda} \in \sigma(A+E)$

$$\left.\begin{aligned} &\bullet \ ((A+E) - \tilde{\lambda} I) x = 0 \\ &\quad (A+E) x = \tilde{\lambda} x \quad \Rightarrow \tilde{\lambda} \in \sigma(A+E) \\ &\bullet \ \|E\| = \sigma_{min} \overset{\text{předp.}}{\leq} \varepsilon \end{aligned}\right\} \Rightarrow \text{platí a)}$$

$\square$

<u>výpočet $\sigma_\varepsilon(A)$</u>: • vzorkování (vykreslení $\sigma(A+E)$ pro 100 náhodných matic A)

• odhad hranice $\sigma_\varepsilon(A)$, viz Trefethen, Higham, Wright

$\rightarrow$ $\boxed{\text{CV: Lab}}$! (Eigtool)

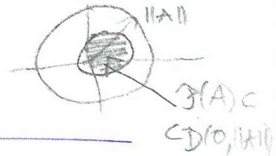
Pole hodnot a pseudospektrum – vlastnosti ③

def: $\mathcal{F}(A) = \{x^*Ax,\ \|x\|=1\} = \{\langle Ax, x\rangle,\ \|x\|=1\}$ ... pole hodnot
(numerical range, field of values)

$r(A) = \max\{|\lambda| : \lambda \in \mathcal{F}(A)\}$ ... num. poloměr

platí:
1, $\mathcal{F}(A)$ je konvexní
2, $sp(A) \subseteq \mathcal{F}(A)$
3, $\mathcal{F}(A) = \mathcal{F}(U^*AU)$, U – unitární
4, $\mathcal{F}(A^*) = \overline{\mathcal{F}(A)}$
5, $\mathcal{F}(\alpha A + \beta I) = \alpha \mathcal{F}(A) + \beta$

D: 1, 2, ... triv   4, 5, ... triv $\mathcal{F}(I) = \{1\}$
3, $x^* U^* A U x = (Ux)^* A Ux$, $\|x\| = \|Ux\| = 1$ □



platí: $\rho(A) \le r(A) \le \|A\| \le 2r(A)$
~~D. těžký~~   (*)   D. těžký

D(*): $r(A) = |y^*Ay| \le \|y\| \cdot \|A\| \cdot \|y\| = \|A\|$ □
$\|y\| = 1$

důsl: A – hermitovská, pak $\rho(A) = r(A) = \|A\|$

D: $\rho(A) = \lambda_{max}(A)$, $\|A\| = \sqrt{\lambda_{max}(A^*A)} = \sqrt{\lambda^2_{max}(A)} = \lambda_{max}(A)$ □

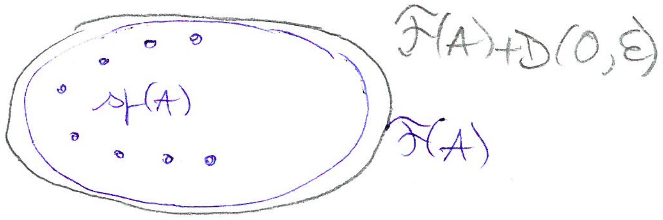
lemma: $sp_\varepsilon(A) \subseteq \mathcal{F}(A) + D(0,\varepsilon)$, kde
$D(0,\varepsilon) = \{z \in \mathbb{C} : |z| \le \varepsilon\}$

D: $\tilde\lambda \in sp_\varepsilon(A) \overset{a)}{\Longrightarrow} \exists E : \|E\| \le \varepsilon$ a $\tilde\lambda \in sp(A+E)$

$\Rightarrow \exists x \neq 0, \|x\| = 1 : (A+E)x = \tilde\lambda x \Rightarrow$

$$\Rightarrow \tilde{\lambda} = \underbrace{x^*(A + \cancel{\;})x}_{\in \mathcal{F}(A)} + \underbrace{x^* E x}_{\|\cdot\| \le \|E\| \le \varepsilon} \qquad \square$$

Př: A; $\tilde{A} = A + E$



$\Rightarrow$ $\mathcal{F}(A)$ nemusí dobře charakterizovat $sp(A)$ a $sp(\tilde{A})$ a zachytit citlivost vl. č.

Lemma: || $A \in \mathbb{C}^{N\times N}$ normální, pak $sp_\varepsilon(A) = \bigcup_{i=1}^{N} D_i(\lambda_i, \varepsilon)$

D: $sp_A(\tilde{A}) \le \overbrace{\kappa(X)}^{=1} \cdot \|E\| = \|E\| \le \varepsilon \Rightarrow$

$$\Rightarrow sp_A(\tilde{A}) \subseteq \bigcup_{i=1}^{N} D_i(\lambda_i, \varepsilon) \qquad \square$$

Př: (nabízí se): $A = 0$, $E = \varepsilon \Rightarrow \|E\| = \varepsilon$

$sp(A) = \{0\}$, $sp(A+E) = \{\varepsilon\}$

Lemma (podobn. transformace): (cvič.)

|| $A = \overset{\text{regul.}}{S} B S^{-1}$, pak $sp_\varepsilon(A) \subseteq sp_{\kappa(S)\cdot\varepsilon}(B)$

D: $\tilde{\lambda} \in sp_\varepsilon(A) \Rightarrow \frac{1}{\varepsilon} \le \|(\tilde{\lambda}\overset{SS^{-1}}{I} - \overset{SBS^{-1}}{A})^{-1}\| = \|S^{-1}(\tilde{\lambda}I - B)^{-1}S^{-1}\| \le$

$$\le \underbrace{\|S\| \cdot \|S\|}_{=\kappa(S)} \cdot \|(\tilde{\lambda}I - B)^{-1}\| \Rightarrow$$

$$\Rightarrow \|(\tilde{\lambda}I - B)^{-1}\| \ge \frac{1}{\kappa(S)\cdot\varepsilon} \Rightarrow \tilde{\lambda} \in sp_{\kappa(S)\cdot\varepsilon}(B) \qquad \square$$

POZOR NA $\kappa(S)$ PŘI TRANSFORMACI