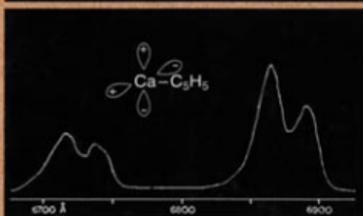
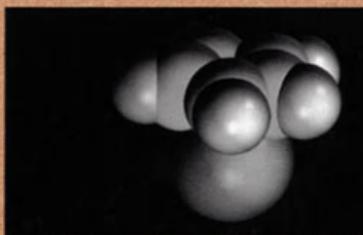


# Spectra of Atoms and Molecules

SECOND EDITION



Peter F. Bernath

## **Spectra of Atoms and Molecules**

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Spectra of Atoms and Molecules  
*Second Edition*

Peter F. Bernath

**OXFORD**  
UNIVERSITY PRESS

2005

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Published by Oxford University Press, Inc.  
198 Madison Avenue, New York, New York 10016

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Library of Congress Cataloging-in-Publication Data  
Bernath, Peter F.

Spectra of atoms and molecules / Peter F. Bernath.—2nd ed.  
p. cm.

Includes bibliographical references and index.

ISBN-13 978-0-19-517759-6

ISBN 0-19-517759-2

1. Spectrum analysis—Textbooks. 2. Atomic spectroscopy.

3. Molecular spectroscopy. I. Title.

QC451.B47 2005

535.8'4—dc22 2004062020

9 8 7 6 5 4 3 2 1

Printed in the United States of America  
on acid-free paper

*For Robin, Elizabeth, and Victoria*

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# Preface

In this second edition I have mainly updated and revised the material presented in the first edition. For example, the 1998 revision of the physical constants has been used throughout, and the use of symbols and units conforms more closely to recommended practice. The level of treatment and spirit of the book have not changed. I still aim to meet the needs of new students of spectroscopy regardless of their background. I have restrained myself and have not introduced spherical tensors, for example, because I believe that too many new concepts at one time are confusing.

A certain amount of new material has been added based on my recent experiences with what is misleadingly called “quantitative spectroscopy.” Spectroscopists are generally divided into two camps: those who interpret the spectral positions of lines and bands, and those who concern themselves more with line and band intensities. The latter camp is populated mainly by analytical chemists, but includes astronomers and atmospheric scientists as well.

Nothing in spectroscopy causes as much confusion as line intensities. Some of the problems seem to originate from the degeneracies inherent in atomic and molecular systems. The usual intensity formulas are derived (as in Chapter 1) for transitions between nondegenerate quantum *states*, while measurements are generally made on transitions between degenerate energy *levels*. The correct inclusion of this degeneracy turns out to be a nontrivial problem and is presented in Chapter 5 for atoms, but the expressions given there also apply to molecular systems. Even the definition of what constitutes a line can be a source of difficulties.

Line intensities are also confusing because of the dozens of different units used to report line and band strengths. The best procedure is to derive and cite all formulas in SI units, and then make any needed conversions to “customary” units in a second step. It is surprisingly difficult to locate line intensity formulas in SI units, with the appropriate degeneracies included. The line intensity formulas listed in this book should prove useful to the modern student.

Other than the addition of material pertaining to line intensities in Chapters 5 to 10, a major change in the second edition is in the discussion of the Raman effect and light scattering (Chapter 8). The standard theoretical treatment of light scattering and the Raman effect, as first presented by Placzek in the 1930s, has been added. Although Placzek’s approach is hardly light reading, the diligent student will find the derivations illuminating. A solid understanding of the classical and quantum mechanical theory of polarizability of molecules is indispensable in the area of nonlinear spectroscopy.

I am very grateful for the comments and helpful criticism from many people, particularly F. R. McCourt, R. J. Le Roy, C. Bissonette, K. Lehmann, A. Anderson, R. Shiell, and J. Hardwick. I also thank my fall 2004 graduate class in molecular spectroscopy

(M. Dick, D. Fu, S. Gunal, T. Peng, and S. Yu) for their comments and corrections. The figures for the second edition have been prepared by S. M. McLeod, T. Nguyen, Y. Bresler, and E. R. Bernath.

Finally, my wife Robin has made the second edition possible through her continuing encouragement and understanding. My special thanks to her.

*Ontario*  
*August 2004*

P.F.B.

# Preface to First Edition

This book is designed as a textbook to introduce advanced undergraduates and, particularly, new graduate students to the vast field of spectroscopy. It presumes that the student is familiar with the material in an undergraduate course in quantum mechanics. I have taken great care to review the relevant mathematics and quantum mechanics as needed throughout the book. Considerable detail is provided on the origin of spectroscopic principles. My goal is to demystify spectroscopy by showing the necessary steps in a derivation, as appropriate in a textbook.

The digital computer has permeated all of science including spectroscopy. The application of simple analytical formulas and the nonstatistical graphical treatment of data are long dead. Modern spectroscopy is based on the matrix approach to quantum mechanics. Real spectroscopic problems can be solved on the computer more easily if they are formulated in terms of matrix operations rather than differential equations. I have tried to convey the spirit of modern spectroscopy, through the extensive use of the language of matrices.

The infrared and electronic spectroscopy of polyatomic molecules makes extensive use of group theory. Rather than assume a previous exposure or try to summarize group theory in a short chapter, I have chosen to provide a more thorough introduction. My favorite book on group theory is the text by Bishop, *Group Theory and Chemistry*, and I largely follow his approach to the subject.

This book is not a monograph on spectroscopy, but it can be profitably read by physicists, chemists, astronomers, and engineers who need to become acquainted with the subject. Some topics in this book, such as parity, are not discussed well in any of the textbooks or monographs that I have encountered. I have tried to take particular care to address the elementary aspects of spectroscopy that students have found to be most confusing.

To the uninitiated, the subject of spectroscopy seems enshrouded in layers of bewildering and arbitrary notation. Spectroscopy has a long tradition so many of the symbols are rooted in history and are not likely to change. Ultimately all notation is arbitrary, although some notations are more helpful than others. One of the goals of this book is to introduce the language of spectroscopy to the new student of the subject. Although the student may not be happy with some aspects of spectroscopic notation, it is easier to adopt the notation than to try to change long-standing spectroscopic habits.

The principles of spectroscopy are timeless, but spectroscopic techniques are more transient. Rather than focus on the latest methods of recording spectra (which will be out of fashion tomorrow), I concentrate on the interpretation of the spectra themselves. This book attempts to answer the question: What information is encoded in the spectra of atoms and molecules?

A scientific subject cannot be mastered without solving problems. I have therefore

provided many spectroscopic problems at the end of each chapter. These problems have been acquired over the years from many people including M. Barfield, S. Kukolich, R. W. Field, and F. McCourt. In addition I have “borrowed” many problems either directly or with only small changes from many of the books listed as general references at the end of each chapter and from the books listed in Appendix D. I thank these people and apologize for not giving them more credit!

Spectroscopy needs spectra and diagrams to help interpret the spectra. Although the ultimate analysis of a spectrum may involve the fitting of line positions and intensities with a computer program, there is much qualitative information to be gained by the inspection of a spectrum. I have therefore provided many spectra and diagrams in this book. In addition to the specific figure acknowledgments at the end of the appendices, I would like to thank a very talented group of undergraduates for their efforts. J. Ogilvie, K. Walker, R. LeBlanc, A. Billyard, and J. Dietrich are responsible for the creation of most of the figures in this book.

I also would like to thank the many people who read drafts of the entire book or of various chapters. They include F. McCourt, M. Dulick, D. Klapstein, R. Le Roy, N. Isenor, D. Irish, M. Morse, C. Jarman, P. Colarusso, R. Bartholomew, and C. Zhao. Their comments and corrections were very helpful. Please contact me about other errors in the book and with any comments you would like to make. I thank Heather Hergott for an outstanding job typing the manuscript.

Finally, I thank my wife Robin for her encouragement and understanding. Without her this book would never have been written.

*Ontario*  
*January 1994*

P.F.B.

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# Spectra of Atoms and Molecules

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# Chapter 1

## Introduction

### 1.1 Waves, Particles, and Units

Spectroscopy is the study of the interaction of light with matter. To begin, a few words about light, matter, and the effect of light on matter are in order.

Light is an electromagnetic wave represented (for the purposes of this book) by the plane waves

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi_0) \quad (1.1)$$

or

$$\mathbf{E}(\mathbf{r}, t) = \text{Re} \left( \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi_0)} \right). \quad (1.2)$$

In this book vectors and matrices are written in bold Roman type, except in certain figures in which vectors are indicated with a half arrow (e.g.,  $\mathbf{E}$ ) for clarity. There is an electric field  $\mathbf{E}$  (in volts per meter) perpendicular to  $\mathbf{k}$  that propagates in the direction  $\mathbf{k}$  and has an angular frequency  $\omega = 2\pi\nu = 2\pi/T$ . The frequency  $\nu$  (in hertz) is the reciprocal of the period  $T$  (in seconds), that is,  $\nu = 1/T$ . The period  $T$  and the wavelength  $\lambda$  are defined in Figure 1.1 with  $\mathbf{k}$  in the  $z$  direction. The wavevector  $\mathbf{k}$  has a magnitude  $|\mathbf{k}| = k = 2\pi/\lambda$  and a direction given by the normal to the plane of constant phase.  $|\mathbf{E}_0|$  is the amplitude of the electric field, while  $\mathbf{k} \cdot \mathbf{r} - \omega t + \phi_0$  is the phase ( $\phi_0$  is an initial phase angle of arbitrary value).

The presence of a magnetic field, also oscillating at angular frequency  $\omega$  and orthogonal to both  $\mathbf{E}$  and  $\mathbf{k}$ , is ignored in this book. Other “complications” such as Maxwell’s equations, Gaussian laser beams, birefringence, and vector potentials are also not considered. These subjects, although part of spectroscopy in general, are discussed in books on optics, quantum optics, lasers, or electricity and magnetism.

Wavelength and frequency are related by the equation  $\lambda\nu = c$ , in which  $c$  is the speed of the electromagnetic wave. In vacuum  $c = c_0$ , but in general  $c = c_0/n$  with  $n$  as the index of refraction of the propagation medium. Since  $\nu$  has the same value in any medium, the wavelength also depends on the index of refraction of the medium. Thus since

$$\lambda\nu = c \quad (1.3)$$

we must have

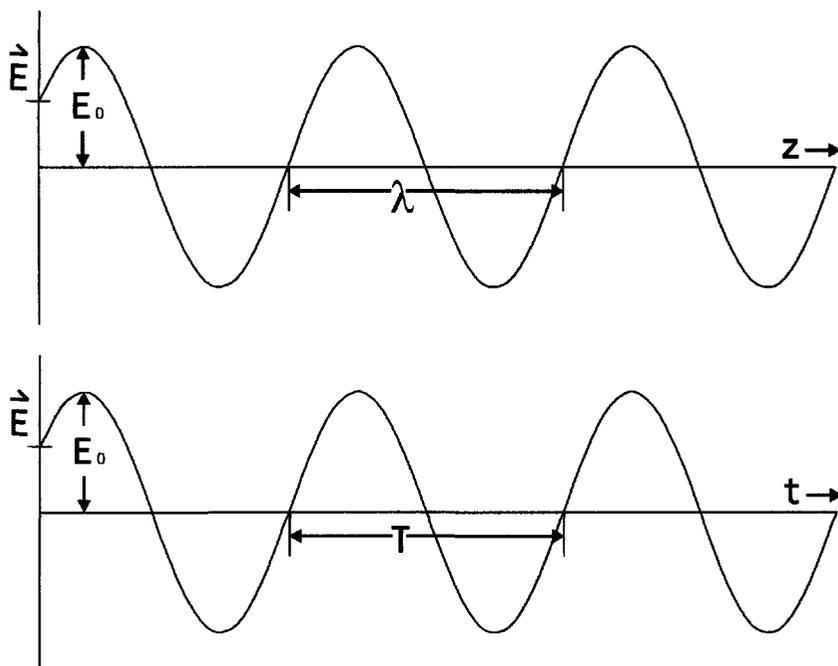


Figure 1.1: The electric field at  $t = 0$  as a function of  $z$  is plotted in the upper panel, while the lower panel is the corresponding plot at  $z = 0$  as a function of time.

$$\frac{\lambda_0}{n} \nu = \frac{c_0}{n}. \quad (1.4)$$

Historically, direct frequency measurements were not possible in the infrared and visible regions of the spectrum. It was therefore convenient to measure (and report)  $\lambda$  in air, correct for the refractive index of air to give  $\lambda_0$ , and then define  $\tilde{\nu} = 1/\lambda_0$ , with  $\lambda_0$  in cm. Before SI units were adopted, the centimeter was more widely used than the meter so that  $\tilde{\nu}$  represents the number of wavelengths in one centimeter in vacuum and as a consequence  $\tilde{\nu}$  is called the wavenumber. The units of the wavenumber are  $\text{cm}^{-1}$  (reciprocal centimeters) but common usage also calls the “ $\text{cm}^{-1}$ ” the “wavenumber.” Fortunately, the SI unit for  $\tilde{\nu}$ , the  $\text{m}^{-1}$ , is almost never used so that this sloppy, but standard, practice causes no confusion.

The oscillating electric field is a function of both spatial ( $\mathbf{r}$ ) and temporal ( $t$ ) variables. If the direction of propagation of the electromagnetic wave is along the  $z$ -axis and the wave is examined at one instant of time, say  $t = 0$ , then for  $\phi_0 = 0$ ,

$$\mathbf{E} = \mathbf{E}_0 \cos(kz) = \mathbf{E}_0 \cos\frac{2\pi z}{\lambda}. \quad (1.5)$$

Alternatively, the wave can be observed at a single point, say  $z = 0$ , as a function of time

$$\mathbf{E} = \mathbf{E}_0 \cos(\omega t) = \mathbf{E}_0 \cos(2\pi \nu t). \quad (1.6)$$

Both equations (1.5) and (1.6) are plotted in Figure 1.1 with arbitrary initial phases.

In contrast to longitudinal waves, such as sound waves, electromagnetic waves are transverse waves. If the wave propagates in the  $z$  direction, then there are two possible independent transverse directions,  $x$  and  $y$ . This leads to the polarization of light, since  $\mathbf{E}$  could lie either along  $x$  or along  $y$ , or more generally, it could lie anywhere in the  $xy$ -plane. Therefore we may write

$$\mathbf{E} = E_x \hat{\mathbf{i}} + E_y \hat{\mathbf{j}}, \quad (1.7)$$

with  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$  representing unit vectors lying along the  $x$ - and  $y$ -axes.

The wave nature of light became firmly established in the nineteenth century, but by the beginning of the twentieth century, light was also found to have a particle aspect. The wave-particle duality of electromagnetic radiation is difficult to visualize since there are no classical, macroscopic analogs. In the microscopic world, electromagnetic waves seem to guide photons (particles) of a definite energy  $E$  and momentum  $p$  with

$$E = h\nu = \hbar\omega = \frac{hc}{\lambda} = 10^2 hc\tilde{\nu} \quad (1.8)$$

and

$$p = \frac{h}{\lambda} = \hbar k. \quad (1.9)$$

The factor of  $10^2$  in equation (1.8) comes from the conversion of  $\text{cm}^{-1}$  for  $\tilde{\nu}$  into  $\text{m}^{-1}$ .

In 1924 it occurred to de Broglie that if electromagnetic waves could display properties associated with particles, then perhaps particles could also display wavelike properties. Using equation (1.9), he postulated that a particle should have a wavelength,

$$\lambda = \frac{h}{p} = \frac{h}{mv}. \quad (1.10)$$

This prediction of de Broglie was verified in 1927 by Davisson and Germer's observation of an electron beam diffracted by a nickel crystal.

In this book SI units and expressions are used as much as possible, with the traditional spectroscopic exceptions of the ångström ( $\text{\AA}$ ) and the wavenumber  $\tilde{\nu}$  ( $\text{cm}^{-1}$ ). The symbols and units used will largely follow the International Union of Pure and Applied Chemistry (IUPAC) recommendations of the "Green Book" by I. M. Mills *et al.*<sup>1</sup> The fundamental physical constants, as supplied in Appendix A, are the 1998 Mohr and Taylor<sup>2</sup> values. Notice that the speed of light in vacuum ( $c_0$ ) is fixed exactly at 299 792 458 m/s. The atomic masses used are the 2003 Audi, Wapstra, and Thibault<sup>3</sup> values and atomic mass units have the recommended symbol, u.<sup>1</sup>

## 1.2 The Electromagnetic Spectrum

There are traditional names associated with the various regions of the electromagnetic spectrum. The radio frequency region (3 MHz–3 GHz) has photons of sufficient energy to flip nuclear spins (nuclear magnetic resonance (NMR)) in magnetic fields of a few tesla. In the microwave region (3 GHz–3 000 GHz) energies correspond to rotational transitions in molecules and to electron spin flips (electron spin resonance (ESR)). Unlike all the spectra discussed in this book, NMR and ESR transitions are induced by

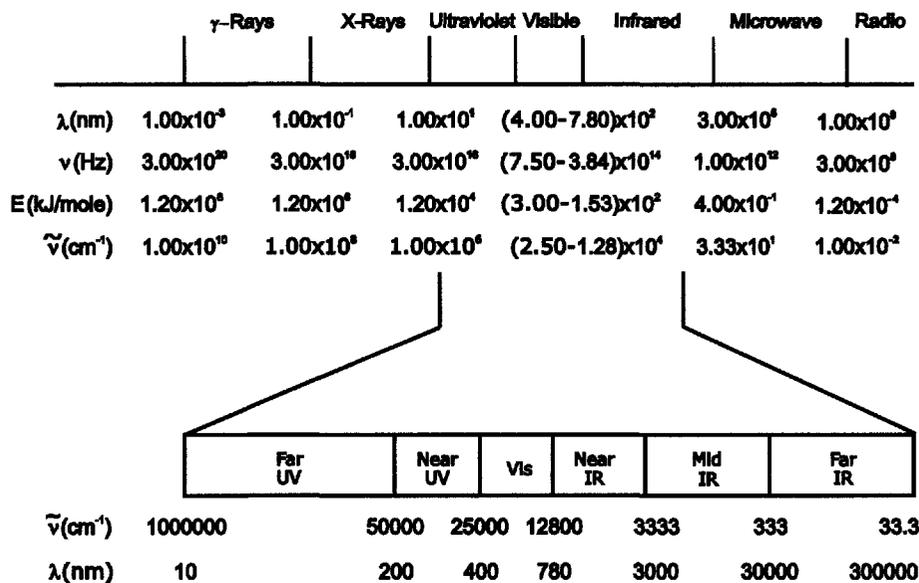


Figure 1.2: The electromagnetic spectrum.

the oscillating magnetic field of the electromagnetic radiation. Infrared quanta ( $100 \text{ cm}^{-1}$ – $13\,000 \text{ cm}^{-1}$ ) excite the vibrational motion in matter. Visible and ultraviolet (UV) transitions ( $10\,000 \text{ \AA}$ – $100 \text{ \AA}$ ) involve valence electron rearrangements in molecules ( $1 \text{ nm} = 10 \text{ \AA}$ ). Core electronic transitions are promoted at x-ray wavelengths ( $100 \text{ \AA}$ – $0.1 \text{ \AA}$ ). Finally, below  $0.1 \text{ \AA}$  in wavelength,  $\gamma$ -rays are associated with nuclear processes. Chemists customarily use the units of MHz or GHz for radio and microwave radiation,  $\text{cm}^{-1}$  for infrared radiation, and nm or  $\text{\AA}$  for visible, UV, and x-ray radiation (Figure 1.2). These customary units are units of frequency (MHz), reciprocal wavelength ( $\text{cm}^{-1}$ ), and wavelength ( $\text{\AA}$ ).

It is worth noting that the different regions of the spectrum do not possess sharp borders and that the type of molecular motion associated with spectroscopy in each region is only approximate. For example, overtone vibrational absorption can be found in the visible region of the spectrum (causing the blue color of the oceans). Infrared electronic transitions are also not rare, for example, the Ballik–Ramsay electronic transition of  $\text{C}_2$ .

A further subdivision of the infrared, visible, and ultraviolet regions of the spectrum is customary. The infrared region is divided into the far-infrared ( $33$ – $333 \text{ cm}^{-1}$ ), mid-infrared ( $333$ – $3333 \text{ cm}^{-1}$ ), and near-infrared ( $3333$ – $13\,000 \text{ cm}^{-1}$ ) regions. In the far-infrared region are found rotational transitions of light molecules, phonons of solids, and metal-ligand vibrations, as well as ring-puckering and torsional motions of many organic molecules. The mid-infrared is the traditional infrared region in which the fundamental vibrations of most molecules lie. The near-infrared region is associated with overtone vibrations and a few electronic transitions. The visible region is divided into the colors of the rainbow from the red limit at about  $7800 \text{ \AA}$  to the violet at  $4000 \text{ \AA}$ . The near-ultraviolet region covers  $4000 \text{ \AA}$ – $2000 \text{ \AA}$ , while the vacuum ultraviolet region is  $2000 \text{ \AA}$ – $100 \text{ \AA}$ . The vacuum ultraviolet region is so named because air is opaque to wavelengths below  $2000 \text{ \AA}$ , so that only evacuated instruments can be used when spectra are taken in this region.

It is a spectroscopic custom to report all infrared, visible, and near-ultraviolet wavelengths as air wavelengths ( $\lambda$ ), rather than as vacuum wavelengths ( $\lambda_0$ ). Of course, below 2000 Å all wavelengths are vacuum wavelengths since measurements in air are not possible. The wavenumber is related to energy,  $E = 10^2 hc\tilde{\nu}$ , and is the reciprocal of the vacuum wavelength in centimeters,  $\tilde{\nu} = 1/\lambda_0$ , but in air  $\tilde{\nu} = 1/\lambda = 1/n\lambda_0$ . For accurate work, it is necessary to correct for the refractive index of air. This can be seen, for example, by considering dry air at 15°C and 760 Torr for which  $n = 1.0002781$  at 5000 Å.<sup>4</sup> Thus  $\lambda = 5000.000$  Å in air corresponds to  $\lambda_0 = 5001.391$  Å in vacuum and  $\tilde{\nu} = 19994.44$  cm<sup>-1</sup> rather than 20000 cm<sup>-1</sup>!

## 1.3 Interaction of Radiation with Matter

### *Blackbody Radiation*

The spectrum of the radiation emitted by a blackbody is important both for historical reasons and for practical applications. Consider a cavity (Figure 1.3) in a material that is maintained at constant temperature  $T$ . The emission of radiation from the cavity walls is in equilibrium with the radiation that is absorbed by the walls. It is convenient to define a radiation density  $\rho$  (with units of joules/m<sup>3</sup>) inside the cavity. The frequency distribution of this radiation is represented by the function  $\rho_\nu$ , which is the radiation density in the frequency interval between  $\nu$  and  $\nu + d\nu$  (Figure 1.4), and is defined so that

$$\rho = \int_0^\infty \rho_\nu d\nu. \quad (1.11)$$

Therefore, the energy density function  $\rho_\nu$  has units of joule-seconds per cubic meter (J s m<sup>-3</sup>). The distribution function characterizing the intensity of the radiation emitted from the hole is labeled  $I_\nu$  (units of watt-seconds per square meter of the hole). In the radiometric literature the quantity  $I = \int I_\nu d\nu$  (W m<sup>-2</sup>) would be called the radiant excittance and  $I_\nu$  (W s m<sup>-2</sup>) would be the spectral radiant excittance.<sup>5</sup> The recommended radiometric symbol for excittance is  $M$ , which is not used here because of possible confusion with the symbol for dipole moment. The functions  $\rho_\nu$  and  $I_\nu$  are universal functions depending only upon the temperature and frequency, and are independent of the shape or size of the cavity and of the material of construction as long as the hole is small.

Planck obtained the universal function,

$$\rho_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}, \quad (1.12)$$

named in his honor. The symbol  $k = 1.3806503 \times 10^{-23}$  J K<sup>-1</sup> (Appendix A) in equation (1.12) represents the Boltzmann constant. Geometrical considerations (Problem 13) then give the relationship between  $I_\nu$  and  $\rho_\nu$  as

$$I_\nu = \rho_\nu \frac{c}{4}. \quad (1.13)$$

Figure 1.5. shows  $\rho_\nu$  as a function of  $\nu$  and the dependence on the temperature  $T$ .

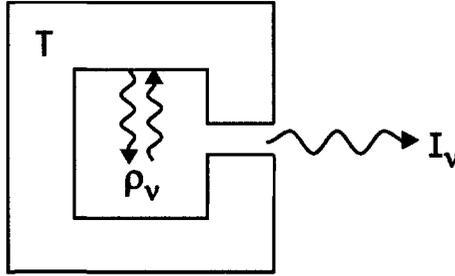


Figure 1.3: Cross section of a blackbody cavity at a temperature  $T$  with a radiation density  $\rho_\nu$  emitting radiation with intensity  $I_\nu$  from a small hole.

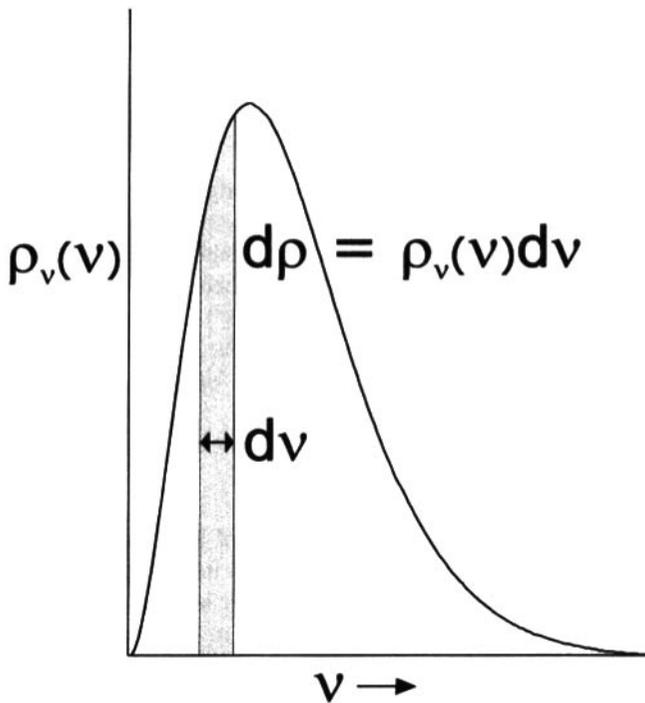


Figure 1.4: The Planck function  $\rho_\nu(\nu)$  is a distribution function defined by  $d\rho/d\nu = \rho_\nu(\nu)$  or  $\rho = \int \rho_\nu d\nu$ .

### ***Einstein A and B Coefficients***

Consider a collection of  $N$  two-level systems (Figure 1.6) in a volume of  $1 \text{ m}^3$  with upper energy  $E_1$  and lower energy  $E_0$ , all at a constant temperature  $T$  and bathed by the radiation density  $\rho_\nu(T)$ . Since the entire collection is in thermal equilibrium, if the number of systems with energy  $E_1$  is  $N_1$  and the number of systems with energy  $E_0$  is  $N_0$ , then the populations  $N_1$  and  $N_0$  ( $N = N_1 + N_0$ ) are necessarily related by

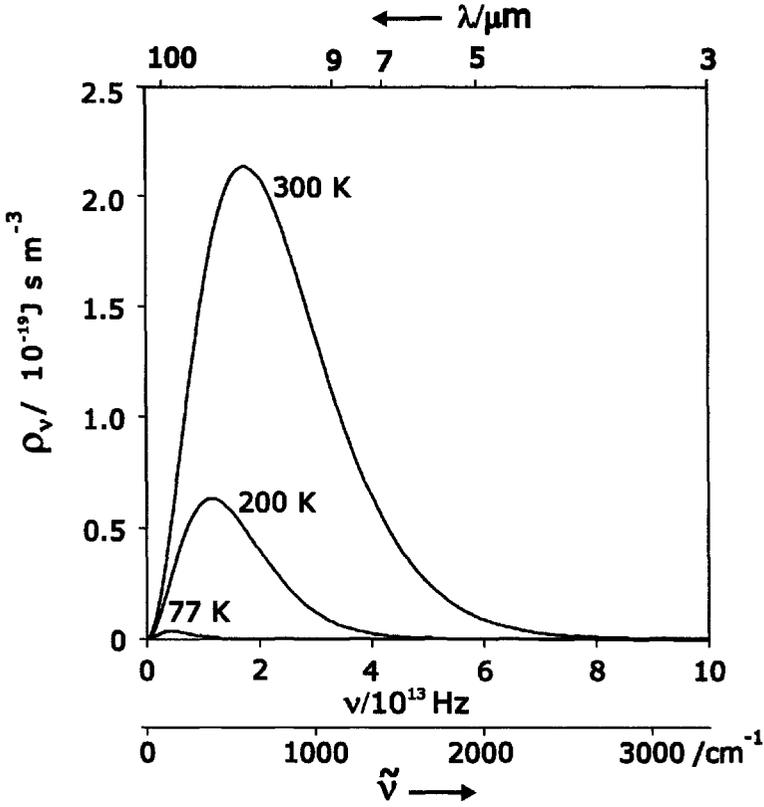


Figure 1.5: The Planck function at 77 K, 200 K, and 300 K.

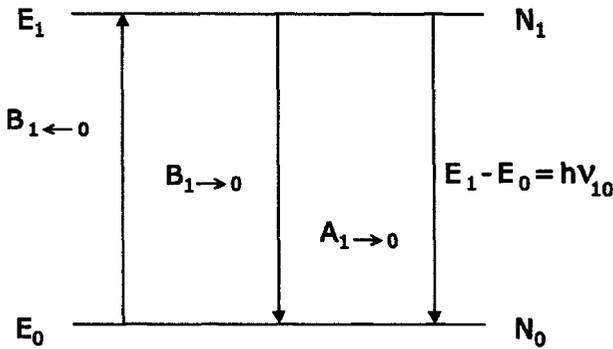


Figure 1.6: A two-level system.

$$\frac{N_1}{N_0} = e^{-h\nu_{10}/kT}, \tag{1.14}$$

in which  $h\nu_{10} = E_1 - E_0$ . This is the well-known Boltzmann expression for thermal equilibrium between nondegenerate levels.

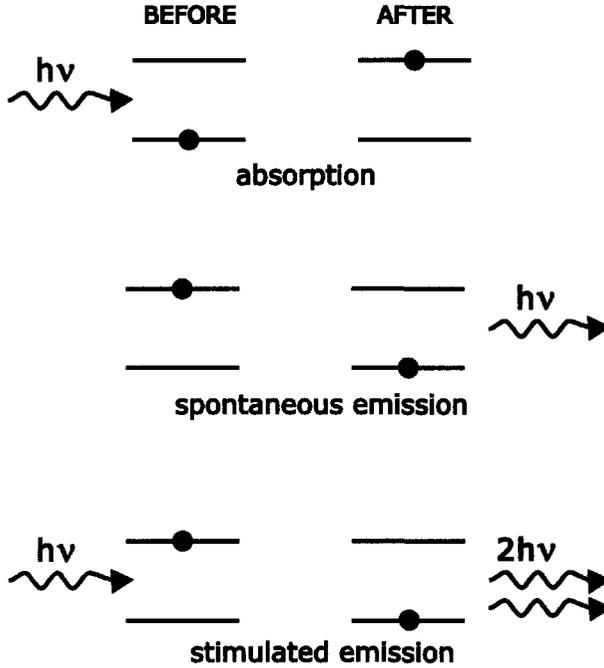


Figure 1.7: Schematic representations of absorption (top), spontaneous emission (middle) and stimulated emission (bottom) processes in a two-level system.

There are three possible processes that can change the state of the system from  $E_0$  to  $E_1$  or from  $E_1$  to  $E_0$ : absorption, spontaneous emission, and stimulated emission (Figure 1.7). Absorption results from the presence of a radiation density  $\rho_\nu(\nu_{10})$  of the precise frequency needed to drive a transition from the ground state to the excited state at the rate

$$\frac{dN_1}{dt} = B_{1\leftarrow 0}\rho_\nu(\nu_{10})N_0. \quad (1.15)$$

The coefficient  $B_{1\leftarrow 0}$  is thus a “rate constant” and is known as the Einstein absorption coefficient or Einstein  $B$  coefficient. Similarly if the system is already in an excited state, then a photon of energy  $h\nu_{10}$  (provided by  $\rho_\nu$ ) can induce the system to make the transition to the ground state. The rate for stimulated emission is given by

$$\frac{dN_1}{dt} = -B_{1\rightarrow 0}\rho_\nu(\nu_{10})N_1, \quad (1.16)$$

in which  $B_{1\rightarrow 0}$  is the stimulated emission coefficient. Finally the system in the excited state can spontaneously emit a photon at a rate

$$\frac{dN_1}{dt} = -A_{1\rightarrow 0}N_1. \quad (1.17)$$

Since the system is at equilibrium, the rate of population of the excited state by absorption must balance the rate of depopulation by stimulated and spontaneous emission, so that

$$N_0 B_{1\leftarrow 0} \rho_\nu = A_{1\rightarrow 0} N_1 + B_{1\rightarrow 0} \rho_\nu N_1 \quad (1.18)$$

and hence using (1.14)

$$\frac{N_1}{N_0} = \frac{B_{1\leftarrow 0} \rho_\nu}{A_{1\rightarrow 0} + B_{1\rightarrow 0} \rho_\nu} = e^{-h\nu_{10}/kT}. \quad (1.19)$$

Solving for  $\rho_\nu$  in equation (1.19) then yields

$$\rho_\nu(\nu_{10}) = \frac{A_{1\rightarrow 0}}{B_{1\leftarrow 0} e^{h\nu_{10}/kT} - B_{1\rightarrow 0}}. \quad (1.20)$$

However,  $\rho_\nu(\nu_{10})$  is also given by the Planck function (1.12)

$$\rho_\nu(\nu_{10}) = \frac{8\pi h\nu_{10}^3}{c^3} \frac{1}{e^{h\nu_{10}/kT} - 1}.$$

For expressions (1.12) and (1.20) both to be valid, it is necessary that

$$B_{1\leftarrow 0} = B_{1\rightarrow 0} \quad (1.21)$$

and that

$$A_{1\rightarrow 0} = \frac{8\pi h\nu_{10}^3}{c^3} B_{1\leftarrow 0}. \quad (1.22)$$

Remarkably, the rate constants for absorption and stimulated emission—two apparently different physical processes—are identical (1.21). Moreover, the spontaneous emission rate (lifetime) can be determined from the absorption coefficient (1.22). Note, however, the  $\nu_{10}^3$  factor in (1.22), which plays an important role in the competition between the induced and spontaneous emission processes.

### *Absorption and Emission of Radiation*

The interaction of electromagnetic radiation with matter can be described by a simple semiclassical model. In the semiclassical treatment the energy levels of molecules are obtained by solution of the time-independent Schrödinger equation

$$\hat{H}\psi_n = E_n\psi_n,$$

while the electromagnetic radiation is treated classically. Consider a two-level system described by lower and upper state wavefunctions,  $\psi_0$  and  $\psi_1$  (Figure 1.8), respectively.

Electromagnetic radiation that fulfills the Bohr condition,  $E_1 - E_0 = h\nu = \hbar\omega$ , is applied to the system in order to induce a transition from the lower energy state at  $E_0$  to the upper energy state at  $E_1$ . The molecule consists of nuclei and electrons at positions  $\mathbf{r}_i$  possessing charges  $q_i$ . The system as a whole thus has a net dipole moment  $\boldsymbol{\mu}$  with Cartesian components

$$\mu_x = \sum x_i q_i, \quad (1.23)$$

$$\mu_y = \sum y_i q_i, \quad (1.24)$$

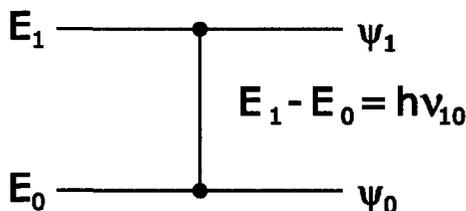
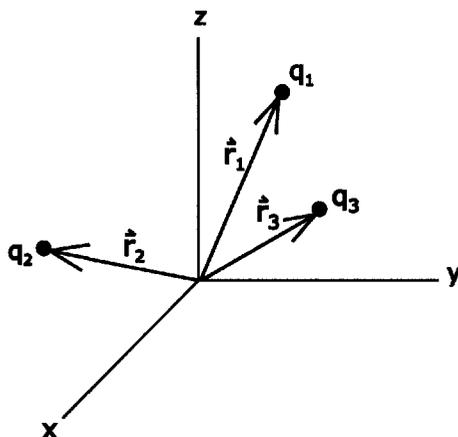


Figure 1.8: Two-level system.

Figure 1.9: Three particles with charges  $q_1$ ,  $q_2$ , and  $q_3$  located at positions  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ , and  $\mathbf{r}_3$ .

$$\mu_z = \sum z_i q_i, \quad (1.25)$$

where  $(x_i, y_i, z_i)$  give the coordinates of particle  $i$  relative to the center of mass of the molecule (Figure 1.9).

The interaction of the radiation with the material system is taken into account by the addition of the time-dependent perturbation (see Chapter 4 and the first section of this chapter for definitions),

$$\begin{aligned} \hat{H}'(t) &= -\boldsymbol{\mu} \cdot \mathbf{E}(t) \\ &= -\boldsymbol{\mu} \cdot \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t). \end{aligned} \quad (1.26)$$

If the oscillating electric field is in the  $z$  direction ( $\mathbf{E}_0 = (0, 0, E_{0z})$ ) and the system is at the origin,  $\mathbf{r} = 0$  (the wavelength  $\lambda$  is greater than the dimensions of the system to avoid having different electric field strengths at different parts of the molecule), then

$$\hat{H}'(t) = -\mu_z E_{0z} \cos(\omega t) \quad (1.27)$$

$$= -\boldsymbol{\mu} \mathbf{E} \cos(\omega t). \quad (1.28)$$

The transition probability is obtained by solving the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = (\hat{H} + \hat{H}'(t))\Psi. \quad (1.29)$$

In the absence of  $\hat{H}'$ , the two time-dependent solutions of equation (1.29) are

$$\begin{aligned} \Psi_0(t) &= \psi_0 e^{-iE_0 t/\hbar} = \psi_0 e^{-i\omega_0 t} \\ \Psi_1(t) &= \psi_1 e^{-iE_1 t/\hbar} = \psi_1 e^{-i\omega_1 t}, \end{aligned}$$

with  $\omega_i = E_i/\hbar$ . An uppercase Greek  $\Psi$  is used for a time-dependent wavefunction, while a lowercase  $\psi$  represents a time-independent wavefunction.

The wavefunction for the perturbed two-level system is given by the linear combination of the complete set of functions  $\Psi_0$  and  $\Psi_1$ :

$$\Psi(t) = a_0(t)\psi_0 e^{-iE_0 t/\hbar} + a_1(t)\psi_1 e^{-iE_1 t/\hbar} = a_0\psi_0 e^{-i\omega_0 t} + a_1\psi_1 e^{-i\omega_1 t}, \quad (1.30)$$

where  $a_0$  and  $a_1$  are time-dependent coefficients. Substitution of the solution (1.30) into the time-dependent Schrödinger equation (1.29) leads to the equation

$$i\hbar(\dot{a}_0\psi_0 e^{-i\omega_0 t} + \dot{a}_1\psi_1 e^{-i\omega_1 t}) = \hat{H}'a_0\psi_0 e^{-i\omega_0 t} + \hat{H}'a_1\psi_1 e^{-i\omega_1 t}, \quad (1.31)$$

where the dot notation  $\dot{a}_0 = da_0/dt$  is used to indicate derivatives with respect to time.

Multiplication by  $\psi_0^* e^{i\omega_0 t}$ , or  $\psi_1^* e^{i\omega_1 t}$ , followed by integration over all space then gives two coupled differential equations

$$i\hbar\dot{a}_0 = a_0\langle\psi_0|\hat{H}'|\psi_0\rangle + a_1\langle\psi_0|\hat{H}'|\psi_1\rangle e^{-i\omega_{10}t} \quad (1.32a)$$

$$i\hbar\dot{a}_1 = a_0\langle\psi_1|\hat{H}'|\psi_0\rangle e^{i\omega_{10}t} + a_1\langle\psi_1|\hat{H}'|\psi_1\rangle \quad (1.32b)$$

using the Dirac bracket notation  $\langle f_1|\hat{A}|f_3\rangle = \int f_1^* \hat{A} f_3 d\tau$ . At this stage no approximations have been made (other than the restriction to the two states  $\psi_1$  and  $\psi_0$ ), and the two equations (1.32) are entirely equivalent to the original Schrödinger equation. Now if  $\hat{H}'$  is taken as  $-\mu E \cos(\omega t)$  in the electric-dipole approximation, then the time-dependent perturbation  $\hat{H}'$  has odd parity (i.e., is an odd function of the spatial coordinates, see also Chapters 5 and 9). In other words  $\hat{H}'$  is an odd function since  $\mu = -ez$ , while the products  $|\psi_1|^2$  or  $|\psi_0|^2$  are even functions so that the integrands  $\psi_1^* \hat{H}' \psi_1$  and  $\psi_0^* \hat{H}' \psi_0$  are also odd functions. All atomic and molecular states that have definite parities (either even or odd) with respect to inversion in the space-fixed coordinate system (see Chapters 5 and 9 for further details) have  $\langle\psi_0|\hat{H}'|\psi_0\rangle = \langle\psi_1|\hat{H}'|\psi_1\rangle = 0$ , and equations (1.32) reduce to

$$i\hbar\dot{a}_0 = -a_1 M_{01} E e^{-i\omega_{10}t} \cos \omega t, \quad (1.33a)$$

$$i\hbar\dot{a}_1 = -a_0 M_{01} E e^{i\omega_{10}t} \cos \omega t. \quad (1.33b)$$

The integral  $M_{01} = M_{10} = \langle\psi_1|\mu|\psi_0\rangle$  is the transition dipole moment and is the most critical factor in determining selection rules and line intensities. In general  $\mathbf{M}_{10}$  is a vector quantity and the symbol  $\boldsymbol{\mu}_{10}$  ( $\equiv \mathbf{M}_{10}$ ) is often used. It is convenient to define

$$\omega_R = \frac{M_{10}E}{\hbar}, \quad (1.34)$$

which is known as the *Rabi frequency*, and to use the identity

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

to rewrite equations (1.33a) and (1.33b) as

$$\dot{a}_0 = \frac{ia_1\omega_R(e^{-i(\omega_{10}-\omega)t} + e^{-i(\omega_{10}+\omega)t})}{2} \quad (1.35a)$$

$$\dot{a}_1 = \frac{ia_0\omega_R(e^{i(\omega_{10}-\omega)t} + e^{i(\omega_{10}+\omega)t})}{2}. \quad (1.35b)$$

The physical meaning of the Rabi frequency will become clear later in this section.

At this stage an approximation can be made by noting that  $\omega_{10} \approx \omega$  since the system with Bohr angular frequency  $(E_1 - E_0)/\hbar = \omega_{10}$  is resonant or nearly resonant with the optical angular frequency  $\omega = 2\pi\nu$ . The terms  $e^{i(\omega_{10}-\omega)t}$  and  $e^{-i(\omega_{10}-\omega)t}$  thus represent slowly varying functions of time compared to the rapidly oscillating nonresonant terms  $e^{i(\omega_{10}+\omega)t}$  and  $e^{-i(\omega_{10}+\omega)t}$ .

In what is known as the rotating wave approximation the nonresonant high-frequency terms can be neglected as their effects essentially average to zero because they are rapidly oscillating functions of time. Upon defining  $\Delta$  as  $\omega - \omega_{10}$ , equations (1.35a) and (1.35b) become

$$\dot{a}_0 = \frac{i\omega_R e^{i\Delta t} a_1}{2} \quad (1.36a)$$

$$\dot{a}_1 = \frac{i\omega_R e^{-i\Delta t} a_0}{2}. \quad (1.36b)$$

Equations (1.36a) and (1.36b) can be solved analytically. The difference  $\Delta$  is often referred to as the *detuning frequency*, since it measures how far the electromagnetic radiation of angular frequency  $\omega$  is tuned away from the resonance frequency  $\omega_{10}$ . The solution (see Problem 14) to these two simultaneous first-order differential equations with initial conditions  $a_0(0) = 1$  and  $a_1(0) = 0$  for the system initially in the ground state at  $t = 0$  is

$$a_0(t) = \left( \cos\left(\frac{\Omega t}{2}\right) - i\left(\frac{\Delta}{\Omega}\right) \sin\left(\frac{\Omega t}{2}\right) \right) e^{i\Delta t/2} \quad (1.37)$$

and

$$a_1(t) = i\left(\frac{\omega_R}{\Omega}\right) \sin\left(\frac{\Omega t}{2}\right) e^{-i\Delta t/2}, \quad (1.38)$$

in which  $\Omega = ((\omega_R)^2 + \Delta^2)^{1/2}$ . These solutions can be checked by substitution into equations (1.36).

The time-dependent probability that the system will be found in the excited state is given by

$$|a_1(t)|^2 = \frac{\omega_R^2}{\Omega^2} \sin^2\left(\frac{\Omega t}{2}\right), \quad (1.39)$$

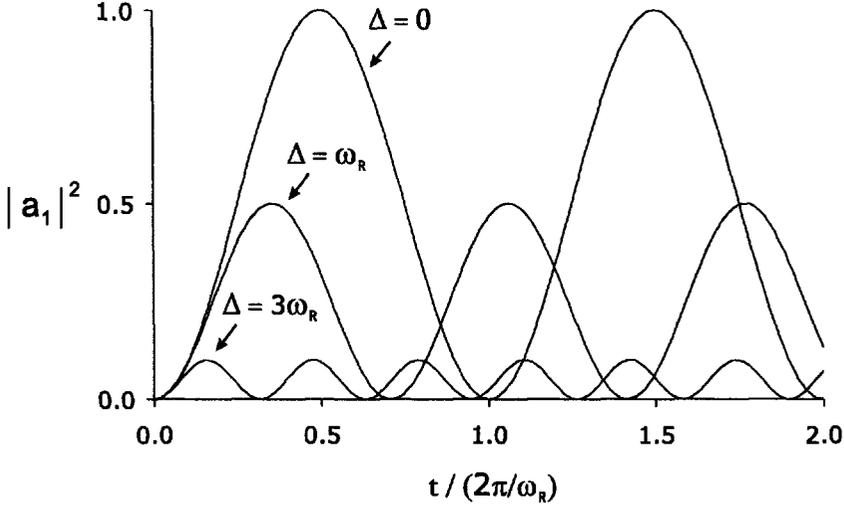


Figure 1.10: The probability for finding the driven two-level system in the excited state for three detunings:  $\Delta = 0$ ,  $\Delta = \omega_R$ , and  $\Delta = 3\omega_R$ .

while the corresponding time-dependent probability that the system will be found in the ground state is given by

$$|a_0|^2 = 1 - |a_1|^2 = 1 - \frac{\omega_R^2}{\Omega^2} \sin^2\left(\frac{\Omega t}{2}\right). \quad (1.40)$$

At resonance  $\Delta = 0$  and  $\Omega = \omega_R$ , so that in this case

$$|a_1|^2 = \sin^2\left(\frac{\omega_R t}{2}\right) \quad (1.41)$$

$$|a_0|^2 = 1 - \sin^2\left(\frac{\omega_R t}{2}\right) = \cos^2\left(\frac{\omega_R t}{2}\right). \quad (1.42)$$

The transition probability  $|a_1|^2$  is plotted in Figure 1.10 for three detuning frequencies.

The meaning of the Rabi frequency becomes clear from equations (1.34), (1.41), and Figure 1.10. The system is coherently cycled (i.e., with no abrupt changes in the phases or amplitudes of the wavefunctions) between the ground and excited state by the electromagnetic radiation. At resonance the system is completely inverted after a time  $t_\pi = \pi/\omega_R$ , while off-resonance there is a reduced probability for finding the system in the excited state.

This simple picture of a coherently driven system has ignored all decay processes such as spontaneous emission from the excited state. Spontaneous emission of a photon would break the coherence of the excitation and reset the system to the ground state (this is referred to as a  $T_1$  process). Similarly, collisions can also cause relaxation in the system. In fact, collisions can reset the phase of the atomic or molecular wavefunction (only the relative phases of  $\psi_1$  and  $\psi_0$  are important) without changing any of the populations (this is referred to as a  $T_2$  process). These phase-changing collisions also interrupt the coherent cycling of the system. Such processes were first studied in NMR (which is the source of the names  $T_1$  and  $T_2$  processes) and are now extensively studied in the field of quantum optics.

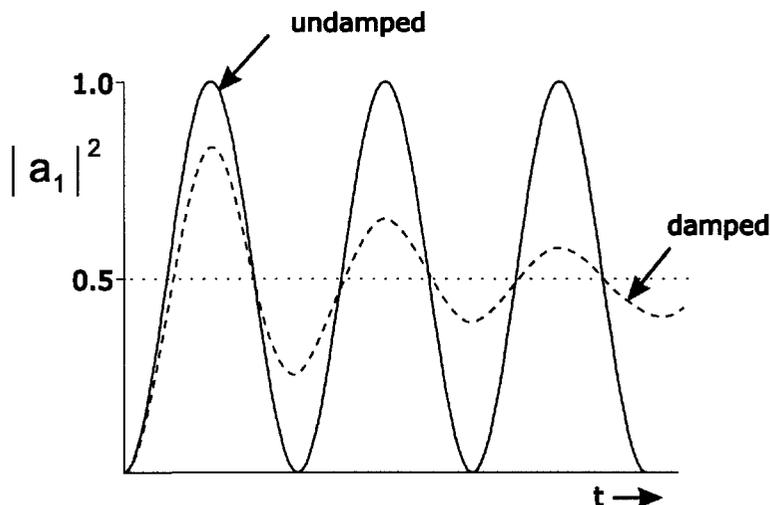


Figure 1.11: The driven two-level system saturates when relaxation processes (damping) are included.

The effect of collisions and other relaxation phenomena is to damp out the coherent cycling of the excited system (called *Rabi oscillations*). However, Rabi oscillations can be observed in any quantum system simply by increasing the intensity of the radiation. This increases the applied electric field  $\mathbf{E}$  so that at some point the Rabi cycling frequency exceeds the relaxation frequency,  $\omega_R \gg \omega_{\text{relaxation}}$ , and coherent behavior will be observed. This is easily achieved in NMR where spin relaxation processes are slow and many watts of radio frequency power can be applied to the system. In the infrared and visible region of the spectrum relaxation processes are much faster and Rabi oscillations are normally damped. For example, a real system would oscillate briefly when a strong field is applied suddenly to it, but it soon loses coherence and saturates (Figure 1.11). When the system is saturated, half of the molecules in the system are in the lower state and half are in the upper state. The rate of stimulated emission (down) matches the rate of absorption (up).

For example, consider a 1-W laser beam, 1 mm in diameter interacting with a two-level system that has a transition dipole moment of 1 debye ( $1 \text{ D} = 3.33564 \times 10^{-30} \text{ C m}$ ). What is the Rabi frequency? The intensity of the laser beam is  $1.3 \times 10^6 \text{ W/m}^2$  and the electric field  $E = |\mathbf{E}_0|$  is calculated from

$$I = \left( \frac{1}{2} \epsilon_0 E^2 \right) c \quad (1.43)$$

with  $\epsilon_0 = 8.8541878 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$  (Appendix A) being the permittivity of free space, from which the electric field can be obtained as

$$E = 27.4\sqrt{I} = 3.1 \times 10^4 \text{ V/m}$$

and

$$\omega_R = \frac{\mu E}{\hbar} = 9.8 \times 10^8 \text{ rad/s} \quad \text{or} \quad \nu_R = 156 \text{ MHz}, \quad \frac{1}{\nu_R} = 6.4 \text{ ns.}$$

Recall that  $\epsilon_0$  appears in Coulomb's law for the magnitude of the force of interaction between two electrical charges,

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}.$$

The factor 1/2 in equation (1.43) applies when the radiation is polarized and the electric field is given by equation (1.1). Since a typical electronic transition may have a natural lifetime of 10 ns ( $\omega_{\text{natural}} = 6.3 \times 10^8$  rad/s), the effects of Rabi cycling are already present at 1 W. At the megawatt or higher power levels of typical pulsed lasers, the coherent effects of strong radiation are even more pronounced, provided electrical breakdown is avoided. At these high electric field strengths, however, the simple two-level model is not a good description of an atomic or molecular system.

There is much confusion between the various terms and symbols used in the subfield of radiometry. For example, in physics the commonly used term "intensity,"  $I$  ( $\text{W m}^{-2}$ ) of a laser beam would be called the "irradiance" in radiometry.<sup>5</sup> In radiometry the terms intensity,  $I$  ( $\text{W sr}^{-1}$ ), and spectral intensity  $I_\nu = dI/d\nu$  ( $\text{W s sr}^{-1}$ ) are instead used for power per steradian and power per steradian per hertz (respectively). (A sphere subtends a solid angle of  $4\pi$  steradians.) In radiometry a distinction is also made between the excittance,  $M$  ( $\text{W m}^{-2}$ ), of power leaving a surface (e.g., equation (1.13)) and the irradiance,  $E$  ( $\text{W m}^{-2}$ ), of power falling on or crossing a surface (e.g., equation (1.43)) although they have the same dimensions. The right subscript  $\nu$  is used to distinguish between "integrated" quantities such as the irradiance,  $E = \int E_\nu d\nu$  and the spectral irradiance,  $E_\nu = dE/d\nu$ . In this book we follow the physics custom of using the single term intensity,  $I$  (or  $I_\nu$ ), for the excittance (or spectral excittance) and the irradiance (or spectral irradiance), with  $I = \int I_\nu d\nu$ . The term radiance,  $L = \int L_\nu d\nu$ , is universally reserved for power per square meter per steradian ( $\text{W m}^{-2} \text{sr}^{-1}$ ), and the spectral radiance,  $L_\nu$ , has dimensions  $\text{W s m}^{-2} \text{sr}^{-1}$  in SI units. The spectral radiance of a blackbody is given by

$$L_\nu^{\text{BB}} = \frac{\rho_\nu c}{4\pi} = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}. \quad (1.44)$$

The case of weak electromagnetic radiation interacting with the system is also common. In fact before the development of the laser in 1960 the weak-field case applied to all regions of the spectrum except the radio frequency and microwave regions, for which powerful coherent sources were available. In the weak-field case there is a negligible buildup of population in the excited state, so that  $a_1 \approx 0$ ,  $a_0 \approx 1$ , and

$$\dot{a}_1 = \frac{i\omega_R}{2} e^{-i\Delta t}. \quad (1.45)$$

Equation (1.45) is readily integrated to give

$$\begin{aligned} a_1 &= \frac{i\omega_R}{2} \int_0^t e^{-i\Delta t} dt \\ &= \frac{\omega_R}{-2\Delta} (e^{-i\Delta t} - 1). \end{aligned} \quad (1.46)$$

The probability of finding the system in the excited state after a time  $t$  is then obtained from equation (1.46) as

$$P_{1\leftarrow 0} = |a_1|^2 = \frac{\omega_R^2}{\Delta^2} \sin^2\left(\frac{\Delta t}{2}\right) = \frac{\mu_{10}^2 E^2}{\hbar^2} \frac{\sin^2((\omega - \omega_{10})t/2)}{(\omega - \omega_{10})^2}. \quad (1.47)$$

This formula is very deceptive because it assumes monochromatic radiation and short interaction times. These two requirements are inconsistent with one other because the Heisenberg energy-time uncertainty principle

$$\Delta E \Delta t \geq \hbar \quad \text{or} \quad \Delta \nu \Delta t \geq \frac{1}{2\pi} \quad (1.48)$$

must always be satisfied. If monochromatic radiation is applied to the system for a time  $\Delta t$ , then the system sees radiation of width  $\Delta \nu = 1/(2\pi\Delta t)$  in frequency space (this is certainly not monochromatic!). For example, a pulse of radiation 10 ns long has an intrinsic width of at least 160 MHz in frequency space.

Before equation (1.47) can be used, the effects of the finite frequency spread of the radiation must be included. Consider the radiation applied to the system to be broadband rather than monochromatic and to have a radiation density  $\rho = \epsilon_0 E^2/2$ . The total transition probability is given by integrating over all frequencies, that is, by

$$\begin{aligned} P_{1\leftarrow 0} &= \frac{2\mu_{10}^2}{\epsilon_0 \hbar^2} \int \rho_\nu(\omega) \frac{\sin^2((\omega - \omega_{10})t/2)}{(\omega - \omega_{10})^2} d\omega \\ &= \frac{2\mu_{10}^2}{\epsilon_0 \hbar^2} \rho_\nu(\omega_{10}) \int \frac{\sin^2((\omega - \omega_{10})t/2)}{(\omega - \omega_{10})^2} d\omega \\ &= \frac{\mu_{10}^2}{\epsilon_0 \hbar^2} \rho_\nu(\omega_{10}) \pi t, \end{aligned} \quad (1.49)$$

in which  $\rho(\omega)$  is assumed to be slowly varying near  $\omega_{10}$  so that it can be removed from the integration. This is indeed the case as  $\sin^2((\omega - \omega_{10})t/2)/(\omega_{10} - \omega)^2$  is sharply peaked at  $\omega = \omega_{10}$  (see Figure 1.26). The absorption rate per molecule is thus given by

$$\frac{dP_{1\leftarrow 0}}{dt} = \frac{\pi\mu_{10}^2}{\epsilon_0 \hbar^2} \rho_\nu(\omega_{10}). \quad (1.50)$$

In order to derive an expression for the absorption coefficient in terms of the transition dipole moment, equation (1.50) needs to be compared with equation (1.15), with  $N_0 \approx N$  for the weak-field case: dividing by  $N$  then gives the transition probability per molecule as

$$\frac{d(N_1/N)}{dt} = B_{1\leftarrow 0} \rho_\nu(\nu_{10}),$$

or as

$$\frac{dP_{1\leftarrow 0}}{dt} = B_{1\leftarrow 0} \rho_\nu(\nu_{10}). \quad (1.51)$$

A factor 3 is missing from equation (1.50) because equation (1.51) has been derived using isotropic radiation traveling in  $x$ ,  $y$ , and  $z$  directions, while equation (1.50) has been derived using a plane wave traveling in the  $z$  direction. Since only the  $z$  component of the isotropic radiation is effective in inducing a transition, and since  $\rho(\nu) = 2\pi\rho(\omega)$ , we have

$$B_{1\leftarrow 0} = \frac{1}{6\varepsilon_0\hbar^2}\mu_{10}^2 = \frac{2\pi^2}{3\varepsilon_0\hbar^2}\mu_{10}^2 \quad (1.52)$$

and

$$A_{1\rightarrow 0} = \frac{16\pi^3\nu^3}{3\varepsilon_0\hbar c^3}\mu_{10}^2. \quad (1.53)$$

These equations, (1.52) and (1.53), are key results because they relate the observed macroscopic transition rates to the microscopic transition dipole moment of an atom or molecule. Upon substitution of the values of the constants, we obtain  $A_{1\rightarrow 0} = 3.136 \times 10^{-7}(\tilde{\nu})^3\mu_{10}^2$ , with  $\tilde{\nu}$  expressed in  $\text{cm}^{-1}$  and  $\mu_{10}$  in debye. Although these equations are essentially correct, one factor that has been ignored is the possibility of relaxation.

Collisions or the spontaneous radiative lifetime of the upper state have all been ignored so far. When these losses are considered, the molecular absorption lineshape changes from a Dirac delta function  $\delta(\nu - \nu_{10})$  that is infinitely sharp and infinitely narrow, but with unit area, to a real molecular lineshape. As described below, the lineshape function  $g(\nu - \nu_{10})$  is typically either a Lorentzian or a Gaussian function with unit area but finite width and height, and now equations (1.52) and (1.53) are replaced by

$$(B_{1\leftarrow 0})_\nu = \frac{2\pi^2}{3\varepsilon_0\hbar^2}\mu_{10}^2g(\nu - \nu_{10}) \quad (1.54)$$

and

$$(A_{1\rightarrow 0})_\nu = \frac{16\pi^3\nu^3}{3\varepsilon_0\hbar c^3}\mu_{10}^2g(\nu - \nu_{10}), \quad (1.55)$$

respectively, in which  $\int(B_{1\leftarrow 0})_\nu d\nu = B_{1\leftarrow 0}$  and  $\int(A_{1\rightarrow 0})_\nu d\nu = A_{1\rightarrow 0}$ . In practice the  $\nu$  subscripts in equations (1.54) and (1.55) are suppressed and the same symbols  $A_{1\rightarrow 0}$  and  $B_{1\leftarrow 0}$  are used with and without lineshape functions, although the dimensions are different. In particular, as  $A_\nu$  and  $A$  are related by  $A_\nu = Ag(\nu - \nu_{10})$ ,  $A$  has dimensions  $\text{s}^{-1}$  and  $g(\nu - \nu_{10})$  has dimensions  $\text{s}$ , so that  $A_\nu$  is dimensionless. Note also that the two levels 1 and 0 are assumed to be nondegenerate. The usual cases of degenerate atomic and molecular energy levels will be considered in later chapters.

### *Beer's Law*

Consider a system (Figure 1.12) with  $N_0$  molecules per cubic meter in the ground state and  $N_1$  in the excited state. A flux of photons  $F_0 = I_0/h\nu$  (units of photons  $\text{m}^{-2} \text{s}^{-1}$ ) is incident upon the system from the left. As these photons travel through the system they can be absorbed or they can induce stimulated emission. What is the intensity of radiation after a distance  $l$ ?

If only absorption and stimulated emission are considered, then at a particular distance  $x$  we can write

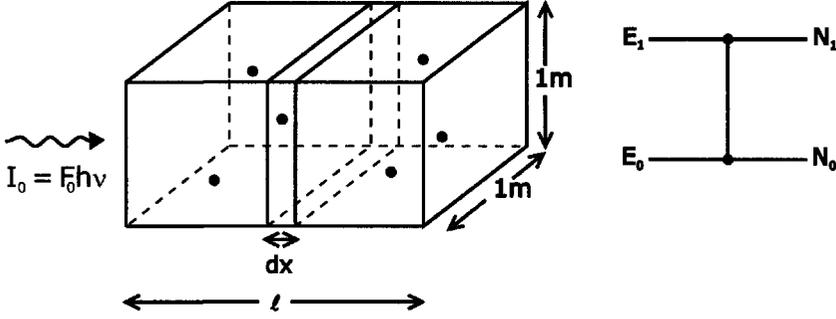


Figure 1.12: A system with dimensions  $1 \text{ m} \times 1 \text{ m} \times l \text{ m}$  that contains molecules.

$$\begin{aligned}
 \frac{dN_1}{dt} &= -B_{1 \rightarrow 0} \rho N_1 + B_{1 \leftarrow 0} \rho N_0 \\
 &= \frac{2\pi^2 \mu_{10}^2}{3\epsilon_0 \hbar^2} (N_0 - N_1) g(\nu - \nu_{10}) \rho \\
 &= \frac{2\pi^2 \mu_{10}^2 \nu}{3\epsilon_0 \hbar c} (N_0 - N_1) g(\nu - \nu_{10}) F \\
 &= \sigma F (N_0 - N_1),
 \end{aligned} \tag{1.56}$$

in which  $\rho = I/c = h\nu F/c$  has been used. The absorption cross section is defined in this way as

$$\sigma = \frac{2\pi^2 \mu_{10}^2}{3\epsilon_0 \hbar c} \nu g(\nu - \nu_{10}), \tag{1.57}$$

with dimensions of  $\text{m}^2$ . The physical interpretation of  $\sigma$  is as the “effective area” that a molecule presents to the stream of photons of flux  $F$ . Notice that (1.57) and (1.53) can be combined to give the convenient equation

$$\sigma = \frac{A\lambda^2 g(\nu - \nu_{10})}{8\pi} = \frac{\lambda^2 g(\nu - \nu_{10})}{8\pi\tau_{\text{sp}}}, \tag{1.58}$$

which relates the cross section to the “radiative lifetime”  $\tau_{\text{sp}} \equiv 1/A_{1 \rightarrow 0}$  of a transition for a two-level system. The subscript sp on  $\tau$  refers to spontaneous emission.

Care is required when radiative lifetimes are used for real multilevel systems because any given level  $n$  can emit spontaneously to all lower levels, so that

$$\tau_{\text{sp}} = \frac{1}{\sum A_{n \rightarrow j}}. \tag{1.59}$$

In other words the lifetime is related to the rates of all radiative rates connecting the upper state  $|n\rangle$  to all lower energy states  $|j\rangle$ , rather than just  $\tau_{\text{sp}} = 1/A_{1 \rightarrow 0}$ . Any nonradiative processes add additional rate terms to the sum in equation (1.59). The individual  $A$  and  $B$  coefficients, however, still obey the equations developed for a two-level system.

If a flux  $F$  is incident to the left of a small element of thickness  $dx$  (Figure 1.12) with cross-sectional area of  $1 \text{ m}^2$ , then the change in flux caused by passing through the element is

$$dF = -\sigma F(N_0 - N_1)dx. \quad (1.60)$$

Upon integrating over the absorption path, this becomes

$$\int_{F_0}^F \frac{dF}{F} = -\sigma(N_0 - N_1) \int_0^l dx$$

or

$$\ln\left(\frac{F}{F_0}\right) = \ln\left(\frac{I}{I_0}\right) = -\sigma(N_0 - N_1)l. \quad (1.61)$$

Expression (1.61) can also be rewritten in the form

$$I = I_0 e^{-\sigma(N_0 - N_1)l}, \quad (1.62)$$

which is equivalent to the commonly encountered decadic version of Beer's law,

$$I = I_0 10^{-\varepsilon cl}. \quad (1.63)$$

It is common to report  $\sigma$  in  $\text{cm}^2$ ,  $N$  in molecules per  $\text{cm}^3$ , and  $l$  in cm rather than the corresponding SI units. The units used in Beer's law (1.63) are customarily moles per liter for  $c$ , cm for  $l$ , and liter  $\text{mole}^{-1} \text{cm}^{-1}$  for the molar absorption coefficient,  $\varepsilon$ . Sometimes the cross section and concentration are combined to define an absorption coefficient  $\alpha = \sigma(N_0 - N_1)$  for a system, in which case we write

$$I = I_0 e^{-\alpha l}, \quad (1.64)$$

rather than (1.62).

### ***Lineshape Functions***

A real spectrum of a molecule, such as that for gaseous  $\text{CO}_2$  (Figure 1.13), contains many absorption features called *lines* organized into a band associated with a particular mode of vibration. For the spectrum illustrated in Figure 1.13 the lines are associated with the antisymmetric stretching mode,  $\nu_3$ , of  $\text{CO}_2$ . At high resolution the spectrum seems to consist of very narrow features, but if the scale is expanded the lines are observed to have definite widths and characteristic shapes. What are the possible lineshape functions  $g(\nu - \nu_{10})$  and what physical processes are responsible for these shapes?

Lineshape functions fall into one of two general categories: homogeneous and inhomogeneous. A homogeneous lineshape occurs when all molecules in the system have identical lineshape functions. For example, if an atomic or molecular absorber in the gas phase is subject to a high pressure, then all molecules in the system are found to have an identical pressure-broadened lineshape for a particular transition. Pressure broadening of a transition is said, therefore, to be a homogeneous broadening.

In contrast, if a molecule is dissolved in a liquid, then the disorder inherent in the structure of the liquid provides numerous different solvent environments for the solute. Each solute molecule experiences a slightly different solvent environment and therefore has a slightly different absorption spectrum. The observed absorption spectrum (Figure 1.14) is made up of all of the different spectra for the different molecular environments; it is said to be inhomogeneously broadened.

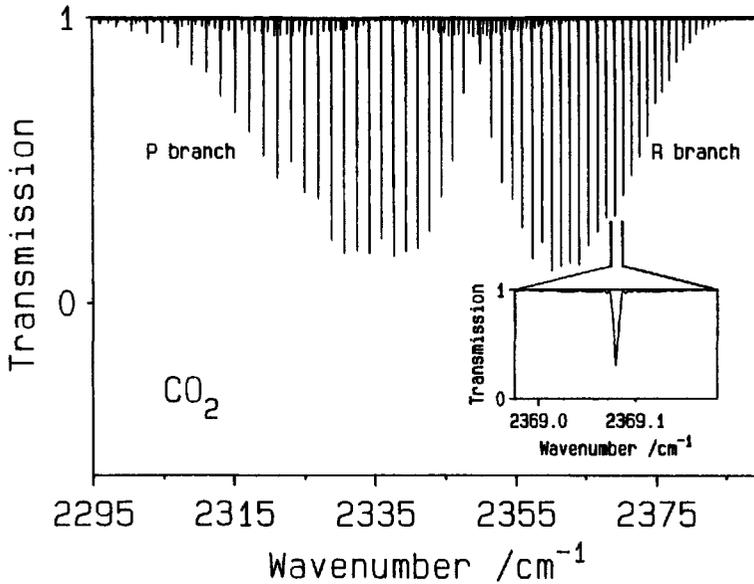


Figure 1.13: A typical molecular spectrum, the antisymmetric stretching mode of carbon dioxide. The weak bending hot band (see Chapter 7) is also present.

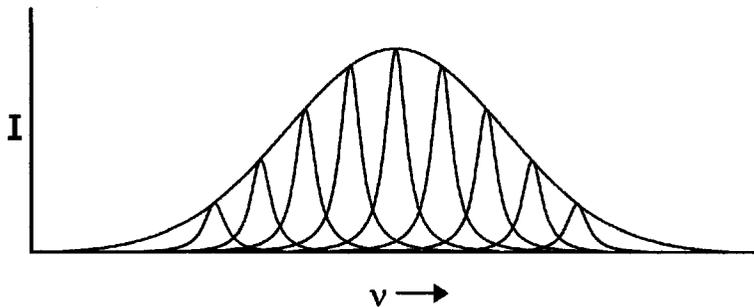


Figure 1.14: An inhomogeneously broadened line made up of many homogeneously broadened components.

The most important example of gas phase inhomogeneous broadening occurs because of the Maxwell-Boltzmann distribution of molecular velocities and is called *Doppler broadening*. The different molecular velocities give the incident radiation a frequency shift of  $\nu = (1 \pm v/c)\nu_0$  in the molecular frame of reference. This results in slightly different spectra for molecules moving at different velocities and results in an inhomogeneous lineshape.

### *Natural Lifetime Broadening*

Consider a two-level system with an intrinsic lifetime  $\tau_{\text{sp}}$  seconds for the level at energy  $E_1$  for the spontaneous emission of radiation (Figure 1.15). The wavefunction that

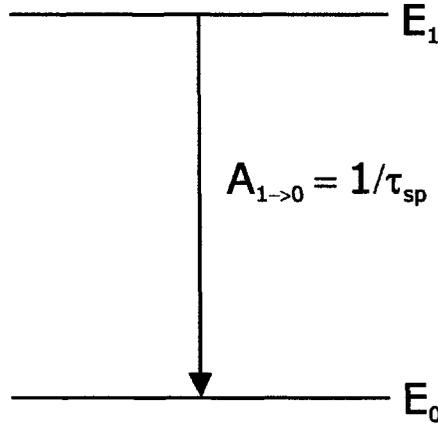


Figure 1.15: Spontaneous emission in a two-level system.

describes the state of the system in the absence of electromagnetic radiation is given by

$$\begin{aligned}\Psi(t) &= a_0\Psi_0(t) + a_1\Psi_1(t) \\ &= a_0\psi_0e^{-iE_0t/\hbar} + a_1\psi_1e^{-iE_1t/\hbar}\end{aligned}\quad (1.65)$$

where  $a_0$  and  $a_1$  are simply constants. Should the system be excited into this superposition state (for example, by a pulse of electromagnetic radiation), the dipole moment of the system in this state is given by the expectation value of the dipole moment operator as

$$\begin{aligned}\mathbf{M}(t) &= \langle \Psi | \boldsymbol{\mu} | \Psi \rangle \\ &= \boldsymbol{\mu}_{10}(a_0^*a_1e^{-i\omega_{10}t} + a_0a_1^*e^{i\omega_{10}t}) \\ &= \text{Re}(2\boldsymbol{\mu}_{10}a_0a_1^*e^{i\omega_{10}t}),\end{aligned}\quad (1.66)$$

assuming that the space-fixed dipole moments  $\langle \psi_0 | \boldsymbol{\mu} | \psi_0 \rangle$  and  $\langle \psi_1 | \boldsymbol{\mu} | \psi_1 \rangle$  both vanish in states  $|1\rangle$  and  $|0\rangle$ . (NB: Nonzero values for the dipole moment are still possible in the molecular frame.) The dipole moment of the system oscillates at the Bohr angular frequency  $\omega_{10}$  as

$$\mathbf{M}(t) = 2a_0a_1\boldsymbol{\mu}_{10}\cos(\omega_{10}t),\quad (1.67)$$

if  $a_0$  and  $a_1$  are chosen to be real numbers. A system in such a superposition state has a macroscopic oscillating dipole in the laboratory frame (Figure 1.16).

Now if the population in the excited state decreases slowly in time (relative to the reciprocal of the Bohr frequency) due to spontaneous emission, then the amplitude of the oscillation will also decrease. This corresponds to a slow decrease in  $a_1$  (equation (1.67)) at a rate of  $\gamma/2$  where  $\gamma = 1/\tau_{sp} = A_{1 \rightarrow 0}$ . Thus the oscillating dipole moment is now

$$\mathbf{M}(t) = \mathbf{M}_0e^{-\gamma t/2}\cos(\omega_{10}t)\quad (1.68)$$

as shown in Figure 1.17.

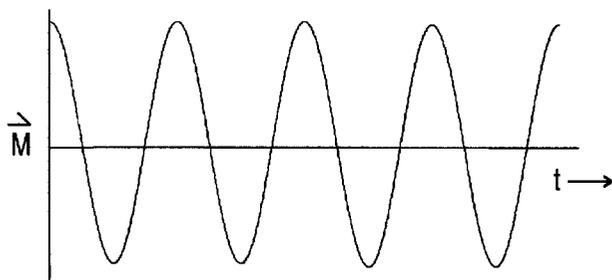


Figure 1.16: Oscillating dipole moment of a system in a superposition state.

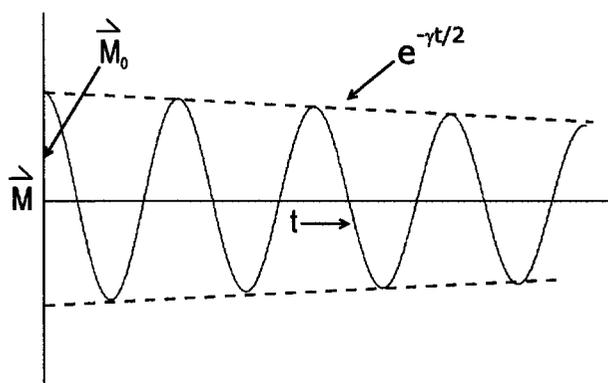


Figure 1.17: Slowly damped oscillating dipole moment.

What frequencies are associated with a damped cosine wave? Clearly the undamped wave oscillates infinitely at exactly the Bohr frequency  $\omega_{10}$ . The distribution of frequencies  $F(\omega)$  present in a waveform  $f(t)$  can be determined by taking a Fourier transform, i.e., as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt. \quad (1.69)$$

Note also that an arbitrary waveform  $f(t)$  can be written as a sum (integral) over plane waves  $e^{i\omega t}$ , each with amplitude  $F(\omega)$ , as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} d\omega, \quad (1.70)$$

which is referred to as the inverse Fourier transform. Thus  $F(\omega)$  measures the “amount” of each “frequency” required to synthesize  $f(t)$  out of sine and cosine functions ( $e^{i\omega t} = \cos \omega t + i \sin \omega t$ ). Taking the Fourier transform of the time-dependent part of  $\mathbf{M}(t)$  gives

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} e^{-\gamma t/2} \cos(\omega_{10}t) e^{-i\omega t} dt \\
&= \frac{1}{2} \int_0^{\infty} e^{-\gamma t/2} (e^{-i(\omega-\omega_{10})t} + e^{-i(\omega+\omega_{10})t}) dt \\
&= \frac{1}{2} \left( \frac{1}{\gamma/2 + i(\omega - \omega_{10})} + \frac{1}{\gamma/2 + i(\omega + \omega_{10})} \right) \quad (1.71)
\end{aligned}$$

for the decay process beginning at  $t = 0$ . The nonresonant term, i.e., the term containing  $\omega + \omega_{10}$ , is dropped because  $\omega \approx \omega_{10}$  and  $\omega_{10} \gg \gamma$ , so that it is negligible in comparison with the resonant term containing  $\omega - \omega_{10}$  (cf. the rotating wave approximation). With this (rather good) approximation,  $F(\omega)$  becomes

$$F(\omega) \approx \frac{1}{2} \frac{1}{\gamma/2 + i(\omega - \omega_{10})}. \quad (1.72)$$

In the semiclassical picture an oscillating dipole moment radiates power at a rate proportional to  $|\mu_{10}|^2$  (i.e.,  $A_{1 \rightarrow 0} \propto |\mu_{10}|^2$ ) and the lineshape function, given by

$$|F(\omega)|^2 = \frac{1}{4} \frac{1}{(\gamma/2)^2 + (\omega - \omega_{10})^2} \quad (1.73)$$

is an unnormalized Lorentzian. Normalization requires that

$$\int_{-\infty}^{\infty} g(\omega - \omega_{10}) d\omega = \int_{-\infty}^{\infty} g(\nu - \nu_{10}) d\nu = 1, \quad (1.74)$$

so that the final normalized Lorentzian lineshape functions are

$$g(\omega - \omega_{10}) = \frac{\gamma/(2\pi)}{(\gamma/2)^2 + (\omega - \omega_{10})^2}, \quad (1.75)$$

and

$$g(\nu - \nu_{10}) = \frac{\gamma}{(\gamma/2)^2 + (2\pi)^2(\nu - \nu_{10})^2}. \quad (1.76)$$

Note that  $g(\omega - \omega_{10})$  and  $g(\nu - \nu_{10})$  are related by

$$2\pi g(\omega - \omega_{10}) = g(\nu - \nu_{10}).$$

Without spontaneous emission the lineshape function would be  $\delta(\nu - \nu_{10})$  since the infinite cosine wave oscillates at a frequency of exactly  $\nu_{10}$ . The decaying cosine wave caused by spontaneous emission gives a Lorentzian function of finite width (Figure 1.18). At the peak center ( $\nu = \nu_{10}$ ) we have  $g(\nu - \nu_{10}) = 4/\gamma$ , and the function drops to half this value when

$$(2\pi)^2(\nu_{1/2} - \nu_{10})^2 = \left(\frac{\gamma}{2}\right)^2 \text{ or } \pm(\nu_{1/2} - \nu_{10}) = \frac{\gamma}{4\pi}.$$

The full width at half maximum (FWHM), represented by  $\Delta\nu_{1/2}$ , is given as

$$\Delta\nu_{1/2} = \frac{\gamma}{2\pi} = \frac{1}{2\pi\tau_{\text{sp}}}, \quad (1.77)$$

since  $\gamma = 1/\tau_{\text{sp}}$ . The Lorentzian lineshape function ( $\nu_0 = \nu_{10}$ ) can thus be expressed as

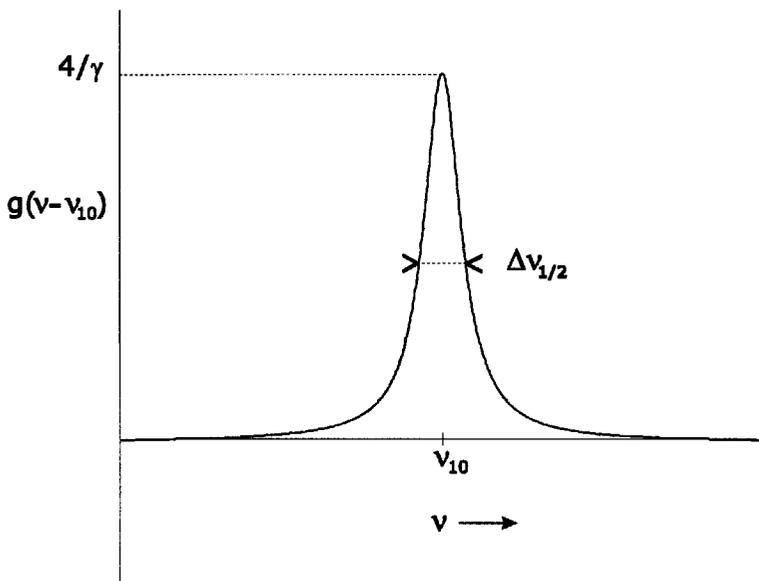


Figure 1.18: A normalized Lorentzian function.

$$g(\nu - \nu_0) = \frac{\Delta\nu_{1/2}/(2\pi)}{(\Delta\nu_{1/2}/2)^2 + (\nu - \nu_0)^2} \quad (1.78)$$

in terms of the full width at half maximum. Note that some authors use the half width at half maximum as a parameter rather than the full width.

The important result  $\Delta\nu_{1/2} = 1/(2\pi\tau_{\text{sp}})$  agrees with the Heisenberg time-energy uncertainty principle  $\Delta E\Delta t \geq \hbar$ , or

$$\frac{\Delta E}{h}\tau_{\text{sp}} \sim \frac{1}{2\pi}, \quad (1.79a)$$

or

$$\Delta\nu \sim \frac{1}{2\pi\tau_{\text{sp}}}. \quad (1.79b)$$

The spontaneous lifetime of the excited state means that the atom or molecule cannot be found at  $E_1$  for more than  $\tau_{\text{sp}}$  on average. This provides a fundamental limit on the linewidth arising from the transition between the two states (Figure 1.19). Formula (1.79b) has been checked experimentally, for example in the case of the sodium  $3^2P_{3/2} \rightarrow 3^2S_{1/2}$  transition (one of the famous sodium *D*-lines) at 5890 Å. The experimentally measured lifetime  $\tau_{\text{sp}} = 16$  ns and the observed homogeneous linewidth  $\Delta\nu_{1/2} = 10$  MHz are consistent with equation (1.79b). The uncertainty principle therefore requires that if an excited state exists for only  $\tau_{\text{sp}}$  seconds on average, then the

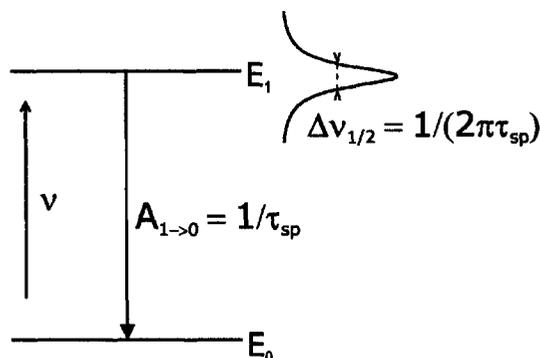


Figure 1.19: The spontaneous lifetime  $\tau_{\text{sp}}$  gives the transition  $E_1 \rightarrow E_0$  a finite linewidth.

energy level  $E_1$  cannot be measured relative to  $E_0$  with an accuracy that is greater than  $\Delta\nu_{1/2}$  Hz.

The expression  $\Delta\nu_{1/2} = 1/(2\pi\tau)$  has widespread use in chemical physics. For example, if  $\text{H}_2\text{O}$  is excited by vacuum ultraviolet light, it can dissociate very rapidly:



If the  $\text{H}_2\text{O}$  molecule exists in a given excited electronic state for only one vibrational period ( $\tilde{\nu} = 3600 \text{ cm}^{-1}$  corresponding to an OH stretch), then according to the Heisenberg uncertainty principle the lifetime  $\tau$  will be given by  $\tau = 9.3 \times 10^{-15} \text{ s} = 9.3 \text{ femtoseconds (fs)}$ . Thus the width (FWHM) of a line in the spectrum will be  $\Delta\nu_{1/2} = 1.7 \times 10^{13} \text{ Hz}$  or  $\Delta\tilde{\nu}_{1/2} = 570 \text{ cm}^{-1}$ . A measurement of the homogeneous width of a particular spectral line can thus provide an estimate of the lifetime of the excited state.

### Pressure Broadening

The derivation of the pressure-broadening lineshape is a difficult problem because it depends on the intermolecular potentials between the colliding molecules. However, a simplified model within the semiclassical picture gives some estimation of the effect.

Consider the two-level system discussed in the previous section with the wavefunction written as a superposition state. The dipole moment oscillates at the Bohr frequency except during a collision. If the collision is sufficiently strong, then the phase of the oscillating dipole moment is altered in a random manner by the encounter. Let the average time between collisions be  $T_2$  (Figure 1.20). The infinite cosine wave is broken by successive collisions into pieces of average length  $T_2$ . The effect of collisions will be to convert the infinitely narrow lineshape associated with an infinitely long cosine wave into a lineshape function of finite width. The application of Fourier transform arguments (using autocorrelation functions<sup>6</sup>) to decompose the broken waveform into frequency components results in a Lorentzian lineshape with a width (FWHM) given by

$$\Delta\nu_{1/2} = \frac{1}{\pi T_2}. \quad (1.81)$$

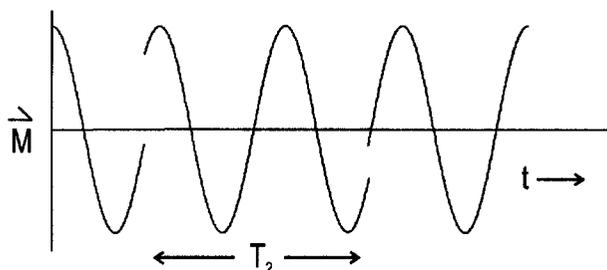


Figure 1.20: The phase of an oscillating dipole moment randomly interrupted by collisions.

Since the average time between collisions is proportional to the reciprocal of the pressure,  $p$ , it therefore follows that the FWHM will be proportional to the pressure, i.e.,

$$\Delta\nu_{1/2} = bp, \quad (1.82)$$

with  $b$  referred to as the pressure-broadening coefficient. The quantitative calculation of  $b$  without recourse to experiment poses a difficult theoretical problem. Experimentally, typical values for  $b$  are about 10 MHz per Torr of the pressure-broadening gas.

In general not only are the lines broadened by increasing pressure but they are also shifted in frequency. These shifts are generally small, often less than 1 MHz/Torr, but they become important when very precise spectroscopic measurements are to be made.

## Doppler Broadening

Doppler broadening results in an inhomogeneous lineshape function. If the transition has an intrinsic homogeneous lineshape  $g_H(\nu - \nu'_0)$  centered at  $\nu'_0$ , then the inhomogeneous distribution function  $g_I(\nu'_0 - \nu_0)$ , centered at  $\nu_0$ , is required to describe the total lineshape function  $g(\nu - \nu_0)$  according to the expression

$$g(\nu - \nu_0) = \int_{-\infty}^{\infty} g_I(\nu'_0 - \nu_0) g_H(\nu - \nu'_0) d\nu'_0. \quad (1.83)$$

The distribution function  $g_I(\nu'_0 - \nu_0)$  gives the probability that a system has a resonance frequency in the interval  $\nu'_0$  to  $\nu'_0 + d\nu'_0$ , i.e.,

$$dp = g_I(\nu'_0 - \nu_0) d\nu'_0. \quad (1.84)$$

The lineshape integral (1.83) is referred to mathematically as a convolution of the two functions  $g_I$  and  $g_H$ , as can be made more apparent by making the substitution  $x = \nu'_0 - \nu_0$ ,

$$g(\nu - \nu_0) = \int_{-\infty}^{\infty} g_I(x) g_H((\nu - \nu_0) - x) dx. \quad (1.85)$$

Commonly the homogeneous lineshape function  $g_H$  is Lorentzian, while the inhomogeneous function  $g_I$  is a Gaussian: the convolution of these two functions is called a Voigt lineshape function (Figure 1.21).

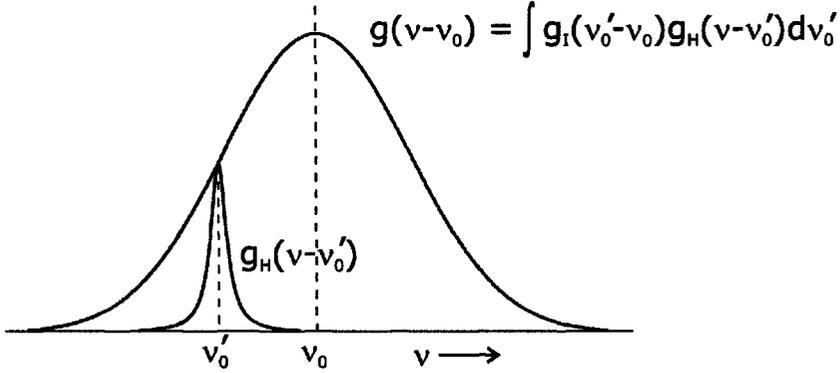


Figure 1.21: The Voigt lineshape is a convolution of an inhomogeneous Gaussian lineshape function with a homogeneous Lorentzian lineshape function.

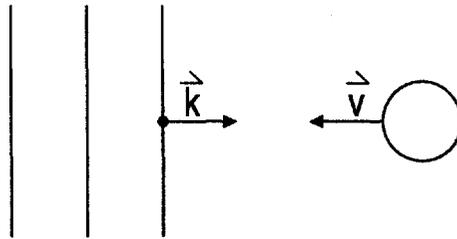


Figure 1.22: Interaction of a plane electromagnetic wave with a moving atom.

The Voigt lineshape function is a general form that can include purely homogeneous or purely inhomogeneous lineshapes as limiting cases. If the width of the inhomogeneous part is much greater than that of the homogeneous part, that is, if  $\Delta\nu_I \gg \Delta\nu_H$ , then  $g_H(\nu - \nu'_0) \approx \delta(\nu - \nu'_0)$  and

$$\begin{aligned}
 g(\nu - \nu_0) &= \int_{-\infty}^{\infty} g_I(\nu'_0 - \nu_0)\delta(\nu - \nu'_0)d\nu'_0 \\
 &= g_I(\nu - \nu_0).
 \end{aligned}
 \tag{1.86}$$

Conversely if  $\Delta\nu_I \ll \Delta\nu_H$ , then  $g_I(\nu'_0 - \nu_0) \approx \delta(\nu'_0 - \nu_0)$  and  $g(\nu - \nu_0) = g_H(\nu - \nu_0)$ .

Consider an atom with velocity  $\mathbf{v}$  interacting with a plane wave with a wave vector  $\mathbf{k}$ . If  $\mathbf{k}$  is parallel to  $\mathbf{v}$ , then the atom sees a Doppler shifted frequency  $\nu' = \nu(1 \pm v/c)$  depending upon whether the atom is moving in a direction that is the same as (-) or opposite to (+) that of the electromagnetic radiation (Figure 1.22). In general, it is only the component of  $\mathbf{v}$  along  $\mathbf{k}$  (i.e.,  $v \cos \theta$ ) that matters, so that

$$\nu' = \nu \left( 1 - \frac{\mathbf{v} \cdot \mathbf{k}}{c|\mathbf{k}|} \right),
 \tag{1.87}$$

neglecting a small relativistic correction<sup>7</sup> (“second-order Doppler effect”).

The Doppler effect can be viewed in two equivalent ways. In the frame of the atom it is the frequency of the electromagnetic wave which has been shifted, with the atom at

rest at the origin of the atomic coordinate system. Alternatively, in the fixed laboratory frame the electromagnetic wave is unshifted at  $\nu$ , but the atomic resonance frequency  $\nu_0$  (of the atom moving at velocity  $v$ ) has been shifted to the new value of

$$\nu'_0 = \frac{\nu_0}{1 \pm v/c}. \quad (1.88)$$

All that is required to obtain a lineshape function is the distribution of velocities.

The distribution of molecular velocity components along a given axis (such as  $\mathbf{k}$ ) in a gaseous system is given by the Maxwell-Boltzmann distribution function

$$p_v dv = \left( \frac{m}{2\pi kT} \right)^{1/2} e^{-mv^2/(2kT)} dv \quad (1.89)$$

for particles of mass  $m$  at a temperature  $T$ . Using equation (1.89) and  $dv = (c/\nu_0)d\nu'_0$  (obtained by taking differentials of equation (1.87)) gives the normalized inhomogeneous lineshape function

$$g_D(\nu - \nu_0) = \frac{1}{\nu_0} \left( \frac{mc^2}{2\pi kT} \right)^{1/2} e^{-mc^2(\nu - \nu_0)^2/(2kT\nu_0^2)}. \quad (1.90)$$

The FWHM,  $\Delta\nu_D$ , is easily shown to be

$$\Delta\nu_D = 2\nu_0 \sqrt{\frac{2kT \ln(2)}{mc^2}}, \quad (1.91)$$

or

$$\Delta\tilde{\nu}_D = 7.2 \times 10^{-7} \tilde{\nu}_0 \sqrt{\frac{T}{M}}, \quad (1.92)$$

in which  $T$  is in K,  $M$  in atomic mass units u,  $\tilde{\nu}_0$  in  $\text{cm}^{-1}$ , and  $\Delta\nu_D$  in  $\text{cm}^{-1}$ . The Doppler lineshape function is thus given by

$$g_D(\nu - \nu_0) = \frac{2}{\Delta\nu_D} \sqrt{\frac{\ln(2)}{\pi}} e^{-4 \ln 2 ((\nu - \nu_0)/\Delta\nu_D)^2} \quad (1.93)$$

in terms of the Doppler FWHM.

The Gaussian function is the bell-shaped curve well known in statistics. The Gaussian lineshape function is more sharply peaked around  $\nu - \nu_0$  than the corresponding Lorentzian lineshape function (Figure 1.23). Notice the much more extensive "wings" on the Lorentzian function in comparison with the Gaussian function.

For example, the Doppler width of the  $3^2P_{3/2} - 3^2S_{1/2}$  transition of the Na atom at 300 K is already  $\Delta\nu_D = 0.044 \text{ cm}^{-1} = 1317 \text{ MHz}$ , and is much larger than the natural linewidth of 10 MHz. In addition, if the Na atom is surrounded by Ar atoms at 1 Torr total pressure, then pressure broadening contributes 27 MHz to the total homogeneous linewidth of 37 MHz.<sup>8</sup> Visible and ultraviolet transitions of gas phase atoms and molecules typically display Doppler broadening at low pressure because the inhomogeneous linewidth greatly exceeds the total homogeneous linewidth.

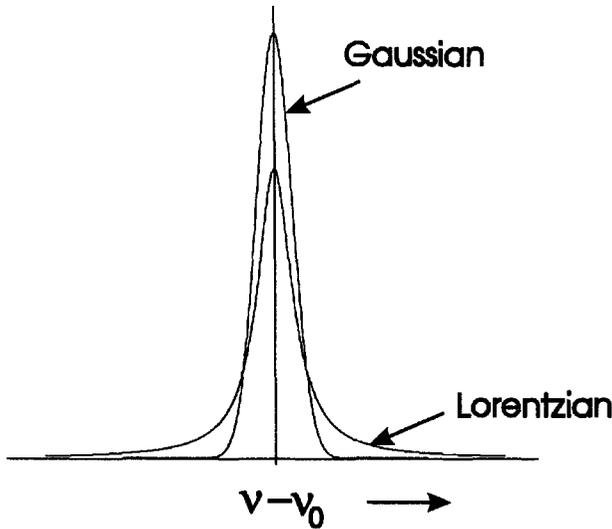


Figure 1.23: Normalized Gaussian and Lorentzian lineshape functions.

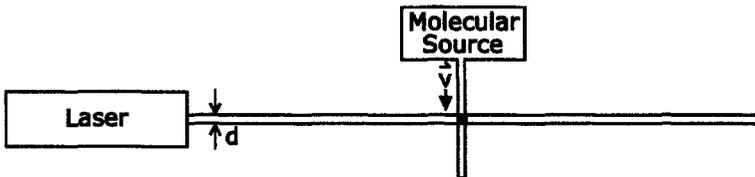


Figure 1.24: A laser beam interacts with a molecular beam inside a vacuum chamber.

### *Transit-Time Broadening*

Consider the experimental arrangement of Figure 1.24 in which a laser beam of width  $d$  is crossed at right angles with a beam of molecules moving at a speed of  $v$  in a vacuum chamber. The molecules can only interact with the radiation for a finite time,  $\tau = d/v$ , called the *transit time*. The time  $\tau$  corresponds to the time required by a molecule in the molecular beam to cross through the laser beam. If the laser is considered to be perfectly monochromatic, with frequency  $\nu$ , then a molecule experiences an electric field as shown in Figure 1.25, assuming a constant weak light intensity from one side of the laser beam to the other. If an intense laser beam is used in these experiments, Rabi oscillations (discussed earlier) are observed.

Suppose an infinitely long (in time) oscillating electric field with an infinitely narrow frequency distribution has been chopped into a finite length with a finite frequency width. The finite time allowed for the laser-molecule interaction has resulted in a broadening of the transition. As far as the observed lineshape is concerned, it does not matter whether the molecular resonance is broadened or the frequency distribution of the applied radiation is increased. The result is the same: a broader line.

The frequency distribution associated with the electric field of Figure 1.25 is determined, as for lifetime broadening, by taking the Fourier transform. Thus we write

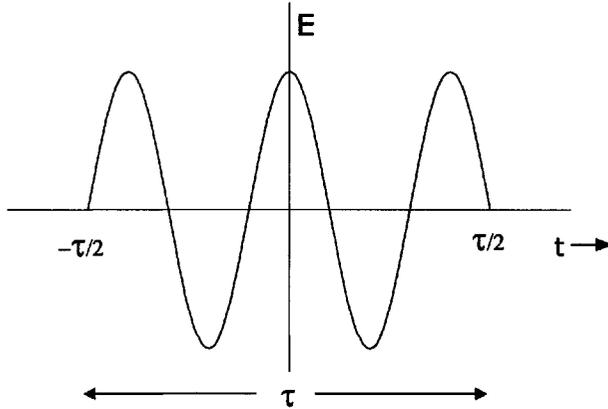


Figure 1.25: A molecule experiences an electromagnetic wave of finite length  $\tau$ .

$$\begin{aligned}
 F(\omega) &= \int_{-\infty}^{\infty} E_0 \cos(\omega_0 t) e^{-i\omega t} dt \\
 &= \frac{E_0}{2} \int_{-\tau/2}^{\tau/2} (e^{i\omega_0 t} + e^{-i\omega_0 t}) e^{-i\omega t} dt \\
 &= \frac{E_0}{2} \left[ \frac{e^{i(\omega_0 - \omega)t}}{i(\omega_0 - \omega)} + \frac{e^{-i(\omega_0 + \omega)t}}{-i(\omega_0 + \omega)} \right]_{-\tau/2}^{\tau/2} \\
 &\approx E_0 \frac{\sin((\omega_0 - \omega)\tau/2)}{(\omega_0 - \omega)}, \tag{1.94}
 \end{aligned}$$

in which the nonresonant term in  $\omega_0 + \omega$  has, as usual, been discarded. Since the intensity of the light is proportional to  $|E|^2$ , the unnormalized lineshape is proportional to

$$|F(\omega)|^2 = \frac{\sin^2((\omega_0 - \omega)\tau/2)}{(\omega_0 - \omega)^2}. \tag{1.95}$$

The corresponding normalized lineshape function

$$g(\omega - \omega_0) = \frac{2}{\pi\tau} \frac{\sin^2((\omega - \omega_0)\tau/2)}{(\omega - \omega_0)^2}, \tag{1.96}$$

or

$$g(\nu - \nu_0) = \frac{1}{\pi^2\tau} \frac{\sin^2(2\pi(\nu - \nu_0)\tau/2)}{(\nu - \nu_0)^2}, \tag{1.97}$$

is shown in Figure 1.26. The FWHM of this function is about  $\Delta\nu_{1/2} = 0.89/\tau$ . For an atom traveling at 500 m/s through a laser beam of 1 mm width,  $\tau = 2 \times 10^{-6}$  s and  $\Delta\nu_{1/2} = 0.45$  MHz. Although transit-time broadening is relatively small, it is not negligible for very precise Doppler-free measurements or for microwave-molecular beam measurements.

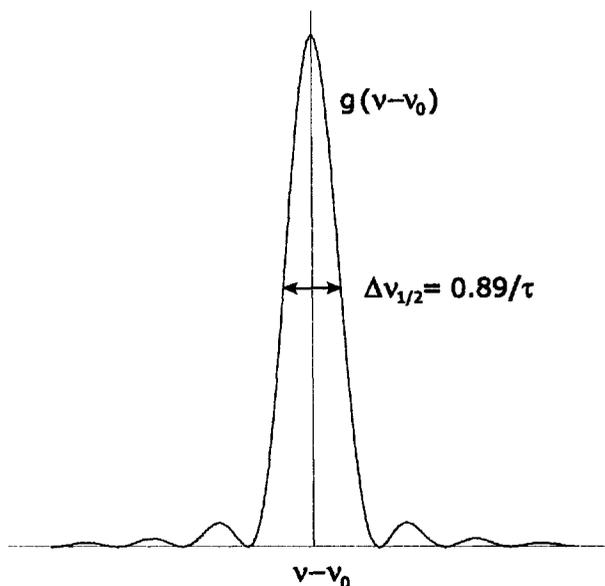


Figure 1.26: A plot of the lineshape function caused by transit-time broadening.

### Power Broadening

The high-power laser has become an ubiquitous tool of modern spectroscopy. The application of intense electromagnetic radiation to a system will cause the spectral lines to broaden and even to split. The detailed calculation of a molecular lineshape at high power is complicated, but a simple estimate of the linewidth is available from the time-energy uncertainty principle of equation (1.48), i.e.,

$$\Delta E \Delta t \geq \hbar.$$

At high powers the molecular system undergoes Rabi oscillations at an angular frequency  $\omega_R = \mu_{10}E/\hbar$ . The system is thus in the excited state  $E_1$  only for a period of about  $\hbar/\mu_{10}E$ , which therefore provides an estimate for  $\Delta t$ . Using this estimate for  $\Delta t$  in the uncertainty relation (1.48) gives

$$\Delta E \frac{\hbar}{\mu_{10}E} \sim \hbar$$

or equivalently, using  $\hbar\Delta\nu = \Delta E$ , a frequency uncertainty  $\Delta\nu$  of

$$\Delta\nu \sim \frac{\mu_{10}E}{2\pi\hbar} = \frac{\omega_R}{4\pi^2}. \quad (1.98)$$

For example, a 1-W laser beam of 1-mm diameter interacting with a two-level system with a transition dipole moment of 1 D and a Rabi frequency  $\omega_R = 9.8 \times 10^8$  rad/s already gives  $\Delta\nu = 25$  MHz. Pulsed lasers can easily achieve peak powers of 1 MW (10 mJ in 10 ns), which will increase  $E$ , equation (1.43), by a factor of 1 000, to  $3.1 \times 10^7$  V/m for the preceding example. The power-broadened linewidth is then of the order 25 000 MHz or  $0.83 \text{ cm}^{-1}$ , which is larger than a typical Doppler width for a visible or UV electronic transition.

In this chapter the interaction of light with matter has been discussed and key equations derived. For spectroscopy perhaps the most important equations are (1.57), (1.58), and (1.62) because they relate the microscopic molecular world to macroscopic absorption and emission rates. The transition dipole moment

$$\mathbf{M}_{10} = \boldsymbol{\mu}_{10} = \int \psi_1^* \boldsymbol{\mu} \psi_0 d\tau \quad (1.99)$$

will be used numerous times to derive selection rules. The absorption strength, equation (1.54), of a transition and the emission rate, equation (1.55), are proportional to the square of the transition dipole moment. The absorption ( $I/I_0$ ) associated with a particular molecular transition depends upon three factors: the absorption cross section  $\sigma$ , the population difference between the two levels,  $N_0 - N_1$ , and the optical path length  $l$  in equation (1.62), namely,

$$\frac{I}{I_0} = e^{-\sigma(N_0 - N_1)l}.$$

Spectroscopists use these relationships constantly.

### Problems

- The helium-neon laser operates at wavelength 6328.165 Å in air. The refractive index of air is 1.0002759 at this wavelength.
  - What is the speed of light in air and the vacuum wavelength?
  - What is the wavenumber  $\tilde{\nu}$  ( $\text{cm}^{-1}$ ) and the magnitude of the wavevector in air.
  - What are the frequency and the period of oscillation?
  - Calculate the energy and momentum of a photon with an air wavelength of 6328.165 Å.
  - What will be the wavelength and frequency in water if the refractive index is 1.3333?
- The refractive index of dry air at 15°C and 760 Torr pressure is given by the Cauchy formula

$$10^7(n - 1) = 2726.43 + 12.288 \frac{10^8}{\lambda^2} + 0.3555 \frac{10^{16}}{\lambda^4}$$

where  $\lambda$  is in Å (*CRC Handbook of Chemistry and Physics*, CRC Press). (The formula due to Edlén, *Metrologia* **2**, 71 (1966) is slightly more accurate but less convenient to use.)

- Calculate the refractive index of air at 4000 Å, 6000 Å, and 8000 Å.
  - Convert the air wavelengths into vacuum wavelengths and calculate the corresponding wavenumbers ( $\text{cm}^{-1}$ ).
- (a) What is the momentum and the de Broglie wavelength associated with a human weighing 150 lb and walking at 4 miles/hr?

- (b) What is the momentum and the de Broglie wavelength of an electron accelerated through a voltage of 100 V?
4. A crystal lattice has a typical spacing of 2 Å.
- (a) What velocity, momentum, and kinetic energy should be used for the electrons in an electron diffraction experiment? (*Hint:*  $\lambda \approx d$  (lattice spacing) for a good diffraction experiment.)
- (b) What voltage needs to be applied to the electron gun for the diffraction experiment?
- (c) Answer part (a) if neutrons were used instead of electrons.
- (d) If the diffraction experiment were carried out with photons, then what wavelength, energy, and momentum would be required?
5. (a) Make the necessary conversions in order to fill in the table:

Wavelength (Å)	420		
Wavenumber (cm <sup>-1</sup> )		100	
Energy (J)			
Energy (kJ/mole)		490	
Frequency (Hz)			8.21 × 10 <sup>13</sup>

- (b) Name the spectral region associated with each of the last four columns of the table.
6. There are two limiting cases associated with the Planck function

$$\rho_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}. \tag{1.100}$$

- (a) Calculate  $kT$  at room temperature (20°C) in J, kJ/mole, eV, and cm<sup>-1</sup>.
- (b) For long wavelengths (microwave frequencies at room temperature)  $h\nu \ll kT$ . In this case, derive a simpler, approximate equation (called the Rayleigh–Jeans law) for  $\rho_\nu(T)$ . This is the formula derived using classical arguments prior to Planck’s quantized oscillator approach.
- (c) For high-energy photons (near infrared wavelengths at room temperature)  $h\nu \gg kT$ . In this case, derive a simpler, approximate expression for  $\rho_\nu(T)$  called Wien’s formula.
7. (a) Differentiate the Planck function to determine the frequency at which  $\rho_\nu$  is a maximum (Figure 1.5).
- (b) Convert the Planck law from a function of frequency to a function of wavelength; that is, derive  $\rho_\lambda d\lambda$  from  $\rho_\nu d\nu$ .
- (c) Derive the Wien displacement law for blackbody radiation

$$\lambda_{\max} T = 2.898 \times 10^{-3} \text{ m K} \tag{1.101}$$

using  $\rho_\lambda d\lambda$ .

- (d) What wavelengths correspond to the maximum of the Planck function in interstellar space at 3 K, at room temperature (20°C), in a flame (2000°C), and in the photosphere of the sun (6000 K)?
8. The total power at all frequencies emitted from a small hole in the wall of a blackbody cavity is given by the Stefan–Boltzmann law

$$I = \sigma T^4,$$

where  $\sigma = 5.6705 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$  is the Stefan–Boltzmann constant.

- (a) Derive the Stefan–Boltzmann law.
- (b) Determine an expression for  $\sigma$  in terms of fundamental physical constants and obtain a numerical value. (*Hint:*  $\int_0^\infty x^3/(e^x - 1)dx = \pi^4/15$ .)
9. Derive the Wien displacement law (Problem 7) using  $\rho_\nu$  rather than  $\rho_\lambda$ .
10. Consider the two-level system (Figure 1.6) at room temperature, 20°C, and in the photosphere of the sun at 6000 K. What are the relative populations  $N_1/N_0$  corresponding to transitions that would occur at 6000 Å, 1000 cm<sup>-1</sup>, 100 GHz, and 1 GHz?
11. A 100-W tungsten filament lamp operates at 2000 K. Assuming that the filament emits as a blackbody, what is the total power emitted between 6000 Å and 6001 Å? How many photons per second are emitted in this wavelength interval?
12. (a) What is the magnitude of the electric field for the beam of a 1-mW helium–neon laser with a diameter of 1 mm?
- (b) How many photons per second are emitted at 6328 Å?
- (c) If the laser linewidth is 1 kHz, what temperature would a blackbody have to be at in order to emit the same number of photons from an equal area over the same frequency interval as the laser?
13. Derive the relationship (1.13) between the energy density  $\rho_\nu$  and the intensity  $I_\nu$  for a blackbody

$$I_\nu = \rho_\nu c_o/4.$$

(*Hint:* The total power passing through the hole and present in the solid angle  $d\Omega$  is  $\rho_\nu c \cos \theta (d\Omega/4\pi)$ .)

14. Solve equations (1.36a) and (1.36b) for the interaction of light with a two-level system.
- (a) First convert the two first-order simultaneous differential equations into a single second-order equation by substituting one equation (1.36a) into the other (1.36b).

- (b) The general solution for a second-order differential equation with constant coefficients is

$$a_0(t) = Ae^{i\alpha t} + Be^{i\beta t}.$$

Show that this form for  $a_0(t)$  implies that  $a_1(t)$  is given by

$$a_1(t) = \frac{2}{i\omega_R} e^{-i\Delta t} (Ai\alpha e^{i\alpha t} + Bi\beta e^{i\beta t}).$$

- (c) To obtain  $\alpha$  and  $\beta$  substitute the general solution into the second-order equation and obtain the *characteristic equation*

$$\alpha^2 - \Delta\alpha - \frac{\omega_R^2}{4} = 0.$$

The two solutions of this equation are  $\alpha$  and  $\beta$ . Find  $\alpha$  and  $\beta$ , and simplify your answer using the definition

$$\Omega \equiv (\omega_R^2 + \Delta^2)^{1/2}.$$

- (d) Determine  $A$  and  $B$  from the initial conditions at  $t = 0$ ,  $a_0(0) = 1$ , and  $a_1(0) = 0$ , and derive equations (1.37) and (1.38).
- (e) Verify that the final answer (1.37) and (1.38) satisfies the differential equations (1.36a) and (1.36b).
15. The  $3^2P_{3/2} - 3^2S_{1/2}$  transition of Na (actually the  $(F = 3) - (F = 2)$  hyperfine transition) has a Rabi frequency of  $4.15 \times 10^8$  rad/s with a laser intensity of  $560$  mW/cm<sup>2</sup>. What is the transition dipole moment in debye?
16. The lifetime of the  $3^2P_{1/2} \rightarrow 3^2S_{1/2}$  transition of the Na atom at  $5896 \text{ \AA}$  is measured to be  $16.4$  ns.
- (a) What are the Einstein  $A$  and  $B$  coefficients for the transition?
- (b) What is the transition dipole moment in debye?
- (c) What is the peak absorption cross section for the transition in  $\text{\AA}^2$ , assuming that the linewidth is determined by lifetime broadening?
17. What are the Doppler linewidths (in  $\text{cm}^{-1}$ ) for the pure rotational transition of CO at  $115$  GHz, the infrared transition of CO<sub>2</sub> at  $667 \text{ cm}^{-1}$ , and the ultraviolet transition of the Hg atom at  $2537 \text{ \AA}$ , all at room temperature ( $20^\circ\text{C}$ )?
18. Calculate the transit-time broadening for hydrogen atoms traversing a  $1\text{-mm}$  diameter laser beam. For the speed of the hydrogen atoms use the rms speed ( $v = (3kT/m)^{1/2}$ ) at room temperature ( $20^\circ\text{C}$ ).
19. At what pressure will the Doppler broadening (FWHM) equal the pressure broadening (FWHM) for a room temperature ( $20^\circ\text{C}$ ) sample of CO gas for a pure rotational transition at  $115$  GHz, a vibration-rotation transition at  $2140 \text{ cm}^{-1}$ , and an electronic transition at  $1537 \text{ \AA}$ ? Use a "typical" pressure-broadening coefficient of  $10 \text{ MHz/Torr}$  in all three cases.

20. What are the minimum spectral linewidths (in  $\text{cm}^{-1}$ ) of pulsed lasers with pulse durations of 10 fs, 1 ps, 10 ns, and 1  $\mu\text{s}$ ?
21. (a) For Na atoms in a flame at 2000 K and 760-Torr pressure, calculate the peak absorption cross section (at line center) for the  $3^2P_{1/2} - 3^2S_{1/2}$  transition at 5896 Å. Use 30 MHz/Torr as the pressure-broadening coefficient and the data in Problem 16.
- (b) If the path length in the flame is 10 cm, what concentration of Na atoms will produce an absorption ( $I/I_0$ ) of  $1/e$  at line center?
- (c) Is the transition primarily Doppler or pressure broadened?
- (d) Convert the peak absorption cross section in  $\text{cm}^2$  to the decadic peak molar absorption coefficient  $\epsilon$  (see equation (1.63)).

22. For Ar atoms at room temperature (20°C) and 1-Torr pressure, estimate a collision frequency for an atom from the van der Waals radius of 1.5 Å. What is the corresponding pressure-broadening coefficient in MHz/Torr?

23. A stationary atom of mass  $m$  emits a photon of energy  $h\nu$  and momentum  $\hbar k$ .
- (a) Use the laws of conservation of energy and momentum to show that the shift in frequency of the emitted photon due to recoil of the atom is given by

$$\frac{\nu - \nu_0}{\nu_0} \approx \frac{-h\nu_0}{2mc^2}.$$

- (b) What is the shift in frequency due to recoil of the atom for the Na  $D$  line at 5890 Å?
- (c) What is the shift in frequency for a  $\gamma$ -ray of energy 1369 keV emitted from  $^{24}\text{Na}$ ?
24. At the top of the earth's atmosphere the solar irradiance is 1368  $\text{W}/\text{m}^2$  (the solar constant). Calculate the magnitude of the electric field, assuming a plane wave at a single frequency for  $E$ .
25. At night, the concentration of the  $\text{NO}_3$  free radical is about  $10^9$  molecules/ $\text{cm}^3$  near the ground.  $\text{NO}_3$  has a visible absorption band near 662 nm, with a peak absorption cross section of  $2.3 \times 10^{-17} \text{ cm}^2 \text{ molecule}^{-1}$  at 298 K. For an absorption path of 1 km, what will be the change in atmospheric transmission ( $1 - I/I_0$ ) at 662 nm due to  $\text{NO}_3$ ?

26. For transit time broadening, consider the typical case (Figure 1.24) of a molecular beam crossing through a Gaussian laser beam, i.e., the applied electric field is given as

$$E = E_0 e^{-(t/a)^2} \cos \omega_0 t.$$

- (a) What is the normalized lineshape function,  $g(\nu - \nu_0)$ ?

$$(\text{Hint: } \int_{-\infty}^{\infty} e^{-p^2 x^2 \pm qx} dx = \frac{\sqrt{\pi}}{p} e^{q^2/4p^2}, \quad p > 0.)$$

- (b) What is the full width at half maximum  $\Delta\nu_{1/2}$  of  $g(\nu - \nu_0)$ ?

- (c) What is the  $\Delta\nu\Delta t$  product for a Gaussian beam? How does this value compare with that of the corresponding Heisenberg uncertainty principle?
- (d) The radial distribution of the electric field of a Gaussian laser beam is proportional to  $e^{-(r/w)^2}$ . What is the relationship between the Gaussian beam width parameter  $w$  and the parameter  $a$  defined above?

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## Chapter 2

# Molecular Symmetry

The language of group theory has become the language of spectroscopy. The concept of molecular symmetry and its application to the study of spectra of atoms and molecules (in the form of group theory) has proved to be of great value. Group theory is used to label and classify the energy levels of molecules. Group theory also provides qualitative information about the possibility of transitions between these energy levels. For example, the vibrational energy levels of a molecule can be labeled quickly by symmetry type and transitions between energy levels sorted into electric-dipole allowed and electric-dipole forbidden categories.

The concept of molecular symmetry is more subtle than expected because of the continuous motion of the atoms. As the molecule vibrates and rotates, which positions of the nuclei should be chosen as representative? In this book only the symmetry of a molecule at its equilibrium geometry will be considered in detail. Only in a few isolated examples, such as in the inversion of ammonia or in bent-linear correlation diagrams, is the possibility of fluxional behavior considered.

In some areas of spectroscopy, such as the study of hydrogen-bonded and van der Waals complexes (for example,  $(\text{H}_2\text{O})_2$ ), fluxional behavior is the norm rather than the exception.<sup>1</sup> The weak intermolecular bonds between the monomeric units in these systems allows many different geometrical isomers to interconvert rapidly. In this case group theory based on the permutations and inversions of nuclei<sup>2</sup> rather than on the customary symmetry operations is more useful.

## 2.1 Symmetry Operations

The idea of molecular symmetry can be quantified by the introduction of symmetry operations. A symmetry operation is a geometrical action (such as a reflection) that leaves the nuclei of a molecule in equivalent positions. These geometrical operations can be classified into four types: reflections ( $\hat{\sigma}$ ), rotations ( $\hat{C}_n$ ), rotation-reflections ( $\hat{S}_n$ ), and inversions ( $\hat{i}$ ). For mathematical reasons a fifth operation, the “do nothing” operation of identity ( $\hat{E}$ ), needs to be added.

Associated with each symmetry operation (except the identity) is a symmetry element. For example, associated with a particular reflection symmetry operation ( $\hat{\sigma}$ ) is a plane of symmetry ( $\sigma$ ) (see Figure 2.1). The distinction between symmetry operators and symmetry elements is quite important, and is a source of some confusion. The symmetry operation is the actual action, while the symmetry element is the point, line,

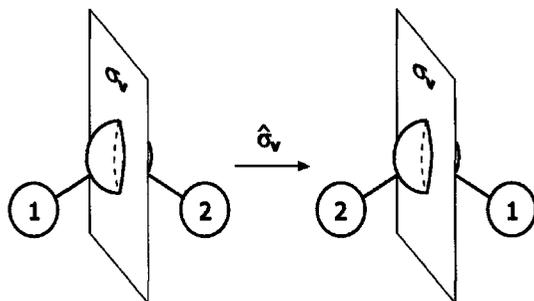


Figure 2.1: For the  $\text{H}_2\text{O}$  molecule, the  $\hat{\sigma}_v$  operation reflects hydrogen atom 1 into hydrogen atom 2 (and 2 into 1) through the associated plane of symmetry,  $\sigma_v$ .

or plane about which the action occurs. In this book operators will be marked with circumflexes (hats) (e.g.,  $\hat{\sigma}$ ) in order to distinguish between symmetry operators and symmetry elements.

### Operator Algebra

An operator is a mathematical prescription for transforming one function into another. For example, if  $\hat{A}$  represents an operator and  $f$  and  $g$  are two functions which are related by

$$\hat{A}f \rightarrow g, \quad (2.1)$$

then  $\hat{A}$  is said to transform  $f$  into  $g$ . For example, for symmetry operations a reflection might move an atom (represented by Cartesian coordinates  $(x, y, z)$ ) to a new location  $(x', y', z')$ ; this operation is represented by the expression

$$\hat{\sigma}(x, y, z) \rightarrow (x', y', z'). \quad (2.2)$$

Other examples of operators include the differential operator,  $\hat{D} \equiv d/dx$ , and the exponential operator,  $\exp \equiv e^{(\ )}$ . Linear operators, such as symmetry operators or differential operators, obey the rule

$$\hat{A}(af_1 + bf_2) = a\hat{A}f_1 + b\hat{A}f_2, \quad (2.3)$$

in which  $a$  and  $b$  are constants and  $f_1$  and  $f_2$  are arbitrary functions.

Operators can be combined together by addition, namely

$$\hat{C}f = (\hat{A} + \hat{B})f \equiv \hat{A}f + \hat{B}f \quad \text{or} \quad \hat{C} = (\hat{A} + \hat{B}) = \hat{A} + \hat{B} \quad (2.4)$$

and multiplication:

$$\hat{Z}f = (\hat{X}\hat{Y})f = \hat{X}(\hat{Y}f) \quad \text{or} \quad \hat{Z} = (\hat{X}\hat{Y}) = \hat{X}\hat{Y}. \quad (2.5)$$

For multiplication of operators (2.5), the operator on the right operates first. Multiplication of operators is simply defined as repeated operations, for example,  $\hat{X} \cdot \hat{X} \cdot \hat{Y} \equiv \hat{X}^2\hat{Y}$ . The inverse of an operator simply undoes the operation, that is

$$\hat{A}\hat{A}^{-1} = \hat{A}^{-1}\hat{A} = \hat{E} = \hat{1}, \quad (2.6)$$

in which  $\hat{E}$  and  $\hat{I}$  are the identity or unit operators. For example,  $\ln$  and  $\exp$  are an operator-inverse operator pair since  $\ln e^{(\cdot)} = \hat{I}(\cdot)$ .

Although operators always have an implied function  $f$  to the right, this function is usually suppressed. This leads to a compact notation but can also lead to confusion or fallacious conclusions—for example, for  $\hat{x} = x$  and  $\hat{D} = d/dx$ ,

$$\hat{D}\hat{x} = \hat{I} + \hat{x}\hat{D} \neq \hat{x}\hat{D}.$$

The addition and multiplication of operators strongly resembles ordinary algebra, with the exception that  $\hat{A}\hat{B}$  is not necessarily equal to  $\hat{B}\hat{A}$ . Operators, in general, do not commute, so that

$$\begin{aligned} (\hat{A} + \hat{B})^2 &= \hat{A}^2 + \hat{A}\hat{B} + \hat{B}\hat{A} + \hat{B}^2 \\ &\neq \hat{A}^2 + 2\hat{A}\hat{B} + \hat{B}^2. \end{aligned} \quad (2.7)$$

The similarity between ordinary algebra and operator algebra occurs because addition and multiplication are defined, and the associative law,

$$\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}, \quad (2.8)$$

and the distributive law,

$$\hat{A}(\hat{B} + \hat{C}) = \hat{A}\hat{B} + \hat{A}\hat{C}, \quad (2.9)$$

both hold.

### $\hat{E}$ Operator

The identity operator leaves a molecule unchanged. The symbol  $\hat{E}$  comes from the German word *Einheit* which means unity.

### $\hat{C}_n$ Operator

The rotation operator rotates a molecule by an angle of  $2\pi/n$  radians in a clockwise direction about a  $C_n$  axis (Figure 2.2). A molecule is said to possess an  $n$ -fold axis of symmetry if a rotation of  $2\pi/n$  radians leaves the nuclei in equivalent positions. Although we are able (as a matter of convenience) to label the nuclei in our drawings of molecules, real nuclei carry no labels. When a molecule has several rotational axes of symmetry the one with the largest value of  $n$  is called the *principal axis*.

Rotations can be repeated, i.e.,

$$\hat{C}_n^k = \hat{C}_n \cdot \hat{C}_n \cdots \hat{C}_n \text{ (} k \text{ times)}, \quad (2.10)$$

and  $\hat{C}_n^n = \hat{E}$ , since a rotation by  $n(2\pi/n) = 2\pi$  radians corresponds to no rotation at all. A rotation in the counterclockwise direction  $\hat{C}_n^{-1}$  (that is, by  $-2\pi/n$  radians) has the effect of undoing the  $\hat{C}_n$  operation, so that

$$\hat{C}_n \hat{C}_n^{-1} = \hat{C}_n^{-1} \hat{C}_n = \hat{E}. \quad (2.11)$$

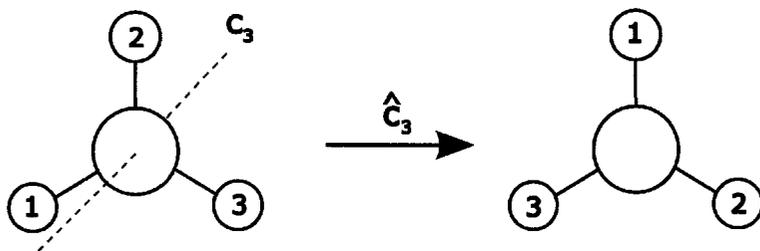


Figure 2.2: For the  $\text{BF}_3$  molecule, the  $\hat{C}_3$  operator rotates the F atoms by  $2\pi/3$  radians =  $120^\circ$  in a clockwise direction about the  $C_3$  axis out of the molecular plane. The clockwise direction is defined by viewing the molecule from the  $+z$  direction toward the  $xy$ -plane containing the molecule.

### $\hat{\sigma}$ Operator

The reflection operator reflects a molecule through a plane passing through the center of the molecule. If the nuclei are in equivalent positions after a reflection operation so that it remains indistinguishable, then the molecule has a plane of symmetry. The use of the Greek letter  $\sigma$  (sigma) originates from the German word *Spiegel* for mirror. Since a second reflection undoes the effect of the first reflection,  $\hat{\sigma}^2 = \hat{E}$ , the  $\hat{\sigma}$  operation is its own inverse.

There are three types of mirror planes. They are labeled by subscripts  $v$ ,  $h$ , and  $d$  (standing for vertical, horizontal, and dihedral). The vertical  $\sigma_v$  and the horizontal  $\sigma_h$  mirror planes are easy to spot because they either contain the principal axis of the molecule ( $\sigma_v$ ) or are perpendicular to it ( $\sigma_h$ ). Dihedral planes are more difficult to find and are special cases of vertical planes. A  $\sigma_d$  plane is a vertical plane that also bisects the angle between two adjacent twofold axes ( $C_2$ ) that are perpendicular to the principal axis. For example, the two types of vertical planes can be seen in Figure 2.3 for benzene; the  $\sigma_v$  planes bisect atoms, while the  $\sigma_d$  planes bisect bonds. In the case of benzene, by convention<sup>3</sup> the three  $C_2$  axes that coincide with the three  $\sigma_v$  planes are chosen to “pass through the greater numbers of atoms.”

### $\hat{S}_n$ Operator

The rotation-reflection operator is made up of a clockwise rotation by  $2\pi/n$  radians about a rotation axis, followed by a reflection in a plane perpendicular to that axis, i.e.,

$$\hat{S}_n = \hat{\sigma}_h \hat{C}_n \quad \text{or} \quad \hat{S}_n = \hat{C}_n \hat{\sigma}_h \quad (2.12)$$

since, in this case, the operations commute (Figure 2.4). If the atomic framework is unchanged by this operation, then the molecule is said to possess a rotation-reflection axis of symmetry or an improper axis of symmetry. All molecules that have a  $C_n$  axis of symmetry and a  $\sigma_h$  plane of symmetry must also have an  $S_n$  axis along the  $C_n$  axis.

### $\hat{i}$ Operator

The inversion operation  $\hat{i}$  is a special case of an improper rotation operation. The inversion operation  $\hat{i}$  inverts all atoms of the molecule through a point. If the molecule is coincident with itself after inversion, then it is said to possess a center (point) of symmetry.

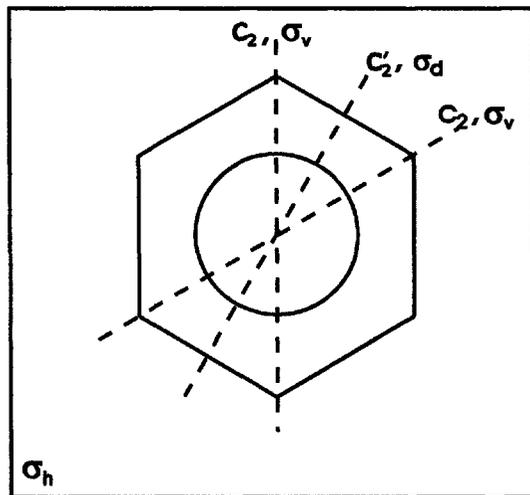


Figure 2.3: The benzene molecule has  $\sigma_h$ ,  $\sigma_v$ , and  $\sigma_d$  planes.

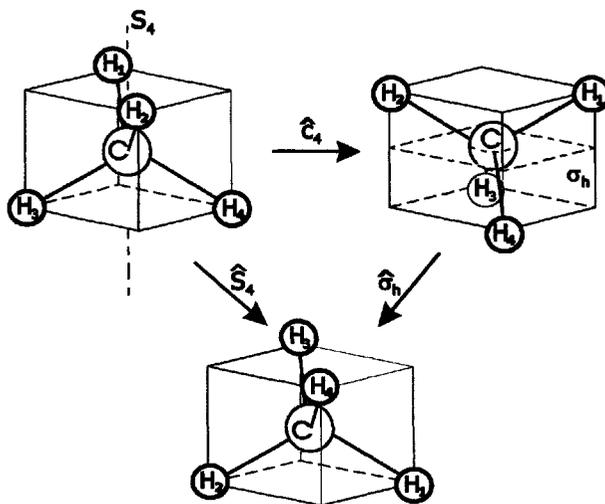


Figure 2.4: The  $\text{CH}_4$  molecule has an  $S_4$  axis that coincides with a  $C_2$  axis but does not have a  $C_4$  axis. The  $\hat{C}_4$  operator is not a symmetry operator, but  $\hat{S}_4 = \hat{\sigma}_h \hat{C}_4$  is one.

If the origin of the molecular coordinate system coincides with the center of symmetry (as is customary) (Figure 2.5), then the inversion operation changes the signs of the coordinates of an atom, that is,

$$\hat{i}(x, y, z) \rightarrow (-x, -y, -z). \quad (2.13)$$

The  $\hat{S}_2 = \hat{\sigma}_h \hat{C}_2 = \hat{C}_2 \hat{\sigma}_h$  operation is equivalent to inversion because

$$\hat{\sigma}_h \hat{C}_2(x, y, z) \rightarrow \hat{\sigma}_h(-x, -y, z) \rightarrow (-x, -y, -z), \quad (2.14)$$

where the  $z$ -axis coincides with the  $C_2$  axis. The inversion operator is also its own

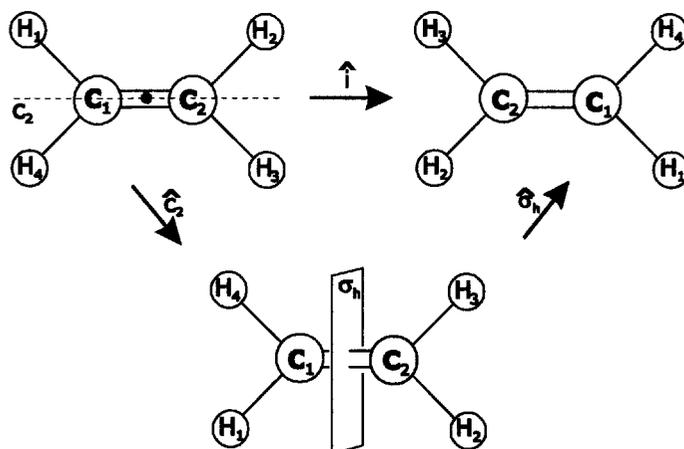


Figure 2.5: The ethylene molecule has a center of symmetry. Note that  $\hat{i} = \hat{\sigma}_h \hat{C}_2$ .

inverse since

$$(\hat{i})^2 = (\hat{C}_2 \hat{\sigma}_h)^2 = \hat{C}_2^2 \cdot \hat{\sigma}_h^2 = \hat{E}. \quad (2.15)$$

### Symmetry Operator Algebra

Symmetry operators can be applied successively to a molecule to produce new operators. For example, consider the ammonia molecule and the rotation and reflection operators. Figure 2.6 shows that the successive application of a rotation operator and a reflection operator generates a new reflection operator. Recall that the operator on the right operates first and that, in general, operators do not commute, for example,

$$\hat{C}_3 \hat{\sigma}_v'' \neq \hat{\sigma}_v'' \hat{C}_3, \quad (2.16)$$

(Figures 2.6 and 2.7).

Although “division” of symmetry operators is not defined, there is always an inverse operator that serves the same function. The inverse operator is useful in algebraic manipulations. For example, consider the product

$$\hat{C}_3 \hat{\sigma}_v'' = \hat{\sigma}_v'. \quad (2.17)$$

If we operate on the left by the inverse operator  $\hat{C}_3^{-1}$ , we obtain

$$\hat{C}_3^{-1} \hat{C}_3 \hat{\sigma}_v'' = \hat{C}_3^{-1} \hat{\sigma}_v', \quad (2.18)$$

or

$$\hat{\sigma}_v'' = \hat{C}_3^{-1} \hat{\sigma}_v'. \quad (2.19)$$

However, if we operate on the right of equation (2.17) by the inverse operator  $(\hat{\sigma}_v'')^{-1} = \hat{\sigma}_v''$ , we obtain the result

$$\hat{C}_3 (\hat{\sigma}_v'')^2 = \hat{\sigma}_v' \hat{\sigma}_v'', \quad (2.20)$$

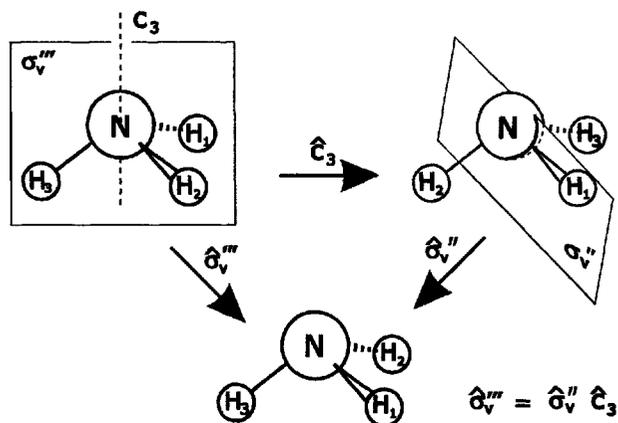


Figure 2.6: The application of  $\hat{C}_3$  and then  $\hat{\sigma}_v''$  to the ammonia molecule is equivalent to  $\hat{\sigma}_v'''$  ( $\hat{\sigma}_v'' \hat{C}_3 = \hat{\sigma}_v'''$ ). Notice that the  $\hat{\sigma}_v'$ ,  $\hat{\sigma}_v''$ ,  $\hat{\sigma}_v'''$  planes contain the original  $H_1$ ,  $H_2$ , and  $H_3$  atoms, respectively. The application of symmetry operators interchanges atoms, but the symmetry planes (and their labels) are unaffected.

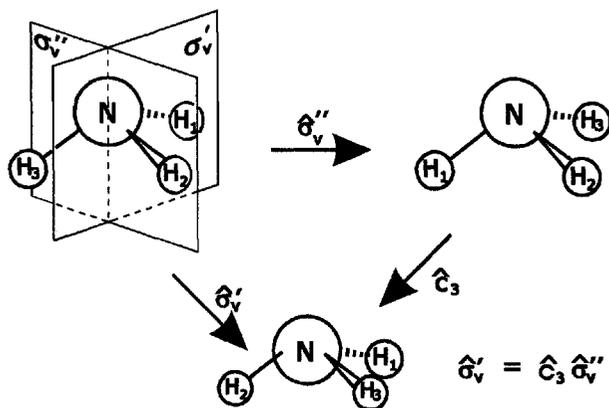


Figure 2.7: The application of  $\hat{\sigma}_v''$  and then  $\hat{C}_3$  to the ammonia molecule is equivalent to  $\hat{\sigma}_v'$  ( $\hat{C}_3 \hat{\sigma}_v'' = \hat{\sigma}_v'$ ).

or

$$\hat{C}_3 = \hat{\sigma}_v' \hat{\sigma}_v'' \quad (2.21)$$

In the preceding examples multiplication from the left by  $\hat{C}_3^{-1}$  (equation (2.18)) or from the right by  $\hat{\sigma}_v'' = (\hat{\sigma}_v'')^{-1}$  (equation (2.20)) produces two new equations, (2.19) and (2.21).

Care must be taken to preserve the order of operators. Taking the inverse of a product operator reverses the order of the component operators, that is

$$(\hat{A}\hat{B}\hat{C})^{-1} = \hat{C}^{-1}\hat{B}^{-1}\hat{A}^{-1} \quad (2.22)$$

To illustrate  $(\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1}$  we can use the symmetry operators associated with the  $\text{NH}_3$  molecule. For instance, from expression (2.17) we obtain

$$(\hat{C}_3 \hat{\sigma}_v'')^{-1} = (\hat{\sigma}_v')^{-1} = \hat{\sigma}_v'. \quad (2.23)$$

Multiplying (2.23) by  $\hat{\sigma}_v''$  from the right

$$(\hat{C}_3 \hat{\sigma}_v'')^{-1} (\hat{\sigma}_v'') = \hat{\sigma}_v' \hat{\sigma}_v'', \quad (2.24)$$

and then  $\hat{C}_3^{-1}$  from the right, we obtain, using equations (2.17) and (2.21),

$$(\hat{C}_3 \hat{\sigma}_v'')^{-1} (\hat{\sigma}_v'') \hat{C}_3^{-1} = \hat{\sigma}_v' \hat{\sigma}_v'' \hat{C}_3^{-1} = \hat{C}_3 \hat{C}_3^{-1} = \hat{E}. \quad (2.25)$$

This expression (2.25) is equivalent to

$$(\hat{C}_3 \hat{\sigma}_v'')^{-1} ((\hat{\sigma}_v'')^{-1} \hat{C}_3^{-1}) = \hat{E} \quad (2.26)$$

or

$$(\hat{C}_3 \hat{\sigma}_v'')^{-1} = (\hat{\sigma}_v'')^{-1} \hat{C}_3^{-1}, \quad (2.27)$$

as required.

## 2.2 Groups

A group is a set of elements along with a combining operation (commonly referred to as “group multiplication”) such that the following four rules are obeyed:

1. *Closure.* The product of any two elements must also be in the group. If  $P$  and  $Q$  are members of the group and  $PQ = R$ , then  $R$  is also a member of the group.
2. *Associative law.* As long as the elements are not interchanged, the order of multiplication is immaterial:  $(PQ)R = P(QR)$ .
3. *Identity element.* There is an identity element  $E$  in the group,  $RE = ER = R$ .
4. *Inverse.* Every element  $R$  has an inverse  $R^{-1}$  in the group,  $RR^{-1} = R^{-1}R = E$ .

This definition of a group is very general. The elements could be, for example, numbers, matrices, or symmetry operators and the combining operators could be addition, multiplication, or matrix multiplication. Note also that the elements of a group do not necessarily commute—that is,  $PQ \neq QP$ . If the group elements *do* commute with one another, then the group is called an *Abelian group*.

An example of a group (with an infinite number of members) is the set of positive and negative integers, including zero, under the operation of addition. The numbers 1,  $-1$ ,  $i = \sqrt{-1}$ , and  $-i$  form a group if the combining operation is multiplication. The number of elements in the group is called the *order*, so that  $\{1, -1, i, -i\}$  forms a group of order 4 under multiplication.

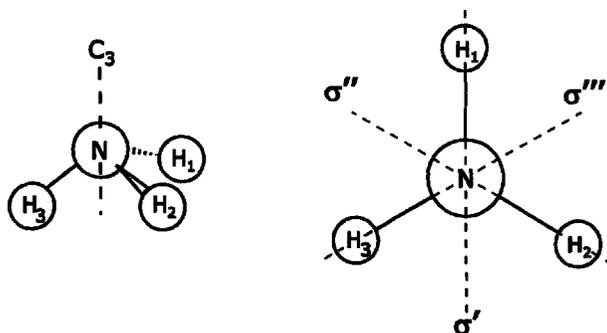


Figure 2.8: The elements of symmetry for the ammonia molecule.

Table 2.1: Group Multiplication Table for the Symmetry Operations Associated with  $\text{NH}_3$ 

$C_{3v}$	$\hat{E}$	$\hat{\sigma}'_v$	$\hat{\sigma}''_v$	$\hat{\sigma}'''_v$	$\hat{C}_3$	$\hat{C}_3^{-1}$
$\hat{E}$	$\hat{E}$	$\hat{\sigma}'_v$	$\hat{\sigma}''_v$	$\hat{\sigma}'''_v$	$\hat{C}_3$	$\hat{C}_3^{-1}$
$\hat{\sigma}'_v$	$\hat{\sigma}'_v$	$\hat{E}$	$\hat{C}_3$	$\hat{C}_3^{-1}$	$\hat{\sigma}''_v$	$\hat{\sigma}'''_v$
$\hat{\sigma}''_v$	$\hat{\sigma}''_v$	$\hat{C}_3^{-1}$	$\hat{E}$	$\hat{C}_3$	$\hat{\sigma}'''_v$	$\hat{\sigma}'_v$
$\hat{\sigma}'''_v$	$\hat{\sigma}'''_v$	$\hat{C}_3$	$\hat{C}_3^{-1}$	$\hat{E}$	$\hat{\sigma}'_v$	$\hat{\sigma}''_v$
$\hat{C}_3$	$\hat{C}_3$	$\hat{\sigma}'''_v$	$\hat{\sigma}'_v$	$\hat{\sigma}''_v$	$\hat{C}_3^{-1}$	$\hat{E}$
$\hat{C}_3^{-1}$	$\hat{C}_3^{-1}$	$\hat{\sigma}''_v$	$\hat{\sigma}'''_v$	$\hat{\sigma}'_v$	$\hat{E}$	$\hat{C}_3$

## Point Groups

Point symmetry groups are groups whose elements are the symmetry operations of molecules. They are called *point groups* because the center of mass of the molecule remains unchanged under all symmetry operations and all of the symmetry elements meet at this point.

For example, the point group associated with the ammonia molecule has six members,  $\{\hat{E}, \hat{C}_3, \hat{C}_3^{-1}, \hat{\sigma}'_v, \hat{\sigma}''_v, \hat{\sigma}'''_v\}$ , associated with the three vertical planes of symmetry and the  $C_3$  axis of symmetry (Figure 2.8). The six symmetry operations can be combined, and the results of all possible products are summarized in the group multiplication table (Table 2.1). The multiplication table is read by picking the column headed by the operator applied first (e.g.,  $\hat{\sigma}'_v$ ) and the row with the second operator (e.g.,  $\hat{\sigma}''_v$ ) and finding the symmetry operator  $\hat{C}_3^{-1}$  at their intersection; thus,  $\hat{\sigma}''_v \hat{\sigma}'_v = \hat{C}_3^{-1}$ . Notice that each operator appears in a given row or column of the multiplication table just once but in a different position. This result is known as the *Rearrangement Theorem*.

Two important terms in group theory are *isomorphic* and *homomorphic*. Two groups are isomorphic if there is a one-to-one correspondence between the elements of the two groups such that  $AB = C$  implies that  $A'B' = C'$  (Figure 2.9). The two groups, therefore, have the same multiplication table except for a change in symbols or in the meaning of the operators. For example, the two groups  $G_1$  and  $G_2$

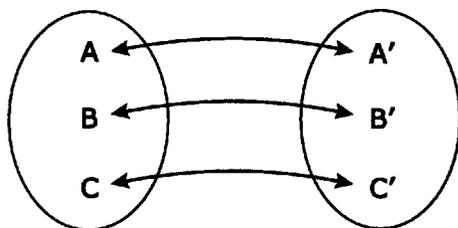


Figure 2.9: An isomorphic mapping.

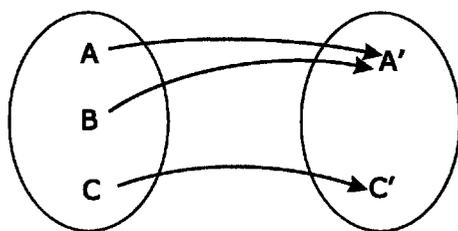


Figure 2.10: A homomorphic mapping.

$$G_1 = \{1, i, -1, -i\}$$

$$G_2 = \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \right\}$$

are isomorphic with the combining operations for  $G_1$  and  $G_2$  being multiplication and matrix multiplication, respectively. Two groups are homomorphic if there is a many-to-one relationship between some of the elements of the two groups. The structure of the two homomorphic groups is no longer identical in form, but the multiplication rules are preserved since  $AB = C$  and  $A'B' = C'$ . A homomorphic mapping allows a many-to-one correspondence between the elements of two groups (Figure 2.10). For example, there is the trivial homomorphic relationship between any group (e.g.,  $G_1$ ) and the group  $G_3$ , containing the number 1 as the only element,

$$G_1 = \{1, i, -1, -i\}$$

$$G_3 = \{1\}.$$

### Classes

The members of a group can be divided into classes. Two members of a group,  $P$  and  $R$ , belong to the same class if another member  $Z$  can be found such that  $P = Z^{-1}RZ$ :  $P$  and  $R$  are said to be conjugate to each other and they form a class.

For example, consider all possible classes associated with the symmetry group of ammonia. Clearly  $\hat{E}$  is in a class by itself since  $\hat{A}^{-1}\hat{E}\hat{A} = \hat{A}^{-1}\hat{A} = \hat{E}$  for any  $\hat{A}$ . For  $\hat{C}_3$  all possible other members of the class are

$$\hat{\sigma}'_v \hat{C}_3 \hat{\sigma}'_v = \hat{\sigma}'_v \hat{\sigma}'''_v = \hat{C}_3^{-1} \quad (2.28)$$

$$\hat{\sigma}''_v \hat{C}_3 \hat{\sigma}''_v = \hat{\sigma}''_v \hat{\sigma}'_v = \hat{C}_3^{-1} \quad (2.29)$$

$$\hat{\sigma}'''_v \hat{C}_3 \hat{\sigma}'''_v = \hat{\sigma}'''_v \hat{\sigma}''_v = \hat{C}_3^{-1} \quad (2.30)$$

$$\hat{C}_3^{-1} \hat{C}_3 \hat{C}_3 = \hat{E} \hat{C}_3 = \hat{C}_3 \quad (2.31)$$

$$\hat{C}_3 \hat{C}_3 \hat{C}_3^{-1} = \hat{C}_3 \hat{E} = \hat{C}_3 \quad (2.32)$$

so that  $\hat{C}_3$  and  $\hat{C}_3^{-1}$  are in the same class. Similarly for  $\hat{\sigma}'_v$  all possible other members of the class are

$$\hat{\sigma}'_v \hat{\sigma}'_v \hat{\sigma}'_v = \hat{\sigma}'_v \hat{E} = \hat{\sigma}'_v \quad (2.33)$$

$$\hat{\sigma}''_v \hat{\sigma}'_v \hat{\sigma}''_v = \hat{\sigma}''_v \hat{C}_3 = \hat{\sigma}'''_v \quad (2.34)$$

$$\hat{\sigma}'''_v \hat{\sigma}'_v \hat{\sigma}'''_v = \hat{\sigma}'''_v \hat{C}_3^{-1} = \hat{\sigma}''_v \quad (2.35)$$

$$\hat{C}_3^{-1} \hat{\sigma}'_v \hat{C}_3 = \hat{C}_3^{-1} \hat{\sigma}''_v = \hat{\sigma}'''_v \quad (2.36)$$

$$\hat{C}_3 \hat{\sigma}'_v \hat{C}_3^{-1} = \hat{C}_3^{-1} \hat{\sigma}'''_v = \hat{\sigma}''_v \quad (2.37)$$

so that  $\hat{\sigma}'_v$ ,  $\hat{\sigma}''_v$ , and  $\hat{\sigma}'''_v$  are in the same class. The symmetry group of order 6 for ammonia thus has three classes:  $\hat{E}$ ;  $\hat{C}_3$  and  $\hat{C}_3^{-1}$ ;  $\hat{\sigma}'_v$ ,  $\hat{\sigma}''_v$ , and  $\hat{\sigma}'''_v$ . Although it is not obvious from the mathematical definition of a class, the members of a class of a point group have a geometrical relationship to each other—for example, all involve reflections or rotations of a certain type.

### Subgroups

A subgroup is a subset of the elements of the full group that also forms a group under the same combination law. For example, the rotational subgroup associated with the  $\text{NH}_3$  point group is  $\{\hat{E}, \hat{C}_3, \hat{C}_3^{-1}\}$ . The order of a subgroup is always a factor of the order of the full group; that is, if  $g$  is the order of the full group and  $g_1$  is the order of the subgroup, then  $g$  is exactly divisible by  $g_1$ .

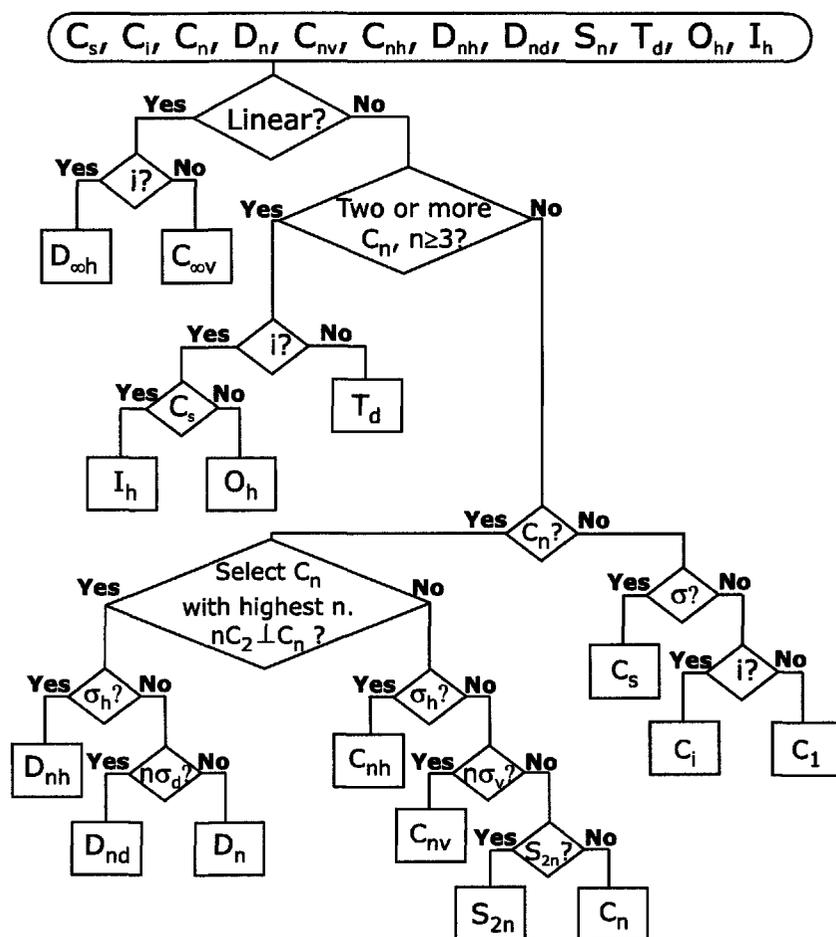
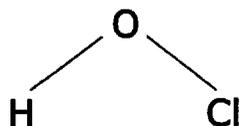


Figure 2.11: Flowchart for determining molecular symmetry.

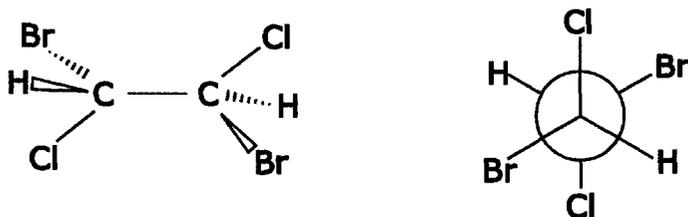
## 2.3 Notation for Point Groups

In this book the *Schoenflies notation* is used to label the possible point groups. In the list below only the essential symmetry elements are specified for the various groups, except for the tetrahedral ( $T_d$ ), octahedral ( $O_h$ ), icosahedral ( $I_h$ ), and spherical ( $K_h$ ) groups, which can be recognized by inspection. Notice that the structures are assumed to be rigid; the possibility of rotation around carbon-carbon single bonds is ignored. A flowchart is provided (Figure 2.11) as an aid for determining molecular symmetry.

$C_s$ . One symmetry plane.

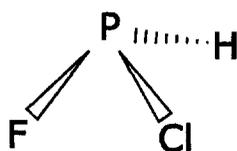


$C_i$ . One center of symmetry.

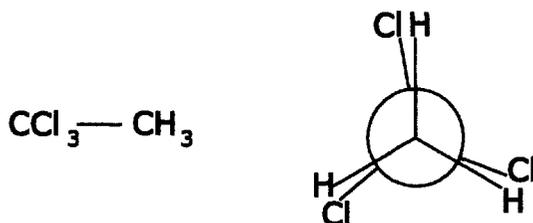


$C_n$ . A simple  $C_n$  axis.

$C_1$ . No elements of symmetry.

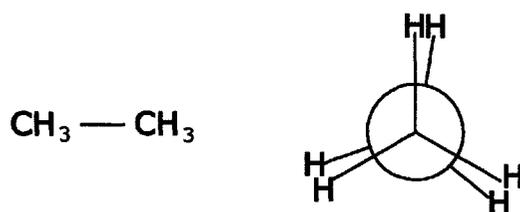


$C_3$ .  $\text{CCl}_3\text{—CH}_3$ . Neither eclipsed nor staggered.



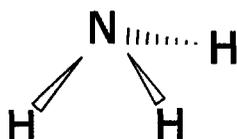
$D_n$ . One  $C_n$  axis and  $nC_2$  axes perpendicular to it.

$D_3$ .  $\text{CH}_3\text{—CH}_3$ . Neither eclipsed nor staggered.



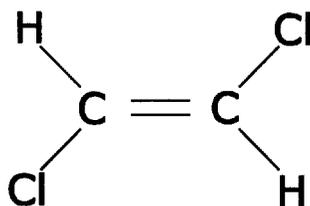
$C_{nv}$ . One  $C_n$  axis with  $n\sigma_v$  planes.

$C_{3v}$ . Ammonia.



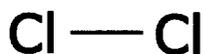
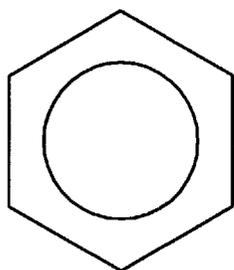
$C_{nh}$ . One  $C_n$  axis with a  $\sigma_h$  plane.

$C_{2h}$ . *trans*-Dichloroethylene.



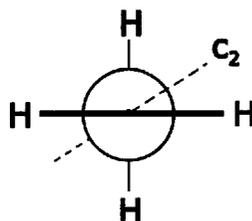
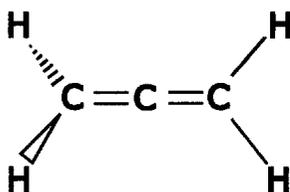
$D_{nh}$ . One  $C_n$  axis,  $nC_2$  axes perpendicular to the  $C_n$  axis, and one  $\sigma_h$  plane.

$D_{6h}$ . Benzene.  $D_{\infty h}$ . Cl—Cl.



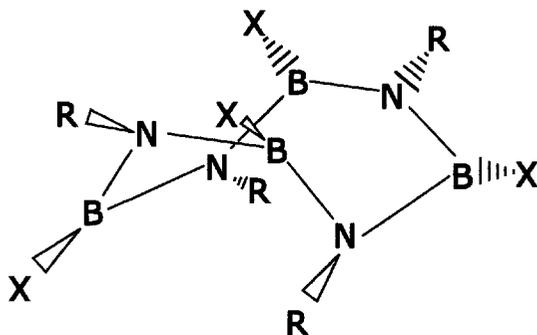
$D_{nd}$ . One  $C_n$  axis,  $nC_2$  axes perpendicular to the  $C_n$  axis and  $n\sigma_d$  planes.

$D_{2d}$ . Allene.

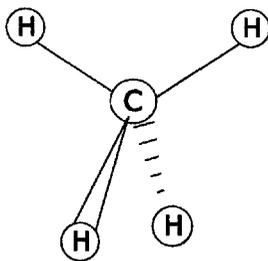


$S_n$ . One  $S_n$  axis.

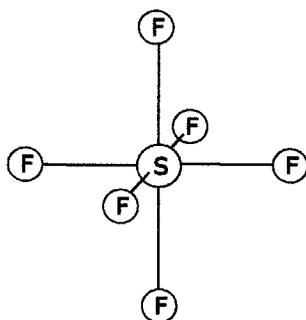
$S_4$ . Puckered octagon.



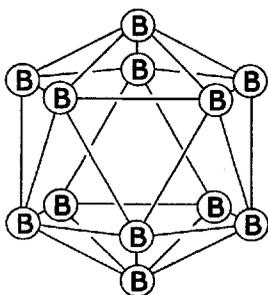
$T_d$ .  $\text{CH}_4$ .



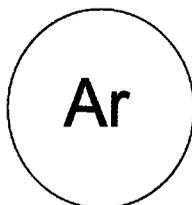
$O_h$ .  $\text{SF}_6$ .



$I_h$ .  $\text{B}_{12}\text{H}_{12}^{2-}$ .



$K_h$ . Ar.



## Problems

- Construct the group multiplication table for the symmetry operations of the  $\text{H}_2\text{CO}$  molecule.
- Construct the group multiplication table for the symmetry operations of the  $\text{BF}_3$  molecule.
- Determine the point groups for the following molecules:
  - $\text{CF}_4$
  - $\text{CH}_2\text{ClBr}$
  - $\text{CH}_3\text{Br}$
  - $\text{PH}_3$
  - $\text{BCl}_3$
  - Cyclohexane, chair
  - Cyclohexane, boat
  - Ferrocene, staggered
  - Acetylene
  - Ethylene
- List all of the symmetry operations for the following molecules:
  - trans*-Dichloroethylene
  - $\text{CH}_3\text{-CH}_3$ , neither eclipsed nor staggered
  - $\text{SF}_6$
  - Allene
  - $\text{NC-CN}$
- Determine the point groups of the following square planar complexes:
  - $[\text{Pt}(\text{Cl})_4]^{2-}$
  - $[\text{Pt}(\text{Cl})_3\text{CN}]^{2-}$
  - cis*- $[\text{Pt}(\text{Cl})_2(\text{CN})_2]^{2-}$
  - trans*- $[\text{Pt}(\text{Cl})_2(\text{CN})_2]^{2-}$
- Determine the point groups of the following octahedral molecules:
  - $\text{MA}_6$
  - $\text{MA}_5\text{B}$
  - cis*- $\text{MA}_4\text{B}_2$
  - trans*- $\text{MA}_4\text{B}_2$
  - fac*- $\text{MA}_3\text{B}_3$
  - mer*- $\text{MA}_3\text{B}_3$

7. It is generally stated that a molecular species is optically active if its mirror image is not superimposable upon the original structure. A more universally applicable test is for the presence of an improper rotation axis. When an improper axis is present, the structure is optically inactive. Using this criterion, determine whether or not the following structures are optically active:

- (a) *trans*-1,2-Dichlorocyclopropane
- (b) Ethane (neither staggered nor eclipsed)
- (c) CHFCIBr
- (d)  $\text{Co}(\text{en})_3^{3+}$
- (e) *cis*- $[\text{Co}(\text{en})_2\text{Cl}_2]^+$
- (f) *trans*- $[\text{Co}(\text{en})_2\text{Cl}_2]^+$

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## Chapter 3

# Matrix Representation of Groups

So far the symmetry operations associated with a molecule have been defined in geometric terms. If the atom locations are represented by their Cartesian coordinates, then the symmetry operators can be represented by matrices. In general a set of matrices homomorphic with the point group operations can be found by the methods described in this chapter. These matrices form a representation of the group. A brief summary of some properties of matrices is made first. Proofs of these properties are left as exercises or can be found in various standard references.

### 3.1 Vectors and Matrices

A point in space can be described by the vector (Figure 3.1)

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \quad (3.1)$$

where in this context the circumflexes are used to denote unit vectors rather than operators. The magnitude of the vector is given by

$$|\mathbf{r}| = r = (x^2 + y^2 + z^2)^{1/2} \quad (3.2)$$

and its direction is specified by the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  in the planes containing  $\mathbf{r}$  and the  $x$ -,  $y$ -, and  $z$ -axes, respectively.

If  $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$  and  $\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$ , then the dot product of two vectors  $\mathbf{A}$  and  $\mathbf{B}$  is the scalar given as

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z = |\mathbf{A}||\mathbf{B}|\cos \theta, \quad (3.3)$$

in which  $\theta$  is the angle between the two vectors. The cross product of two vectors  $\mathbf{A}$  and  $\mathbf{B}$  gives a third vector,  $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ , with magnitude  $|\mathbf{C}|$  given by

$$|\mathbf{C}| = |\mathbf{A}||\mathbf{B}|\sin \theta, \quad (3.4)$$

and with direction perpendicular to the plane defined by  $\mathbf{A}$  and  $\mathbf{B}$ . In terms of components  $\mathbf{C}$  can be written as

$$\mathbf{C} = (A_y B_z - A_z B_y)\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\hat{\mathbf{k}}. \quad (3.5)$$

Alternatively,  $\mathbf{C}$  can be expressed in the form of a determinant (defined later) as

$$\mathbf{C} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \quad (3.6)$$

Spherical polar coordinates  $r, \theta, \phi$  are illustrated in Figure 3.2 and  $\mathbf{r}$  can be represented by

$$\mathbf{r} = r(\sin \theta \cos \phi \hat{\mathbf{i}} + \sin \theta \sin \phi \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}}). \quad (3.7)$$

A matrix is in general an  $m \times n$  rectangular array of numbers (real or complex)

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix} \quad (3.8)$$

with a typical element  $A_{ij}$ . A vector is in general a one-dimensional matrix—either an  $n \times 1$  matrix (column vector) or a  $1 \times n$  matrix (row vector). However, in this book only square  $n \times n$  matrices will be used and vectors will be assumed to be column vectors, for example

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \quad (3.9)$$

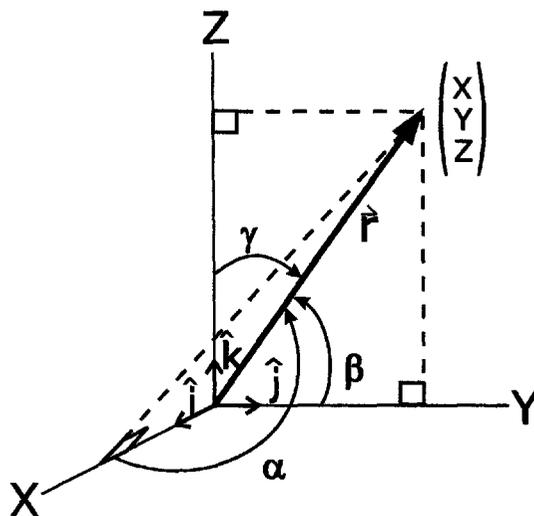


Figure 3.1: Cartesian coordinates.

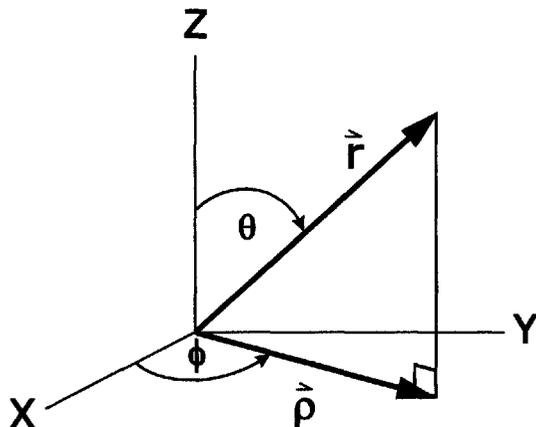


Figure 3.2: Spherical polar coordinates.

A row vector  $\mathbf{x}^t = (x_1 x_2 \cdots x_n)$  is the transpose of the corresponding column vector.

A number of matrix operations will be employed, including:

- complex conjugation, giving the matrix  $\mathbf{A}^*$ , with

$$(A^*)_{ij} = (A_{ij})^*; \quad (3.10)$$

- transposition, giving the transpose matrix  $\mathbf{A}^t$ , with

$$(A^t)_{ij} = A_{ji}; \quad (3.11)$$

- Hermitian conjugation defined via  $\mathbf{A}^\dagger \equiv (\mathbf{A}^t)^* = (\mathbf{A}^*)^t$ , with

$$(A^\dagger)_{ij} = (A_{ji})^*; \quad (3.12)$$

- inversion, giving the inverse matrix  $\mathbf{A}^{-1}$  (see below), with

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{1}; \quad (3.13)$$

- formation of the trace, designated  $\text{tr}(\mathbf{A})$ , with

$$\text{tr}(\mathbf{A}) \equiv \sum_{i=1}^n A_{ii}; \quad (3.14)$$

- formation of the determinant, designated  $|\mathbf{A}|$ , with

$$|\mathbf{A}| = \sum_{j=1}^n A_{ij} M_{ij}. \quad (3.15)$$

The quantities  $M_{ij}$  in equation (3.15) are known as *cofactors*, and they are obtained by striking out the  $i$ th row and the  $j$ th column of the original  $n \times n$  determinant to give an  $(n-1) \times (n-1)$  determinant and multiplying by  $(-1)^{i+j}$ :

$$M_{ij} = (-1)^{i+j} \begin{vmatrix} A_{11} & \cdots & A_{1j} & \cdots & A_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{i1} & \cdots & A_{ij} & \cdots & A_{in} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{n1} & \cdots & A_{nj} & \cdots & A_{nn} \end{vmatrix}. \tag{3.16}$$

The smaller determinants  $((n - 1) \times (n - 1)$  in size) in the cofactors can be expanded in a similar fashion to create in the end after repeated expansion, the sum of  $n!$  terms. For example, consider the simple case of a  $3 \times 3$  determinant:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{vmatrix} &= 1 \begin{vmatrix} 5 & 6 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} \\ &= 1(5 - 12) - 2(4 - 18) + 3(8 - 15) \\ &= 0. \end{aligned}$$

Matrix addition of two matrices **A** and **B** to give a third matrix **C**, written

$$\mathbf{A} + \mathbf{B} = \mathbf{C},$$

is defined only if **A** and **B** have the same number of rows and columns, so that

$$A_{ij} + B_{ij} = C_{ij}. \tag{3.17}$$

The matrix product **C** of two matrices **A** and **B** is written as

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{AB} = \mathbf{C},$$

with the matrix element  $C_{ij}$  of **C** given by

$$C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}. \tag{3.18}$$

As an example, consider the product of two  $2 \times 2$  matrices:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}.$$

Matrix multiplication, however, is not commutative in general since

$$(\mathbf{BA})_{ij} = \sum_{k=1}^n B_{ik} A_{kj} \quad \text{while} \quad (\mathbf{AB})_{ij} = \sum_{k=1}^n A_{ik} B_{kj}. \tag{3.19}$$

In some particular cases these two products will be equal, in which case we say that these particular matrices commute. The noncommutativity of the product of two matrices can be illustrated by interchanging the order of multiplication in the example preceding equation (3.19), resulting in a different product matrix:

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}.$$

It will also be useful to take another look at the inner product  $\mathbf{a} \cdot \mathbf{b}$  between the two column vectors  $\mathbf{a}$  and  $\mathbf{b}$  in the context of the matrix product defined by equation (3.19). To utilize this format, in which the elements of the product matrix are determined via multiplication of the rows of the matrix on the left-hand side of the product with the columns of the matrix on the right-hand side of the product (by multiplying each pair of corresponding elements and summing), we write  $\mathbf{a}$  as a  $1 \times n$  (row) matrix, which we designate as  $\mathbf{a}^t$  (the transpose of the  $n \times 1$  column matrix  $\mathbf{a}$ ) and form the matrix product  $\mathbf{a}^t \mathbf{b} = \sum a_i b_i$ . Notice that this matrix product is equal to the matrix product  $\mathbf{b}^t \mathbf{a} = \sum b_i a_i$  in the reverse order, so that the inner product of two vectors of real numbers, looked at in this context, is a commutative product. We shall return to this just below, when we need to generalize this idea to allow for vectors of complex numbers rather than real numbers, as at present. We shall also introduce here a notation for the inner product of two vectors that can be generalized readily later: we shall designate the inner (dot) product between two vectors

$$\langle \mathbf{a} | \mathbf{b} \rangle \equiv \mathbf{a}^t \mathbf{b} = \mathbf{b}^t \mathbf{a} = \langle \mathbf{b} | \mathbf{a} \rangle. \quad (3.20)$$

If we allow the elements of the (column) vectors  $\mathbf{a}$  and  $\mathbf{b}$  to be complex numbers, we must generalize the definition of the inner product between  $\mathbf{a}$  and  $\mathbf{b}$  to

$$\langle \mathbf{a} | \mathbf{b} \rangle \equiv \mathbf{a}^\dagger \mathbf{b}, \quad (3.21)$$

in which  $\mathbf{a}^\dagger$  is called the *adjoint* (Hermitian conjugate) vector to  $\mathbf{a}$ . This generalization for complex-valued elements is necessitated by the requirement that the length of a vector be a real number: if  $\mathbf{a}$  is a vector with complex-valued components, then its length will be defined via  $|\mathbf{a}|^2 \equiv \sum a_k^* a_k$ , which is a real number (rather than as  $\sum a_k a_k$ , which is not a real number). Notice, however, that for complex vectors, the inner product  $\langle \mathbf{a} | \mathbf{b} \rangle$  is not the same as the inner product  $\langle \mathbf{b} | \mathbf{a} \rangle$ : they are, in fact, related by

$$\langle \mathbf{b} | \mathbf{a} \rangle = \langle \mathbf{a} | \mathbf{b} \rangle^*, \quad (3.22)$$

so that they are complex conjugates of one another.

We shall encounter a number of special types of matrix: in addition to the identity matrix, designated by  $\mathbf{1}$  and with elements given by

$$(\mathbf{1})_{ij} = \delta_{ij}, \quad (3.23)$$

(the Kronecker delta,  $\delta_{ij}$ , has a value of 1 for  $i = j$  and 0 for  $i \neq j$ ), we have symmetric matrices, which have the property that they are equal to their transposes, i.e.,

$$\mathbf{A}^t = \mathbf{A}; \quad (3.24)$$

Hermitian matrices, which have the property that they are equal to their Hermitian conjugates, i.e.,

$$\mathbf{A}^\dagger = \mathbf{A}; \quad (3.25)$$

orthogonal matrices, which have the property that their inverses are equal to their transposes,

$$\mathbf{A}^t = \mathbf{A}^{-1}; \quad (3.26)$$

and unitary matrices, which have the property that their inverses are equal to their Hermitian conjugates, i.e.,

$$\mathbf{A}^\dagger = \mathbf{A}^{-1}. \quad (3.27)$$

Matrices are useful in transforming the magnitude and direction of a vector. Thus if  $\mathbf{y}$  is an  $n \times 1$  vector and  $\mathbf{A}$  is an  $n \times n$  matrix, then a new  $n \times 1$  vector  $\mathbf{y}'$  is related to  $\mathbf{y}$  by the matrix equation  $\mathbf{y}' = \mathbf{A}\mathbf{y}$ , or

$$\begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{pmatrix} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}. \quad (3.28)$$

Unitary (or orthogonal) transformations of vectors do not alter the magnitude of a vector but do change its direction. Matrices can also be used in inverting a set of linear equations. Thus, if a set of inhomogeneous linear equations is represented by

$$\mathbf{A}\mathbf{x} = \mathbf{c}, \quad (3.29)$$

with  $\mathbf{c}$  a column vector of constants, then multiplication from the left by  $\mathbf{A}^{-1}$  gives  $\mathbf{x}$  in terms of  $\mathbf{c}$  via

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{A}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{c}, \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{c}. \end{aligned} \quad (3.30)$$

The inverse matrix can be calculated in a number of ways, such as

$$(\mathbf{A}^{-1})_{ij} = \frac{M_{ji}}{|\mathbf{A}|} \quad (3.31)$$

or

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} M_{11} & \cdots & M_{n1} \\ \vdots & & \vdots \\ M_{1n} & \cdots & M_{nn} \end{pmatrix} \quad (3.32)$$

where  $M_{ij}$  is a cofactor of  $\mathbf{A}$  defined in equation (3.16).

### ***Matrix Eigenvalue Problem***

The  $n \times n$  matrix  $\mathbf{A}$  has  $n$  *eigenvalues* and  $n$  *eigenvectors* defined by the equation

$$\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i, \quad i = 1, \dots, n.$$

The eigenvectors are special vectors whose directions are unchanged (but are stretched or shrunk by the factor  $\lambda$ ) when transformed by  $\mathbf{A}$ . These  $n$  eigenvectors are determined by solving a set of  $n$  linear homogeneous equations

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \quad (3.33)$$

or

$$(\mathbf{A} - \lambda\mathbf{1})\mathbf{x} = \mathbf{0}. \quad (3.34)$$

These equations have a trivial solution,  $\mathbf{x} = \mathbf{0}$ , and a set of  $n$  nontrivial solutions obtained from the  $n$ th-order polynomial in  $\lambda$  (also referred to as the secular determinant or secular equation),

$$|\mathbf{A} - \lambda\mathbf{1}| = 0. \quad (3.35)$$

Each eigenvector  $\mathbf{x}_i$  associated with each  $\lambda_i$  can be determined by substituting  $\lambda_i$  in equation (3.34) and solving for the components of  $\mathbf{x}$ . Usually the eigenvectors are normalized to unity:

$$|\mathbf{x}| = (\mathbf{x}^\dagger \mathbf{x})^{1/2} = \sqrt{\sum_{i=1}^n x_i^* x_i} = \sqrt{\sum_{i=1}^n |x_i|^2}. \quad (3.36)$$

These  $n$  eigenvectors, expressed as column vectors, can be arranged in a matrix  $\mathbf{X} = (\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n)$ , and the  $n$  eigenvalue equations written as

$$\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{\Lambda}, \quad (3.37)$$

in which  $\mathbf{\Lambda}$  is the diagonal matrix,

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

made up of the eigenvalues  $\lambda_i$ . From relation (3.37), we see that  $\mathbf{\Lambda}$  can be determined from  $\mathbf{A}$  as

$$\mathbf{\Lambda} = \mathbf{X}^{-1} \mathbf{A} \mathbf{X}. \quad (3.38)$$

For example, if the matrix  $\mathbf{A}$  is

$$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix},$$

then from equation (3.35) the secular determinant is

$$\begin{vmatrix} 4 - \lambda & 1 \\ 2 & 3 - \lambda \end{vmatrix} = 0,$$

from which the second degree eigenvalue equation is

$$\lambda^2 - 7\lambda + 10 = 0,$$

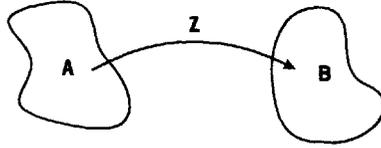


Figure 3.3: A matrix transformation.

with solutions

$$\lambda_1 = 2, \quad \lambda_2 = 5.$$

Substitution of the eigenvalues  $\lambda_1$  and  $\lambda_2$  in turn into equation (3.34) and solving for the components of  $\mathbf{x}_1$  and  $\mathbf{x}_2$  gives

$$\mathbf{x}_1 = \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{and} \quad \mathbf{X} = \begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

If  $\mathbf{A}$  is Hermitian,  $\mathbf{A}^\dagger = \mathbf{A}$ , then the eigenvalues are real and the eigenvectors can always be made to be orthogonal to one another. In this case  $\mathbf{X}$  is a unitary matrix since, in general, the elements of the  $\mathbf{x}_i$  will be complex numbers. If  $\mathbf{A}$  is real and symmetric, then the  $\mathbf{x}_i$  will also be real and  $\mathbf{X}$  is then an orthogonal matrix. The determination of eigenvectors is equivalent to finding a matrix  $\mathbf{X}$  which transforms  $\mathbf{A}$  into the diagonal form (3.38). Surprisingly, this is a simpler problem (for a computer!) than finding the zeros of the  $n$ th-order polynomial generated by the secular equation (3.35).

### Similarity Transformations

The matrix  $\mathbf{A}$  is transformed into the matrix  $\mathbf{B}$  via a matrix  $\mathbf{Z}$  by the relationship

$$\mathbf{Z}^{-1}\mathbf{A}\mathbf{Z} = \mathbf{B}.$$

This is a similarity transformation (Figure 3.3). If  $\mathbf{Z}$  is a unitary (or orthogonal) matrix, then  $\mathbf{A}$  and  $\mathbf{B}$  are related via a unitary (or orthogonal) transformation. Notice that if  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{Z}$  were matrix representations of symmetry operators, then  $\mathbf{A}$  and  $\mathbf{B}$  would be in the same class. If  $\mathbf{A}$  and  $\mathbf{B}$  are similar then the eigenvalues of  $\mathbf{A}$  are the same as  $\mathbf{B}$  (but the eigenvectors are different), and  $|\mathbf{A}| = |\mathbf{B}|$ , and  $\text{tr}(\mathbf{A}) = \text{tr}(\mathbf{B})$ . If  $\mathbf{Z}$  transforms  $\mathbf{A}$  into the diagonal matrix  $\mathbf{B}$ , then the matrix  $\mathbf{Z}$  diagonalizes  $\mathbf{A}$  and the eigenvalues are found in  $\mathbf{B}$ . A Hermitian (or symmetric) matrix is diagonalized by a unitary (or orthogonal) transformation.

## 3.2 Symmetry Operations and Position Vectors

Perhaps the simplest way to generate a matrix representation of a group is to consider the effect of a symmetry operation on a point

$$\mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

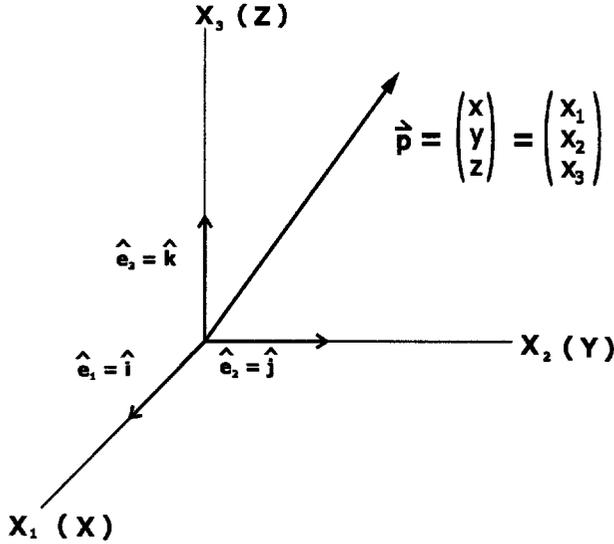


Figure 3.4: Cartesian coordinate system.

in space. The point is located in real three-dimensional space (Figure 3.4).

It is convenient to replace the familiar  $X$ ,  $Y$ , and  $Z$  axes by  $X_1$ ,  $X_2$ , and  $X_3$ , respectively, as well as the usual cartesian unit vectors  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  by  $\hat{e}_1$ ,  $\hat{e}_2$ , and  $\hat{e}_3$  (Figure 3.4). In this case, we can write  $\mathbf{p}$  formally as

$$\mathbf{p} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 = \sum_{i=1}^3 x_i\hat{e}_i.$$

### Reflection

A reflection in the  $X_1$ - $X_3$  ( $X$ - $Z$ ) plane changes the  $x_2$  component of  $\mathbf{p}$  to  $-x_2$ , so that

$$\hat{\sigma}_{13} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (3.39)$$

so that  $\hat{\sigma}_{13}$  may be represented by the matrix

$$\mathbf{D}(\hat{\sigma}_{13}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, matrix representations for  $\hat{\sigma}_{12}$  and  $\hat{\sigma}_{23}$  are

$$\mathbf{D}(\hat{\sigma}_{12}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{D}(\hat{\sigma}_{23}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

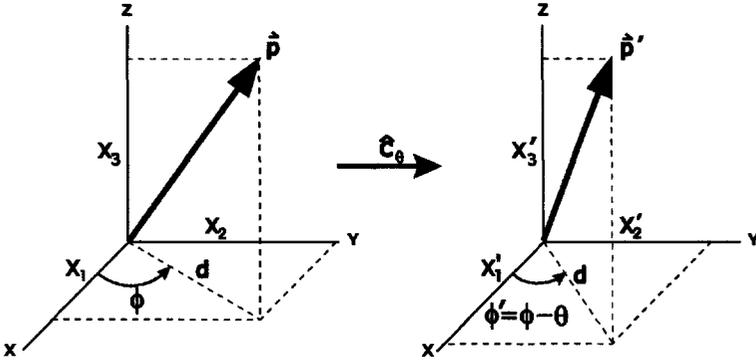


Figure 3.5: The  $\hat{C}_\theta^{(Z)}$  operation rotates the point  $\mathbf{p}$  by  $\theta$  radians about the  $X_3$ -axis.

**Rotation**

The clockwise rotation by  $\theta$  radians about the  $X_3(Z)$ -axis produces a new set of coordinates (Figure 3.5):

$$\hat{C}_\theta^{(Z)} \mathbf{p} = \mathbf{p}', \tag{3.40}$$

or

$$\hat{C}_\theta \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}, \tag{3.41}$$

in which

$$\begin{aligned} x'_1 = d \cos \phi' &= d \cos(\phi - \theta) \\ &= d \cos \phi \cos \theta + d \sin \phi \sin \theta \\ &= x_1 \cos \theta + x_2 \sin \theta, \end{aligned} \tag{3.42}$$

$$\begin{aligned} x'_2 = d \sin \phi' &= d \sin(\phi - \theta) \\ &= d \sin \phi \cos \theta - d \cos \phi \sin \theta \\ &= x_2 \cos \theta - x_1 \sin \theta \\ &= -x_1 \sin \theta + x_2 \cos \theta, \end{aligned} \tag{3.43}$$

and

$$x'_3 = x_3. \tag{3.44}$$

These equations can be expressed compactly in matrix form as

$$\begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \tag{3.45}$$

with the matrix representation  $\mathbf{D}(\hat{C}_\theta)$  of a clockwise rotation by an angle  $\theta$  about the  $X_3$ -axis given by

$$\mathbf{D}(\hat{C}_\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.46)$$

The matrix for  $\hat{C}_\theta^{-1} = \hat{C}_{-\theta}$  is given by

$$\begin{aligned} \mathbf{D}(\hat{C}_\theta^{-1}) &= \begin{pmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (3.47)$$

$\mathbf{D}(\hat{C}_\theta)$  is an orthogonal matrix since  $\mathbf{D}(\hat{C}_\theta^{-1}) = (\mathbf{D}(\hat{C}_\theta))^t$  as required.

### *Inversion*

The operation of inversion  $\hat{i}$  changes the signs of all coordinates

$$\hat{i} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} \quad (3.48)$$

so that its matrix representation is

$$\mathbf{D}(\hat{i}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3.49)$$

### *Rotation-Reflection*

The improper rotation operation,  $\hat{S}_\theta$ , corresponds to a  $\hat{C}_\theta$  operation about the  $X_3$ -axis followed by a horizontal reflection  $\hat{\sigma}_{12}$  in the  $X_1$ - $X_2$  ( $X$ - $Y$ ) plane. As the matrix representations of  $\hat{C}_\theta$  and  $\hat{\sigma}_{12}$  have already been derived, the matrix for  $\hat{S}_\theta$  can be determined simply by matrix multiplication: thus,

$$\mathbf{D}(\hat{S}_\theta) = \mathbf{D}(\hat{\sigma}_{12})\mathbf{D}(\hat{C}_\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.50)$$

$$= \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3.51)$$

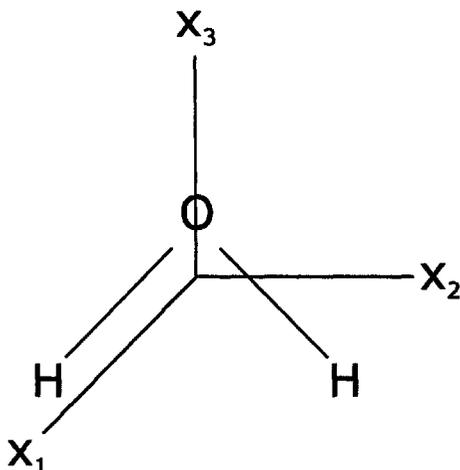


Figure 3.6: The coordinate system for the water molecule.

### Identity

The “do-nothing” operation  $\hat{E}$  is represented by the unit matrix **1** or

$$\mathbf{D}(\hat{E}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.52)$$

For example, the  $C_{2v}$  point group of water (Figure 3.6) has four symmetry operators  $\{\hat{E}, \hat{C}_2, \hat{\sigma}_{13}, \hat{\sigma}_{23}\}$ , so that the corresponding matrix representations of the symmetry operations are

$$\begin{aligned} \mathbf{D}(\hat{E}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \mathbf{D}(\hat{C}_2) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{D}(\hat{\sigma}_{13}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \mathbf{D}(\hat{\sigma}_{23}) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

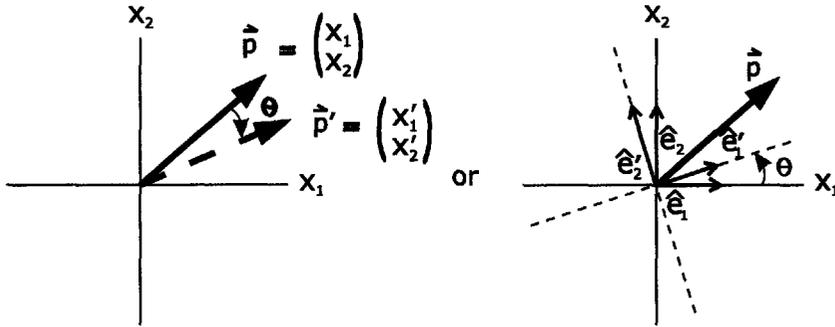
A multiplication table for the  $C_{2v}$  point group (Table 3.1) is obtained by calculating all possible products between the four **D** matrices just given. For example,  $\mathbf{D}(\hat{C}_2)\mathbf{D}(\hat{\sigma}_{13}) = \mathbf{D}(\hat{\sigma}_{23})$ .

### 3.3 Symmetry Operators and Basis Vectors

A matrix representation of a group can also be generated by considering the effect of symmetry operations on basis vectors. For example,  $\hat{C}_\theta$ , the rotation operation about the  $X_3$ -axis, can occur either by rotating the point **p** in the clockwise direction or by leaving **p** fixed but rotating the coordinate system in the opposite direction (Figure 3.7). For example, the matrix representation for the  $\hat{C}_3$  operation generated from the rotation of the point **p** is

Table 3.1. Multiplication Table for the  $C_{2v}$  Point Group

$C_{2v}$	$\hat{E}$	$\hat{C}_2$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{23}$
$\hat{E}$	$\hat{E}$	$\hat{C}_2$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{23}$
$\hat{C}_2$	$\hat{C}_2$	$\hat{E}$	$\hat{\sigma}_{23}$	$\hat{\sigma}_{13}$
$\hat{\sigma}_{13}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{23}$	$\hat{E}$	$\hat{C}_2$
$\hat{\sigma}_{23}$	$\hat{\sigma}_{23}$	$\hat{\sigma}_{13}$	$\hat{C}_2$	$\hat{E}$

Figure 3.7: The clockwise rotation by  $\theta$  radians about the  $Z$ -axis,  $\hat{C}_\theta$ , can be accomplished by rotating the point by  $+\theta$  or by rotating the coordinate system by  $-\theta$ .

$$\mathbf{D}(\hat{C}_3) = \begin{pmatrix} \cos \frac{2\pi}{3} & \sin \frac{2\pi}{3} & 0 \\ -\sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that  $\mathbf{x}' = \mathbf{D}(\hat{C}_3)\mathbf{x}$  has components  $x'_i$  given in terms of the components  $x_i$  by

$$\begin{aligned} x'_1 &= -\frac{x_1}{2} + \frac{\sqrt{3}}{2}x_2 \\ x'_2 &= \frac{\sqrt{3}}{2}x_1 - \frac{x_2}{2} \\ x'_3 &= x_3. \end{aligned}$$

However, rotation of the basis vectors in the opposite direction requires that the rotated basis vectors be given in terms of the original basis vectors by

$$\begin{aligned}\hat{\mathbf{e}}'_1 &= -\frac{\hat{\mathbf{e}}_1}{2} - \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_2, \\ \hat{\mathbf{e}}'_2 &= \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_1 - \frac{\hat{\mathbf{e}}_2}{2}, \\ \hat{\mathbf{e}}'_3 &= \hat{\mathbf{e}}_3,\end{aligned}$$

or, expressed in a matrix format, by

$$\begin{pmatrix} \hat{\mathbf{e}}'_1 \\ \hat{\mathbf{e}}'_2 \\ \hat{\mathbf{e}}'_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{pmatrix}. \tag{3.53}$$

For matrix manipulations it is convenient to introduce the basis vectors  $\hat{\mathbf{e}}_1 = \hat{\mathbf{i}}$ ,  $\hat{\mathbf{e}}_2 = \hat{\mathbf{j}}$ , and  $\hat{\mathbf{e}}_3 = \hat{\mathbf{k}}$ . The matrix representation of  $\hat{C}_3$  cannot depend on whether the representation is generated using the coordinates  $x_i$  or the basis functions  $\hat{\mathbf{e}}_i$  as the two are equivalent. This condition requires that we reformat the transformation (3.53) in terms of a row vector representation, rather than in terms of the more commonly used column vector representation. That is, we must replace (3.53) by

$$(\hat{\mathbf{e}}'_1 \hat{\mathbf{e}}'_2 \hat{\mathbf{e}}'_3) = (\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_3) \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that  $(\mathbf{e}')^t = (\mathbf{D}(\hat{C}_3^{-1})\mathbf{e})^t = \mathbf{e}^t(\mathbf{D}(\hat{C}_3^{-1}))^t = \mathbf{e}^t\mathbf{D}(\hat{C}_3)$ . This formulation allows the matrix representations of the group symmetry operators to preserve the order of multiplication of the group elements, as required by the homomorphism.

A general operation  $\hat{R}$  can operate equivalently on the point or on the basis functions:

hence, if we write  $\mathbf{p}$  as

$$\mathbf{p} = \sum \hat{\mathbf{e}}_i x_i, \tag{3.54}$$

then  $\mathbf{p}'$  can be expressed either by

$$\mathbf{p}' = \hat{R}\mathbf{p} = \sum \hat{\mathbf{e}}_i(\hat{R}x_i) = \sum \hat{\mathbf{e}}'_i x'_i, \tag{3.55}$$

or by

$$\mathbf{p}' = \sum (\hat{R}\hat{\mathbf{e}}_i)x_i = \sum \hat{\mathbf{e}}'_i x_i. \tag{3.56}$$

These equations can be written more compactly in matrix form as

$$\mathbf{p} = \mathbf{e}^t \mathbf{x} \tag{3.57}$$

$$\begin{aligned}\mathbf{p}' = \hat{R}\mathbf{p} = \mathbf{e}^t \mathbf{D}(\hat{R})\mathbf{x} &= \mathbf{e}^t(\mathbf{D}(\hat{R})\mathbf{x}) = \mathbf{e}'^t \mathbf{x}' \\ &= (\mathbf{e}^t \mathbf{D}(\hat{R}))\mathbf{x} = (\mathbf{e}')^t \mathbf{x}.\end{aligned} \tag{3.58}$$

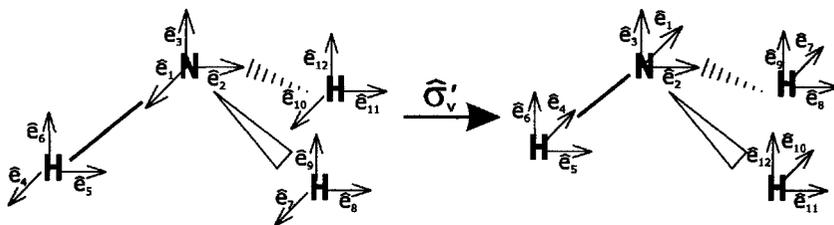


Figure 3.8: The  $\hat{\sigma}'_v$  operator reflects the twelve basis functions through the plane of the paper (the plane containing  $\hat{e}_5$  and  $\hat{e}_6$ ).

The use of row vectors for basis functions seems surprising but, as mentioned above, it is necessary in order to obtain a consistent matrix representation.

The use of basis functions rather than a single point is very useful for generating matrix representations of dimension larger than 3. For example, if each atom of  $\text{NH}_3$  is given a set of three Cartesian basis functions, then a  $12 \times 12$  matrix representation is generated. Consider the effect of  $\hat{\sigma}'_v$  on the 12 basis functions shown in Figure 3.8: the matrix representative of  $\hat{\sigma}'_v$  is given in this basis by

$$D(\hat{\sigma}'_v) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The derivation and use of such matrix representations will be discussed in Chapter 7 on vibrational spectroscopy.

### 3.4 Symmetry Operators and Basis Functions

The ordinary three-dimensional Cartesian vector space can be generalized to an  $n$ -dimensional vector space. In the previous section a 12-dimensional vector space was used with the 12 orthogonal basis functions associated with the four atoms of the  $\text{NH}_3$  molecule. It is possible to generalize even further by allowing the basis vectors to be functions. In this case the vector space is usually called a *function space*. The properties of a function space parallel those of ordinary vector spaces.

#### Function Spaces

A function space is a set of functions  $\{f_1, f_2, \dots, f_n\}$  ( $n$  may be infinite) with the following properties:

1. The addition of any two functions  $f_i$  and  $f_j$  produces a third function  $f_k$  that is also a member of the function space. Thus,  $f_i + f_j = f_k$  is analogous to the addition of two vectors  $\mathbf{a} + \mathbf{b} = \mathbf{c}$ .
2. Multiplication of any function  $f_i$  by a constant  $k$  produces a new function  $kf_i$  that is also a member of the function space. Again, this is analogous to changing the length of a vector  $\mathbf{a}$  to  $k\mathbf{a}$  by multiplication by a constant  $k$ .
3. The scalar or inner product of two complex-valued functions is given by

$$\langle f_i | f_j \rangle = \int f_i^* f_j d\tau, \quad (3.59)$$

in which  $d\tau$  represents the differential volume element for the independent variables upon which the functions  $\{f_i\}$  depend. This may be compared with the definition of the dot product of two vectors  $\mathbf{a}$  and  $\mathbf{b}$ :

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3. \quad (3.60)$$

4. If there are  $n$  linearly independent functions in the function space,  $f_1, f_2, \dots, f_n$ , then

$$a_1 f_1 + a_2 f_2 + \dots + a_n f_n = 0 \quad \text{only if} \quad a_1 = a_2 = \dots = a_n = 0,$$

and any function in the space can be represented by a linear combination of these  $n$  linearly independent functions—that is,

$$f = \sum_{i=1}^n a_i f_i. \quad (3.61)$$

The  $n$  linearly independent functions are said to span the function space of dimension  $n$ . It is always possible to find a set of  $n$  orthogonal basis functions—that is, functions  $f_i, f_j$ —such that

$$\langle f_i | f_j \rangle = \delta_{ij},$$

which spans the space. These  $n$  basis functions are like the three orthogonal Cartesian basis functions  $\hat{\mathbf{e}}_1 (= \hat{\mathbf{i}})$ ,  $\hat{\mathbf{e}}_2 (= \hat{\mathbf{j}})$ , and  $\hat{\mathbf{e}}_3 (= \hat{\mathbf{k}})$ . Any vector in three-dimensional space can be expressed in terms of these basis functions via

$$\mathbf{a} = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}} + a_z \hat{\mathbf{k}}. \quad (3.62)$$

Integrals of the type  $\langle f_i | f_j \rangle$  are often called *overlap* integrals because they measure how much the two functions overlap in space.

Function spaces are commonly used in solving the Schrödinger equation  $\hat{H}\psi = E\psi$ . For example, a set of degenerate wavefunctions associated with a single energy forms a function space. More specifically, consider the five degenerate  $d$ -orbitals represented by  $d_i$ , that form a five-dimensional function space associated with an atom. In this case,

any  $d$ -orbital, represented by  $\psi_d$ , can be expressed in terms of the five orthogonal  $d_i$  functions via

$$\psi_d = \sum_{i=1}^5 a_i d_i. \quad (3.63)$$

### *Gram-Schmidt Procedure*

Given a set of  $n$  linearly independent (but not orthogonal) basis functions, it is often desirable to find a set of  $n$  orthogonal basis functions. This can always be accomplished by the Gram-Schmidt procedure. Let  $\{f_1, f_2, \dots, f_n\}$  be a set of linearly independent functions: then according to the Gram-Schmidt procedure an appropriate set of orthogonal functions, represented by  $\{\phi_i\}$ , can be generated as

$$\phi_1 = \frac{f_1}{(\langle f_1 | f_1 \rangle)^{1/2}}, \quad (3.64)$$

$$\phi_2 = (f_2 - \phi_1 \langle \phi_1 | f_2 \rangle) / (\langle f_2 - \phi_1 \langle \phi_1 | f_2 \rangle | f_2 - \phi_1 \langle \phi_1 | f_2 \rangle \rangle)^{1/2}, \quad (3.65)$$

$$\begin{aligned} \phi_3 = & (f_3 - \phi_1 \langle \phi_1 | f_3 \rangle - \phi_2 \langle \phi_2 | f_3 \rangle) / \\ & (\langle f_3 - \phi_1 \langle \phi_1 | f_3 \rangle - \phi_2 \langle \phi_2 | f_3 \rangle | f_3 - \phi_1 \langle \phi_1 | f_3 \rangle \\ & - \phi_2 \langle \phi_2 | f_3 \rangle \rangle)^{1/2}, \end{aligned} \quad (3.66)$$

⋮

$$\begin{aligned} \phi_n = & (f_n - \phi_1 \langle \phi_1 | f_n \rangle \cdots - \phi_{n-1} \langle \phi_{n-1} | f_n \rangle) / \\ & (\langle f_n - \phi_1 \langle \phi_1 | f_n \rangle \cdots | f_n - \phi_1 \langle \phi_1 | f_n \rangle \cdots \rangle)^{1/2}. \end{aligned} \quad (3.67)$$

The set of functions  $\{\phi_i\}$  now has the property that its members are both orthogonal and normalized, which we express in the form

$$\langle \phi_i | \phi_j \rangle = \delta_{ij}, \quad (3.68)$$

for all  $i$  and  $j$ . This procedure works because at each step the new  $\phi_i$  function is made orthogonal to the preceding  $i-1$  functions by subtracting the overlap integrals and then normalizing the new function. The Gram-Schmidt procedure is useful in vector analysis and in quantum mechanics, where a set of orthogonal functions makes calculations easier because there are no overlap integrals between the basis functions of an orthogonal set.

### *Transformation Operators*

A set of functions in a function space, like a set of basis vectors, can also be used to generate a matrix representation of a group. For this purpose it is necessary to define a set of linear operators  $\{\hat{O}_R\}$  isomorphic with the group of symmetry operators  $\{\hat{R}\}$ . These operators operate on functions and are defined by the equation

$$(\hat{O}_R)f_i(x'_1, x'_2, x'_3) = f_i(x_1, x_2, x_3), \quad (3.69)$$

in which  $f_i$  is a member of the set of  $n$  linearly independent basis functions, and  $x'_i$  results from the operation of  $\hat{R}$  on  $x_i$ . Since the basis functions span the function space, this operation must produce a linear combination of the basis functions:

$$\hat{O}_R f_j = \sum_i f_i (\mathbf{D}(\hat{R}))_{ij}. \tag{3.70}$$

This “backwards” definition of  $\hat{O}_R$  is again necessary in order to obtain a consistent  $\mathbf{D}(\hat{R})$  matrix when the  $\hat{O}_R$  operates on functions that are like basis vectors in a vector space.

As the  $\hat{O}_R$  operators are unitary operators, their matrix representations will necessarily be unitary matrices. Consider unitary transformations of two complex vectors,

$$\mathbf{y}' = \mathbf{U}\mathbf{y}, \tag{3.71}$$

$$\mathbf{x}' = \mathbf{U}\mathbf{x}, \tag{3.72}$$

and examine the dot product between the vectors: as we have seen earlier the dot product for complex-valued vectors will be given as

$$\begin{aligned} (\mathbf{x}')^\dagger \mathbf{y}' &= \sum_i (x'_i)^* y'_i = (\mathbf{U}\mathbf{x})^\dagger (\mathbf{U}\mathbf{y}) \\ &= \mathbf{x}^\dagger \mathbf{U}^\dagger \mathbf{U}\mathbf{y} = \mathbf{x}^\dagger \mathbf{y} \end{aligned} \tag{3.73}$$

since  $\mathbf{U}^\dagger = \mathbf{U}^{-1}$  by definition. As required, a unitary transformation leaves the dot product of two vectors unchanged, and in particular, the length (norm) of a vector  $|\mathbf{x}| = (\mathbf{x}^\dagger \mathbf{x})^{1/2}$  is unchanged by a unitary transformation.

Similarly, the  $\hat{O}_R$  operators work in a function space and they do not change the scalar product of two functions, so that

$$\langle f|g \rangle = \langle \hat{O}_R f | \hat{O}_R g \rangle.$$

The proof is based on the definition of the scalar product

$$\langle f|g \rangle = \int f^* g d\tau = \int f^*(x_1, x_2, x_3) g(x_1, x_2, x_3) dx_1 dx_2 dx_3. \tag{3.74}$$

Now the  $\hat{O}_R$  operator will move the element  $dx_1 dx_2 dx_3$  to  $dx'_1 dx'_2 dx'_3$  and  $f(x_1, x_2, x_3) = \hat{O}_R f(x'_1, x'_2, x'_3)$  and  $g(x_1, x_2, x_3) = \hat{O}_R g(x'_1, x'_2, x'_3)$  by definition. Therefore

$$\langle f|g \rangle = \int \hat{O}_R f^*(x'_1, x'_2, x'_3) \hat{O}_R g(x'_1, x'_2, x'_3) dx'_1 dx'_2 dx'_3 = \langle \hat{O}_R f | \hat{O}_R g \rangle \tag{3.75}$$

since the integration variables are dummy variables. Thus the  $\hat{O}_R$  operators are unitary operators that can be represented by unitary matrices.

As an example of using the  $\hat{O}_R$  operators consider the effect of the  $\hat{C}_3^{(z)}$  operation on the function  $f = xyz = x_1 x_2 x_3$ . In this case we have

$$\hat{O}_{C_3} f' = f, \tag{3.76}$$

and

$$\hat{O}_{C_3} f(x'_1, x'_2, x'_3) = x_1 x_2 x_3, \tag{3.77}$$

but we already know that

$$x'_1 = -\frac{x_1}{2} + \sqrt{3}\frac{x_2}{2}, \quad (3.78a)$$

$$x'_2 = -\sqrt{3}\frac{x_1}{2} - \frac{x_2}{2}, \quad (3.78b)$$

$$x'_3 = x_3, \quad (3.78c)$$

or

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (3.79)$$

Inverting this matrix relation (3.79) gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix},$$

or

$$x_1 = -\frac{x'_1}{2} - \sqrt{3}\frac{x'_2}{2}, \quad (3.80a)$$

$$x_2 = \sqrt{3}\frac{x'_1}{2} - \frac{x'_2}{2}, \quad (3.80b)$$

$$x_3 = x'_3. \quad (3.80c)$$

Substitution of equations (3.80a-c) into equation (3.77) then gives

$$\begin{aligned} (\hat{O}_{C_3})f(x'_1, x'_2, x'_3) &= \left(-\frac{x'_1}{2} - \sqrt{3}\frac{x'_2}{2}\right)\left(\sqrt{3}\frac{x'_1}{2} - \frac{x'_2}{2}\right)x'_3 \\ &= \left(-\frac{\sqrt{3}}{4}(x'_1)^2 + \frac{\sqrt{3}}{4}(x'_2)^2 - \frac{3}{4}(x'_1x'_2) + \frac{1}{4}x'_1x'_2\right)x'_3 \\ &= x'_3 \frac{\left(-\frac{\sqrt{3}}{2}(x'_1)^2 + \frac{\sqrt{3}}{2}(x'_2)^2 - x'_1x'_2\right)}{2} \end{aligned} \quad (3.81)$$

From this result, we may therefore conclude that

$$\hat{O}_{C_3}f = \frac{\left(-\frac{\sqrt{3}}{2}x_1^2 + \frac{\sqrt{3}}{2}x_2^2 - x_1x_2\right)x_3}{2}. \quad (3.82)$$

In this way the effect of any symmetry operator on any function can be determined.

## 3.5 Equivalent, Reducible, and Irreducible Representations

Evidently an infinite number of matrix representations of a point group are possible. The time has come to limit the possibilities.

### *Equivalent Representations*

If two sets of linearly independent basis functions exist and are related by a linear transformation  $\mathbf{g} = \mathbf{A}\mathbf{f}$ , then the matrix representation generated by  $\mathbf{f}$  is said to be equivalent to the matrix representation generated by  $\mathbf{g}$ . In fact, it is found that the two representations are related by a similarity transformation. Consider the effect of a symmetry operator  $\hat{R}$  on the basis functions  $\mathbf{g}$ , with

$$(\mathbf{g}')^t = \mathbf{g}^t \mathbf{D}^g(\hat{R}). \quad (3.83)$$

However, as  $\mathbf{g}$  is related to  $\mathbf{f}$  via

$$\mathbf{g} = \mathbf{A}\mathbf{f},$$

it follows that  $\mathbf{g}^t$  is related to  $\mathbf{f}^t$  via

$$\mathbf{g}^t = (\mathbf{A}\mathbf{f})^t = \mathbf{f}^t \mathbf{A}^t \quad (3.84)$$

and that  $(\mathbf{g}')^t$  and  $(\mathbf{f}')^t$  are related by

$$(\mathbf{g}')^t = (\mathbf{A}(\mathbf{f}'))^t = (\mathbf{f}')^t \mathbf{A}^t. \quad (3.85)$$

Thus comparison between equation (3.83) and equation (3.85) gives

$$(\mathbf{f}')^t \mathbf{A}^t = \mathbf{f}^t \mathbf{A}^t \mathbf{D}^g(\hat{R}) \quad (3.86)$$

or, equivalently,

$$(\mathbf{f}')^t = \mathbf{f}^t (\mathbf{A}^t \mathbf{D}^g(\hat{R}) (\mathbf{A}^t)^{-1}). \quad (3.87)$$

Now if we define  $\mathbf{B}^{-1}$  as

$$\mathbf{A}^t = \mathbf{B}^{-1}$$

and relate  $(\mathbf{f}')^t$  to  $\mathbf{f}^t$  by

$$(\mathbf{f}')^t = \mathbf{f}^t \mathbf{D}^f(\hat{R}), \quad (3.88)$$

then we see that  $\mathbf{D}^f$  and  $\mathbf{D}^g$  are related by

$$\mathbf{B}^{-1} \mathbf{D}^g \mathbf{B} = \mathbf{D}^f. \quad (3.89)$$

Consequently, the matrix representations generated by  $\mathbf{f}$  and  $\mathbf{g}$  are related by a similarity transformation. Equivalent representation matrices thus have the same eigenvalues, traces, and determinants so that they cannot really be considered to be “different” from the point of view of representing symmetry operators.

### Unitary Representations

In general, many different types of matrices could be used to represent symmetry operations. However, if an orthonormal set of basis functions is used in an  $n$ -dimensional function space, then the matrices generated will be unitary. Since it is always possible to find a set of  $n$  orthogonal functions that span the function space for finite groups, it is always possible to construct a unitary representation of a finite group. It is also possible to employ unitary matrices for some continuous groups—in particular, the spherical group,  $K_h$ . The properties of unitary matrices are so convenient we shall use them exclusively to represent symmetry operators.

### Reducible and Irreducible Representations

So far the matrix representations of symmetry operators have been nonequivalent and unitary. However, the dimension of the matrices in a given representation could be very large. Fortunately, it is always possible to find a similarity transformation that reduces the representation to block diagonal form where the nonzero elements occur in blocks along the principal diagonal. For example, the three-dimensional representation of the group  $C_{3v}$  generated by using a point is

$$\begin{aligned}
 \mathbf{D}(\hat{E}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0 & -0 & 1 \end{pmatrix} & \mathbf{D}(\hat{\sigma}'_v) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -0 & -0 & 1 \end{pmatrix} \\
 \mathbf{D}(\hat{C}_3) &= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \mathbf{D}(\hat{C}_3^{-1}) &= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 \mathbf{D}(\hat{\sigma}''_v) &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \mathbf{D}(\hat{\sigma}'''_v) &= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
 \end{aligned} \tag{3.90}$$

Notice that all of the matrices have the same form and that the subblocks (highlighted in the matrices with dashed lines) also form representations of the group since multiplication does not change the block structure. In the  $C_{3v}$  example, the six matrices cannot be reduced further (i.e., no extra zero can be introduced) by the application of a single similarity transformation to all matrices. These simple  $2 \times 2$  and  $1 \times 1$  blocks are therefore called *irreducible representations*, while the three-dimensional representation is termed *reducible*.

The conventional symbol for a representation is  $\Gamma$ , which stands for all members of a representation. A superscript is used to label different representations:

$$\begin{aligned}
 \Gamma^{\text{red}} &= \{(\mathbf{D}(\hat{R})), (\mathbf{D}(\hat{S})), \dots\} \\
 \Gamma^{\text{red}} &= \Gamma^1 \oplus \Gamma^2 \oplus \dots \Gamma^n \\
 &= \oplus \sum_{\nu} a_{\nu} \Gamma^{\nu}.
 \end{aligned}$$

The decomposition of a reducible representation into irreducible representations is symbolized with a  $\oplus$ . The circle indicates that this is a “direct” sum of the representations

rather than ordinary addition, such as for numbers or matrices. A given irreducible representation  $\Gamma^\nu$  may occur several ( $a_\nu$ ) times in the reducible representation. Note that the superscript on  $\Gamma$  always serves as a label and never indicates repeated multiplication. It turns out that while the number of reducible representations of a group is infinite, the number of irreducible representations is small. In particular, the number of irreducible representations of a point group turns out to be equal to the number of classes in that group.

### 3.6 Great Orthogonality Theorem

Since all matrix representations can be reduced to the direct sum of a small number of irreducible representations, these irreducible representations must be very important. The central theorem about irreducible representations of point groups is appropriately called the Great Orthogonality Theorem. The Great Orthogonality Theorem requires that

$$\sum_{\hat{R}} D_{ik}^\mu(\hat{R}) D_{mj}^\nu(\hat{R}^{-1}) = \frac{g}{n_\nu} \delta_{\mu\nu} \delta_{ij} \delta_{km}, \tag{3.91}$$

in which  $\mathbf{D}^\mu(\hat{R})$  and  $\mathbf{D}^\nu(\hat{R})$  are matrix representations of two irreducible representations  $\Gamma^\mu$  and  $\Gamma^\nu$  of the same group,  $g$  is the order of the group, and  $n_\nu$  is the dimension of the  $\nu$ th irreducible representation. The sum is over all operations,  $\hat{R}$ , in the group. As the proof of this theorem is rather involved, it will not be reproduced here.

If unitary matrices are used to represent the group elements, then

$$\mathbf{D}^\nu(\hat{R}^{-1}) = (\mathbf{D}^\nu(\hat{R}))^{-1} = (\mathbf{D}^\nu(\hat{R}))^\dagger, \tag{3.92}$$

from which we see that

$$D_{mj}^\nu(\hat{R}^{-1}) = D_{jm}^{\nu*}(\hat{R}). \tag{3.93}$$

The Great Orthogonality Theorem can therefore be restated in the form

$$\sum_{\hat{R}} D_{ik}^\mu(\hat{R}) (D_{jm}^\nu(\hat{R}))^* = \frac{g}{n_\mu} \delta_{\mu\nu} \delta_{ij} \delta_{km}. \tag{3.94}$$

This theorem states that the corresponding matrix elements in the various irreducible representations can be formed into vectors that are orthogonal to one another. This may best be illustrated with a specific example.

Consider the  $3 \times 3$  matrices which represent the  $C_{3v}$  point group generated earlier (see equations (3.90)). Since the number of symmetry operators (i.e., the order  $g$  of the point group) is six, each vector will be of dimension six. If the matrix element in the upper left corner is used, then the vector, arbitrarily labeled  $\mathbf{v}_1$ , given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

is obtained; four other vectors can be generated in a similar manner to yield

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

We note that the vector  $\mathbf{v}_5$  is associated with a one-dimensional representation of  $C_{3v}$ , while the other four vectors are all associated with the two-dimensional irreducible representation.

The vectors constructed in this fashion are all orthogonal: if  $\mu \neq \nu$  (i.e., the vectors originate from different irreducible representations), then (in the present case)

$$\mathbf{v}_i \cdot \mathbf{v}_5 = 0, \quad i = 1, 2, 3, 4,$$

as required. If  $\mu = \nu$  (i.e., the vectors originate from the same representation), then the vectors arising from different rows and columns are orthogonal: for example,

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \quad \text{for} \quad i \neq j, \quad i, j = 1, 2, 3, 4.$$

Finally the vectors are normalized to the value  $g/n_\mu$  (i.e., the order of the group divided by the dimension of the irreducible representation from which the vectors originate).

In this example, when  $\mu = \nu$ ,  $i = j$ ,  $k = m$ :

$$\mathbf{v}_5 \cdot \mathbf{v}_5 = 6$$

while

$$\frac{g}{n_\mu} = \frac{6}{1} = 6$$

as required and

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 3$$

while

$$\frac{g}{n_\mu} = \frac{6}{2} = 3$$

as required.

If a group is of order  $g$ , then each vector will be  $g$ -dimensional. The maximum number of linearly independent vectors in a  $g$ -dimensional vector space is also  $g$ . If an irreducible representation is an  $n_\mu \times n_\mu$  matrix it must contribute  $n_\mu^2$  orthogonal vectors, and the total number of orthogonal vectors in the vector space cannot exceed  $g$ . Therefore

$$\sum_{\mu} n_{\mu}^2 \leq g, \quad (3.95)$$

where the sum is over all irreducible representations. In fact it can be shown that the equality holds and consequently

$$\sum_{\mu=1}^r n_{\mu}^2 = g, \quad (3.96)$$

in which  $r$  is the number of irreducible representations.

### Characters

A character is the trace of a matrix which serves as a representation of a symmetry operation,

$$\chi(\hat{R}) = \text{tr}(\mathbf{D}(\hat{R})). \quad (3.97)$$

Characters are represented by the symbol  $\chi$  and they serve to represent (i.e., characterize) a matrix. They are convenient because a single number, rather than the entire matrix, can be used in most applications in spectroscopy.

For example, the characters of the three-dimensional representation of the  $C_{3v}$  point group used previously in (3.90) are:

	$\hat{E}$	$\hat{C}_3$	$\hat{C}_3^{-1}$	$\hat{\sigma}'_v$	$\hat{\sigma}''_v$	$\hat{\sigma}'''_v$
$\chi^{\text{red}}(\hat{R})$	3	0	0	1	1	1

Notice that although the matrices that represent the  $\hat{E}$  operator, the three reflection planes, and the two rotation operators are all different, the characters of all symmetry operators in the same class are the same. This is because the members of a class are related by a similarity transformation, for example,

$$\mathbf{D}(\hat{\sigma}''_v) = \mathbf{D}(\hat{C}_3^{-1})\mathbf{D}(\hat{\sigma}'_v)\mathbf{D}(\hat{C}_3)$$

and the traces (characters) of the matrix representations are unchanged by a similarity transformation. This is also convenient because the character of just one member of each class needs to be worked out.

The Great Orthogonality Theorem can be used for characters as well as matrices since

$$\sum_{\hat{R}} D_{ii}^{\mu}(\hat{R}) D_{jj}^{\nu*}(\hat{R}) = \frac{g}{n_{\mu}} \delta_{\mu\nu} \delta_{ij} \delta_{ij} = \frac{g}{n_{\mu}} \delta_{\mu\nu} \delta_{ij}, \quad (3.98)$$

which allows us to sum over  $i$  and  $j$  to obtain

$$\begin{aligned} \sum_{\hat{R}} \sum_{i=1}^{n_{\mu}} D_{ii}^{\mu}(\hat{R}) \sum_{j=1}^{n_{\nu}} D_{jj}^{\nu*}(\hat{R}) &= \frac{g}{n_{\mu}} \delta_{\mu\nu} \sum_{i=1}^{n_{\mu}} \sum_{j=1}^{n_{\nu}} \delta_{ij} \\ &= \frac{g}{n_{\mu}} \delta_{\mu\nu} \min(n_{\mu}, n_{\nu}) \\ &= \frac{g}{n_{\mu}} \delta_{\mu\nu} n_{\mu} \quad (n_{\mu} < n_{\nu}). \end{aligned} \quad (3.99)$$

Upon employing the definition of the trace of a matrix, we obtain the relation

$$\sum_{\hat{R}} \chi^\mu(\hat{R})(\chi^\nu(\hat{R}))^* = g\delta_{\mu\nu}, \quad (3.100)$$

so that the characters of the irreducible representations are also orthogonal in the same sense that the matrix elements of a matrix representation are orthogonal.

For example, the characters of the two irreducible representations generated from the three-dimensional representation of  $C_{3v}$  can be presented in tabular form as

	$\hat{E}$	$\hat{C}_3$	$\hat{C}_3^{-1}$	$\hat{\sigma}'_v$	$\hat{\sigma}''_v$	$\hat{\sigma}'''_v$
$\chi^{\text{red}}(\hat{R})$	3	0	0	1	1	1
$\chi^1(\hat{R})$	1	1	1	1	1	1
$\chi^2(\hat{R})$	2	-1	-1	0	0	0

in which the characters of the three-dimensional reducible representation are also listed. Notice the way in which the characters add, namely,

$$\chi^{\text{red}}(\hat{R}) = \chi^1(\hat{R}) + \chi^2(\hat{R}),$$

while the corresponding representations are added using the  $\oplus$  symbol,

$$\Gamma^{\text{red}} = \Gamma^1 \oplus \Gamma^2.$$

Upon checking, we note that

$$\begin{aligned} \sum_{\hat{R}} \chi^1(\hat{R})\chi^2(\hat{R})^* &= 1(2) + 1(-1) + 1(-1) + 1(0) + 1(0) + 1(0) = 0, \\ \sum_{\hat{R}} \chi^1(\hat{R})\chi^1(\hat{R})^* &= 1 + 1 + 1 + 1 + 1 + 1 = 6, \\ \sum_{\hat{R}} \chi^2(\hat{R})\chi^2(\hat{R})^* &= 2(2) + (-1)(-1) + (-1)(-1) = 6, \end{aligned}$$

as required by the orthogonality relationship (3.100).

Let the number of classes in a group be  $k$  and let the number of members of each class be  $g_i$ . For the preceding example,  $k = 3$  and  $g_1 = 1$ ,  $g_2 = 2$ ,  $g_3 = 3$ , referring to the classes  $\{\hat{E}\}$ ,  $\{\hat{C}_3, \hat{C}_3^{-1}\}$ , and  $\{\hat{\sigma}'_v, \hat{\sigma}''_v, \hat{\sigma}'''_v\}$ , respectively, with

$$\sum_{i=1}^k g_i = g. \quad (3.101)$$

The sum over group operations in the orthogonality theorem can be replaced by a sum over classes:

$$\sum_{i=1}^k g_i \chi^\mu(\hat{R}_i)\chi^\nu(\hat{R}_i)^* = \sum_{i=1}^k (\sqrt{g_i}\chi^\mu(\hat{R}_i))(\sqrt{g_i}\chi^\nu(\hat{R}_i))^* = g\delta_{\mu\nu}. \quad (3.102)$$

This is now an orthogonality relationship in a  $k$ -dimensional ( $k = \text{number of classes}$ ) vector space. The maximum number of independent vectors that can be found in a  $k$ -dimensional space is also  $k$ , so that the number of irreducible representations  $r$  has to

be less than or equal to the number of classes,  $r \leq k$ . In fact, it can be proved that  $r = k$  so that the number of classes is identical to the number of irreducible representations.

A reducible matrix representation can be written as a direct sum of irreducible matrix representations, that is,

$$\Gamma^{\text{red}} = a_1\Gamma^1 \oplus a_2\Gamma^2 \oplus \cdots \oplus a_n\Gamma^n. \tag{3.103}$$

There is a parallel equation for characters, namely

$$\chi^{\text{red}}(\hat{R}) = \sum_{\nu} a_{\nu}\chi^{\nu}(\hat{R}), \tag{3.104}$$

for all  $\hat{R}$  where this is now an arithmetic sum. This equation (3.104) for characters holds true because the sum of the diagonal elements of the reducible matrix must equal the sum of the diagonal elements of the submatrices of irreducible representations of the block diagonal form.

The orthogonality theorem for characters can be used to determine quickly the number of each type of irreducible representation contained within a given reducible representation, provided that the characters are all known (or have been determined). Let us examine the following sequence of steps:

$$\begin{aligned} \sum_{\hat{R}} \chi^{\text{red}}(\hat{R})\chi^{\mu}(\hat{R})^* &= \sum_{\hat{R}} \sum_{\nu} a_{\nu}\chi^{\nu}(\hat{R})\chi^{\mu}(\hat{R})^* \\ &= \sum_{\nu} a_{\nu} \sum_{\hat{R}} \chi^{\nu}(\hat{R})\chi^{\mu}(\hat{R})^* \\ &= \sum_{\nu} a_{\nu}g\delta_{\mu\nu} = a_{\mu}g, \end{aligned}$$

from which we can determine the number of times,  $a_{\mu}$ , that the  $\mu$ th irreducible representation is contained in the reducible representation. Upon solving for  $a_{\mu}$ , we obtain

$$a_{\mu} = \frac{1}{g} \sum_{\hat{R}} \chi^{\text{red}}(\hat{R}_i)\chi^{\mu}(\hat{R}_i)^* = \frac{1}{g} \sum_{i=1}^k g_i\chi^{\text{red}}(\hat{R}_i)\chi^{\mu}(\hat{R}_i)^*. \tag{3.105}$$

### 3.7 Character Tables

The characters of the irreducible representations of the point groups are used in many applications of group theory. It is therefore very helpful to have tables of the characters available (Appendix B). The character table for the  $C_{3v}$  point group, in particular, is

$C_{3v}$	$\hat{E}$	$2\hat{C}_3$	$3\hat{\sigma}_v$
$A_1$	1	1	1
$A_2$	1	1	-1
$E$	2	-1	0

Since the number of classes equals the number of irreducible representations, this table is square. The symmetry operations are listed along the top row with only one character provided for each class. The number of members  $g_i$  in each class is also provided. Along

the leftmost column the names of each of the irreducible representations are provided using *Mulliken notation* (discussed below).

Character tables can be constructed using various properties of characters without finding the actual irreducible matrices. The properties include the following:

1. The number of irreducible representations  $r$  is equal to the number of classes  $k$ , making a square table.
2. The sum of the squares of the dimensions of the irreducible representations is equal to the order of the group,

$$\sum n_{\mu}^2 = g. \quad (3.106)$$

For every group there exists the totally symmetric representation consisting of all ones. These characters form the first entries along the second row of the table.

3. The rows are orthogonal to each other and normalized according to the equation

$$\sum_{\substack{i=1 \\ \text{classes}}}^k g_i \chi^{\mu}(\hat{R}_i) \chi^{\nu}(\hat{R}_i)^* = g \delta_{\mu\nu}. \quad (3.107)$$

4. The columns are also orthogonal and are normalized according to the equation

$$\sum_{\nu=1}^{r(=k)} \chi^{\nu}(\hat{R}_i) \chi^{\nu}(\hat{R}_j)^* = \frac{g}{g_j} \delta_{ij}. \quad (3.108)$$

Equation (3.108) has not been derived here, but proves very useful in the construction of character tables.

### *Mulliken Notation*

Each of the irreducible representations could be numbered in order—for example

$C_{3v}$	$\hat{E}$	$2\hat{C}_3$	$3\hat{\sigma}_v$
$\Gamma^1 (= A_1)$	1	1	1
$\Gamma^2 (= A_2)$	1	1	-1
$\Gamma^3 (= E)$	2	-1	0

but this labeling scheme is not very informative. Mulliken<sup>1</sup> proposed a labeling scheme that provides some additional information about the symmetry properties of the irreducible representation. A modern compilation of recommended notation for spectroscopy and group theory was been published in 1997.<sup>2</sup> One-dimensional representations are labeled  $A$  or  $B$ , depending on whether the irreducible representation is symmetric  $\chi(\hat{C}_n \text{ or } \hat{S}_n) = +1$  or antisymmetric  $\chi(\hat{C}_n \text{ or } \hat{S}_n) = -1$  with respect to rotation (or improper rotation) about the highest order symmetry axis in the molecule (except for a few highly symmetric groups). If there is no rotational axis of symmetry, then the one-dimensional irreducible representations are labeled  $A$ . All two-dimensional irreducible representations are labeled  $E$  (unrelated to the operator  $\hat{E}$ ). Three-dimensional

irreducible representations are labeled  $T$  by most workers, except for some infrared spectroscopists who use  $F$  to label triply degenerate vibrations. Finally the fourfold and fivefold degenerate irreducible representations found in  $I_h$  are labeled  $G$  and  $H$ , respectively.

If a center of symmetry is present in a molecule, then  $g$  or  $u$  is used as a subscript to identify even ( $g$ ) and odd ( $u$ ) irreducible representations. The  $g$  and  $u$  stand for *gerade* and *ungerade*, the German words for even and odd. The irreducible representations are of  $g$  symmetry if  $\chi(\hat{i}) > 0$  and  $u$  symmetry if  $\chi(\hat{i}) < 0$ . The point groups that contain  $\hat{i}$  ( $C_{nh}$  ( $n$  even),  $D_{nh}$  ( $n$  even),  $D_{nd}$  ( $n$  odd),  $O_h$ ,  $D_{\infty h}$ , and  $I_h$ ) can be written as "direct product" groups  $G \otimes C_i$ . The direct product operation is discussed in more detail in the section on direct product representations in Chapter 4 and below. Each direct product group  $G \otimes C_i$  has a character table twice as large as  $G$ . There are twice the number of irreducible representations (now labeled by  $g$  and  $u$ ) and twice the number of symmetry operations of  $G$ :  $G = \{\hat{R}\}$ ,  $G \otimes C_i = \{\hat{R}, \hat{i}\hat{R}\}$ . If the character tables are considered to be square matrices, then the direct product groups  $G \otimes C_i$  have character tables that are direct products (see below) of the character tables for  $G$  and  $C_i$ .

A similar situation arises for point groups with a  $\hat{\sigma}_h$  operation but no  $\hat{i}$  operation ( $C_{2h}$  and  $D_{nh}$  with  $n$  odd). In this case, the group can be written as  $G \otimes C_s$  and all of the characters are either single prime or double prime depending on whether  $\chi(\hat{\sigma}_h) > 0$  (single prime) or  $\chi(\hat{\sigma}_h) < 0$  (double prime). It is useful to recognize direct product groups because the amount of work can be greatly decreased in most applications. For example, one trick is to use the appropriate subgroup for a problem, such as  $O$  rather than the full group  $O_h$ , and then add  $g$  and  $u$  at the end by inspection. As an example, consider the character table for the point group of the octahedron  $O_h = O \otimes C_i$ . The subgroup  $O$ , made up of the pure rotations of  $O_h$ , has the character table

$O$	$\hat{E}$	$8\hat{C}_3$	$3\hat{C}_2$	$6\hat{C}_4$	$6\hat{C}'_2$
$A_1$	1	1	1	1	1
$A_2$	1	1	1	-1	-1
$E$	2	-1	2	0	0
$T_1$	3	0	-1	1	-1
$T_2$	3	0	-1	-1	1

while the character table for  $C_i$  is

$C_i$	$\hat{E}$	$\hat{i}$
$A_g$	1	1
$A_u$	1	-1

From what has been said above about the relationship between the direct product group character table and the character tables of the component groups, we obtain for  $O_h$  the character table

$O_h$	$\hat{E}$	$8\hat{C}_3$	$3\hat{C}_2$	$6\hat{C}_4$	$6\hat{C}'_2$	$\hat{i}$	$8\hat{S}_6$	$3\hat{\sigma}_h$	$6\hat{S}_4$	$6\hat{\sigma}_d$
$A_{1g}$	1	1	1	1	1	1	1	1	1	1
$A_{2g}$	1	1	1	-1	-1	1	1	1	-1	-1
$E_g$	2	-1	2	0	0	2	-1	2	0	0
$T_{1g}$	3	0	-1	1	-1	3	0	-1	1	-1
$T_{2g}$	3	0	-1	-1	1	3	0	-1	-1	1
$A_{1u}$	1	1	1	1	1	-1	-1	-1	-1	-1
$A_{2u}$	1	1	1	-1	-1	-1	-1	-1	1	1
$E_u$	2	-1	2	0	0	-2	1	-2	0	0
$T_{1u}$	3	0	-1	1	-1	-3	0	1	-1	1
$T_{2u}$	3	0	-1	-1	1	-3	0	1	1	-1

The definition of a direct product of two matrices  $\mathbf{A}^{(n \times n)}$  and  $\mathbf{B}^{(m \times m)}$  is

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} A_{11}\mathbf{B} & \cdots & A_{1n}\mathbf{B} \\ \vdots & \cdots & \vdots \\ A_{n1}\mathbf{B} & \cdots & A_{nn}\mathbf{B} \end{pmatrix}, \quad (3.109)$$

so that  $\mathbf{A} \otimes \mathbf{B}$  is a new super-matrix of dimension  $(n \times m) \times (n \times m)$ .

If none of the rules for labeling irreducible representations are sufficient to provide a unique label, then numeric subscripts are added to distinguish among the irreducible representations. As an example of this rule, we could take  $A_1$  and  $A_2$  in the  $C_{3v}$  point group.

Some of the character tables contain characters that are complex numbers, such as the cyclic group  $C_n$ . The cyclic group of order  $n$  is made up of the rotation operators  $\{\hat{C}_n, \hat{C}_n^2, \dots, \hat{C}_n^n = \hat{E}\}$ . Clearly these groups are all Abelian, since any operator commutes with itself and each symmetry operator is in its own class. The number of classes and the number of irreducible representations are therefore equal to  $g$ , the order of the group. All irreducible representations must be one-dimensional since  $\sum n_\nu^2 = g$  is satisfied only for  $n_\nu = 1, \nu = 1, \dots, g$ . The characters that are complex must be paired with their complex conjugates and thereby give rise to a double degeneracy.

As an example, consider the  $C_5$  point group, which has the character table

$C_5$	$\hat{E}$	$\hat{C}_5$	$\hat{C}_5^2$	$\hat{C}_5^3$	$\hat{C}_5^4$
$A_1$	1	1	1	1	1
$E_1$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$	$\varepsilon$	$\varepsilon^2$	$\varepsilon^{2*}$	$\varepsilon^*$
		$\varepsilon^*$	$\varepsilon^{2*}$	$\varepsilon^2$	$\varepsilon$
$E_2$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$	$\varepsilon^2$	$\varepsilon^*$	$\varepsilon$	$\varepsilon^{2*}$
		$\varepsilon^{2*}$	$\varepsilon$	$\varepsilon^*$	$\varepsilon^2$

with  $\varepsilon = e^{2\pi i/5}$ . The characters for the irreducible representations that are complex pairs are labeled as  $E$ . The sums of the complex conjugate pairs are real numbers and can be used in most applications instead of the individual complex components.

The point groups  $C_{\infty v}$  and  $D_{\infty h}$  ( $= C_{\infty v} \otimes C_i$ ) for linear molecules are of infinite order so that the methods discussed so far cannot be used to derive their character tables.  $C_{\infty v}$  has the rotational symmetry operators  $\hat{C}(\phi)$  and their inverses  $\hat{C}(-\phi)$  in the same class (Figure 3.9). There are an infinite number of pairs each with a different  $\phi$  and each pair is in a different class. There are also an infinite number of reflection operators but they all belong to the same class as shown in the character table

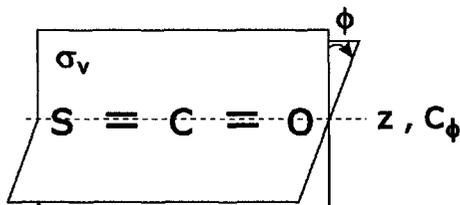


Figure 3.9: The symmetry elements for a linear molecule of  $C_{\infty v}$  symmetry.

$C_{\infty v}$	$E$	$2\hat{C}(\phi)$	$\infty\hat{\sigma}_v$
$A_1 = \Sigma^+$	1	1	1
$A_2 = \Sigma^-$	1	1	-1
$E_1 = \Pi$	2	$2 \cos \phi$	0
$E_2 = \Delta$	2	$2 \cos 2\phi$	0
$E_3 = \Phi$	2	$2 \cos 3\phi$	0

Mulliken's notation for the characters is on the left, but this notation is not normally used by spectroscopists. They use instead the traditional set of Greek labels, also shown in the character table above.

### Problems

1. For the vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix},$$

- (a) Calculate  $|\mathbf{a}|$ ,  $|\mathbf{b}|$ ,  $\mathbf{a} \cdot \mathbf{b}$ ,  $\mathbf{a} \times \mathbf{b}$ .  
 (b) Convert  $\mathbf{a}$  and  $\mathbf{b}$  to spherical polar coordinates.

2. For the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 7 \\ 1 & 1 & 1 \end{pmatrix}$$

obtain:

- (a)  $\mathbf{A}^t$   
 (b)  $|\mathbf{A}|$   
 (c)  $\mathbf{A}^{-1}$

3. For the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(a) Verify that  $\sigma_x \sigma_y = i\sigma_z$ .

(b) Obtain the product matrix  $\sigma_y \sigma_x$ , and evaluate the commutator  $[\sigma_x, \sigma_y]$ .

4. If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & -2 \\ 2 & 3 & 2 \end{pmatrix},$$

obtain  $\mathbf{A} + \mathbf{B}$ ,  $\mathbf{A} - \mathbf{B}$ ,  $\mathbf{AB}$ , and  $\mathbf{BA}$ .

5. Show that

(a)  $(\mathbf{AB})^t = \mathbf{B}^t \mathbf{A}^t$

(b)  $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger$

6. Show that  $\mathbf{A}^\dagger \mathbf{A}$  and  $\mathbf{AA}^\dagger$  are Hermitian for any matrix  $\mathbf{A}$ .

7. Verify that

$$\mathbf{R}_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

is an orthogonal matrix.

8. Find  $\mathbf{A}^{-1}$  for

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 5 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

and verify that  $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{1}$ .

9. Solve the set of linear equations

$$4x - 3y + z = 11,$$

$$2x + y - 4z = -1,$$

$$x + 2y - 2z = 1,$$

using matrix methods.

10. From the characteristic polynomial show that:

(a) The product of the eigenvalues of a matrix equals the determinant,  $|\mathbf{A}| = \lambda_1 \lambda_2 \dots \lambda_n$ .

(b) The sum of eigenvalues equals the trace of a matrix,  $\text{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 + \dots + \lambda_n$ .  
(*Hint*: consider an equivalent diagonal matrix.)

11. Show that the eigenvalues of a Hermitian matrix are real and that the eigenvectors can be made orthogonal to each other.

12. Find the eigenvalues and normalized eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -8 \\ 2 & 11 \end{pmatrix}.$$

13. (a) Find the eigenvalues and normalized eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 4 - i \\ 4 + i & -14 \end{pmatrix}.$$

- (b) Construct the matrix  $\mathbf{X}$  that diagonalizes  $\mathbf{A}$  and verify that it works.
14. (a) Construct the nine-dimensional matrix representation generated by the unit Cartesian vectors associated with each atom of the  $\text{H}_2\text{O}$  molecule. (Choose  $x_i$  out of the plane.)
- (b) What are the characters of the reducible representation?
- (c) How many times does each irreducible representation occur in this reducible representation?
15. Consider the four out-of-plane  $p_z$  orbitals of cyclobutadiene.
- (a) Assuming  $D_{4h}$  symmetry, construct the four-dimensional matrix representation of the  $D_4$  subgroup with the  $p_z$  orbitals.
- (b) What are the characters of this reducible representation?
- (c) How many times does each irreducible representation occur in this representation?
16. Given the set of polynomials  $\{1, x, x^2, x^3, \dots\}$  construct the first three members of a new set of orthonormal polynomials on the interval  $-1 \leq x \leq 1$  using the Gram-Schmidt procedure. They are proportional to the Legendre polynomials.
17. (a) For the point group  $D_{2h}$  construct a three-dimensional matrix representation using the set of three real  $p$  orbitals.
- (b) To what irreducible representations do these orbitals belong?
18. Construct the character table for the  $C_{4v}$  point group, without consulting tables.
19. For the  $D_{3h}$  point group verify equations (3.106), (3.107), and (3.108).
20. Construct the  $D_{2h}$  character table by taking a direct product of the  $C_i$  and  $D_2$  character tables.
21. One matrix representation that can easily be constructed for any group is called the *regular representation*. This representation is obtained by writing the group multiplication table in such a form that the identity element  $E$  lies along the main diagonal. Then the matrix representative for a particular group element  $R$  is obtained by replacing that element everywhere in the multiplication table by unity and all other group elements by zero.
- (a) Do so for the group  $C_{3v}$  and obtain the corresponding reducible matrix representation group.
- (b) What are the characters of this representation?

(c) What irreducible representations make up the regular representation of  $C_{3v}$ ?

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## Chapter 4

# Quantum Mechanics and Group Theory

### 4.1 Matrix Representation of the Schrödinger Equation

The application of quantum mechanics to spectroscopic problems involves solving the appropriate time-independent Schrödinger equation,  $\hat{H}\psi = E\psi$ . The solutions of this eigenvalue problem are a set of wavefunctions  $\{\psi_i\}$  and a corresponding set of energy eigenvalues  $\{E_i\}$ . Although a solution of the Schrödinger equation is, in general, a difficult mathematical problem, steady progress has been made over the years. Spectroscopists, however, do not suffer from such mathematical difficulties—they simply measure the difference between two eigenvalues,  $E_i - E_j = h\nu$ , and the intensity of the transition.

The most appropriate formulation of quantum mechanics for spectroscopy is based upon the Heisenberg matrix mechanics approach. Although simple spectroscopic models, such as the rigid rotor and the harmonic oscillator, are customarily solved using differential equations, any application of quantum mechanics to real systems is usually best handled by matrix mechanics. The general spectroscopic problem is handled by selection of an appropriate Hamiltonian operator and selection of a basis set, followed by diagonalization of the Hamiltonian matrix to obtain the wavefunctions and energy levels.

The solution of  $\hat{H}\psi = E\psi$  (after selection of  $\hat{H}$ ) proceeds by expanding the wavefunction in terms of a set of appropriate basis functions—that is,

$$\psi = \sum_i c_i f_i, \quad (4.1)$$

or using Dirac notation,

$$|\psi\rangle = \sum_i c_i |f_i\rangle. \quad (4.2)$$

For example, if  $\hat{H}$  represents an anharmonic molecular oscillator then  $\{|f_i\rangle\}$  might be the harmonic oscillator wavefunctions. Using this basis set, arbitrary wavefunctions  $|\psi\rangle$  and  $|\phi\rangle$  are represented by column vectors of expansion coefficients, such as

$$|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} \quad \text{and} \quad |\phi\rangle = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \end{pmatrix}.$$

In vector notation the basis functions can be represented by

$$|f_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad |f_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

and so on. The scalar product of two wavefunctions is expressed in the form

$$\langle\psi|\phi\rangle = \int \psi^* \phi d\tau = \sum c_i^* d_i = (c_1^* c_2^* \cdots) \begin{pmatrix} d_1 \\ d_2 \\ \vdots \end{pmatrix}. \quad (4.3)$$

The operation of Hermitian conjugation converts the ket vectors ( $|\psi\rangle$ ) to bra vectors ( $\langle\psi|$ ), that is,

$$|\psi\rangle^\dagger = \langle\psi| = (c_1^* c_2^* \cdots), \quad (4.4)$$

so that the scalar product can be interpreted as a matrix product, as in (4.3).

Dirac notation is particularly useful for algebraic manipulations. Notice, for example,  $\langle f_i | f_j \rangle = \delta_{ij}$  is a number, but  $\hat{P}_{ij} = |f_i\rangle\langle f_j|$  is a matrix operator. If  $i = j$  then, for example, in a five-dimensional space

$$\hat{P}_{33} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for  $i = j = 3$ , and so  $\hat{P}_{ii} \equiv \hat{P}_i$  is represented by a matrix with a 1 in the  $i$ th position on the diagonal and zeros elsewhere. This is an example of a projection operator, since

$$\hat{P}_i |b\rangle = |f_i\rangle\langle f_i | b \rangle, \quad (4.5)$$

$\hat{P}_i$  projects the  $i$ th component out of an arbitrary vector  $|b\rangle$ . A useful identity is that of completeness of the basis set, namely that

$$\sum_i \hat{P}_i = \mathbf{1} = \sum_i |f_i\rangle\langle f_i|, \quad (4.6)$$

which can be used to derive the expansion coefficients  $\langle f_i | \psi \rangle$  of a wavefunction. Thus, we can write

$$|\psi\rangle = \mathbf{1}|\psi\rangle = \left(\sum_i |f_i\rangle\langle f_i|\right)\psi = \sum_i |f_i\rangle\langle f_i|\psi\rangle, \quad (4.7)$$

so that if

$$\psi = \sum_i c_i f_i,$$

then

$$c_i = \langle f_i | \psi \rangle. \quad (4.8)$$

An operator such as  $\hat{H}$  has an eigenvalue equation  $\hat{H}|\psi\rangle = E|\psi\rangle$ , in which  $\hat{H}$  is represented by a matrix in terms of the basis set  $\{f_i\}$ . Although it is not necessary, the  $\{f_i\}$  are assumed to be orthogonal functions ( $\langle f_i | f_j \rangle = \delta_{ij}$ ) and the matrix elements of  $\hat{H}$  are given as

$$H_{ij} = \langle f_i | \hat{H} | f_j \rangle = \int f_i^* \hat{H} f_j d\tau. \quad (4.9)$$

In quantum mechanics the operators associated with observables are Hermitian ( $\hat{A}^\dagger = \hat{A}$ ) so that the corresponding matrices, including  $\hat{H}$ , are also Hermitian ( $\hat{\mathbf{H}}^\dagger = \hat{\mathbf{H}}$ ). The solution of the Schrödinger equation thus requires that the orthogonal eigenvectors and the real eigenvalues of the Hamiltonian matrix  $\hat{\mathbf{H}}$  be determined.

The solution of the secular equation

$$|\hat{\mathbf{H}} - E\mathbf{1}| = 0 \quad (4.10)$$

provides a set of  $n$  energies, while the associated  $n$  eigenvectors can be determined from the corresponding set of homogeneous equations. These eigenvectors can be used as the columns of a unitary matrix  $\mathbf{X}$  and the eigenvalue equation written as

$$\hat{\mathbf{H}}\mathbf{X} = \mathbf{X}\mathbf{E}, \quad (4.11)$$

or

$$\hat{\mathbf{E}} = \mathbf{X}^{-1}\hat{\mathbf{H}}\mathbf{X}. \quad (4.12)$$

The matrix  $\mathbf{X}$  corresponds to a coordinate transformation of the original basis functions  $\{f_i\}$ ,

$$\mathbf{f} = \mathbf{X}\mathbf{f}', \quad (4.13)$$

or

$$\mathbf{f}' = \mathbf{X}^{-1}\mathbf{f}. \quad (4.14)$$

The matrix  $\hat{\mathbf{E}}$  has the energy eigenvalues along the diagonal and zeros elsewhere. In the representation provided by the new set of basis functions the Hamiltonian matrix is diagonal, that is,

$$\langle f'_i | \hat{H} | f'_j \rangle = E_i \delta_{ij}. \quad (4.15)$$

Exact and approximate solutions for a  $2 \times 2$  Hamiltonian matrix prove to be very useful in providing an example, and for simple applications. Let  $\hat{\mathbf{H}}$  be a two-dimensional matrix represented by

$$\hat{\mathbf{H}} = \begin{pmatrix} E_1^0 & V \\ V & E_2^0 \end{pmatrix} \quad (4.16)$$

in terms of the basis set  $\{|f_1\rangle, |f_2\rangle\}$ . Then the solution of the secular equation,

$$(E_1^0 - E)(E_2^0 - E) - V^2 = 0, \quad (4.17)$$

leads to the eigenvalues

$$E = \frac{E_1^0 + E_2^0}{2} \pm \frac{((E_1^0 - E_2^0)^2 + 4V^2)^{1/2}}{2}. \quad (4.18)$$

The transformation matrix  $\mathbf{X}$  which diagonalizes  $\hat{\mathbf{H}}$  can be represented as a rotation of the basis functions, namely

$$\mathbf{X} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (4.19)$$

The angle  $\theta$  of this orthogonal matrix is chosen in order to satisfy the condition that  $H'_{12} = H'_{21} = 0$  for the transformed Hamiltonian matrix,

$$\hat{\mathbf{H}}' = \mathbf{X}^{-1} \hat{\mathbf{H}} \mathbf{X}. \quad (4.20)$$

This condition is satisfied if  $\theta$  is such that

$$\tan 2\theta = \frac{-2V}{E_1^0 - E_2^0}. \quad (4.21)$$

The new basis functions are hence given by

$$\mathbf{f} = \mathbf{X} \mathbf{f}' \quad \text{or} \quad \mathbf{f}' = \mathbf{X}^{-1} \mathbf{f} \quad \text{or} \quad (\mathbf{f}')^t = (\mathbf{f})^t \mathbf{X}, \quad (4.22)$$

or

$$|f'_1\rangle = \cos \theta |f_1\rangle - \sin \theta |f_2\rangle \quad (4.23a)$$

$$|f'_2\rangle = \sin \theta |f_1\rangle + \cos \theta |f_2\rangle. \quad (4.23b)$$

In terms of this new basis set the transformed Hamiltonian matrix  $\hat{\mathbf{H}}'$  is diagonal:

$$\hat{\mathbf{H}}' = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \quad (4.24)$$

The two energies  $E_1$  and  $E_2$  are the two solutions of the secular equation (4.17).

Perturbation theory is also often used to solve a spectroscopic problem approximately. In the  $2 \times 2$  example just given, the Hamiltonian matrix is written as a sum of a zeroth-order term plus an interaction term, so that

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}^{(0)} + \hat{\mathbf{H}}^{(1)}, \quad (4.25)$$

with

$$\hat{\mathbf{H}}^{(0)} = \begin{pmatrix} E_1^0 & 0 \\ 0 & E_2^0 \end{pmatrix} \quad (4.26)$$

and

$$\hat{\mathbf{H}}^{(1)} = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix} \quad (4.27)$$

with

$$\hat{H}^{(0)}\psi_n^{(0)} = E_n^{(0)}\psi_n^{(0)}, \quad n = 1, 2, \quad (4.28)$$

and

$$V = \langle f_1 | \hat{H} | f_2 \rangle, \quad (4.29)$$

with  $f_1 = \psi_1^{(0)}$  and  $f_2 = \psi_2^{(0)}$ . According to perturbation theory the energy for the  $n$ th eigenvalue is given by the zeroth-order energy plus an infinite sum of successive corrections, that is,

$$E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \dots, \quad (4.30)$$

in which

$$E_n^{(1)} = \langle \psi_n^{(0)} | \hat{H}^{(1)} | \psi_n^{(0)} \rangle, \quad (4.31)$$

and

$$E_n^{(2)} = \sum_{m \neq n} \frac{\langle \psi_n^{(0)} | \hat{H}^{(1)} | \psi_m^{(0)} \rangle \langle \psi_m^{(0)} | \hat{H}^{(1)} | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} = \sum_{m \neq n} \frac{|\langle \psi_n^{(0)} | \hat{H}^{(1)} | \psi_m^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}}, \quad (4.32)$$

with higher-order contributions being defined similarly.

A corresponding expansion is used for the wavefunctions, namely

$$\psi_n = \psi_n^{(0)} + \psi_n^{(1)} + \dots, \quad (4.33)$$

in which the first-order correction is given by

$$\psi_n^{(1)} = \sum_{m \neq n} \frac{\langle \psi_m^{(0)} | \hat{H}^{(1)} | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} \psi_m^{(0)}, \quad (4.34)$$

with higher-order corrections being defined analogously. Since the diagonal elements of  $\hat{\mathbf{H}}^{(1)}$  are zero in this example, the first-order correction to the energy  $E^{(1)} = 0$ , and the second-order correction to the energies are given by

$$E_1^{(2)} = \frac{\langle \psi_1^{(0)} | \hat{H}^{(1)} | \psi_2^{(0)} \rangle \langle \psi_2^{(0)} | \hat{H}^{(1)} | \psi_1^{(0)} \rangle}{E_1^{(0)} - E_2^{(0)}} = \frac{V^2}{E_1^{(0)} - E_2^{(0)}} = -\frac{V^2}{\Delta E} \quad (4.35)$$

and

$$E_2^{(2)} = \frac{V^2}{E_2^{(0)} - E_1^{(0)}} = \frac{V^2}{\Delta E}. \quad (4.36)$$

The corresponding wavefunctions (not normalized) are

$$\psi_1 = \psi_1^{(0)} + \frac{V}{E_1^{(0)} - E_2^{(0)}} \psi_2^{(0)} = \psi_1^{(0)} - \frac{V}{\Delta E} \psi_2^{(0)} \quad (4.37)$$

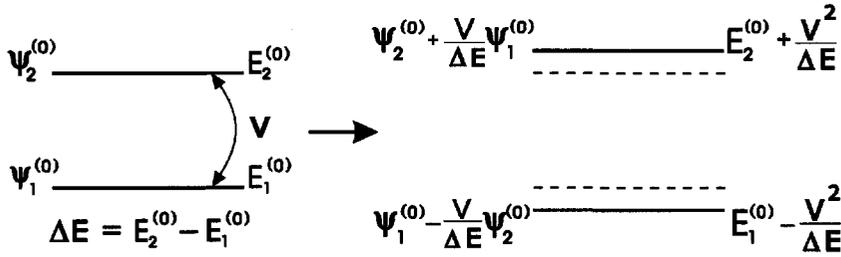


Figure 4.1: The interaction of two states using second-order perturbation theory.

and

$$\psi_2 = \psi_2^{(0)} + \frac{V}{E_2^{(0)} - E_1^{(0)}} \psi_1^{(0)} = \psi_2^{(0)} + \frac{V}{\Delta E} \psi_1^{(0)}. \tag{4.38}$$

The effect of the interaction

$$V = \langle \psi_2^{(0)} | \hat{H} | \psi_1^{(0)} \rangle \tag{4.39}$$

is thus to mix the wavefunctions and shift the energy levels  $E_1^0$  and  $E_2^0$  in opposite directions by the amount  $+V^2/\Delta E$  for  $E_2^0$  and  $-V^2/\Delta E$  for  $E_1^0$ , as shown in Figure 4.1. The degree to which the two wavefunctions can mix depends on both the magnitude of  $V$  and the initial energy difference  $\Delta E = E_2^0 - E_1^0$  as indicated by equations (4.37) and (4.38).

The perturbation theory result can be compared to the exact result for the two-level system by expanding the square root in the expression for the exact solution (4.18), first by rewriting it in the form

$$E_{\pm} = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{\Delta E}{2} \left( 1 + \frac{4V^2}{(\Delta E)^2} \right)^{1/2} \tag{4.40}$$

and then expanding the square root to obtain

$$E_{\pm} = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{\Delta E}{2} \left( 1 + 4 \frac{V^2}{2(\Delta E)^2} + \frac{(1/2)(-1/2)}{2!} \left( \frac{4V^2}{(\Delta E)^2} \right)^2 + \dots \right). \tag{4.41}$$

The two energies are therefore

$$E_2 = E_+ = E_2^{(0)} + \frac{V^2}{\Delta E} - \frac{V^4}{(\Delta E)^3} + \dots, \tag{4.42}$$

and

$$E_1 = E_- = E_1^{(0)} - \frac{V^2}{\Delta E} + \frac{V^4}{(\Delta E)^3} + \dots. \tag{4.43}$$

Comparing equations (4.35) and (4.36) with equations (4.42) and (4.43) indicates that second-order perturbation theory gives accurate results if the terms  $\pm V^4/(\Delta E)^3$  can be neglected.

## 4.2 Born–Oppenheimer Approximation

The central approximation in molecular spectroscopy is the separation of electronic and nuclear motion. The nonrelativistic molecular Hamiltonian operator is given by

$$\hat{H} = \frac{-\hbar^2}{2} \sum_{\alpha} \frac{\nabla_{\alpha}^2}{M_{\alpha}} - \frac{\hbar^2}{2m_e} \sum_i \nabla_i^2 + \sum_{\alpha} \sum_{\beta>\alpha} \frac{Z_{\alpha}Z_{\beta}e^2}{4\pi\epsilon_0 r_{\alpha\beta}} - \sum_{\alpha} \sum_i \frac{Z_{\alpha}e^2}{4\pi\epsilon_0 r_{i\alpha}} + \sum_i \sum_{j>i} \frac{e^2}{4\pi\epsilon_0 r_{ij}}, \quad (4.44)$$

$$\hat{H} = \hat{T}_N + \hat{T}_e + \hat{V}_{NN} + \hat{V}_{eN} + \hat{V}_{ee}, \quad (4.45)$$

in which Greek subscripts in equation (4.44) refer to the nuclei in a molecule, while Roman subscripts refer to the electrons. The various terms in this expression are:

- the nuclear kinetic energy

$$\hat{T}_N = \frac{-\hbar^2}{2} \sum_{\alpha} \frac{\nabla_{\alpha}^2}{M_{\alpha}}, \quad (4.46)$$

- the electronic kinetic energy

$$\hat{T}_e = \frac{-\hbar^2}{2m_e} \sum_i \nabla_i^2, \quad (4.47)$$

- the nuclear-nuclear repulsion energy

$$\hat{V}_{NN} = \sum_{\alpha} \sum_{\beta>\alpha} \frac{Z_{\alpha}Z_{\beta}e^2}{4\pi\epsilon_0 r_{\alpha\beta}}, \quad (4.48)$$

- the electron-nuclear attraction energy

$$V_{eN} = - \sum_{\alpha} \sum_i \frac{Z_{\alpha}e^2}{4\pi\epsilon_0 r_{i\alpha}}, \quad (4.49)$$

- the electron-electron repulsion energy

$$V_{ee} = \sum_i \sum_{j>i} \frac{e^2}{4\pi\epsilon_0 r_{ij}}. \quad (4.50)$$

The Schrödinger equation  $\hat{H}\psi = E\psi$  with this Hamiltonian operator is much easier to write down than it is to solve. The solution of the appropriate Schrödinger equation can in principle explain all of chemistry and spectroscopy. The first step in solving the Schrödinger equation is to invoke the Born–Oppenheimer approximation.<sup>1</sup>

The Coulombic forces acting on the nuclei and on the electrons are similar in magnitude, but the electrons are much lighter. The electrons therefore move much faster than the nuclei and, as a consequence, the electronic motion can be separated from the nuclear motion. The electronic structure is solved by “clamping” the nuclei at fixed positions and solving the purely electronic equation

$$\hat{H}_{\text{el}}\psi_{\text{el}} = E_{\text{el}}\psi_{\text{el}}, \quad (4.51)$$

in which

$$\hat{H}_{\text{el}} = \hat{T}_e + \hat{V}_{eN} + \hat{V}_{ee}. \quad (4.52)$$

If the nuclei are fixed in space, then

$$V_{NN} = \sum_{\alpha} \sum_{\beta > \alpha} \frac{Z_{\alpha} Z_{\beta} e^2}{4\pi\epsilon_0 r_{\alpha\beta}} \quad (4.53)$$

is just a constant that can be added in at the end to form the total electronic energy

$$U = E_{\text{el}} + V_{NN}. \quad (4.54)$$

The separation of the Schrödinger equation into electronic and nuclear motion (vibration-rotation) parts means that  $\psi$  can be approximated as the product function

$$\psi \approx \psi_{\text{el}} \chi_N, \quad (4.55)$$

and that two equations

$$\left( \hat{H}_{\text{el}} + \hat{V}_{NN} \right) \psi_{\text{el}}(\mathbf{r}_i; \mathbf{r}_{\alpha}) = U(\mathbf{r}_{\alpha}) \psi_{\text{el}}(\mathbf{r}_i; \mathbf{r}_{\alpha}), \quad (4.56)$$

and

$$(\hat{T}_N + U(\mathbf{r}_{\alpha})) \chi_N(\mathbf{r}_{\alpha}) = E_N \chi_N(\mathbf{r}_{\alpha}), \quad (4.57)$$

now need to be solved. In equation (4.56) the value of the total electronic energy depends in a parametric way on the particular nuclear positions,  $\mathbf{r}_{\alpha}$ . Clamping the nuclei at different positions will result in different numerical values for  $U$  and different functions for  $\psi_{\text{el}}$ . As the nuclei move, the electrons move so quickly that the  $U(\mathbf{r}_{\alpha})$  derived from (4.56) serves as the potential energy for the nuclear motion (4.57). The vibrational and rotational motions can also be approximately separated in equation (4.57). Although the vibration-rotation separation is conceptually similar to the Born–Oppenheimer approximation, it is a separate step that is not part of the Born–Oppenheimer separation of nuclear and electronic motion.

The terms neglected in the Born–Oppenheimer approximation can be examined by substituting the equation

$$\psi = \psi_{\text{el}}(\mathbf{r}_i; \mathbf{r}_{\alpha}) \chi_N(\mathbf{r}_{\alpha}) \quad (4.58)$$

into the full Schrödinger equation and remembering that  $\chi_N$  depends only upon the nuclear coordinates ( $\mathbf{r}_{\alpha}$ ), while  $\psi_{\text{el}}$  depends upon the electronic coordinates ( $\mathbf{r}_i$ ) and, parametrically, also upon the nuclear coordinates, so that

$$-\frac{\hbar^2}{2} \sum_{\alpha} \frac{\nabla_{\alpha}^2}{M_{\alpha}} \psi_{\text{el}} \chi_N - \frac{\hbar^2}{2m_e} \sum_i \nabla_i^2 \psi_{\text{el}} \chi_N + V \psi_{\text{el}} \chi_N = E \psi_{\text{el}} \chi_N. \quad (4.59)$$

By employing the identity

$$\nabla^2 fg = g \nabla^2 f + 2 \nabla f \cdot \nabla g + f \nabla^2 g, \quad (4.60)$$

we see that (4.59) becomes

$$\begin{aligned}
& -\frac{\hbar^2}{2} \sum_{\alpha} \chi_N \frac{\nabla_{\alpha}^2 \psi_{\text{el}}}{M_{\alpha}} - \hbar^2 \sum_{\alpha} \frac{(\nabla_{\alpha} \psi_{\text{el}})}{M_{\alpha}} \cdot (\nabla_{\alpha} \chi_N) \\
& -\frac{\hbar^2}{2} \sum_{\alpha} \psi_{\text{el}} \frac{\nabla_{\alpha}^2 \chi_N}{M_{\alpha}} - \frac{\hbar^2}{2m_e} \sum_i \chi_N \nabla_i^2 \psi_{\text{el}} + V \psi_{\text{el}} \chi_N = E \psi_{\text{el}} \chi_N. \quad (4.61)
\end{aligned}$$

By neglecting the first and second terms in equation (4.61), the remaining terms can be separated to yield equations (4.56) and (4.57). The neglect of these two terms is equivalent to neglecting the first and second derivatives of the electronic wavefunction with respect to the nuclear coordinates,  $\nabla_{\alpha} \psi_{\text{el}}$  and  $\nabla_{\alpha}^2 \psi_{\text{el}}$ . Indeed, the first-order correction (diagonal correction) for the effects of the breakdown of the Born–Oppenheimer approximation requires that these derivatives be evaluated and used to deduce the energy correction from  $E^{(1)} = \langle \psi^{(0)} | \hat{H}^{(1)} | \psi^{(0)} \rangle$ , in which  $\psi^{(0)}$  is the Born–Oppenheimer wavefunction and  $\hat{H}^{(1)}$  represents the two neglected terms.

### 4.3 Symmetry of the Hamiltonian Operator

The application of symmetry in quantum mechanics makes use of a key theorem: if the operators for two observables  $\hat{A}$  and  $\hat{B}$  commute, then it is possible to find a common set of orthogonal eigenfunctions. In mathematical terms if  $[\hat{A}, \hat{B}] = 0$ , then  $\hat{A}\psi = a\psi$  and  $\hat{B}\psi = b\psi$ , where  $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ , and the functions  $\psi$  in the two eigenvalue equations are the same. This abstract theorem has far-reaching consequences. For example, if the two observables are the total energy and the square of the total angular momentum, then it can be proven that  $[\hat{H}, \hat{J}^2] = 0$ . Therefore a set of common wavefunctions can be found for the two equations  $\hat{H}\psi_{nJ} = E\psi_{nJ}$  and  $\hat{J}^2\psi_{nJ} = E_J\psi_{nJ}$  and, most importantly, the energies and wavefunctions of the system can be labeled with  $J : \{E_{nJ}\}, \{\psi_{nJ}\}$ . The “good quantum number”  $J$  is very useful in characterizing the eigenstates of molecular or atomic systems.

An “almost good quantum number” is associated with an observable of the system that “almost” commutes with the Hamiltonian operator:  $\hat{H}\psi_n = E_n\psi_n$  and  $\hat{A}\psi_n \approx a\psi_n$  (i.e., the wavefunction of the system is an approximate eigenfunction of some other observable). For example, spin-orbit coupling can couple the spin  $\hat{S}$  and orbital angular momentum  $\hat{L}$  in an atom. Although  $J(\hat{J} = \hat{L} + \hat{S})$  is still a good quantum number,  $L$  and  $S$  are only approximate quantum numbers so that, for example,  $\hat{S}^2\psi \approx S(S+1)\hbar^2\psi$ . Although spin is no longer a good quantum number, it is almost good for light atoms and it is still useful to speak of, for example, triplet states.

The Hamiltonian operator for a system,  $\hat{H} = \hat{T} + \hat{V}$ , has certain symmetry properties. For example, the kinetic energy part of the Hamiltonian operator always has the symmetry of a sphere,  $K_h$ , because the Laplacian operator  $\nabla^2$  is invariant under all reflections and rotations which contain the origin. This can be verified by applying the transformation operators, such as  $\hat{O}_{C_{\theta}}$ , to the Laplacian. The potential energy part of the Hamiltonian operator therefore displays the particular point group symmetry of the molecule.

Since the operators  $\hat{O}_R$  leave the Hamiltonian operator unchanged, they must commute with the Hamiltonian operator, that is,  $\hat{O}_R \hat{H} f = \hat{H} \hat{O}_R f$  or  $[\hat{H}, \hat{O}_R] = 0$ . In fact it is possible to show that  $\hat{O}_R$  commutes with  $\hat{T}_e$ ,  $\hat{T}_N$ , and  $\hat{V}$  individually. This means that a common set of eigenfunctions for  $\hat{H}\psi = E\psi$  and  $\hat{O}_R\psi = a\psi$  can be found. The wavefunctions can therefore be classified by their behavior with respect to the set of

symmetry operators  $\{\hat{O}_R\}$ . The wavefunctions have the same symmetry properties as the irreducible representations, which can thus be used to label the wavefunctions. For example, if the molecule has a center of symmetry, then  $\hat{H}_{el}\psi = E_{el}\psi$  and  $\hat{i}\psi_{el} = \pm\psi_{el}$  have a common set of eigenfunctions. Thus  $\psi_{el}$  behaves either like the  $A_g$  row or like the  $A_u$  row of the  $C_i$  character table

$C_i$	$\hat{E}$	$\hat{i}$
$A_g$	1	1
$A_u$	1	-1

The electronic wavefunction is either even or odd, so that  $g$  and  $u$  can be used to classify the wavefunctions as  $\psi_{el,g}$  or  $\psi_{el,u}$ .

For degenerate wavefunctions the symmetry operator  $\hat{O}_R$  changes one wavefunction into a linear combination of the members of the degenerate set. If there is an  $n$ -fold degeneracy, that is, if

$$\begin{aligned}\hat{H}\psi_1 &= E\psi_1 \\ \hat{H}\psi_2 &= E\psi_2 \\ &\vdots \\ \hat{H}\psi_n &= E\psi_n,\end{aligned}$$

then  $\{\psi_1, \psi_2 \dots \psi_n\}$  will form an  $n$ -dimensional function space spanned by the  $n$  orthogonal wavefunctions. Thus the action of  $\hat{O}_R$  on a single member  $\psi_i$  of the set of  $n$  degenerate wavefunctions can be represented by

$$\hat{O}_R\psi_i = \sum_j \psi_j D_{ji}(\hat{R}), \quad (4.62)$$

in which  $\mathbf{D}(\hat{R})$  is the matrix representation of  $\hat{R}$  in the  $n$ -dimensional wavefunction space. The new wavefunction

$$\psi' = \sum_i c_i \psi_i \quad (4.63)$$

produced by the action of a symmetry operator is also a solution of the Schrödinger equation having the same energy eigenvalue. The  $n$  degenerate wavefunctions form a basis for the matrix representation of the point group operations of the molecule. Thus, for example, the electronic wavefunctions of the  $\text{NH}_3$  molecule may be totally symmetric with respect to the six symmetry operators (and have  $A_1$  symmetry), may behave like the  $A_2$  line of the character table, or may be doubly degenerate (with  $E$  symmetry). The electronic states of ammonia are thus said to belong to the  $A_1$ ,  $A_2$ , or  $E$  representations. Basis functions that possess these symmetries are said to be symmetry-adapted basis functions.

## 4.4 Projection Operators

Projection operators are useful in generating functions of the proper symmetry for the solution of a molecular problem. Since a wavefunction must belong to a particular irreducible representation, it is very helpful to construct solutions of the correct symmetry

for the problem. Consider a set of  $n_\nu$  orthogonal functions  $\{f_1^\nu, f_2^\nu \dots f_{n_\nu}^\nu\}$  that belongs to a function space that forms the  $\nu$ th irreducible representation in a point group. The result of operating with the operator  $\hat{O}_R$  on an arbitrary member of this set of functions can be written as

$$\hat{O}_R f_j^\nu = \sum_i f_i^\nu D_{ij}^\nu(\hat{R}), \quad (4.64)$$

in which  $\Gamma^\nu = \{\mathbf{D}^\nu(\hat{R})\}$  is the unitary matrix representation. Let us define the projection operator

$$\hat{P}_{ij}^\mu = \sum_{\hat{R}} D_{ij}^\mu(\hat{R})^* \hat{O}_R \quad (4.65)$$

and apply  $\hat{P}_{ij}^\mu$  to a member of the function space  $\{f^\nu\}$ . We obtain the result

$$\begin{aligned} \hat{P}_{ij}^\mu f_k^\nu &= \sum_{\hat{R}} D_{ij}^\mu(\hat{R})^* \hat{O}_R f_k^\nu = \sum_{\hat{R}} D_{ij}^\mu(\hat{R})^* \sum_m D_{mk}^\nu(\hat{R}) f_m^\nu \\ &= \sum_m \left( \sum_{\hat{R}} D_{ij}^\mu(\hat{R})^* D_{mk}^\nu(\hat{R}) \right) f_m^\nu \\ &= \sum_m \left( \frac{g}{n_\nu} \delta_{\mu\nu} \delta_{im} \delta_{jk} \right) f_m^\nu = \frac{g}{n_\nu} \delta_{\nu\mu} \delta_{jk} f_i^\nu \end{aligned} \quad (4.66)$$

by employing the Great Orthogonality Theorem. Notice that if  $\mu \neq \nu$ , then  $\hat{P}_{ij}^\mu f_k^\nu = 0$ ; if  $j \neq k$ , then  $\hat{P}_{ij}^\mu f_k^\nu = 0$ ; but  $\hat{P}_{ik}^\mu f_k^\mu = (g/n_\mu) f_i^\mu$ . This means that if one member of a set of basis functions belonging to an irreducible representation is known, then it is possible by using projection operators to generate all other members of that representation. The only catch with this type of projection operator is that the representation matrices  $\{\mathbf{D}^\mu(\hat{R})\}$  are needed, not just their traces or characters.

A simpler, but still useful, projection operator can be defined using only the characters of the matrix representation. Let the projection operator  $\hat{P}^\mu$  be defined as

$$\hat{P}^\mu = \sum_{i=1}^{n_\mu} \hat{P}_{ii}^\mu = \sum_{i=1}^{n_\mu} \sum_{\hat{R}} D_{ii}^\mu(\hat{R})^* \hat{O}_R = \sum_{\hat{R}} \sum_{i=1}^{n_\mu} D_{ii}^\mu(\hat{R})^* \hat{O}_R \quad (4.67)$$

or

$$\hat{P}^\mu = \sum_{\hat{R}} \chi^\mu(\hat{R})^* \hat{O}_R \quad (4.68)$$

since the summation over the diagonal elements of  $\mathbf{D}^\mu(\hat{R})$  simply represents the character of  $\mathbf{D}^\mu(\hat{R})$ . Thus we see that when  $\hat{P}^\mu$  acts on a member of the  $\nu$ th function space, we obtain

$$\begin{aligned}
\hat{P}^\mu f_j^\nu &= \sum_i \hat{P}_{ii}^\mu f_j^\nu = \sum_{i=1}^{n_\mu} \sum_{\hat{R}} D_{ii}^\mu(\hat{R})^* \hat{O}_R f_j^\nu \\
&= \sum_{i=1}^{n_\mu} \sum_{\hat{R}} D_{ii}^\mu(\hat{R})^* \sum_m D_{mj}^\nu(\hat{R}) f_m^\nu \\
&= \sum_{i=1}^{n_\mu} \sum_{m=1}^{n_\nu} \frac{g}{n_\mu} \delta_{\mu\nu} \delta_{im} \delta_{ij} f_m^\nu \\
&= \frac{g}{n_\mu} \delta_{\mu\nu} f_j^\nu.
\end{aligned} \tag{4.69}$$

If  $\mu \neq \nu$ , then  $\hat{P}^\mu f_j^\nu = 0$ , and if  $\mu = \nu$  and  $i = j$ , then  $\hat{P}^\mu f_j^\nu = (g/n_\mu) f_j^\mu$ .

At first sight the  $\hat{P}^\mu$  operators do not seem to be very useful. Notice, however, that  $\hat{P}^\mu$  annihilates all functions, or parts of functions, that do not belong to the  $\mu$ th irreducible representation and leaves behind a function that possesses the correct symmetry.

For example, consider a set of  $1s$  functions on the hydrogen atoms of  $\text{NH}_3$  and suppose that a linear combination of the three atomic hydrogen orbitals is needed to make approximate molecular orbitals to bond with the nitrogen atomic orbitals (Figure 4.2). Individually the three hydrogen orbitals do not have the correct symmetry since, for example,  $\hat{C}_3(1s_A) = 1s_C$ . However, projection operators can be used to construct a set of symmetry-adapted linear combinations of the hydrogen  $1s$  orbitals. As we have seen earlier the appropriate symmetry point group is  $C_{3v}$ , so that the projection operators, given by

$$\hat{P}^{A_1} = 1 \cdot \hat{O}_E + 1 \cdot \hat{O}_{C_3} + 1 \cdot \hat{O}_{C_3^{-1}} + 1 \cdot \hat{O}_{\sigma'} + 1 \cdot \hat{O}_{\sigma''} + 1 \cdot \hat{O}_{\sigma'''}, \tag{4.70a}$$

$$\hat{P}^{A_2} = 1 \cdot \hat{O}_E + 1 \cdot \hat{O}_{C_3} + 1 \cdot \hat{O}_{C_3^{-1}} - 1 \cdot \hat{O}_{\sigma'} - 1 \cdot \hat{O}_{\sigma''} - 1 \cdot \hat{O}_{\sigma'''}, \tag{4.70b}$$

$$\hat{P}^E = 2 \cdot \hat{O}_E - 1 \cdot \hat{O}_{C_3} - 1 \cdot \hat{O}_{C_3^{-1}} + 0 \cdot \hat{O}_{\sigma'} + 0 \cdot \hat{O}_{\sigma''} + 0 \cdot \hat{O}_{\sigma'''}, \tag{4.70c}$$

can be applied to a typical  $1s$  function, say  $1s_A$ , with the result being

$$\hat{P}^{A_1} 1s_A = 1s_A + 1s_C + 1s_B + 1s_A + 1s_C + 1s_B, \tag{4.71a}$$

$$\hat{P}^{A_2} 1s_A = 1s_A + 1s_C + 1s_B - 1s_A - 1s_C - 1s_B = 0, \tag{4.71b}$$

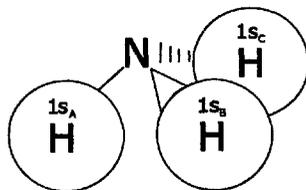
$$\hat{P}^E 1s_A = 2(1s_A) - 1s_C - 1s_B. \tag{4.71c}$$

If the hydrogenic basis functions are assumed to be orthogonal to one another, then we obtain the orthonormalized symmetry-adapted basis functions

$$\phi_1 = \frac{1}{\sqrt{3}}(1s_A + 1s_B + 1s_C), \tag{4.72a}$$

with  $A_1$  symmetry and

$$\phi_2 = \frac{1}{\sqrt{6}}(2(1s_A) - 1s_B - 1s_C) \tag{4.72b}$$

Figure 4.2: The three  $1s$  hydrogen functions in the  $\text{NH}_3$  molecule.

with  $E$  symmetry. Notice that there is a missing linear combination since the application of a single  $\hat{P}^E$  operator generates only a single function of  $E$  symmetry and to complete the  $E$  representation a second basis function is needed. This second function can be generated by applying  $\hat{P}^E$  to another atomic orbital, say the  $1s_B$  orbital, to obtain

$$\hat{P}^E 1s_B = 2(1s_B) - 1s_C - 1s_A. \quad (4.73)$$

This second function can then be made orthogonal to the original  $E$  function by subtracting out the overlapping part using the Gram-Schmidt procedure (see section 3.4). The new function has the form

$$\begin{aligned} \phi'_3 &= \frac{1}{\sqrt{6}} (2(1s_B) - 1s_C - 1s_A) - \frac{1}{\sqrt{6}} \langle \phi_2 | 2(1s_B) - 1s_C - 1s_A \rangle \phi_2 \\ &= \frac{1}{\sqrt{6}} (2(1s_B) - 1s_C - 1s_A) + \phi_2/2 \\ &= \frac{1}{\sqrt{6}} \left( \frac{3}{2}(1s_B) - \frac{3}{2}(1s_C) \right), \end{aligned} \quad (4.74)$$

which becomes upon normalizing,

$$\phi_3 = \frac{1}{\sqrt{2}} (1s_B - 1s_C). \quad (4.75)$$

The functions  $\phi_1$ ,  $\phi_2$ , and  $\phi_3$  have the appropriate symmetry for the  $\text{NH}_3$  molecule. Although these simple functions may not be the best functions for a realistic calculation of the electronic energy of  $\text{NH}_3$ , they are the appropriate functions to use within the simple linear combination of atomic orbitals model. Since the final electronic wavefunction must belong to the  $A_1$ ,  $A_2$ , or  $E$  irreducible representations, the calculation is simplified if the initial trial wavefunctions also belong to one of these representations.

## 4.5 Direct Product Representations

The total wavefunction is often written as a product  $\psi = \psi_{\text{el}} \chi_N$ , in which  $\psi_{\text{el}}$  and  $\chi_N$  each belong to particular irreducible representations of the molecular point group. To which representation does the product  $\psi = \psi_{\text{el}} \chi_N$  belong? In order to determine this, let  $\{f_1^\nu f_2^\nu \cdots f_{n_\nu}^\nu\}$  be a set of  $n_\nu$  functions belonging to the  $\nu$ th representation  $\Gamma^\nu$ , while  $\{f_1^\mu \cdots f_{n_\mu}^\mu\}$  belongs to  $\Gamma^\mu$ . A new function space with  $n_\nu \times n_\mu$  members can be formed by taking all possible products

$$\begin{array}{cccc} f_1^\nu f_1^\mu & f_1^\nu f_2^\mu & \cdots & f_1^\nu f_{n_\mu}^\mu \\ \vdots & & & \vdots \\ f_{n_\nu}^\nu f_1^\mu & \cdots & \cdots & f_{n_\nu}^\nu f_{n_\mu}^\mu \end{array}$$

These product functions form a new representation  $\Gamma^{\nu\otimes\mu} = \Gamma^\nu \otimes \Gamma^\mu$ , in which the symbol  $\otimes$  is used to represent the direct product in order to distinguish the operation from ordinary multiplication. A new set of matrices,  $n_\nu n_\mu \times n_\nu n_\mu$  in dimension, is formed by taking direct products of the matrix representatives in  $\Gamma^\nu$  and  $\Gamma^\mu$ —that is,  $\mathbf{D}^{\nu\otimes\mu} = \mathbf{D}^\nu \otimes \mathbf{D}^\mu$ . For example, the direct product of two  $2 \times 2$  matrices  $\mathbf{A}$  and  $\mathbf{B}$  is represented by

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= \begin{pmatrix} A_{11}\mathbf{B} & A_{12}\mathbf{B} \\ A_{21}\mathbf{B} & A_{22}\mathbf{B} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}. \end{aligned} \quad (4.76)$$

The characters of the direct product matrices

$$\begin{aligned} \sum_i D_{ii}^{\nu\otimes\mu}(\hat{R}) &= \sum_i \sum_j^{n_\nu} D_{ii}^\nu(\hat{R}) D_{jj}^\mu(\hat{R}) = \left( \sum_i D_{ii}^\nu(\hat{R}) \right) \left( \sum_j D_{jj}^\mu(\hat{R}) \right) \\ &= \chi^\nu(\hat{R}) \chi^\mu(\hat{R}) \end{aligned} \quad (4.77)$$

are just the product of the characters of the individual matrix representations in  $\Gamma^\mu$  and  $\Gamma^\nu$ . Of course, these direct product representations are reducible in terms of the irreducible representations of the point group, that is,

$$\Gamma^{\nu\otimes\mu} = \Gamma^\nu \otimes \Gamma^\mu = a_1 \Gamma^1 \oplus a_2 \Gamma^2 \cdots = \oplus \sum a_i \Gamma^i, \quad (4.78)$$

in which the  $a_i$  are determined from

$$a_i = \frac{1}{g} \sum_{\hat{R}} \chi^{\nu\otimes\mu}(\hat{R}) (\chi^i(\hat{R}))^*. \quad (4.79)$$

For example, consider the product  $E \otimes E$  obtained for the product wavefunction  $\psi = \psi_{e, E\chi_{N,E}}$ . The appropriate characters are given by

$C_{3v}$	$\hat{E}$	$2\hat{C}_3$	$3\hat{\sigma}_v$
$A_1$	1	1	1
$A_2$	1	1	-1
$E$	2	-1	0
$\chi^{E \otimes E}$	4	1	0

and the use of equation (4.79) leads to the results

$$a_{A_1} = \frac{1}{6}(4(1) + 2(1)(1)) = 1,$$

$$a_{A_2} = \frac{1}{6}(4(1) + 2(1)(1)) = 1,$$

$$a_E = \frac{1}{6}(4(2) + 2(-1)(1)) = 1,$$

so that the direct product representation is decomposed as

$$\Gamma^{E \otimes E} = \Gamma^{A_1} \oplus \Gamma^{A_2} \oplus \Gamma^E. \quad (4.80)$$

## 4.6 Integrals and Selection Rules

The intensity  $I$  of a transition between two states, designated by  $|i\rangle$  and  $|j\rangle$ , is proportional to the square of a transition dipole moment, that is,

$$I \propto \left| \int \psi_i^* \boldsymbol{\mu} \psi_j d\tau \right|^2, \quad (4.81)$$

so that integrals of the type

$$\int \psi_i^* x \psi_j d\tau, \quad (4.82)$$

or, more generally,

$$\int \psi_i^* f \psi_j d\tau, \quad (4.83)$$

with  $f$  as an arbitrary function, are of interest. The integrand is a product of three functions, each of which belongs to a particular irreducible representation. What is the overall symmetry of the integrand? If  $\psi_i$  belongs to the  $\Gamma^\mu$  irreducible representation,  $\psi_j$  to  $\Gamma^\nu$ , and  $f$  to  $\Gamma^\lambda$ , then the triple product  $\psi_i^* f \psi_j$  belongs to the direct product representation

$$\Gamma^\mu \otimes \Gamma^\lambda \otimes \Gamma^\nu = \oplus \sum_i a_i \Gamma^i, \quad (4.84)$$

which can be reduced to the direct sum of the irreducible representations of the point group. If this reduction does not contain the  $A_1$  irreducible (totally symmetric) representation, then the integral over all space is exactly zero. This is just a generalization of the fact that the integral over all space of an odd function,  $f(-x) = -f(x)$ , is zero, or

$$\int_{-\infty}^{\infty} f_{\text{odd}}(x) dx = 0.$$

The proof of the assertion that a nonzero integral

$$\int \psi_i^* f \psi_j d\tau$$

must have an integrand that belongs to a direct product representation that contains  $A_1$  requires the use of the projection operator

$$\hat{P}^{A_1} = \sum_{\hat{R}} \chi^{A_1}(\hat{R})^* \hat{O}_R = \sum_{\hat{R}} \hat{O}_R. \quad (4.85)$$

If  $\psi_i^* f \psi_j$  does not contain a function of  $A_1$  symmetry, then

$$\hat{P}^{A_1}(\psi_i^* f \psi_j) = 0. \quad (4.86)$$

The  $\hat{O}_R$  symmetry operators have no effect on the integral

$$\int \hat{O}_R(\psi_i^* f \psi_j) d\tau = \int \psi_i^* f \psi_j d\tau, \quad (4.87)$$

since the integration is over all space. Thus, summing over all  $\hat{R}$ , we find that

$$\begin{aligned} g \int \psi_i^* f \psi_j d\tau &= \sum_{\hat{R}} \int \hat{O}_R(\psi_i^* f \psi_j) d\tau \\ &= \int \hat{P}^{A_1}(\psi_i^* f \psi_j) d\tau, \end{aligned} \quad (4.88)$$

with  $g$  the order of the group, and hence

$$\int \psi_i^* f \psi_j d\tau = \frac{1}{g} \int \hat{P}^{A_1}(\psi_i^* f \psi_j) d\tau. \quad (4.89)$$

Thus, if  $\hat{P}^{A_1}(\psi_i^* f \psi_j) = 0$ , then the integral must vanish, and it is a necessary (but not sufficient) condition that the integrand contain a function of  $A_1$  symmetry in order that the integral not vanish.

An important application of this rule (in addition to deriving selection rules) is in the construction of Hamiltonian matrices. It is possible to let  $f$  be an operator such as  $\hat{H}$ , which belongs to the  $A_1$  irreducible representation, since the Hamiltonian operator is unchanged under all symmetry operations. Therefore

$$H_{ij} = \int \psi_i^* \hat{H} \psi_j d\tau \quad (4.90)$$

will be nonzero depending on the symmetry properties of  $\psi_i$  and  $\psi_j$ . The number of times that the direct product representation contains an irreducible representation of  $A_1$  symmetry is given by

$$\begin{aligned} a_{A_1} &= \frac{1}{g} \sum_{\hat{R}} \chi^{\mu \otimes \nu}(\hat{R}) \chi^{A_1}(\hat{R})^* \\ &= \frac{1}{g} \sum_{\hat{R}} \chi^{\mu \otimes \nu}(\hat{R}) \cdot 1 \\ &= \frac{1}{g} \sum_{\hat{R}} \chi^\mu(\hat{R}) \chi^\nu(\hat{R}) \\ &= \frac{1}{g} (g \delta_{\mu\nu}) = \delta_{\mu\nu}, \end{aligned} \quad (4.91)$$

since  $\Gamma^\mu$  and  $\Gamma^\nu$  are irreducible representations. This means that matrix elements between functions belonging to different irreducible representations ( $\mu \neq \nu$ ) will be identically zero. The Hamiltonian matrix becomes block diagonal with each block corresponding to a different irreducible representation. Each block can now be diagonalized separately since there are no matrix elements connecting blocks of different symmetry

$$\hat{\mathbf{H}} = \begin{pmatrix} A_1 & & \\ \text{block} & 0 & 0 \\ & \mu\text{th} & \\ & \text{block} & 0 \\ & & \nu\text{th} \\ & 0 & \text{block} \end{pmatrix}. \tag{4.92}$$

**Problems**

- Given the matrices **A** and **B** as

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{3} & \sqrt{\frac{2}{3}} & \frac{\sqrt{2}}{3} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{3} & \frac{1}{\sqrt{3}} & -\frac{2}{3} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \frac{5}{3} & \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{3}{2} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{3\sqrt{2}} & \frac{1}{2\sqrt{3}} & \frac{11}{6} \end{pmatrix}.$$

Show that **A** and **B** commute. Find their eigenvalues and eigenvectors, and obtain a unitary transformation matrix **U** which diagonalizes both **A** and **B**.

- Obtain eigenvalues to second order and eigenvectors to first order of the matrix

$$\hat{\mathbf{H}} = \begin{pmatrix} 1 & 2\alpha & 0 \\ 2\alpha & 2 + \alpha & 3\alpha \\ 0 & 3\alpha & 3 + 2\alpha \end{pmatrix}$$

using the small parameter  $\alpha$ .

- A particle of mass  $m$  is confined to an infinite potential box with potential

$$V(x) = \begin{cases} \infty, & x < 0, x > L, \\ k(1 - \frac{x}{L}), & 0 \leq x \leq L. \end{cases}$$

Calculate the ground and fourth excited-state energies of the particle in this box using first-order perturbation theory. Obtain the ground and fourth excited-state wavefunctions to first order, and sketch their appearance. How do they differ from the corresponding unperturbed wavefunctions?

- A matrix representation of the Hamiltonian operator for a two-dimensional system is given by  $\hat{\mathbf{H}} = \hat{\mathbf{H}}^{(0)} + \hat{\mathbf{H}}^{(1)}$ , with

$$\hat{\mathbf{H}}^{(0)} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \hat{\mathbf{H}}^{(1)} = \alpha \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad \alpha \ll 1.$$

- Obtain eigenvectors to first order and eigenvalues to second order for the problem

$$\hat{\mathbf{H}}\mathbf{x}_k = \lambda_k\mathbf{x}_k, \quad k = 1, 2,$$

using perturbation theory.

(b) For comparison solve the problem exactly, first for  $\hat{H}^{(0)}$  and then for  $\hat{H}$ .

5. Consider the Hamiltonian matrix constructed in the  $\{\phi_i\}$  basis ( $\alpha, \beta$  are real numbers):

$$\hat{H} = \begin{pmatrix} -\alpha & \beta & 0 \\ \beta & -\alpha & \beta \\ 0 & \beta & 2\alpha \end{pmatrix}.$$

Obtain the eigenvalues, their corresponding eigenvectors, and the unitary transformation that brings  $\hat{H}$  to diagonal form. (*Hint:* There is a trigonometric solution to certain cubic equations.)

6. Write out the characters for the following direct products and then determine which irreducible representations they decompose into:

(a)  $A_1 \otimes A_1, A_2 \otimes A_2, A_2 \otimes E, E \otimes E, E \otimes E \otimes E$  for  $C_{3v}$ .

(b)  $A_{2g} \otimes A_{2g}, E_g \otimes E_g, T_{2g} \otimes T_{2g}, T_{1u} \otimes T_{2g}, E_u \otimes T_{1u}$  for  $O_h$ .

(c)  $E'' \otimes A_2', A_2' \otimes A_2', E' \otimes E'', E'' \otimes E''$  for  $D_{3h}$ .

7. Consider the transition dipole moment integral

$$\int \psi_1^* \boldsymbol{\mu} \psi_0 d\tau$$

(a) For the  $C_{2v}$  point group, if  $\psi_0$  belongs to the  $A_1$  irreducible representation and  $\mu_z, \mu_x,$  and  $\mu_y$  have  $A_1, B_1,$  and  $B_2$  symmetry, respectively, what are the possible symmetries of  $\psi_1$  in order to make the integral nonzero?

(b) Repeat (a) for  $D_{6h}$  where  $\mu_z$  and  $(\mu_x, \mu_y)$  have  $A_{2u}$  and  $E_{1u}$  symmetry, and  $\psi_0$  has  $A_{1g}$  symmetry.

(c) Repeat (a) for  $T_d$  where  $(\mu_x, \mu_y, \mu_z)$  have  $T_2$  symmetry and  $\psi_0$  has  $E$  symmetry.

8. Show that the eigenvalues and eigenvectors of a symmetric  $2 \times 2$  matrix (4.16) are given by equations (4.18), (4.21), and (4.23).

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## Chapter 5

# Atomic Spectroscopy

### 5.1 Background

Historically, atomic spectroscopy was developed before molecular spectroscopy. The discovery of the Fraunhofer absorption lines in the spectrum of the sun and the observation by Herschel of the colors emitted when metal salts are introduced into flames occurred in the early 1800s. It was not, however, until the 1850s that Kirchhoff and Bunsen clearly established that each atom had a characteristic spectral signature. These ideas led to the identification of the elements rubidium and cesium by emission spectroscopy, and in the discovery of helium in the sun in advance of its isolation on earth.

Atomic spectroscopy was used simply as a diagnostic tool in these early measurements, although Balmer noted mathematical regularities in the spectrum of the hydrogen atom in 1885. It was not until the work of Bohr in 1913 that the spectrum of the hydrogen atom was explained. The Bohr model was unable to account for the spectra of atoms with more than one electron and was soon superseded by the development of quantum mechanics in the 1920s. In fact the desire to explain atomic spectra was one of the primary motivations for the development of quantum mechanics.

When the hydrogen molecule is excited in an electrical discharge, a regular series of atomic hydrogen emission lines is observed (Figure 5.1). The line positions seem to converge to a limit for the Balmer series. This pattern repeats itself in other regions of the spectrum, for example, in the near infrared (Paschen) series and the vacuum ultraviolet (Lyman) series. These lines are customarily labeled with the series name and with a Greek letter to indicate the member of the series (Figure 5.2). For example, Balmer  $\alpha$  ( $H_\alpha$ ) and Balmer  $\beta$  ( $H_\beta$ ) denote the first and second members of the Balmer series at  $15\,233\text{ cm}^{-1}$  ( $6\,562.7\text{ \AA}$ ) and  $20\,565\text{ cm}^{-1}$  ( $4\,861.3\text{ \AA}$ ), respectively.

Balmer discovered that the wavelengths of the series that now bears his name could be represented by the empirical formula,

$$\lambda = \frac{3\,645.6n^2}{n^2 - n_0^2}\text{ \AA}, \quad n = 3, 4, 5, \dots; \quad n_0 = 2 \quad (5.1)$$

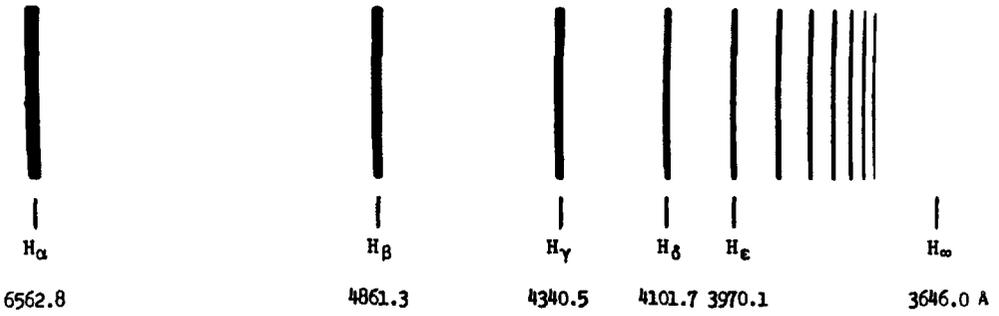


Figure 5.1: The Balmer series of the hydrogen atom.

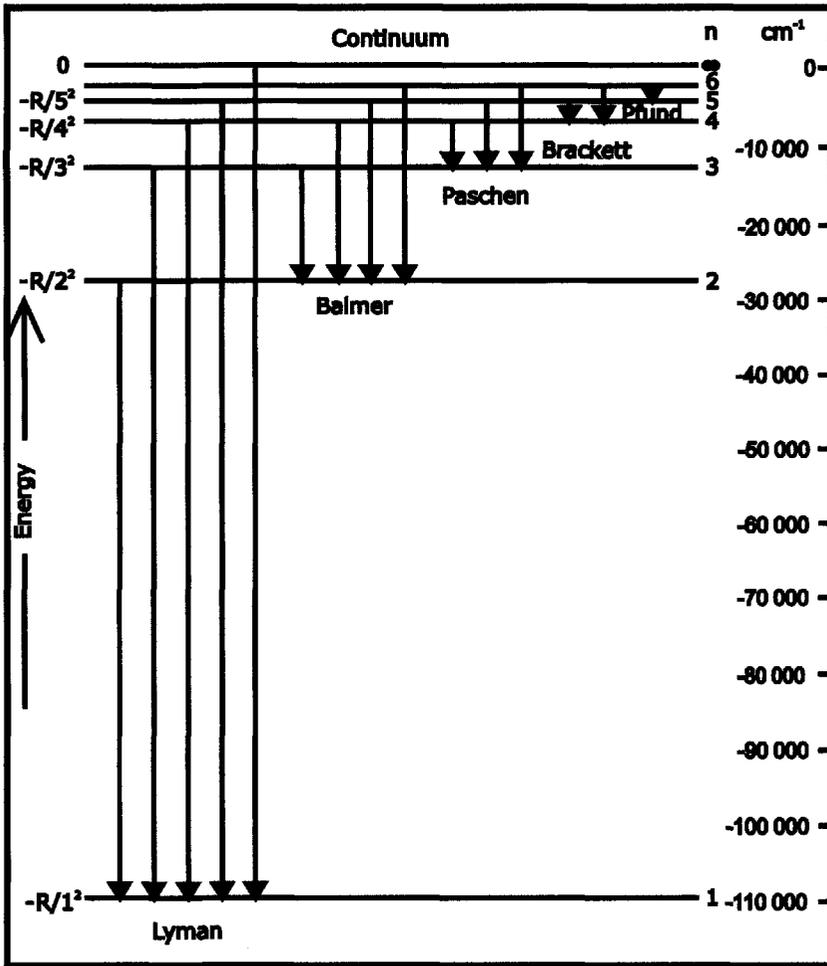


Figure 5.2: Energy levels of the hydrogen atom.

in which  $3645.6 \text{ \AA}$  is the series limit. In terms of  $\text{cm}^{-1}$  the formula becomes

$$\begin{aligned}\tilde{\nu} &= 109\,678 \left( \frac{1}{2^2} - \frac{1}{n^2} \right) \text{cm}^{-1}, \quad n = 3, 4, \dots \\ &= \tilde{R}_{\text{H}} \left( \frac{1}{2^2} - \frac{1}{n^2} \right),\end{aligned}\tag{5.2}$$

with  $\tilde{R}_{\text{H}}$  called the Rydberg constant (in  $\text{cm}^{-1}$ ). The other series were found to obey formulas similar to the Balmer formula (5.1), with each transition given as the difference between two terms, but with  $n_0 = 1, 3, 4, 5, \dots$

Remarkably, the spectra of the alkali atoms provided similar patterns in both emission and absorption. Although the emission and absorption spectra had some lines in common, the emission spectra were more complex. The emission spectra could also be organized into series which were given the names sharp (*S*), principal (*P*), diffuse (*D*), and fundamental (*F*). The names sharp and diffuse were based on the appearance of the lines, while the principal series appeared in both absorption and emission. The fundamental series was thought to be more fundamental because it occurred to the red (to longer wavelength) of the others and was most like the hydrogen series. Moreover, simple formulas similar to equation (5.2) were found to represent the various series of lines, namely

$$\tilde{\nu} = \tilde{T} - \frac{\tilde{R}}{(n - \delta)^2},\tag{5.3}$$

in which  $\tilde{T}$  is the series limit,  $\tilde{R}$  is the Rydberg constant, and  $n$  is an integer. Unfortunately, a small non-integer  $\delta$ , referred to as the quantum defect, had to be introduced, since the use of integer quantum numbers could not reproduce the series of alkali line positions. Elements other than hydrogen and the alkalis had even more complex spectra.

## 5.2 Angular Momentum

The interpretation of atomic spectra is closely related to the concept of angular momentum. The classical angular momentum  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$  can be transformed into a quantum mechanical operator by the usual substitution

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}, \quad \hat{p}_y = -i\hbar \frac{\partial}{\partial y}, \quad \hat{p}_z = -i\hbar \frac{\partial}{\partial z}.$$

This leads to the following expressions for the components and square of  $\hat{L}$  in Cartesian coordinates, as well as in terms of the polar angle  $\theta$  and the azimuthal angle  $\phi$  associated with the position of the electron in spherical polar coordinates:

$$\hat{L}_x = -i\hbar \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) = i\hbar \left( \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) \quad (5.4)$$

$$\hat{L}_y = -i\hbar \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) = i\hbar \left( -\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right) \quad (5.5)$$

$$\hat{L}_z = -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -i\hbar \frac{\partial}{\partial \phi} \quad (5.6)$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2 \left( \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \right). \quad (5.7)$$

With the use of the definition for the commutator,  $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ , one obtains

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y, \quad (5.8)$$

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0, \quad (5.9)$$

so that a simultaneous set of eigenfunctions for  $\hat{L}^2$  and  $\hat{L}_z$  can be found. These eigenfunctions are the spherical harmonics  $Y_{LM}(\theta, \phi)$  where

$$\hat{L}^2 Y_{LM}(\theta, \phi) = \hbar^2 L(L+1) Y_{LM}(\theta, \phi), \quad (5.10)$$

and

$$\hat{L}_z Y_{LM}(\theta, \phi) = \hbar M Y_{LM}(\theta, \phi). \quad (5.11)$$

The  $Y_{LM}$  can be further separated into a product of two functions

$$Y_{LM}(\theta, \phi) = \Theta_{LM}(\theta) \Phi_M(\phi), \quad (5.12)$$

where  $\Theta_{LM}(\theta)$  is an associated Legendre function, and  $\Phi_M(\theta)$  is given by

$$\Phi_M(\phi) = \frac{e^{iM\phi}}{\sqrt{2\pi}}. \quad (5.13)$$

The first few spherical harmonics for  $L \leq 2$  as well as a formula for generating the  $Y_{LM}$  for  $L > 2$  are provided in Table 5.1, using the ‘‘Condon and Shortley’’ phase convention.

A simple geometric interpretation of the  $\hat{L}^2$  and  $\hat{L}_z$  operators is of a quantized vector of length  $\sqrt{L(L+1)}\hbar$  units precessing about the  $z$ -axis so that  $\hat{L}_x$  and  $\hat{L}_y$  have undefined values, but a definite projection of  $M\hbar$  units along the  $z$ -axis (see below). The  $z$ -axis is arbitrarily chosen as the reference axis appropriate to the experimental situation in which observations are made.

Raising and lowering operators, defined as

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y \quad (5.14)$$

and

$$\hat{L}_- = \hat{L}_x - i\hat{L}_y, \quad (5.15)$$

are so named because they raise and lower the  $M$  values for a given value of  $L$ . Specifically, their actions on the  $Y_{LM}(\theta, \phi)$  are represented by

$$\hat{L}_\pm Y_{LM}(\theta, \phi) = \hbar \sqrt{L(L+1) - M(M \pm 1)} Y_{LM \pm 1}. \quad (5.16)$$

Table 5.1: Spherical Harmonics

---


$$Y_{0,0}(\theta, \phi) = \sqrt{\frac{1}{4\pi}}$$

$$Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{1,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$$Y_{2,0}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$$

$$Y_{2,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$$

$$Y_{2,\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$$

$$Y_{LM}(\theta, \phi) = (-1)^M \left( \frac{(2L+1)(L-M)!}{4\pi(L+M)!} \right)^{1/2} P_L^M(\cos \theta) e^{iM\phi}, \quad M \geq 0;$$

$$Y_{L,-M}(\theta, \phi) = (-1)^M Y_{LM}^*$$

$$P_L^M(x) = (1-x^2)^{M/2} \frac{d^M}{dx^M} P_L(x), \quad M > 0;$$

$$P_L(x) = \frac{1}{2^L L!} \frac{d^L}{dx^L} (x^2 - 1)^L.$$

Normalization:  $\int_0^{2\pi} \int_0^\pi Y_{LM}^*(\theta, \phi) Y_{LM}(\theta, \phi) \sin \theta d\theta d\phi = 1.$

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The spherical harmonics  $Y_{LM}(\theta, \phi)$  can also be represented in Dirac notation simply as  $|LM\rangle$ . The set of spherical harmonic functions  $\{Y_{LM}(\theta, \phi), -L \leq M \leq L\}$  for a given value of  $L$  provide a convenient set of basis functions for the construction of a matrix representation of the angular momentum operators. In this basis  $\hat{L}^2$  and  $\hat{L}_z$  are represented by diagonal matrices with matrix elements

$$\langle L'M' | \hat{L}^2 | LM \rangle = L(L+1) \hbar^2 \delta_{L'L} \delta_{M'M} \quad (5.17)$$

$$\langle L'M' | \hat{L}_z | LM \rangle = M \hbar \delta_{L'L} \delta_{M'M}. \quad (5.18)$$

The raising and lowering operators  $\hat{L}_+$  and  $\hat{L}_-$  are represented in this basis by non-Hermitian matrices with matrix elements

$$\langle L'M' | \hat{L}_\pm | LM \rangle = \hbar \sqrt{L(L+1) - M(M \pm 1)} \delta_{L',L} \delta_{M',M \pm 1}.$$

Since  $\hat{L}_+$  and  $\hat{L}_-$  do not correspond to observables, they do not require Hermitian representations. The  $\hat{L}_+$  and  $\hat{L}_-$  matrices can be used to construct  $\hat{L}_x$  and  $\hat{L}_y$  matrices from the relationships

$$\hat{L}_x = (\hat{L}_+ + \hat{L}_-)/2 \quad (5.19)$$

and

$$\hat{L}_y = -i(\hat{L}_+ - \hat{L}_-)/2. \quad (5.20)$$

For the specific case when  $L = 1$ ,  $M = 1, 0, -1$ , we can choose for the three basis functions the unit vectors for the three Cartesian directions, i.e.,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1, 1\rangle, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |1, 0\rangle, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |1, -1\rangle.$$

In terms of these basis vectors the matrix representations of  $\hat{L}^2$ ,  $\hat{L}_z$ ,  $\hat{L}_+$ ,  $\hat{L}_-$ ,  $\hat{L}_x$ , and  $\hat{L}_y$  are

$$\hat{\mathbf{L}}^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.21)$$

$$\hat{\mathbf{L}}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (5.22)$$

$$\hat{\mathbf{L}}_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (5.23)$$

$$\hat{\mathbf{L}}_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (5.24)$$

$$\hat{\mathbf{L}}_x = \frac{\sqrt{2}}{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (5.25)$$

and

$$\hat{\mathbf{L}}_y = \frac{\sqrt{2}}{2}i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (5.26)$$

Electron orbital angular momentum in the hydrogen atom depends on the  $\theta$  and  $\phi$  spherical polar coordinates of the electron. In this case,  $L$  is restricted to integral values but, in general, angular momentum can assume half-integral values as well. For example, the spin angular momentum of an electron is found to be  $\frac{1}{2}\hbar$ . In this case, the angular momentum of the electron is *defined* in terms of the commutation relations (5.8) and the associated matrices. The letter  $S$  is used to designate electron spin angular momentum. More generally, the preceding equations must be transformed by  $L \rightarrow J$  and  $M \rightarrow M_J$ ,

where  $J$  and  $M_J$  always symbolize the total electronic angular momentum (spin plus orbital) in a system and its projection on the laboratory  $z$ -axis.

For the simple spin- $\frac{1}{2}$  case let

$$|\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |S = \frac{1}{2}, M_S = \frac{1}{2}\rangle, \quad |\beta\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |S = \frac{1}{2}, M_S = -\frac{1}{2}\rangle.$$

The corresponding matrix representations are

$$\hat{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (5.27)$$

$$\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \frac{\hbar}{2}\hat{\sigma}_z, \quad (5.28)$$

$$\hat{S}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (5.29)$$

$$\hat{S}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (5.30)$$

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \frac{\hbar}{2}\hat{\sigma}_x, \quad (5.31)$$

and

$$\hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{\hbar}{2}\hat{\sigma}_y. \quad (5.32)$$

The matrices  $\hat{\sigma}_x$ ,  $\hat{\sigma}_y$ , and  $\hat{\sigma}_z$  are known as the Pauli spin matrices.

Matrix representations of operators are vital in spectroscopy because they provide a quantitative description of the system. The Hamiltonian operator  $\hat{H}$  for the system is expressed in terms of various operators such as the spin and orbital angular momentum operators. To transform the Schrödinger equation into a matrix equation, we choose a basis set and evaluate the matrix elements of  $\hat{H}$ . Finally, the matrix form of the Schrödinger equation in this basis,  $\hat{H}\psi = E\psi$ , is solved by transforming  $\hat{H}$  to diagonal form in order to find the eigenvalues  $\{E_n\}$  and the associated eigenvectors  $\{\psi_n\}$ .

### 5.3 The Hydrogen Atom and One-Electron Spectra

The energy level structure of the hydrogen atom and hydrogen-like ions can be explained by solving the Schrödinger equation

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{Ze^2\psi}{4\pi\epsilon_0 r} = E\psi, \quad (5.33)$$

with  $\mu$  the reduced mass of the hydrogen atom or the hydrogen-like ion. This differential equation is most easily solved in terms of spherical polar coordinates. Thus, the wavefunction is written as  $\psi = \psi(r, \theta, \phi)$  and the Schrödinger equation becomes

$$\frac{-\hbar^2}{2\mu} \left( \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{r^2} \cot \theta \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right) - \frac{Ze^2 \psi}{4\pi\epsilon_0 r} = E\psi, \quad (5.34)$$

or

$$\frac{-\hbar^2}{2\mu r^2} \frac{\partial}{\partial r} r^2 \frac{\partial \psi}{\partial r} + \frac{1}{2\mu r^2} \hat{L}^2 \psi - \frac{Ze^2 \psi}{4\pi\epsilon_0 r} = E\psi. \quad (5.35)$$

This form of the Schrödinger equation is used because the partial differential equation (5.34) can then be separated into three ordinary differential equations involving  $r$ ,  $\theta$ , and  $\phi$  alone. As usual, boundary conditions force quantization and the energy eigenvalues are found to be

$$E_n = \frac{-\mu Z^2 (e^2/4\pi\epsilon_0)^2}{2n^2 \hbar^2} = \frac{-R}{n^2}, \quad n = 1, 2, 3, \dots, \quad (5.36)$$

in SI units, and for  $\text{cm}^{-1}$  units,  $\tilde{R} = \tilde{R}_H = 109\,677.421\,2 \text{ cm}^{-1}$  for the hydrogen atom. Note that most tables (Appendix A) report  $R_\infty$ , appropriate for a stationary, infinitely heavy nucleus, rather than  $R_H$ . In fact  $R_\infty$  and  $R_H$  are connected by the relationship

$$R_H = \frac{R_\infty}{1 + m_e/m_p}, \quad (5.37)$$

which is obtained from the definition of the reduced mass of two particles, namely  $\mu = m_e m_p / (m_e + m_p)$ . The solution of the Schrödinger equation yields three quantum numbers: the principal quantum number  $n$ , the azimuthal quantum number  $l$ , and the magnetic quantum number  $m$ , that can only assume the values

$$\begin{aligned} n &= 1, 2, 3, \dots, \infty, \\ l &= 0, 1, \dots, n-1, \end{aligned}$$

and

$$m = 0, \pm 1, \dots, \pm l.$$

The  $l$  values of 0, 1, 2, 3, ... are usually labeled  $s, p, d, f, g, h, i, k, l, m, n, o$ , etc., for the historical reasons that were touched upon in the introduction to this chapter. The wavefunction is a product of a radial part and an angular part, namely

$$\psi(r, \theta, \phi) = R_{nl}(r) Y_{lm}(\theta, \phi),$$

in which the radial part  $R_{nl}(r)$  is an associated Laguerre function and  $Y_{lm}(\theta, \phi)$  is a spherical harmonic. A few of the  $Y_{lm}(\theta, \phi)$  and  $R_{nl}(r)$  functions are listed in Tables 5.1 and 5.2. The constant  $a$  in Tables 5.2 and 5.3 is the Bohr radius  $a_0$ , which can be expressed in terms of fundamental constants (Appendix A) as

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} = 0.529\,177\,208\,3 \text{ \AA}.$$

Table 5.2: Radial Functions of the Hydrogen Atom

$$\begin{aligned}
 R_{10}(r) &= \left(\frac{Z}{a}\right)^{3/2} 2e^{-Zr/a} \\
 R_{20}(r) &= \left(\frac{Z}{2a}\right)^{3/2} 2\left(1 - \frac{Zr}{2a}\right) e^{-Zr/2a} \\
 R_{21}(r) &= \left(\frac{Z}{2a}\right)^{3/2} \frac{2}{\sqrt{3}} \left(\frac{Zr}{2a}\right) e^{-Zr/2a} \\
 R_{30}(r) &= \left(\frac{Z}{3a}\right)^{3/2} 2\left(1 - 2\frac{Zr}{3a} + \frac{2}{3}\left(\frac{Zr}{3a}\right)^2\right) e^{-Zr/3a} \\
 R_{31}(r) &= \left(\frac{Z}{3a}\right)^{3/2} \frac{4\sqrt{2}}{3} \left(\frac{Zr}{3a}\right) \left(1 - \frac{1}{2}\frac{Zr}{3a}\right) e^{-Zr/3a} \\
 R_{32}(r) &= \left(\frac{Z}{3a}\right)^{3/2} \frac{2\sqrt{2}}{3\sqrt{5}} \left(\frac{Zr}{3a}\right)^2 e^{-Zr/3a} \\
 \text{Normalization: } &\int_0^\infty R_{nl}^* R_{nl} r^2 dr = 1
 \end{aligned}$$

Note that if  $\mu$  replaces  $m_e$  in the definition of the Bohr radius, then one obtains  $a_H$ , the actual Bohr radius of the hydrogen atom.

The angular parts of the hydrogen eigenfunctions  $Y_{lm}(\theta, \phi)$  are complex when  $|m| > 0$ . These complex functions are not very useful when one tries to visualize the shape of the orbitals in real space. Since the energy does not depend on the magnetic quantum number  $m$ , the wavefunctions are degenerate and any linear combination of them is also a solution to the Schrödinger equation. Hence, the linear combinations

$$\frac{1}{\sqrt{2}}(Y_{l,|m|} + Y_{l,-|m|}) \quad \text{and} \quad \frac{1}{i\sqrt{2}}(Y_{l,|m|} - Y_{l,-|m|}) \quad (5.38)$$

are used when plotting the orbitals in real space. These real linear combinations give the Cartesian forms for the solutions (referred to as orbitals). Some of the real-valued hydrogen orbitals are listed in Table 5.3 and their plots are illustrated in Figure 5.3.

The orbital angular momentum operators  $\hat{l}^2$  and  $\hat{l}_z$  commute with the hydrogenic Hamiltonian operator, so that

$$[\hat{H}, \hat{l}_z] = [\hat{H}, \hat{l}^2] = 0. \quad (5.39)$$

From now on, the customary notation of lowercase letters will be used to represent one-electron properties and uppercase letters will be used to represent many-electron properties. Simultaneous eigenfunctions can be constructed for the three equations

$$\hat{H}\psi_{nlm} = E_n\psi_{nlm}, \quad (5.40)$$

$$\hat{l}^2\psi_{nlm} = l(l+1)\hbar^2\psi_{nlm}, \quad (5.41)$$

and

$$\hat{l}_z\psi_{nlm} = m\hbar\psi_{nlm}. \quad (5.42)$$

Table 5.3: Some of the Real Hydrogen Wavefunctions

$$\begin{aligned}
\psi_{1s} &= \frac{1}{\pi^{1/2}} \left(\frac{Z}{a}\right)^{3/2} e^{-Zr/a} \\
\psi_{2s} &= \frac{1}{4(2\pi)^{1/2}} \left(\frac{Z}{a}\right)^{3/2} \left(2 - \frac{Zr}{a}\right) e^{-Zr/2a} \\
\psi_{2p_z} &= \frac{1}{4(2\pi)^{1/2}} \left(\frac{Z}{a}\right)^{5/2} r e^{-Zr/2a} \cos \theta \\
\psi_{2p_x} &= \frac{1}{4(2\pi)^{1/2}} \left(\frac{Z}{a}\right)^{5/2} r e^{-Zr/2a} \sin \theta \cos \phi \\
\psi_{2p_y} &= \frac{1}{4(2\pi)^{1/2}} \left(\frac{Z}{a}\right)^{5/2} r e^{-Zr/2a} \sin \theta \sin \phi \\
\psi_{3s} &= \frac{1}{81(3\pi)^{1/2}} \left(\frac{Z}{a}\right)^{3/2} \left(27 - 18\frac{Zr}{a} + 2\frac{Z^2 r^2}{a^2}\right) e^{-Zr/3a} \\
\psi_{3p_z} &= \frac{2^{1/2}}{81\pi^{1/2}} \left(\frac{Z}{a}\right)^{5/2} \left(6 - \frac{Zr}{a}\right) r e^{-Zr/3a} \cos \theta \\
\psi_{3p_x} &= \frac{2^{1/2}}{81\pi^{1/2}} \left(\frac{Z}{a}\right)^{5/2} \left(6 - \frac{Zr}{a}\right) r e^{-Zr/3a} \sin \theta \cos \phi \\
\psi_{3p_y} &= \frac{2^{1/2}}{81\pi^{1/2}} \left(\frac{Z}{a}\right)^{5/2} \left(6 - \frac{Zr}{a}\right) r e^{-Zr/3a} \sin \theta \sin \phi \\
\psi_{3d_{z^2}} &= \frac{1}{81(6\pi)^{1/2}} \left(\frac{Z}{a}\right)^{7/2} r^2 e^{-Zr/3a} (3 \cos^2 \theta - 1) \\
\psi_{3d_{xz}} &= \frac{2^{1/2}}{81\pi^{1/2}} \left(\frac{Z}{a}\right)^{7/2} r^2 e^{-Zr/3a} \sin \theta \cos \theta \cos \phi \\
\psi_{3d_{yz}} &= \frac{2^{1/2}}{81\pi^{1/2}} \left(\frac{Z}{a}\right)^{7/2} r^2 e^{-Zr/3a} \sin \theta \cos \theta \sin \phi \\
\psi_{3d_{x^2-y^2}} &= \frac{1}{81(2\pi)^{1/2}} \left(\frac{Z}{a}\right)^{7/2} r^2 e^{-Zr/3a} \sin^2 \theta \cos 2\phi \\
\psi_{3d_{xy}} &= \frac{1}{81(2\pi)^{1/2}} \left(\frac{Z}{a}\right)^{7/2} r^2 e^{-Zr/3a} \sin^2 \theta \sin 2\phi
\end{aligned}$$

The complex form of the  $Y_{lm}(\theta, \phi)$  functions listed in Table 5.1 satisfy equations (5.40) – (5.42) and the corresponding quantum numbers  $n$ ,  $l$ , and  $m$  are used to label the wavefunctions  $\psi_{nlm}$ .

### Vector Model

Since angular momenta are so widely used in spectroscopy, it is useful to have a simple pictorial model (Figure 5.4). This model summarizes the mathematical results of quantum mechanics. An angular momentum  $\hat{\mathbf{J}}$  is represented in this picture by a vector of length  $\sqrt{J(J+1)}\hbar$  units. While  $\hat{\mathbf{J}}$  has a definite projection  $M_J\hbar$  along the laboratory  $z$ -axis, the components along the  $x$ - and  $y$ -axes do not have definite values. This means that the vector  $\hat{\mathbf{J}}$  is inclined at an angle

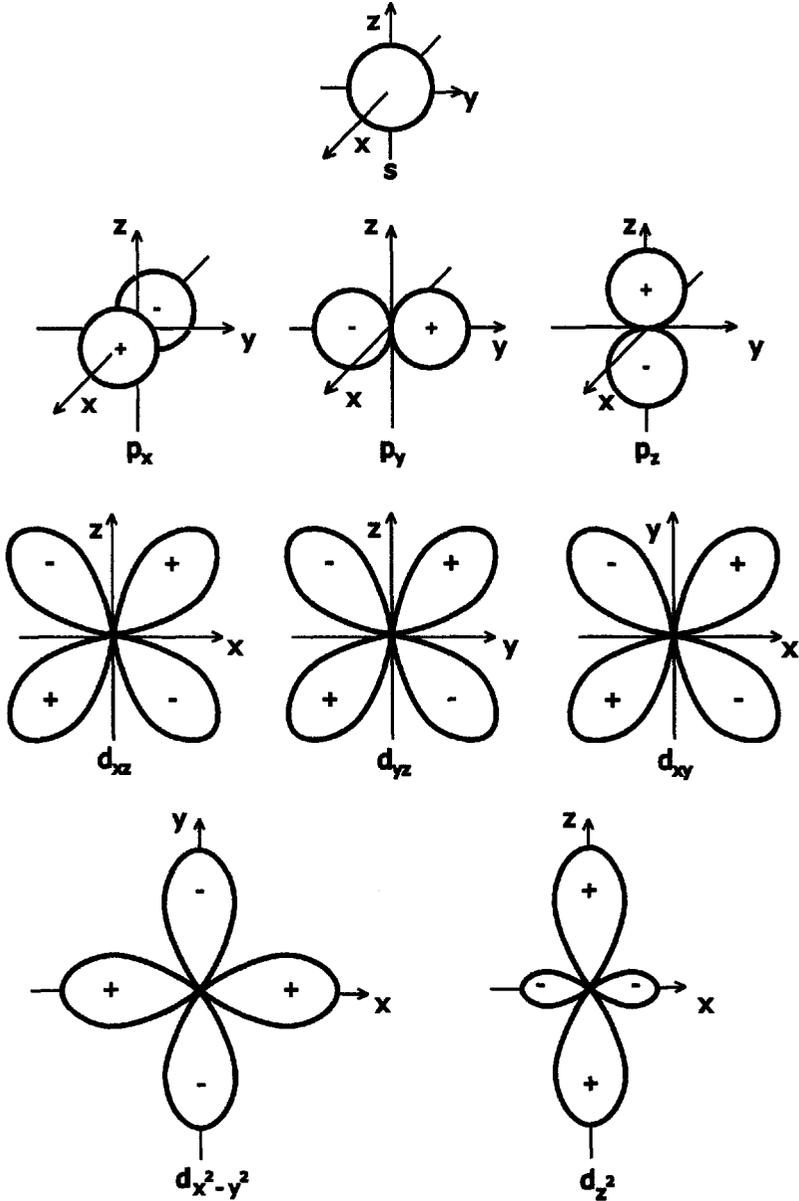


Figure 5.3: The angular part of the real forms of the  $1s$ ,  $2p$ , and  $3d$  orbitals.

$$\theta = \cos^{-1} \left( \frac{M_J}{\sqrt{J(J+1)}} \right) \tag{5.43}$$

with respect to the  $z$ -axis and precesses at a constant angular velocity. This precessional motion ensures that the  $x$  and  $y$  components of  $\hat{\mathbf{J}}$  have undetermined values (until an external measurement forces one of these components to have a definite value). The different values of  $M_J$  thus correspond to different spatial orientations of  $\hat{\mathbf{J}}$ . If space is

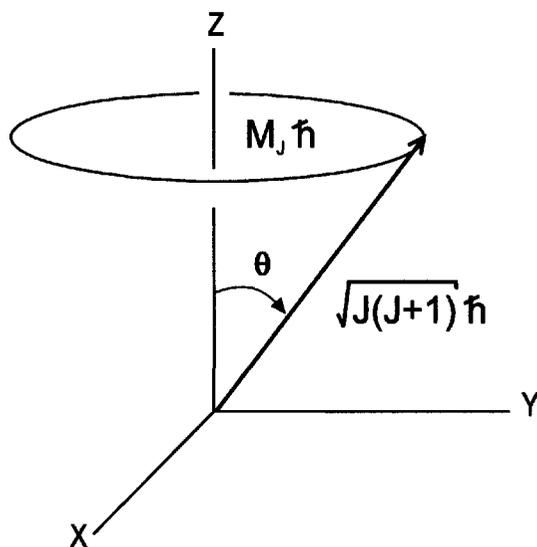


Figure 5.4: The vector model for angular momentum.

isotropic (i.e., there are no electric or magnetic fields present), then the energy cannot depend on the orientation of the total angular momentum in space so that there must be a  $(2J + 1)$ -fold degeneracy corresponding to all allowed  $M_J$  values.

### Spin-Orbit Coupling

Because the electron is a charged particle, the orbital motion of an electron produces a current. Associated with this current is a magnetic field which affects the orientation of the magnetic moment of the electron associated with the presence of electron spin. This phenomenon is known as spin-orbit coupling and is responsible for the “fine structure” in the spectrum of the hydrogen atom.

The strength of the magnetic field at the electron is proportional to  $\hat{\mathbf{l}}$ , while the magnitude of the electronic spin magnetic moment is proportional to  $\hat{\mathbf{s}}$ , given by

$$\hat{\boldsymbol{\mu}}_s = -g_e \mu_B \hat{\mathbf{s}},$$

in which  $g_e$  is a numerical constant and  $\mu_B$  is the Bohr magneton (see section 5.8 on Zeeman effects later in this chapter). Since the energy of interaction of a magnetic moment with a magnetic flux density  $\mathbf{B}$  is

$$E = -\boldsymbol{\mu} \cdot \mathbf{B} \quad (5.44)$$

the spin-orbit coupling is written as

$$\hat{H}_{so} = \xi(r) \hat{\mathbf{l}} \cdot \hat{\mathbf{s}}. \quad (5.45)$$

The function  $\xi(r)$  is defined by

$$\xi(r) = \frac{1}{2\mu^2 c^2} \frac{1}{r} \frac{\partial V}{\partial r} = \frac{1}{2\mu^2 c^2} \left( \frac{Ze^2}{4\pi\epsilon_0 r^3} \right), \quad (5.46)$$

in which

$$V = -\frac{Ze^2}{4\pi\epsilon_0 r} \tag{5.47}$$

is the potential energy due to the Coulombic attraction between the electron and the nucleus. A detailed derivation of equations (5.45) and (5.46) requires the use of relativistic quantum electrodynamics,<sup>1</sup> which is beyond the level of this book.

The inclusion of spin-orbit coupling transforms the eigenvalue equation into

$$(\hat{H}^{(0)} + \xi(r)\hat{\mathbf{l}} \cdot \hat{\mathbf{s}})\psi = E\psi, \tag{5.48}$$

where  $\hat{H}^{(0)}$  is the simple hydrogenic Hamiltonian operator. Provided that the spin-orbit coupling term is small, first-order perturbation theory can be applied to solve equation (5.48) approximately. Letting the perturbation Hamiltonian operator  $\hat{H}'$  be

$$\hat{H}' = \xi(r)\hat{\mathbf{l}} \cdot \hat{\mathbf{s}} \tag{5.49}$$

leads to the energy having the form

$$E = E^{(0)} + \int \psi^{(0)*} \hat{H}' \psi^{(0)} d\tau = -\frac{R_H}{n^2} + \int \psi_n^{(0)*} \xi(r) \hat{\mathbf{l}} \cdot \hat{\mathbf{s}} \psi_n^{(0)} d\tau \tag{5.50}$$

to first order. Degenerate perturbation theory must be used, however, to find the correct  $\psi_n^{(0)}$ , as spin-orbit coupling removes some of the  $\hat{\mathbf{l}}$  and  $\hat{\mathbf{s}}$  degeneracy in the hydrogen atom.

The quantum numbers  $l$  and  $s$  are no longer good when spin-orbit coupling is taken into account since the operators  $\hat{l}_z$  and  $\hat{s}_z$  do not commute with the Hamiltonian operator due to the presence of the term  $\xi(r)\hat{\mathbf{l}} \cdot \hat{\mathbf{s}}$ . However, the total vector angular momentum

$$\hat{\mathbf{j}} = \hat{\mathbf{l}} + \hat{\mathbf{s}} \tag{5.51}$$

is still a constant of motion which means that  $\hat{j}^2$  and  $\hat{j}_z$  must commute with the full Hamiltonian operator. The matrix elements of the  $\xi(r)\hat{\mathbf{l}} \cdot \hat{\mathbf{s}}$  term can be evaluated by using the simple direct product basis set

$$\psi_{lm_l sm_s} = |l, m_l\rangle |s, m_s\rangle. \tag{5.52}$$

This basis set is referred to as the uncoupled representation because  $\hat{\mathbf{l}}$  and  $\hat{\mathbf{s}}$  are not coupled to give  $\hat{\mathbf{j}}$ . For example, consider the set of  $2p$  functions for the hydrogen atom:  $l = 1, s = \frac{1}{2}$ , which yield a total of six basis functions, namely

$$\begin{aligned} |l = 1, m_l = 1\rangle |s = \frac{1}{2}, m_s = \pm \frac{1}{2}\rangle, \\ |l = 1, m_l = 0\rangle |s = \frac{1}{2}, m_s = \pm \frac{1}{2}\rangle, \\ |l = 1, m_l = -1\rangle |s = \frac{1}{2}, m_s = \pm \frac{1}{2}\rangle. \end{aligned}$$

The diagonal matrix elements of  $\hat{H}$ , written for convenience as

$$\hat{H} = \hat{H}^{(0)} + \xi(r)\hat{\mathbf{l}} \cdot \hat{\mathbf{s}} = (\hat{H}^{(0)} + \xi(r)\hat{l}_z \hat{s}_z) + \frac{\xi(r)(\hat{l}_+ \hat{s}_- + \hat{l}_- \hat{s}_+)}{2}, \tag{5.53}$$

are

$$\begin{aligned}
 \langle l = 1, m_l | \langle s = \frac{1}{2}, m_s | \hat{H} | s = \frac{1}{2}, m_s \rangle | l = 1, m_l \rangle \\
 = E_{2p}^{(0)} + \zeta_{2p} m_l m_s,
 \end{aligned}
 \tag{5.54}$$

in which  $E_{2p}^{(0)} = -R_H/4$  gives the energy of a  $2p$  electron in the absence of spin-orbit interactions, and

$$\zeta_{2p} = \hbar^2 \int R_{2p}^*(r) \xi(r) R_{2p}(r) r^2 dr
 \tag{5.55}$$

represents the spin-orbit interaction energy. The integral over  $r$  comes from the radial parts of the  $\psi_{nlm}(r, \theta, \phi)$  eigenfunctions.

The off-diagonal matrix elements are due to the  $\hat{l}_+ \hat{s}_-$  term which connects basis functions with the same  $m_j = m_l + m_s = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ , resulting in a matrix with mostly zeros. Diagonalizing the  $2 \times 2$  blocks yields two energy levels with energies  $E_{2p}^{(0)} - \zeta_{2p}$ , and  $E_{2p}^{(0)} + \zeta_{2p}/2$  from the Hamiltonian matrix,  $\hat{H}$ :

$$\begin{pmatrix}
 E^{(0)} + \zeta/2 & 0 & 0 & 0 & 0 & 0 \\
 0 & E^{(0)} - \zeta/2 & \sqrt{2}\zeta/2 & 0 & 0 & 0 \\
 0 & \sqrt{2}\zeta/2 & E^{(0)} & 0 & 0 & 0 \\
 0 & 0 & 0 & E^{(0)} & \sqrt{2}\zeta/2 & 0 \\
 0 & 0 & 0 & \sqrt{2}\zeta/2 & E^{(0)} - \zeta/2 & 0 \\
 0 & 0 & 0 & 0 & 0 & E^{(0)} + \zeta/2
 \end{pmatrix}
 \begin{matrix}
 |1, \frac{1}{2}\rangle \\
 |1, -\frac{1}{2}\rangle \\
 |0, \frac{1}{2}\rangle \\
 |0, -\frac{1}{2}\rangle \\
 |-1, \frac{1}{2}\rangle \\
 |-1, -\frac{1}{2}\rangle
 \end{matrix}$$

An alternative and easier way to solve the problem involves the use of the coupled basis functions  $|lsjm_j\rangle$  where  $j$  and  $m_j$  are the good quantum numbers. For the  $2p$  functions  $\hat{\mathbf{j}} = \hat{\mathbf{l}} + \hat{\mathbf{s}}$ , corresponding to  $j = \frac{3}{2}$  and  $j = \frac{1}{2}$ . Notice that the two values of  $j$  correspond to a vector addition of  $\hat{\mathbf{l}}$  and  $\hat{\mathbf{s}}$  (see Figure 5.5). Note that the usual (confusing) shorthand notation of using  $l = 1, s = \frac{1}{2}$ , and  $j = \frac{3}{2}$  to represent the lengths of vectors is used. In the coupled basis there are again six  $2p$  functions:

$$\begin{aligned}
 &|2p, j = \frac{3}{2}, m_j = \frac{3}{2}\rangle, \\
 &|2p, j = \frac{3}{2}, m_j = \frac{1}{2}\rangle, \\
 &|2p, j = \frac{3}{2}, m_j = -\frac{1}{2}\rangle, \\
 &|2p, j = \frac{3}{2}, m_j = -\frac{3}{2}\rangle, \\
 &|2p, j = \frac{1}{2}, m_j = \frac{1}{2}\rangle, \\
 &|2p, j = \frac{1}{2}, m_j = -\frac{1}{2}\rangle.
 \end{aligned}$$

In this coupled basis set the Hamiltonian matrix is already diagonal since

$$\hat{j}^2 = (\hat{\mathbf{l}} + \hat{\mathbf{s}}) \cdot (\hat{\mathbf{l}} + \hat{\mathbf{s}}) = \hat{l}^2 + \hat{s}^2 + 2\hat{\mathbf{l}} \cdot \hat{\mathbf{s}}
 \tag{5.56}$$

commutes with the Hamiltonian operator. The spin-orbit operator can be expressed in terms of  $\hat{j}^2$ ,  $\hat{l}^2$ , and  $\hat{s}^2$  as

$$\xi \hat{\mathbf{l}} \cdot \hat{\mathbf{s}} = \frac{\xi(\hat{j}^2 - \hat{l}^2 - \hat{s}^2)}{2}.
 \tag{5.57}$$

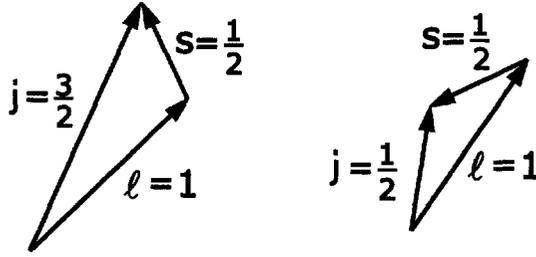


Figure 5.5: Vector addition of  $l$  and  $s$ .

First-order perturbation theory now gives

$$\begin{aligned}
 E_{njs} &= E_n^{(0)} + \int (\psi_{nlm}^{(0)})^* \xi(r) \hat{l} \cdot \hat{s} \psi_{nlm}^{(0)} d\tau \\
 &= E_n^{(0)} + \frac{1}{2} \int (\psi_{nlm}^{(0)})^* \xi(r) (\hat{j}^2 - \hat{l}^2 - \hat{s}^2) \psi_{nlm}^{(0)} d\tau \\
 &= E_n^{(0)} + \frac{\zeta_{nl}}{2} (j(j+1) - l(l+1) - s(s+1)), \tag{5.58}
 \end{aligned}$$

using the definition of  $\zeta$ , equation (5.55). When  $s = \frac{1}{2}$ , then  $j = l \pm \frac{1}{2}$  and the energy levels are

$$E_{nl} = E^{(0)} + \frac{(\zeta_{nl})l}{2} \quad \text{for} \quad j = l + \frac{1}{2} \tag{5.59a}$$

or

$$E_{nl} = E^{(0)} - \frac{\zeta_{nl}(l+1)}{2} \quad \text{for} \quad j = l - \frac{1}{2}. \tag{5.59b}$$

For the case of the hydrogen  $2p$  states ( $l = 1$ ) the energy levels are

$$E = E^{(0)} + \frac{\zeta_{2p}}{2} \quad \text{for} \quad 2p_{3/2} \tag{5.60a}$$

and

$$E = E^{(0)} - \zeta_{2p} \quad \text{for} \quad 2p_{1/2}, \tag{5.60b}$$

where the good quantum number  $j$  is now used to label the wavefunctions. Notice that the  $2p$  energy levels do not depend on whether the coupled or the uncoupled basis set is chosen to construct the Hamiltonian matrix. This is due to the fact that the coupled  $|j_1 j_2 J M_J\rangle$  and uncoupled  $|j_1 m_1\rangle |j_2 m_2\rangle$  basis sets are related by a linear transformation

$$|j_1, j_2, J, M_J\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1, j_2; m_1, m_2 | J M_J \rangle |j_1 m_1\rangle |j_2 m_2\rangle, \tag{5.61}$$

in which the coupling coefficients  $\langle j_1, j_2; m_1, m_2 | J M \rangle$  are known as the Clebsch–Gordan coefficients. For the  $2p$  hydrogen atom orbitals the transformation is

$$|2p_{3/2}M_J = \frac{3}{2}\rangle = |m_l = 1\rangle|m_s = \frac{1}{2}\rangle, \quad (5.62a)$$

$$|2p_{3/2}M_J = \frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|m_l = 0\rangle|m_s = \frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|m_l = 1\rangle|m_s = -\frac{1}{2}\rangle, \quad (5.62b)$$

$$|2p_{3/2}M_J = -\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|m_l = -1\rangle|m_s = \frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|m_l = 0\rangle|m_s = -\frac{1}{2}\rangle, \quad (5.62c)$$

$$|2p_{3/2}M_J = -\frac{3}{2}\rangle = |m_l = -1\rangle|m_s = -\frac{1}{2}\rangle, \quad (5.62d)$$

and

$$|2p_{1/2}M_J = \frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|m_l = 0\rangle|m_s = \frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|m_l = 1\rangle|m_s = -\frac{1}{2}\rangle, \quad (5.62e)$$

$$|2p_{1/2}M_J = -\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|m_l = -1\rangle|m_s = \frac{1}{2}\rangle - \sqrt{\frac{1}{3}}|m_l = 0\rangle|m_s = -\frac{1}{2}\rangle. \quad (5.62f)$$

The transformation can be derived by using the operators

$$\hat{j}_{\pm} = \hat{l}_{\pm} + \hat{s}_{\pm} \quad (5.63)$$

and the orthogonality of the wavefunctions.

## 5.4 Many-Electron Atoms

The nonrelativistic Schrödinger equation for an  $N$ -electron atom with a nucleus of charge  $Z$  at the origin is

$$\left( \frac{-\hbar^2}{2m_e} \sum_{i=1}^N \nabla_i^2 - \sum_{i=1}^N \frac{Ze^2}{4\pi\epsilon_0 r_i} + \sum_{i,j>i}^N \frac{e^2}{4\pi\epsilon_0 r_{ij}} \right) \psi = E\psi. \quad (5.64)$$

By invoking the orbital approximation, the wavefunction is represented by a Slater determinant, namely

$$\begin{aligned} \psi &= (N!)^{-1/2} \begin{vmatrix} \phi_1(1)\alpha(1) & \phi_1(2)\alpha(2) & \cdots & \phi_1(N)\alpha(N) \\ \phi_1(1)\beta(1) & \phi_1(2)\beta(2) & \cdots & \phi_1(N)\beta(N) \\ \phi_2(1)\alpha(1) & \phi_2(2)\alpha(2) & \cdots & \phi_2(N)\alpha(N) \\ \vdots & \vdots & & \vdots \\ \phi_{N/2}(1)\beta(1) & \phi_{N/2}(2)\beta(2) & \cdots & \phi_{N/2}(N)\beta(N) \end{vmatrix} \\ &\equiv |\phi_1\bar{\phi}_1\phi_2\cdots\bar{\phi}_{N/2}|. \end{aligned} \quad (5.65)$$

In shorthand notation a bar represents a  $\beta$  or spin-down ( $m_s = -\frac{1}{2}$ ) electron, while the absence of a bar represents an  $\alpha$  or spin-up ( $m_s = +\frac{1}{2}$ ) electron. The Slater determinant automatically satisfies the Pauli exclusion principle since the interchange of any two columns, which corresponds to the exchange of two electrons, changes the sign of the determinant. Since electrons are fermions the Pauli exclusion principle requires that

$$\hat{P}_{12}\psi = -\psi \quad (5.66)$$

hold for the exchange of two electrons, where  $\hat{P}_{12}$  is the permutation operator that exchanges the coordinates of two electrons.

The Pauli exclusion principle requires that each orbital  $\phi_i$  can contain, at most, two electrons with opposite spin. The orbitals are approximated by the product function

$$\phi_i(r_i, \theta_i, \phi_i) = R(r_i)Y_{lm}(\theta_i, \phi_i).$$

By choosing this form we are assuming that the orbitals possess a hydrogen-like angular shape, but the radial functions need not be the associated Laguerre polynomials of the hydrogen atom. Instead, the radial functions associated with each  $\phi_i$  are usually determined by minimizing the total energy of the atom using the variational principle.

The configuration of a multi-electron atom is constructed by placing electrons in the lowest energy orbitals in accordance with the *Aufbau* principle. For example, the lowest energy configuration of the Li atom is  $(1s)^22s$  which corresponds to the Slater determinant

$$\psi = 6^{-1/2} \begin{vmatrix} 1s(1)\alpha(1) & 1s(2)\alpha(2) & 1s(3)\alpha(3) \\ 1s(1)\beta(1) & 1s(2)\beta(2) & 1s(3)\beta(3) \\ 2s(1)\alpha(1) & 2s(2)\alpha(2) & 2s(3)\alpha(3) \end{vmatrix} = |1s\bar{1}s2s|. \quad (5.67)$$

The tasks of either calculating atomic energy levels and wavefunctions or experimentally measuring energy level differences by atomic spectroscopy are active areas of research. Calculation or measurement of atomic energy levels can be a complex task, but the labeling of atomic energy levels using the theory of angular momentum coupling is relatively straightforward.

All the various orbital and spin angular momenta of an atom must add vectorially to make  $\hat{\mathbf{J}}$  the total angular momentum, which must remain a constant of the motion. For light atoms where spin-orbit coupling is small, it is convenient to use the Russell-Saunders coupling scheme. A coupling scheme is no more than a prescription that describes the order in which angular momenta are coupled. In the Russell-Saunders scheme the orbital angular momenta of all electrons are coupled to give a total orbital angular momentum of the atom, that is,

$$\hat{\mathbf{L}} = \hat{\mathbf{l}}_1 + \hat{\mathbf{l}}_2 + \cdots + \hat{\mathbf{l}}_N = \sum_{i=1}^N \hat{\mathbf{l}}_i. \quad (5.68)$$

Similarly, for electron spin,

$$\hat{\mathbf{S}} = \sum_{i=1}^N \hat{\mathbf{s}}_i \quad (5.69)$$

and the total angular momentum is given by their vector sum, namely

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}. \quad (5.70)$$

The usual convention in which capital letters are used to designate many-electron angular momenta and lowercase letters are used to designate single electron angular momenta is also employed here.

As required by equations (5.68), (5.69), and (5.70), the individual operator components add to give the total component, for example

$$\hat{L}_z = \hat{l}_{z1} + \hat{l}_{z2} + \cdots + \hat{l}_{zN}, \quad (5.71)$$

$$\hat{S}_z = \hat{s}_{z1} + \hat{s}_{z2} + \cdots + \hat{s}_{zN}, \quad (5.72)$$

and

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \quad (5.73)$$

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2. \quad (5.74)$$

The components of  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  all commute with the nonrelativistic Hamiltonian operator in equation (5.64). The commutation of the spin operators with the atomic Schrödinger equation is obvious since there are no spin variables present in equation (5.64). For the multi-electron atom  $\hat{L}_z$  and  $\hat{H}$  are

$$\hat{L}_z = -i\hbar \left( \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2} + \cdots + \frac{\partial}{\partial \phi_N} \right) \quad (5.75)$$

and

$$\hat{H} = \frac{-\hbar^2}{2m_e} \sum_i \nabla_i^2 - \sum_i \frac{Ze^2}{4\pi\epsilon_0 r_i} + \sum_{i,j>i} \frac{e^2}{4\pi\epsilon_0 r_{ij}} \quad (5.76)$$

with

$$\nabla_i^2 = \frac{1}{r_i^2} \frac{\partial}{\partial r_i} r_i^2 \frac{\partial}{\partial r_i} + \frac{1}{r_i^2 \sin^2 \theta_i} \frac{\partial}{\partial \theta_i} \left( \sin^2 \theta_i \frac{\partial}{\partial \theta_i} \right) + \frac{1}{r_i^2 \sin^2 \theta_i} \frac{\partial^2}{\partial \phi_i^2}. \quad (5.77)$$

$\hat{L}_z$  commutes with the kinetic energy operator because the  $\phi_i$  variables appear as simple second derivatives in the Laplacian and as simple first derivatives in  $\hat{L}_z$ . The Coulomb attraction term is only a function of  $r_i$ , so it also commutes with  $\hat{L}_z$ . Finally, the electron-electron repulsion term is an implicit function of  $\phi_i$  because of the presence of  $r_{ij}$ . Consideration of Figure 5.6, however, leads one to the conclusion that  $r_{12}$  depends only on  $\phi_1 - \phi_2$  so

$$\begin{aligned} \hat{L}_z \left( \frac{1}{r_{12}} \right) &= -i\hbar \left( \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2} + \cdots \right) \frac{1}{r_{12}} \\ &= -i\hbar \left( \frac{\partial(1/r_{12})}{\partial(\phi_1 - \phi_2)} \right) \left( \frac{\partial(\phi_1 - \phi_2)}{\partial \phi_1} + \frac{\partial(\phi_1 - \phi_2)}{\partial \phi_2} \right) + \cdots \\ &= 0 \end{aligned} \quad (5.78)$$

using the chain rule. Thus  $\hat{L}_z$  commutes with  $\sum e^2/r_{ij}$  since all of the azimuthal angles occur as differences,  $\phi_i - \phi_j$ .

Notice that although individual electron quantum numbers  $m_l$  cannot be defined because the individual  $\hat{l}_{zi}$  operators do not commute with  $\hat{H}$ , i.e.,

$$[\hat{l}_{zi}, \hat{H}] \neq 0, \quad (5.79)$$

so that

$$\hat{l}_{zi}\psi \neq m_{li}\hbar\psi, \quad (5.80)$$

but  $M_L$  given by

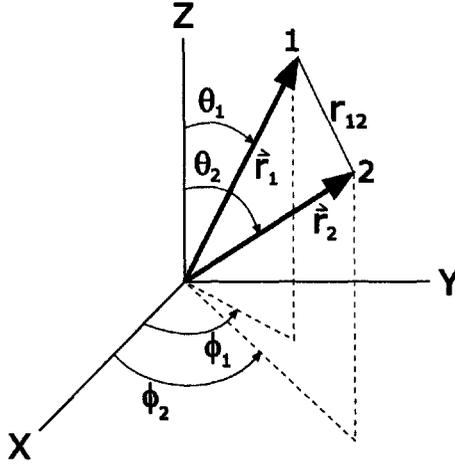


Figure 5.6: Polar coordinates of two electrons in an atom.

$$M_L = m_{l1} + m_{l2} + \dots + m_{lN}, \tag{5.81}$$

is a good quantum number in the absence of spin-orbit coupling, since  $\hat{L}_z$  operating on  $\psi$  has the property

$$\hat{L}_z \psi = M_L \hbar \psi. \tag{5.82}$$

Analogous equations hold for the electron spins, so that

$$\hat{S}_z \psi = M_S \hbar \psi, \tag{5.83}$$

with

$$M_S = m_{s1} + m_{s2} + \dots + m_{sN}. \tag{5.84}$$

Because an atom is spherically symmetric, the orientation of the  $z$ -axis is arbitrary so that if

$$[\hat{L}_z, \hat{H}] = 0,$$

then

$$[\hat{L}_x, \hat{H}] = 0 \quad \text{and} \quad [\hat{L}_y, \hat{H}] = 0.$$

Further, if  $[\hat{L}_z, \hat{H}] = 0$ , so does

$$[\hat{L}_z^2, \hat{H}] = 0, \quad [\hat{L}_x^2, \hat{H}] = 0, \quad \text{and} \quad [\hat{L}_y^2, \hat{H}] = 0,$$

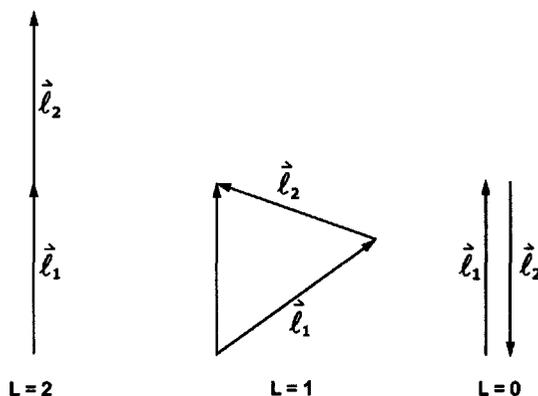
then

$$[\hat{L}^2, \hat{H}] = 0$$

by the properties of commutators. A comparison between the properties of multi-electron atoms and one-electron atoms is presented in Table 5.4.

Table 5.4: One-electron and Multi-electron Atoms

Multi-electron Atoms	One-electron Atoms
$\hat{L}^2\psi = L(L+1)\hbar^2\psi$	$\hat{l}^2\psi = l(l+1)\hbar^2\psi$
$\hat{L}_z\psi = M_L\hbar\psi$	$\hat{l}_z\psi = m_l\hbar\psi$
$\hat{S}^2\psi = S(S+1)\hbar^2\psi$	$\hat{s}^2\psi = s(s+1)\hbar^2\psi$
$\hat{S}_z\psi = M_S\hbar\psi$	$\hat{s}_z\psi = m_s\hbar\psi$
$L = 0, 1, 2, 3, 4, 5$	$l = 0, 1, 2, 3, 4, 5$
$S P D F G H$	$s p d f g h$

Figure 5.7: Vector addition of two  $l = 1$  angular momenta.

The operators  $\hat{H}$ ,  $\hat{L}^2$ ,  $\hat{S}^2$ ,  $\hat{L}_z$ ,  $\hat{S}_z$  all commute with one another so that the wavefunction  $\psi$  is a simultaneous eigenfunction of all five operators. The corresponding quantum numbers  $n, L, S, M_L$ , and  $M_S$  can therefore be used to label the wavefunctions, and one can write  $\psi = |nLM_LSM_S\rangle$ . In the absence of external electric fields, magnetic fields, and spin-orbit coupling the energy levels associated with  $\psi$  possess a  $(2S+1)$ -fold degeneracy due to the different  $M_S$  states and a  $(2L+1)$ -fold degeneracy due to the different  $M_L$  states. It is therefore convenient to label these energy levels by the term symbol  $^{2S+1}L$ , which gives rise to a total degeneracy

$$g = (2L+1)(2S+1). \quad (5.85)$$

The angular momenta  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  are vector quantities, made up of individual  $\hat{\mathbf{I}}_i$  and  $\hat{\mathbf{s}}_i$  vectors: this means that when any two angular momenta, for example,  $\hat{\mathbf{I}}_1$  and  $\hat{\mathbf{I}}_2$ , are coupled together, they must be added vectorially. The possible values of the quantum number  $L$  are then given by  $l_1 + l_2, l_1 + l_2 - 1, \dots, |l_1 - l_2|$  from the vector coupling rules. The vector coupling of  $\hat{\mathbf{I}}_1$  and  $\hat{\mathbf{I}}_2$  can be visualized with the aid of vector coupling diagrams, as shown in Figure 5.7 for the case of  $l_1 = 1$  and  $l_2 = 1$ .

Consider the carbon atom with configuration  $1s^2 2s^2 2p^2$ . What terms arise from this configuration? Filled orbitals such as  $1s^2$  or  $2s^2$  have no net spin or orbital angular momentum, hence they may be ignored. The possible terms can be derived by considering all possible distributions of the two  $p$  electrons among the six spin-orbitals

Table 5.5: Slater Determinants for Configuration  $p^2$ 

	$M_S = 1$	0	-1
$M_L = 2$	—	$ 1, \bar{1}\rangle$	—
$M_L = 1$	$ 1, 0\rangle$	$ 1, \bar{0}\rangle \bar{1}, 0\rangle$	$ \bar{1}, \bar{0}\rangle$
$M_L = 0$	$ 1, -1\rangle$	$ 1, -\bar{1}\rangle \bar{1}, -1\rangle 0, \bar{0}\rangle$	$ \bar{1}, -\bar{1}\rangle$
$M_L = -1$	$ 0, -1\rangle$	$ \bar{0}, -1\rangle 0, -\bar{1}\rangle$	$ \bar{0}, -\bar{1}\rangle$
$M_L = -2$	—	$ -1, -\bar{1}\rangle$	—

$(p_1\alpha, p_0\alpha, p_{-1}\alpha, p_1\beta, p_0\beta, p_{-1}\beta)$  in a manner consistent with the Pauli exclusion principle (Table 5.5). These possible states are often referred to as microstates and, in fact, correspond to individual Slater determinants.

A microstate designated as  $|1, \bar{0}\rangle$  means  $m_{l1} = 1, m_{s1} = \frac{1}{2}, m_{l2} = 0, m_{s2} = -\frac{1}{2}$  with the orbitals arranged in some standard (arbitrary) order. Starting with the maximum values of  $M_L$  and  $M_S$ , one deduces that there are nine microstates corresponding to  ${}^3P$ , five microstates corresponding to  ${}^1D$ , and one microstate corresponding to  ${}^1S$ , consistent with the  $(2S + 1)(2L + 1)$ -fold total degeneracy of each Russell-Saunders term.

The microstate  $|1, \bar{1}\rangle$  clearly belongs to  ${}^1D$  while  $|1, 0\rangle$  belongs to  ${}^3P$ , but to which terms do  $|1, \bar{0}\rangle$  and  $|\bar{1}, 0\rangle$  belong? Neither one is a proper eigenfunction of  $\hat{L}^2$  and  $\hat{L}_z$ . The correct linear combination of determinants can be deduced, however, by the application of the lowering operator  $\hat{L}_-$  to  $|{}^1D, M_L = 2\rangle = |1, \bar{1}\rangle$ , i.e., via

$$\hat{L}_-|1, \bar{1}\rangle = (\hat{l}_{1-} + \hat{l}_{2-})|1, \bar{1}\rangle \quad (5.86)$$

$$\begin{aligned} & \sqrt{L(L+1) - M_L(M_L - 1)}|{}^1D, M_L = 1\rangle \\ &= \sqrt{l_1(l_1 + 1) - m_{l1}(m_{l1} - 1)}|0, \bar{1}\rangle + \sqrt{(l_2(l_2 + 1) - m_{l2}(m_{l2} - 1))}|1, \bar{0}\rangle \end{aligned} \quad (5.87)$$

$$2|{}^1D, M_L = 1\rangle = \sqrt{2}|0, \bar{1}\rangle + \sqrt{2}|1, \bar{0}\rangle \quad (5.88)$$

$$|{}^1D, M_L = 1\rangle = \frac{1}{\sqrt{2}}(|0, \bar{1}\rangle + |1, \bar{0}\rangle) = \frac{1}{\sqrt{2}}(|1, \bar{0}\rangle - |\bar{1}, 0\rangle), \quad (5.89)$$

where the final determinant has been put in standard order. Similarly, the state

$$|{}^3P, M_L = 1, M_S = 0\rangle = \frac{1}{\sqrt{2}}(|1, \bar{0}\rangle + |\bar{1}, 0\rangle) \quad (5.90)$$

is orthogonal to  $|{}^1D, M_L = 1, M_S = 0\rangle$ .

The different terms arising from a configuration have different energies because of the electron-electron repulsion term in the Hamiltonian operator. If

$$\hat{H}_0 = \frac{-\hbar^2}{2m_e} \sum_i \nabla_i^2 - \sum_i \frac{Ze^2}{4\pi\epsilon_0 r_i} \quad (5.91)$$

Table 5.6: Atomic Terms Arising from the  $p^n$  and  $d^n$  Configurations

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$p^1$ : $^2P$	$d^1, d^9$ : $^2D$
$p^2$ : $^1S, ^1D, ^3P$	$d^2, d^8$ : $^1S, ^1D, ^1G, ^3P, ^3F$
$p^3$ : $^2P, ^2D, ^4S$	$d^3, d^7$ : $^2P, ^2D(\text{twice}), ^2F, ^2G, ^2H, ^4P, ^4F$
$p^4$ : $^1S, ^1D, ^3P$	$d^4, d^6$ : $^1S(\text{twice}), ^1D(\text{twice}), ^1F, ^1G(\text{twice}), ^1I, ^3P(\text{twice})$
$p^5$ : $^2P$	$^3D, ^3F(\text{twice}), ^3G, ^3H, ^5D$
	$d^5$ : $^2S, ^2P, ^2D(\text{three times}), ^2F(\text{twice}), ^2G(\text{twice}),$ $^2H, ^2I, ^4P, ^4D, ^4F, ^4G, ^6S$

---

were the only terms present in the Hamiltonian operator, then the orbital approximation would be exact because the electronic coordinates would be separable. Since  $\hat{H}_0$  is comprised of one-electron, hydrogen-like operators, the wavefunction associated with  $\hat{H}_0$  is a product of one-electron orbitals. The electron-electron repulsion term

$$\hat{H}_{ee} = \sum_{i,j>i} \frac{e^2}{4\pi\epsilon_0 r_{ij}} \quad (5.92)$$

in the complete Hamiltonian operator prevents the separation of the total wavefunction into a product of one-electron orbitals. Nevertheless, it is still conceptually useful to retain the orbital approximation.

A set of empirical rules first proposed by Hund in 1927 is useful in predicting the lowest energy term arising from a configuration. Hund's first rule states that the term with the highest multiplicity  $2S + 1$  lies lowest in energy. If this rule does not select a unique term, then Hund's second rule comes into play: of terms of having the same (maximum) multiplicity, the term with the highest  $L$  value lies lowest in energy. For example, Hund's rules predict that from a  $p^2$  configuration the  $^3P$  term lies lower in energy than the  $^1D$  and  $^1S$  terms. Experimentally the ground state of the carbon atom is indeed found to be  $^3P$ . Hund's third rule is given below in the discussion of spin-orbit coupling.

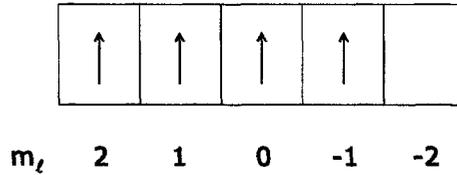
The terms for many common configurations with equivalent electrons are given in Table 5.6. Notice that an electron (e.g.,  $p^1$ ) in a subshell (charge  $-e$ ) has the same terms as a hole (charge  $+e$ ) in a full subshell (e.g.,  $p^5$ ). Similarly the terms arising from  $d^n$  and  $d^{10-n}$  are the same.

The enumeration of all of the microstates arising from a  $d^5$  configuration is quite a task ( $f^7$  is even worse!), but the direct application of Hund's rules will give the lowest energy term without having to determine all remaining terms. As a pictorial representation of the method,  $2l + 1$  boxes are drawn to represent the different orbitals. Each box is labeled with an  $m_l$  value, and the electrons are placed into the boxes to maximize  $M_L = \sum m_l$  and  $M_S = \sum m_s$ . The term that has these maximum  $M_L$  and  $M_S$  values can then be written down by inspection. For example  $d^4$  has a diagram (Figure 5.8) corresponding to  $M_L(\text{max}) = 2$  and  $M_S(\text{max}) = 2$  arising from a  $^5D$  term. The  $2S + 1$  multiplicity of a term such as  $^5D$  is read as "quintet" rather than as "five." By custom all multiplicities in term symbols for atoms and molecules are read in this way (Table 5.7).

A term symbol also indicates the parity of an atomic state by the presence of a small following superscript " $^{\circ}$ " for an odd-parity term. The behavior of an atomic

Table 5.7: Names for the First Ten Multiplicities

$2S + 1$	Name	$2S + 1$	Name
1	Singlet	6	Sextet
2	Doublet	7	Septet
3	Triplet	8	Octet
4	Quartet	9	Nonet
5	Quintet	10	Decet

Figure 5.8: The microstate of  $d^4$  with maximum  $M_S$  and  $M_L$  values.

wavefunction under the operation of inversion can be determined from the parity of the orbitals since

$$\hat{E}^* \psi = \hat{E}^* |1s\bar{1}s2s \dots| = (-1)^{\sum l_i} \psi \quad (5.93)$$

where the inversion operator  $\hat{E}^*$  inverts the coordinates of all of the electrons through the origin. The parity of a state is determined simply by adding all of the  $l_i$  values of the configuration from which the state arises. Thus, for example, the lowest energy term of nitrogen with a configuration  $1s^2 2s^2 2p^3$  is written as  $^4S^\circ$ .

The effects of spin-orbit coupling must be incorporated into the Hamiltonian operator of the multi-electron atom. The spin-orbit term in the Hamiltonian operator is just the sum of the one-electron terms,

$$\hat{H}_{\text{so}} = \sum \xi(r_i) \hat{\mathbf{l}}_i \cdot \hat{\mathbf{s}}_i. \quad (5.94)$$

This form is not very convenient because of the presence of the individual  $\hat{\mathbf{l}}_i$  and  $\hat{\mathbf{s}}_i$  angular momenta. For a given term, however, an equivalent  $\hat{H}_{\text{so}}$  can be derived using the properties of vector operators<sup>2</sup> (Wigner-Eckart theorem):

$$\hat{H}_{\text{so}} = \zeta \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}. \quad (5.95)$$

The numerical factor  $\zeta = \zeta(S, L)$  (typically with units of  $\text{cm}^{-1}$ ) is referred to as the spin-orbit coupling constant of the  $L$ - $S$  term. This form of the spin-orbit interaction is only applicable within a single, isolated  $^{2S+1}L$  term and assumes that there are no interactions with other terms.

When  $\hat{H}_{\text{so}} = \zeta \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$  is added to the atomic Hamiltonian operator (5.64), the components of  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  no longer commute with  $\hat{H}$ . The components of the total angular momentum  $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$  commute with  $\hat{H}$ , however, as do  $\hat{L}^2$  and  $\hat{S}^2$ . The set of commuting observables is now  $\{\hat{H}, \hat{L}^2, \hat{S}^2, \hat{J}^2, \hat{J}_z\}$  rather than the set  $\{\hat{H}, \hat{L}^2, \hat{L}_z, \hat{S}^2,$

Table 5.8: Eigenvalue Equations for the Multi-electron Atom when Spin-Orbit Coupling Is Included

$\hat{H} = \hat{H}_0 + \hat{H}_{ee}$	$\hat{H} = \hat{H}_0 + \hat{H}_{ee} + \zeta \mathbf{L} \cdot \mathbf{S}$	$\hat{H} = \hat{H}_0 + \hat{H}_{ee} + \sum \xi(r_i) \mathbf{l}_i \cdot \mathbf{s}_i$
$\hat{H}\psi = E\psi$	$\hat{H}\psi = E\psi$	$\hat{H}\psi = E\psi$
$\hat{L}^2\psi = L(L+1)\hbar^2\psi$	$\hat{L}^2\psi = L(L+1)\hbar^2\psi$	$\hat{L}^2\psi \approx L(L+1)\hbar^2\psi$
$\hat{L}_z\psi = M_L\hbar\psi$	$\hat{S}^2\psi = S(S+1)\hbar^2\psi$	$\hat{S}^2\psi \approx S(S+1)\hbar^2\psi$
$\hat{S}^2\psi = S(S+1)\hbar^2\psi$	$\hat{J}^2\psi = J(J+1)\hbar^2\psi$	$\hat{J}^2\psi = J(J+1)\hbar^2\psi$
$\hat{S}_z\psi = M_S\hbar\psi$	$\hat{J}_z\psi = M_J\hbar\psi$	$\hat{J}_z\psi = M_J\hbar\psi$

$\hat{S}_z$  used in absence of the spin-orbit coupling term,  $\zeta \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$  (Table 5.8). In fact, as  $\zeta$  increases the various terms begin to interact with each other, since the full spin-orbit Hamiltonian operator  $\sum \xi(r) \hat{\mathbf{l}}_i \cdot \hat{\mathbf{s}}_i$  has additional matrix elements with  $\Delta L = 0, \pm 1$  and  $\Delta S = 0, \pm 1$ . The individual terms can no longer be considered isolated when spin-orbit coupling becomes large. This then means that the true wavefunctions are no longer eigenfunctions of  $\hat{L}^2$  and  $\hat{S}^2$ . Nevertheless they are approximately so, i.e.,

$$\hat{L}^2\psi \approx L(L+1)\hbar^2\psi, \quad \hat{S}^2\psi \approx S(S+1)\hbar^2\psi$$

and hence it is useful to retain the approximate quantum numbers  $L$  and  $S$ . The term symbol  $^{2S+1}L$  is also often used for heavy atoms with large spin-orbit coupling, but then  $J$  is always added as a subscript,  $^{2S+1}L_J$ .

The permissible values of  $J$  determined by vector coupling of  $L$  and  $S$  are  $L+S, L+S-1, \dots, |L-S|$ . For example, the states of the  $p^2$  configuration are  $^3P_2, ^3P_1, ^3P_0, ^1D_2$ , and  $^1S_0$  from the  $^3P, ^1D$ , and  $^1S$  terms. The energy separation of the  $J=2, J=1$ , and  $J=0$  levels of the  $^3P$  term (referred to as the multiplet splitting) is readily determined from the relationship

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \frac{j^2 - \hat{L}^2 - \hat{S}^2}{2} \quad (5.96)$$

derived from

$$j^2 = (\hat{\mathbf{L}} + \hat{\mathbf{S}}) \cdot (\hat{\mathbf{L}} + \hat{\mathbf{S}}) = \hat{L}^2 + \hat{S}^2 + 2\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}. \quad (5.97)$$

If  $L$  and  $S$  are nearly good quantum numbers (in an isolated term), then perturbation theory gives

$$\begin{aligned} \langle \hat{H}_{so} \rangle &= \zeta \langle \hat{\mathbf{L}} \cdot \hat{\mathbf{S}} \rangle = \zeta \langle nJM_JLS | \hat{\mathbf{L}} \cdot \hat{\mathbf{S}} | nJM_JLS \rangle \\ &= \frac{1}{2} \zeta (J(J+1) - L(L+1) - S(S+1)). \end{aligned} \quad (5.98)$$

The intervals are given by

$$E_{J+1} - E_J = \frac{1}{2} \zeta ((J+1)(J+2) - J(J+1)) = \zeta(J+1). \quad (5.99)$$

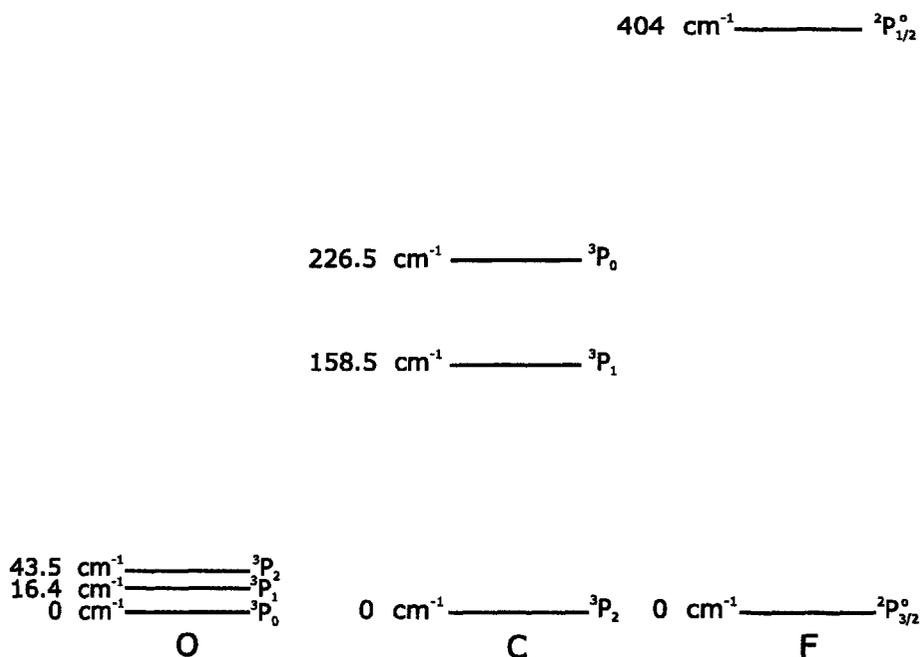


Figure 5.9: The multiplet splittings for the lowest energy term of the C, O, and F atoms.

This is the *Landé interval rule*: the spin-orbit splitting between sequential  $J$  levels in a term is proportional to the larger of the  $J$  values. Whether the level with the largest value of  $J$  lies highest in energy or lowest in energy is determined by the sign of  $\zeta$  (Figure 5.9). If  $\zeta > 0$ , then the term is said to be regular (as for C), and if  $\zeta < 0$  then, the term is inverted (as for O). Hund's third rule predicts whether the lowest energy term will be regular or inverted. If the ground term arises from an electron configuration for which the valence electrons make up a less than half-filled subshell (e.g., C), then the lowest energy term will be regular, while if the configuration is more than half-filled (e.g., O), then the lowest energy term will be inverted. Both C and O have ground  $^3P$  terms arising from  $1s^22s^22p^2$  and  $1s^22s^22p^4$  configurations, respectively; therefore the  $^3P$  term of C is regular while the  $^3P$  term of O is inverted (Figure 5.9). If an incomplete subshell is exactly half full (e.g.,  $p^3$ ), then the lowest energy term is always  $S$  and no spin-orbit splittings are possible.

The total degeneracy  $2J + 1$  of a level  $J$  arises from the  $M_J$  degeneracy. For a term such as  $^3D$  the total degeneracy  $g = (2L + 1)(2S + 1) = 5(3) = 15$ . The presence of spin-orbit coupling lifts this degeneracy and gives rise to the levels  $^3D_3$ ,  $^3D_2$ , and  $^3D_1$ ; the total degeneracy remains 15 (Figure 5.10).

In summary, if only the hydrogen-like terms  $\hat{H}_0$  (equation (5.91)) are retained in the atomic Hamiltonian operator, then the terms in a given configuration are degenerate (Figure 5.11). If the electron-electron repulsion term  $\hat{H}_{ee}$  (equation (5.92)) is added, then the orbital approximation begins to break down and the different terms in a configuration split (Figure 5.11). Finally, when the spin-orbit term  $\hat{H}_{so}$  (equation (5.95)) is added, the degeneracy of the levels in a term is lifted.

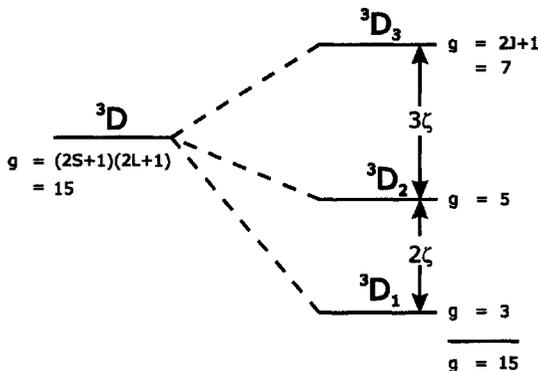


Figure 5.10: Degeneracy in the  ${}^3D$  term.

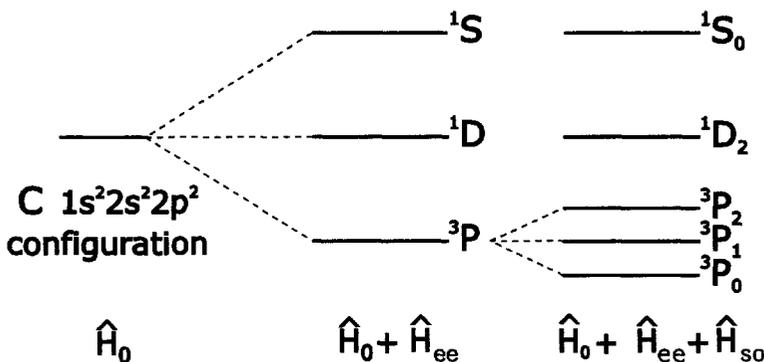


Figure 5.11: The qualitative effect of the  $\hat{H}_0$ ,  $\hat{H}_{ee}$ , and  $\hat{H}_{so}$  terms on the energy level pattern of the  $1s^2 2s^2 2p^2$  configuration of the C atom.

### 5.5 Selection Rules

Explaining the appearance of a spectrum requires detailed knowledge of the energy-level structure and selection rules that govern transitions between levels. To begin, the one-electron selection rules of the hydrogen atom<sup>3</sup> are determined from the transition moment integral (Chapter 1)

$$M_{fi} = \int \psi_f^* \hat{\mu} \psi_i d\tau. \tag{5.100}$$

Because the parity of an atomic wavefunction is determined by  $l$  and the dipole moment operator is of odd parity, the selection rule for one-electron atomic transitions requires that  $\Delta l$  be odd. More restrictive selection rules can be derived by considering the atomic wavefunctions. With the nucleus at the origin of the coordinate system (Figure 5.12), the electric dipole moment operator  $\hat{\mu}$  can be written as  $\hat{\mu} = -er$ , in which case  $M_{n'l'm',nlm}$  is given by

$$M_{n'l'm',nlm} = -e \int \int \int R_{n'l'}^*(r) Y_{l'm'}^*(\theta, \phi) r R_{nl}(r) Y_{lm}(\theta, \phi) r^2 \sin \theta dr d\theta d\phi. \tag{5.101}$$

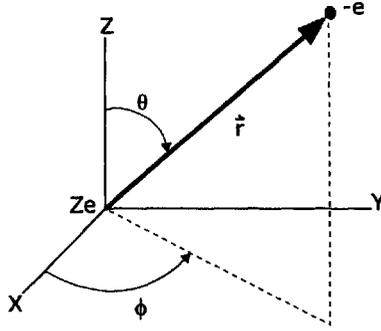


Figure 5.12: Coordinate system for the hydrogen atom.

Each coordinate can be considered separately with

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = r(\sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}). \tag{5.102}$$

For the  $z$  component we have

$$M_z = -eN \int_0^\infty r^3 R_{n'l'} R_{nl} dr \int_0^\pi P_{l'}^{m'}(\theta) \cos\theta P_l^m(\theta) \sin\theta d\theta \int_0^{2\pi} e^{-im'\phi} e^{im\phi} d\phi, \tag{5.103}$$

(with  $N$  a collection of constants from normalization) so that for  $M_z \neq 0$ ,  $m' = m$  or  $\Delta m = 0$ . Also from the properties of associated Legendre polynomials, we can show that

$$\cos\theta P_l^m(\theta) = \frac{(l-m+1)P_{l+1}^m + (l+m)P_{l-1}^m}{2l+1} \tag{5.104}$$

and from the orthogonality of these polynomials, as expressed by

$$\frac{(2l+1)(l-m)!}{2(l+m)!} \int_0^\pi P_{l'}^{m'} P_l^m \sin\theta d\theta = \delta_{l'l} \delta_{m'm}, \tag{5.105}$$

it is required that  $\Delta l = \pm 1$ . For the  $x$  component of the transition moment, we find that

$$M_x = -eN \int_0^\infty r^3 R_{n'l'} R_{nl} dr \int_0^\pi P_{l'}^{m'} \sin\theta P_l^m \sin\theta d\theta \int_0^{2\pi} e^{-im'\phi} \cos\phi e^{im\phi} d\phi, \tag{5.106}$$

but since

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}, \tag{5.107}$$

we see that  $\Delta m = \pm 1$  for nonzero  $M_x$ . In addition we can show that

$$\sin\theta P_l^{m-1} = \frac{P_{l+1}^m - P_{l-1}^m}{2l+1}, \tag{5.108}$$

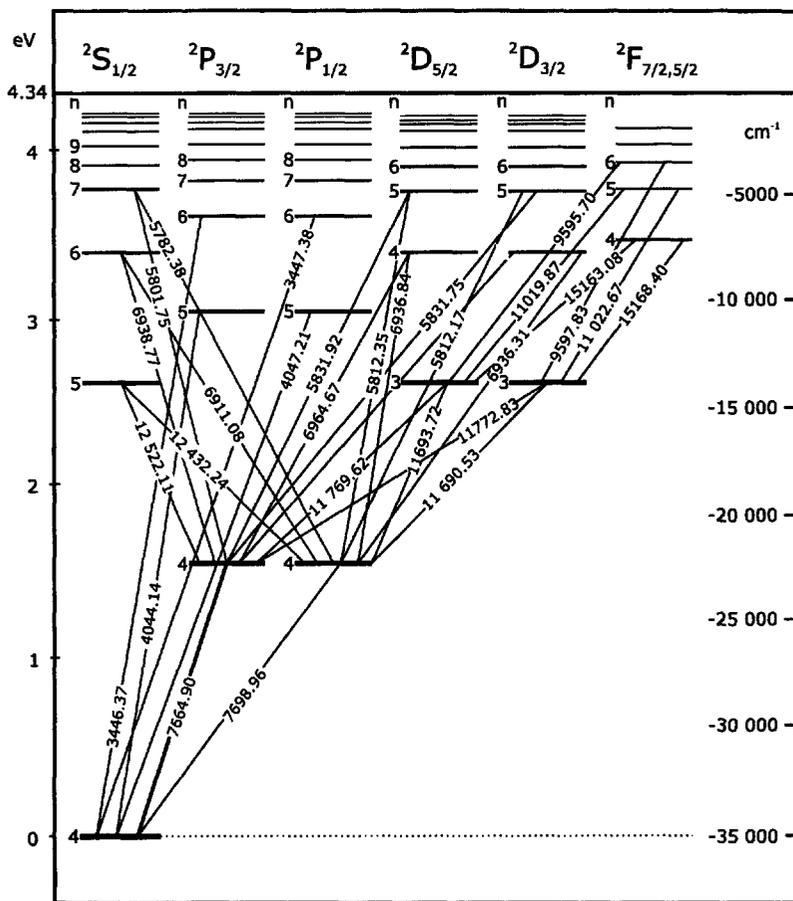


Figure 5.13: Energy-level diagram for the K atom, with wavelengths (in  $\text{\AA}$ ) of transitions.

so that once again  $\Delta l = \pm 1$ . The  $y$  component of  $\mathbf{M}$  gives the same selection rules as  $M_x$ . Further, no restriction arises for  $n'$  and  $n$  so that  $\Delta n = 0, \pm 1, \pm 2, \dots$  transitions are possible.

Thus single-photon, electric-dipole-allowed selection rules in hydrogenic atoms are  $\Delta l = \pm 1$ ,  $\Delta m = 0, \pm 1$ , and  $\Delta n = \text{any integer}$ . In reality, restrictions on  $\Delta n$  do exist for  $l' \leftarrow l$  transitions, due to the decreasing overlap between the  $R_{n'l'}$  and  $R_{nl}$  radial wavefunctions with increasing  $\Delta n$ . Values of the radial part of equation (5.103) and of the transition probabilities are tabulated, for example, in Condon and Shortley.<sup>4</sup>

Selection rules for multi-electron atoms are much more difficult to derive than are the one-electron selection rules, and consequently, only the results will be quoted here. Within the orbital approximation only a single electron can make a jump from one orbital to another, with  $\Delta l = \pm 1$  during an electronic transition. All electrons other than the one making the transition remain in their original orbitals.

The parity selection rule of even  $\leftrightarrow$  odd applies to multi-electron atoms. This parity of a multi-electron atomic state can easily be determined by evaluating  $(-1)^{\sum l_i}$ . The parity selection rule, often referred to as the *Laporte rule*, remains valid in all cases for electric-dipole transitions, even when the  $l_i$  are no longer good quantum numbers.

The selection rule for  $J$ ,  $\Delta J = 0, \pm 1$  but  $J' = 0 \not\leftrightarrow J'' = 0$ , also always remains valid for one-photon, electric-dipole-allowed transitions. The selection rules  $\Delta L = 0, \pm 1$  and  $\Delta S = 0$  for  $L$  and  $S$  only remain valid for small spin-orbit coupling. For the heavier elements the Russell–Saunders coupling scheme is no longer useful because the large spin-orbit coupling allows mixing between terms with different  $L$  and  $S$  values so that these selection rules break down. For example, the Hg  ${}^3P_1 - {}^1S_0$  transition at 253.7 nm becomes quite strong, while the analogous transition for He is very weak.

It would seem that the multi-electron selection rule  $\Delta L = 0$  conflicts with the one-electron selection rule  $\Delta l = \pm 1$ . The Ti atom transition  ${}^3F^\circ(3d^24s4p) - {}^3F(3d^24s^2)$  is an example that illustrates that a  $\Delta L = 0, \Delta l = 1$  transition is possible.

## 5.6 Atomic Spectra

The alkali atoms Li, Na, K, Rb, and Cs, as well as hydrogen, are the prototypes for one-electron atom transitions. In Figure 5.13 the energy levels and transitions of K are displayed. In atomic spectroscopy, energy-level diagrams are called *Grotrian diagrams*.

The  ${}^2P_{3/2} - {}^2S_{1/2}$  (5 890 Å) and  ${}^2P_{1/2} - {}^2S_{1/2}$  (5 896 Å) transitions of Na are known as the Na *D* lines. The letter designation was made by Fraunhofer when he first observed them in absorption in the sun. (Fraunhofer began at the red end of the visible spectrum and the *A*-band at 7619 Å turned out to be the 0–0 band of the forbidden  $b^1\Sigma_g^+ - X^3\Sigma_g^-$  transition of O<sub>2</sub> in the earth's atmosphere, the *B*-band at 6892 Å is the corresponding 1–0 band, while the *C* line is the solar Balmer H<sub>α</sub> line at 6563 Å.) The corresponding transitions of potassium are at 7664.90 Å and 7698.96 Å (Figure 5.13). They are examples of “resonance lines” because they originate from the ground state and are strongly allowed transitions. They are also the so-called persistent lines used for analytical purposes because they are readily observed in emission even when the K atom concentration is very low. There are several series of potassium transitions in Figure 5.13 that converge to the ionization limit of 4.34 eV.

He (Figure 5.14) and Ca (Figure 5.15) are examples of atoms with two valence electrons. The spectra are organized into separate singlet and triplet manifolds of states with only weak intercombination transitions connecting them.

Spin-orbit interaction causes many atomic lines to split into multiplets such as the Na *D* lines or the six-line pattern of the Ca  ${}^3D - {}^3P$  transition at 442.5 – 445.6 nm (Figure 5.16). The splitting into multiplets is called *fine structure*.

The transition elements, lanthanides, and actinides have very complex energy-level patterns because of the many terms and levels arising from open *d* and *f* subshells.

### *Hyperfine Structure*

The presence of nuclear spin produces further splittings in the lines of many elements. Since the splittings caused by electron spin are called fine structure, the much smaller splittings due to the nuclear spin are called *hyperfine structure*.

The nuclear spin  $\hat{\mathbf{I}}$  couples with  $\hat{\mathbf{J}}$  to form the total angular momentum  $\hat{\mathbf{F}}$  via vector coupling, namely

$$\hat{\mathbf{F}} = \hat{\mathbf{J}} + \hat{\mathbf{I}}. \quad (5.109)$$

When nuclear spin is present, the only strictly good quantum number is  $F$ . Splittings due to hyperfine structure are relatively small, typically less than 1 cm<sup>-1</sup>, so  $J$  remains

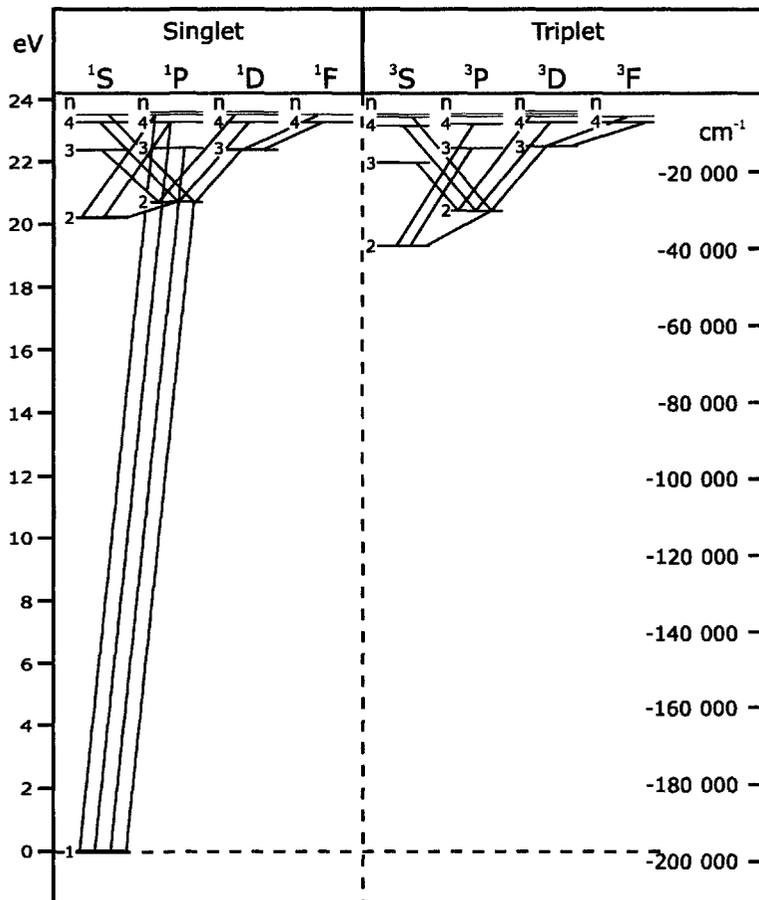


Figure 5.14: Energy-level diagram for the He atom.

a nearly good quantum number. The selection rules for  $F$  are the same as for  $J$ ,  $\Delta F = 0, \pm 1$ , but  $F' = 0 \not\leftrightarrow F'' = 0$ , and  $\Delta M_F = 0, \pm 1$ .

For example, consider the energy-level pattern<sup>5</sup> for the  $^{87}\text{Rb } 5^2P_{3/2} - 5^2S_{1/2}$  transition (one of the resonance lines) of  $^{87}\text{Rb}$  at  $7800 \text{ \AA}$  displayed in Figure 5.17.  $^{87}\text{Rb}$  has a nuclear spin of  $3/2$ . The separation between the two lowest hyperfine levels in the ground electronic state of Rb is used for a frequency standard. The similar ground-state hyperfine transition at  $9\,192.631\,770 \text{ GHz}$  in  $^{133}\text{Cs}$  ( $I = 7/2$ ) is used in an atomic clock to provide the national time standard in many countries.<sup>6</sup>

### Hydrogen Atom

The hydrogen atom continues to fascinate scientists. For example, precise frequency measurements in the hydrogen atom spectrum have led to a refinement in the value of the Rydberg constant as well as practical applications such as a hydrogen maser (microwave laser) that can also serve as an atomic clock. The Rydberg constant is the most accurately known fundamental physical constant,<sup>7</sup> with a value  $R_\infty = 109\,737.315\,685 \text{ cm}^{-1}$  (Appendix A).

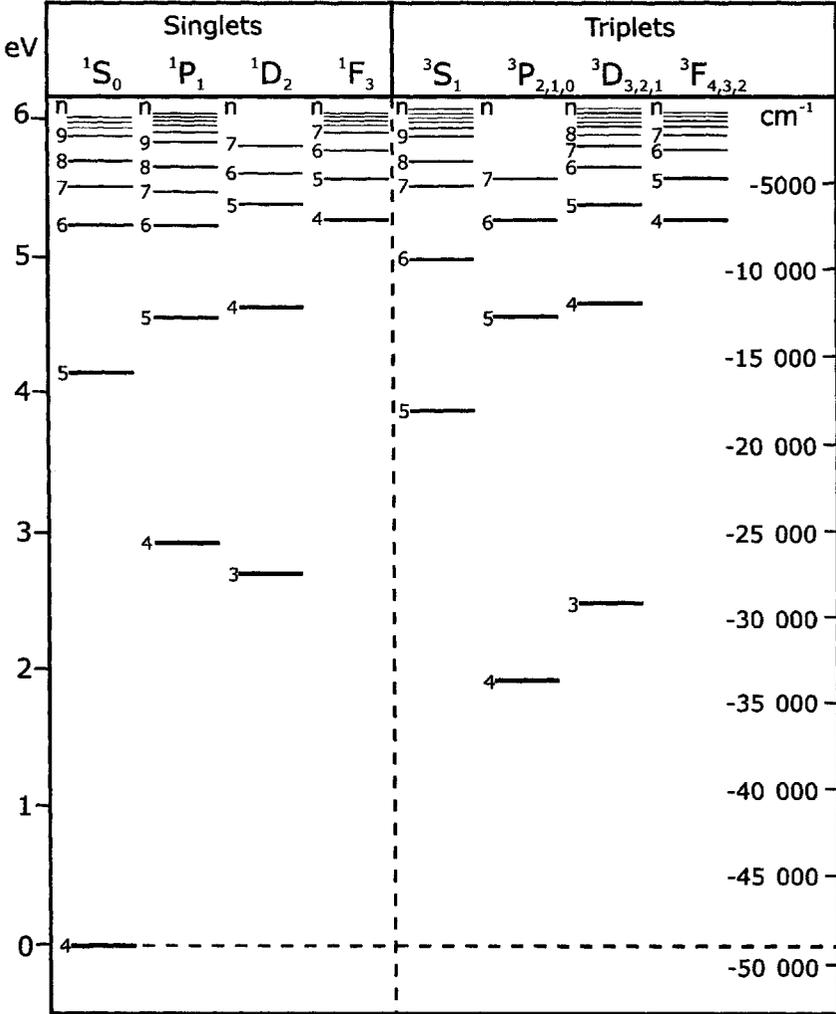


Figure 5.15: Energy-level diagram for the Ca atom.

When spin-orbit coupling is included,<sup>8</sup> the Lyman  $\alpha$  line is comprised of two components,  $2^2P_{3/2} - 1^2S_{1/2}$  and  $2^2P_{1/2} - 1^2S_{1/2}$ . Dirac's relativistic model of the hydrogen atom, however, predicts that the  $2^2P_{1/2}$  and  $2^2S_{1/2}$  levels have the same energy. In fact Lamb and Retherford experimentally determined that these two levels are split by about 1 058 MHz. The theory of quantum electrodynamics, developed by Feynman and others, is able to account for this Lamb shift.

In addition to spin-orbit coupling, hyperfine structure<sup>8</sup> is also present in the spectrum of the hydrogen atom because the proton has a nuclear spin of 1/2. Hyperfine structure doubles all of the energy levels (Figure 5.18). In the ground state the  $F = 1 - F = 0$  splitting is 1 420 MHz, which corresponds to a wavelength of 21 cm. This 21-cm radiation was first detected in interstellar space, leading to the development of radio astronomy. The hydrogen maser also oscillates on this 1 420 MHz hyperfine transition, which is electric-dipole forbidden but magnetic-dipole allowed.

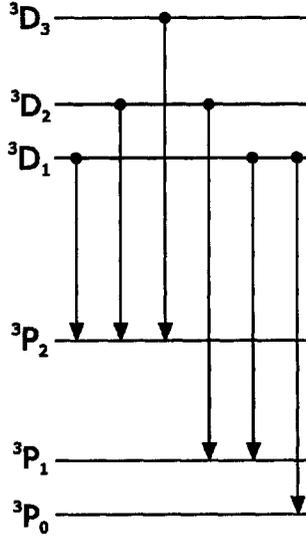


Figure 5.16: Multiplet structure of a  ${}^3D - {}^3P$  transition.

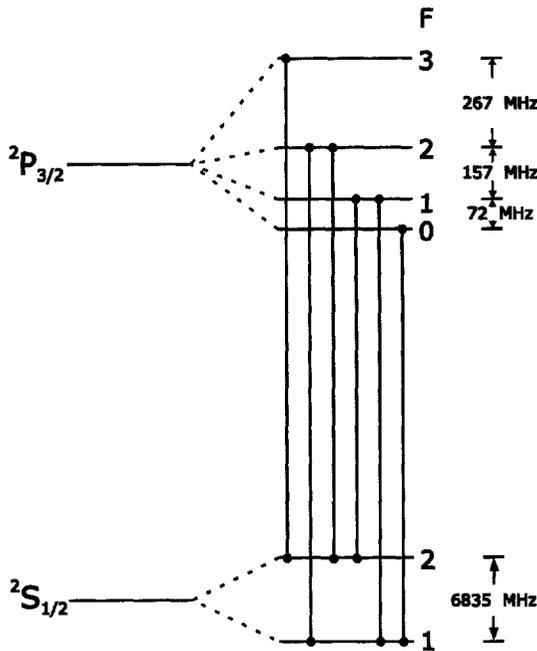


Figure 5.17: The hyperfine energy-level pattern of the  $5^2P_{3/2} - 5^2S_{1/2}$  transition of  ${}^{87}\text{Rb}$  near  $7800 \text{ \AA}$ .

### 5.7 Intensity of Atomic Lines

The intensity of a dipole-allowed transition between two atomic energy levels labeled by 1 and 0 (Figure 1.8) is governed by the equations developed in Chapter 1. For emission,

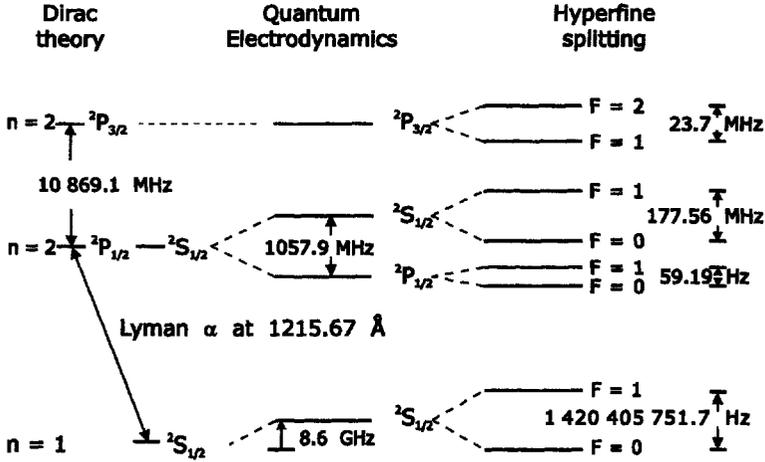


Figure 5.18: Fine and hyperfine structure of the  $n = 1$  and  $n = 2$  levels of the hydrogen atom.

the rate (equation 1.17) in photons per second is determined entirely by the upper state population density (atoms/m<sup>3</sup>) and the Einstein  $A_{1 \rightarrow 0}$  factor, which depends on the transition dipole moment. For absorption, the analogous equation (1.56) exhibits a similar dependence on transition dipole moment but is proportional to the population difference  $N_0 - N_1$  between the levels. In spite of these simple governing equations, line intensities can be a confusing subject, mainly because of the usual degeneracy of energy levels.

The key point is that the equations of Chapter 1 apply to a transition between quantum states  $|J'M' \rangle$  and  $|J''M'' \rangle$ , while measurements are generally made, in the absence of magnetic and electric fields, between the energy levels  $|J' \rangle$  and  $|J'' \rangle$ . Hyperfine structure does not alter this situation because each line is then split into hyperfine components whose relative strength is determined by angular momentum coupling rules.<sup>4</sup> The sum of the intensities of the hyperfine components equals the intensity of the hypothetical unsplit line. As long as the expressions in this section are applied to the integrated intensity of the hyperfine components of a line, then no errors result.

The upper level (with population density  $N_1 = N'$ ) of an atomic transition has an  $M_J$ -degeneracy of  $2J' + 1$  and the lower level (population  $N_0 = N''$ ) a degeneracy of  $2J'' + 1$ . Each  $|JM_J \rangle$  state has a population density  $N_1 / (2J' + 1)$  or  $N_0 / (2J'' + 1)$  for the upper or lower state, respectively, because the  $M_J$ -components are equally occupied. For emission, equation (1.17) becomes

$$\frac{dN_1}{dt} = - \sum_{M'} \sum_{M''} A_{M' \rightarrow M''} \frac{N_1}{2J' + 1} \tag{5.110}$$

and

$$\frac{dN_1}{dt} = - \frac{16\pi^3 \nu^3}{3\epsilon_0 hc^3} \frac{N_1}{(2J' + 1)} \sum_{M'} \sum_{M''} |\langle J'M' | \mu | J''M'' \rangle|^2 . \tag{5.111}$$

The double sum over  $M_J$ -states appearing in (5.111) is defined as the atomic line strength,  $S_{J'J''}$ , i.e.,

$$S_{J'J''} \equiv \sum_{M'} \sum_{M''} |\langle J'M' | \boldsymbol{\mu} | J''M'' \rangle|^2. \quad (5.112)$$

The Einstein  $A$  factor in equation (1.17) can also be written in terms of the atomic line strength as

$$A_{J' \rightarrow J''} = \frac{16\pi^3 \nu^3 S_{J'J''}}{3\varepsilon_0 \hbar c^3 (2J' + 1)}. \quad (5.113)$$

If a lineshape function is to be included, then  $S_{J'J''} / (2J' + 1)$  replaces  $\mu_{10}^2$  in equation (1.55). The  $\boldsymbol{\mu}$  operator in equation (5.111) is just

$$\boldsymbol{\mu} = -e \sum_{i=1}^N \mathbf{r}_i \quad (5.114)$$

for an atom with  $N$  electrons, with the origin of the coordinate system at the nucleus.

For absorption (including the stimulated emission term) equation (1.56) becomes

$$-\frac{dN_0}{dt} = \frac{dN_1}{dt} = \sum_{M'} \sum_{M''} B_{M'M''} \rho \left( \frac{N_0}{2J'' + 1} - \frac{N_1}{2J' + 1} \right) \quad (5.115)$$

when degeneracy is included. Using the definition of atomic line strength leads to

$$-\frac{dN_0}{dt} = \frac{2\pi^2 S_{J'J''} \nu}{3\varepsilon_0 \hbar c} \left( \frac{N_0}{2J'' + 1} - \frac{N_1}{2J' + 1} \right) g(\nu - \nu_{10}) F \quad (5.116)$$

$$= \frac{2\pi^2 \nu}{3\varepsilon_0 \hbar c} \frac{S_{J'J''}}{2J'' + 1} \left( N_0 - N_1 \frac{2J'' + 1}{2J' + 1} \right) g(\nu - \nu_{10}) F \quad (5.117)$$

with  $F$  the photon flux.

The absorption cross section, given by equation (1.57), becomes

$$\sigma = \frac{2\pi^2 \nu S_{J'J''}}{3\varepsilon_0 \hbar c (2J'' + 1)} g(\nu - \nu_{10}) \quad (5.118)$$

in terms of the atomic line strength, so that the absorption equation is

$$-\frac{dN_0}{dt} = \sigma F \left( N_0 - N_1 \frac{2J'' + 1}{2J' + 1} \right) \quad (5.119)$$

and Beer's law becomes

$$I = I_0 e^{-\sigma (N_0 - N_1 \frac{2J'' + 1}{2J' + 1}) l}. \quad (5.120)$$

The relationship between the cross section  $\sigma$  and the Einstein  $A$  value from equations (5.113) and (5.118) becomes

$$\sigma = \frac{A_{J' \rightarrow J''} \lambda^2 g(\nu - \nu_{10})}{8\pi} \frac{2J' + 1}{2J'' + 1}. \quad (5.121)$$

The degeneracy of atomic energy levels also alters the relationship between absorption and stimulated emission. Thermodynamic equilibrium now demands that equations (1.14) and (1.21) be replaced by

$$\frac{N_1}{N_0} = \frac{2J' + 1}{2J'' + 1} e^{-h\nu_{10}/kT} \quad (5.122)$$

and

$$(2J'' + 1)B_{J' \leftarrow J''} = (2J' + 1)B_{J' \rightarrow J''}. \quad (5.123)$$

Further, the Einstein relationship (1.22) between emission and absorption is altered to

$$A_{J' \rightarrow J''} = \frac{2J'' + 1}{2J' + 1} \frac{8\pi h\nu_{10}^3}{c^3} B_{J' \leftarrow J''}, \quad (5.124)$$

with the Einstein absorption and stimulated emission coefficients now expressed by

$$B_{J' \leftarrow J''} = \frac{2\pi^2}{3\varepsilon_0 h^2} \frac{S_{J' J''}}{2J'' + 1} \quad (5.125)$$

and

$$B_{J' \rightarrow J''} = \frac{2\pi^2}{3\varepsilon_0 h^2} \frac{S_{J' J''}}{2J' + 1}. \quad (5.126)$$

The expressions for  $B$  (equations (5.125) and (5.126)) depend on the form of the associated energy density  $\rho_\nu$  used in the defining equations (1.15) and (1.16). In particular, equations (5.125) and (1.126) imply the use of  $\rho_\nu$  for the energy density at frequency  $\nu$ , rather than  $\rho_\omega$  or  $\rho_\nu$ .

If thermodynamic equilibrium applies, then the population density  $N_0$  of the lower state can be replaced by the total population density  $N$ . First take the natural logarithm of equation (5.120) to give

$$-\ln\left(\frac{I}{I_0}\right) = \sigma N_0 \left(1 - e^{-h\nu_{10}/kT}\right) l. \quad (5.127)$$

The term in parentheses on the right-hand side of equation (5.127) is called the stimulated emission correction and can be ignored if there is negligible population in the excited state. At equilibrium the population  $N_0$  can be calculated from equation:

$$N_0 = \frac{N(2J'' + 1)e^{-E_0/kT}}{q}, \quad (5.128)$$

in which  $N$  is the total population of the system,  $q \equiv \sum (2J_i + 1)e^{-E_i/kT}$  is the partition function, and the energy level at  $E_0$  with degeneracy  $2J'' + 1$  has a population density  $N_0$ . Thus the absorbance,  $-\ln(I/I_0)$ , becomes

$$-\ln\left(\frac{I}{I_0}\right) = \frac{2\pi^2\nu_{10} S_{J' J''}}{3\varepsilon_0 hc q} e^{-E_0/kT} \left(1 - e^{-h\nu_{10}/kT}\right) g(\nu - \nu_{10}) N l \quad (5.129)$$

in SI units.

Atomic spectroscopists and astronomers<sup>9</sup> are fond of a concept called the oscillator strength  $f$  for a transition, defined by comparison with the radiation emitted by a classical electron oscillator. The  $f$ -value for absorption (we will not consider negative  $f$  emission values<sup>9</sup>) for a transition is defined as

$$f = f_{\text{abs}} = f_{J' \leftarrow J''} = \frac{8\pi^2 m_e \nu}{3he^2} \frac{S_{J'J''}}{2J'' + 1}, \quad (5.130)$$

so the Einstein  $B$  coefficient is

$$B_{J' \leftarrow J''} = \frac{e^2}{4\varepsilon_0 m_e h \nu} f_{J' \leftarrow J''} \quad (5.131)$$

in terms of the oscillator strength, and the Einstein  $A$  value is

$$A_{J' \rightarrow J''} = \frac{2\pi e^2 \nu^2}{\varepsilon_0 m_e c^3} f_{J' \leftarrow J''} \frac{2J'' + 1}{2J' + 1}. \quad (5.132)$$

The absorption cross section  $\sigma$  can also be related to the oscillator strength  $f$  as

$$\sigma = \frac{e^2}{4\varepsilon_0 m_e c} f_{J' \leftarrow J''} g(\nu - \nu_{10}). \quad (5.133)$$

Note also that, as in Chapter 1, the Einstein  $A$  and  $B$  coefficients might also include a lineshape function  $g(\nu - \nu_{10})$ , and the cross section  $\sigma$  might have the lineshape function integrated over frequency to unity. It would be less confusing if the symbol for a quantity that included a lineshape function always had a subscript  $\nu$ , e.g.,  $(A_{1 \rightarrow 0})_\nu = A_{1 \rightarrow 0} g(\nu - \nu_{10})$  because  $(A_{1 \rightarrow 0})_\nu$  and  $A_{1 \rightarrow 0}$  have different dimensions. The unfortunate custom, however, is often to suppress the subscript  $\nu$  and use the same symbol for quantities that include a lineshape function as for “integrated” quantities (i.e., integrated over frequency so that  $g(\nu - \nu_{10})$  has been suppressed because  $\int g(\nu - \nu_0) d\nu = 1$ ). Even more confusing is the occasional use of formulas in which  $g(\nu - \nu_0)$  has been evaluated for  $\nu = \nu_0$  for a particular lineshape function.

The Einstein  $A$ , Einstein  $B$ , line strength  $S$ , integrated absorption cross section  $\int \sigma d\nu$ , and oscillator strength  $f$  are all related:

$$\begin{aligned}
 A_{J' \rightarrow J''} &= \frac{8\pi h\nu^3}{c^3} \frac{2J''+1}{2J'+1} B_{J' \leftarrow J''} = \frac{16\pi^3 \nu^3 S_{J' J''}}{3\varepsilon_0 h c^3 (2J'+1)} \\
 &= \frac{8\pi \nu^2}{c^2} \frac{2J''+1}{2J'+1} \int \sigma d\nu = \frac{2\pi e^2 \nu^2}{\varepsilon_0 m_e c^3} \frac{2J''+1}{2J'+1} f_{J' \leftarrow J''}, \quad (5.134)
 \end{aligned}$$

$$\begin{aligned}
 B_{J' \leftarrow J''} &= \frac{c^3}{8\pi h\nu^3} \frac{2J'+1}{2J''+1} A_{J' \rightarrow J''} = \frac{2\pi^2}{3\varepsilon_0 h^2} \frac{S_{J' J''}}{2J''+1} \\
 &= \frac{c}{h\nu} \int \sigma d\nu = \frac{e^2}{4\varepsilon_0 m_e h\nu} f_{J' \leftarrow J''}, \quad (5.135)
 \end{aligned}$$

$$\begin{aligned}
 S_{J' J''} &= \frac{3\varepsilon_0 h c^3 (2J'+1)}{16\pi^3 \nu^3} A_{J' \rightarrow J''} = \frac{3\varepsilon_0 h^2 (2J''+1)}{2\pi^2} B_{J' \leftarrow J''}, \\
 &= \frac{3\varepsilon_0 h c (2J''+1)}{2\pi^2 \nu} \int \sigma d\nu = \frac{3h e^2 (2J''+1)}{8\pi^2 m_e \nu} f_{J' \leftarrow J''}, \quad (5.136)
 \end{aligned}$$

$$\begin{aligned}
 \int \sigma d\nu &= \frac{c^2}{8\pi \nu^2} \frac{2J'+1}{2J''+1} A_{J' \rightarrow J''} = \frac{h\nu}{c} B_{J' \leftarrow J''} \\
 &= \frac{2\pi^2 \nu S_{J' J''}}{3\varepsilon_0 h c (2J''+1)} = \frac{e^2}{4\varepsilon_0 m_e c} f_{J' \leftarrow J''}, \quad (5.137)
 \end{aligned}$$

$$\begin{aligned}
 f_{J' \leftarrow J''} &= \frac{\varepsilon_0 m_e c^3}{2\pi e^2 \nu^2} \frac{2J'+1}{2J''+1} A_{J' \leftarrow J''} = \frac{4\varepsilon_0 m_e h\nu}{e^2} B_{J' \leftarrow J''} \\
 &= \frac{8\pi^2 m_e \nu S_{J' J''}}{3h e^2 (2J''+1)} = \frac{4\varepsilon_0 m_e c}{e^2} \int \sigma d\nu. \quad (5.138)
 \end{aligned}$$

SI units are used in equations (5.134) to (5.138), and the Einstein stimulated emission coefficient  $B_{J' \rightarrow J''}$  can be added to the list of interrelated quantities with

$$B_{J' \rightarrow J''} = \frac{2J''+1}{2J'+1} B_{J' \leftarrow J''} \quad (5.139)$$

obtained from equation (5.123).

Oscillator strengths,  $f$ , are dimensionless,  $A$  has units of  $\text{s}^{-1}$ ,  $B$  has dimensions of  $\text{J}^{-1} \text{m}^3 \text{s}^{-2}$  using  $\rho_\nu$ , while the integrated cross section  $\int \sigma d\nu$  is measured in  $\text{m}^2 \text{Hz} = \text{m}^2 \text{s}^{-1}$ . The line strength  $S_{J' J''}$  has SI units of coulomb<sup>2</sup> meter<sup>2</sup> ( $\text{C}^2 \text{m}^2$ ) and the conversion to debye<sup>2</sup> ( $\text{D}^2$ ) and atomic units ( $e^2 a_0^2$ ) is given by

$$S/(\text{C}^2 \text{m}^2) = 1.11265006 \times 10^{-59} S/(\text{D}^2) = 7.1882465 \times 10^{-59} S/(e^2 a_0^2). \quad (5.140)$$

One debye is defined in cgs units as  $10^{-18}$  esu cm. The separation of  $+e$  and  $-e$  by  $1 \text{ \AA}$  results in a dipole moment of  $4.8032042 \text{ D}$  in magnitude. The separation of  $\pm e$  by  $1$  Bohr radius,  $a_0$ , gives the atomic unit of dipole moment ( $ea_0$ ) as  $2.5417462 \text{ D}$ .

In terms of numerical values equations (5.134) to (5.138) become

$$\begin{aligned}
A_{J' \rightarrow J''} / (\text{s}^{-1}) &= 1.665\,312\,7 \times 10^{-26} \tilde{\nu}^3 \frac{2J''+1}{2J'+1} B_{J' \leftarrow J''} / (\text{J m}^{-3} \text{s}^{-2}) \\
&= 3.136\,189\,3 \times 10^{-7} \frac{\tilde{\nu}^3}{(2J'+1)} S_{J' J''} / (\text{D}^2) \\
&= 7.534\,606\,3 \times 10^{11} \tilde{\nu}^2 \frac{2J''+1}{2J'+1} \int \sigma d\tilde{\nu} / (\text{cm/molecule}) \\
&= 0.667\,025\,17 \tilde{\nu}^2 \frac{2J''+1}{2J'+1} f_{J' \leftarrow J''}, \tag{5.141}
\end{aligned}$$

$$\begin{aligned}
B_{J' \leftarrow J''} / (\text{J}^{-1} \text{m}^3 \text{s}^{-2}) &= \frac{6.004\,878\,2 \times 10^{25}}{\tilde{\nu}^3} \frac{2J'+1}{2J''+1} A_{J' \rightarrow J''} / (\text{s}^{-1}) \\
&= \frac{1.883\,243\,5 \times 10^{19}}{2J''+1} S_{J' J''} / (\text{D}^2) \\
&= \frac{4.524\,439\,3 \times 10^{37}}{\tilde{\nu}} \int \sigma d\tilde{\nu} / (\text{cm/molecule}) \\
&= \frac{4.005\,404\,9 \times 10^{25}}{\tilde{\nu}} f_{J' \leftarrow J''}, \tag{5.142}
\end{aligned}$$

$$\begin{aligned}
S_{J' J''} / (\text{D}^2) &= \frac{3.188\,583\,0 \times 10^6}{\tilde{\nu}^3} (2J'+1) A_{J' \rightarrow J''} / (\text{s}^{-1}) \\
&= 5.309\,987\,8 \times 10^{-20} (2J''+1) B_{J' \leftarrow J''} / (\text{J m}^{-3} \text{s}^{-2}) \\
&= \frac{2.402\,471\,7 \times 10^{18}}{\tilde{\nu}} (2J''+1) \int \sigma d\tilde{\nu} / (\text{cm/molecule}) \\
&= \frac{2.126\,865\,1 \times 10^6}{\tilde{\nu}} (2J''+1) f_{J' \leftarrow J''}, \tag{5.143}
\end{aligned}$$

$$\begin{aligned}
\int \sigma d\tilde{\nu} / (\text{cm/molecule}) &= \frac{1.327\,209\,4 \times 10^{-12}}{\tilde{\nu}^2} \frac{2J'+1}{2J''+1} A_{J' \rightarrow J''} / (\text{s}^{-1}) \\
&= 2.210\,218\,6 \times 10^{-38} \tilde{\nu} B_{J' \leftarrow J''} / (\text{J m}^{-3} \text{s}^{-2}) \\
&= 4.162\,379\,9 \times 10^{-19} \frac{\tilde{\nu}}{2J''+1} S_{J' J''} / (\text{D}^2) \\
&= 8.852\,820\,5 \times 10^{-13} f_{J' \leftarrow J''}, \tag{5.144}
\end{aligned}$$

$$\begin{aligned}
f_{J' \leftarrow J''} &= \frac{1.499\,193\,8 \cdot 2J''+1}{\tilde{\nu}^2 \cdot 2J'+1} A_{J' \rightarrow J''} / (\text{s}^{-1}) \\
&= 2.496\,626\,5 \times 10^{-26} \tilde{\nu} B_{J' \leftarrow J''} / (\text{J m}^{-3} \text{s}^{-2}) \\
&= 4.701\,755\,6 \times 10^{-7} \frac{\tilde{\nu}}{2J''+1} S_{J' J''} / (\text{D}^2) \\
&= 1.129\,583\,5 \times 10^{12} \int \sigma d\tilde{\nu} / (\text{cm/molecule}). \tag{5.145}
\end{aligned}$$

In equations (5.141) to (5.145)  $\tilde{\nu}$  is in  $\text{cm}^{-1}$ ,  $A_{J' \rightarrow J''}$  is in  $\text{s}^{-1}$ ,  $B_{J' \leftarrow J''}$  in  $\text{J}^{-1} \text{m}^3 \text{s}^{-2}$  using  $\rho_\nu$ ,  $S_{J' J''}$  in  $\text{D}^2$ ,  $f_{J' \leftarrow J''}$  is dimensionless, and  $\int \sigma d\tilde{\nu}$  is in HITRAN units (see Chapter 7) of  $\text{cm}^2 \text{cm}^{-1}/\text{molecule}$  or  $\text{cm}/\text{molecule}$ . The conversion from HITRAN units of  $\text{cm}/\text{molecule}$  (or  $\text{cm}/\text{atom}$ ) to SI units of  $\text{m}^2 \text{s}^{-1}/\text{molecule}$  requires multiplication by  $10^{-2} c_0 = 2\,997\,924.58$  (section 7.6).

Oscillator strengths obey a sum rule, namely

$$\sum_i f_{i \leftarrow 0} = 1, \quad (5.146)$$

for the  $i \leftarrow 0$  series of transitions (including an integral for the continuum) involving electronic transitions associated with a single electron connected to the ground state  $|0\rangle$ .

Consider the  $3^2P_{3/2} \rightarrow 3^2S_{1/2}$   $D$  line of Na at 5890 Å with a lifetime of about 16 ns (or  $A = 6.25 \times 10^7 \text{ s}^{-1}$ ). This corresponds to a line strength of  $S = 1.81 \times 10^{57} \text{ C}^2 \text{ m}^2 = (12.7 \text{ D})^2$  (1 debye =  $3.3356410 \times 10^{-30} \text{ C m}$ ) and an absorption oscillator strength of 0.65. Atomic data compilations often list “ $gf$ ” =  $(2J'' + 1)f_{\text{abs}}$  values and the accepted  $gf$  value<sup>9</sup> is 1.27 or  $f_{\text{abs}} = 0.635$  for this transition of Na.

In this book only electric dipole transitions are considered, and the intensities and selection rules are based on the use of the electric dipole operator as the perturbation operator, equation (1.26), and in the transition dipole moment integral (1.99). In fact, the interaction of electromagnetic radiation with a molecule generates a number of additional terms in the Hamiltonian operator, of which the dipole moment part, equation (1.26), is the largest. Also present are a magnetic dipole interaction term and an electric quadrupole term: these terms are typically  $10^{-5}$  and  $10^{-6}$  times smaller, respectively, than the electric dipole term. Magnetic dipole transitions obey the same selection rules on  $J$  and  $M$  as do electric dipole transitions (i.e.,  $\Delta J = 0, \pm 1$  but  $J = 0 \leftrightarrow J = 0$  and  $\Delta M = 0, \pm 1$ ), but the parity selection rule is  $+$   $\leftrightarrow$   $+$  and  $- \leftrightarrow -$  because the magnetic dipole operator has even parity. Electric quadrupole selection rules allow  $\Delta J = 0, \pm 1, \pm 2$  (but  $J = 0 \leftrightarrow J = 0$ ,  $J = 1/2 \leftrightarrow 1/2$ , and  $J = 1 \leftrightarrow 0$ ),  $\Delta M = 0, \pm 1, \pm 2$ , and the parity selection rule is also  $+$   $\leftrightarrow$   $+$  and  $- \leftrightarrow -$ . Atomic spectra are dominated by electric dipole transitions (called  $E1$ ), but weaker lines due to magnetic dipole and electric quadrupole transitions (called  $M1$  and  $E2$ , respectively) can also be seen. Indeed even higher-order atomic transitions such as magnetic quadrupole ( $M2$ ) and electric hexadecapole ( $E3$ ) have been detected, particularly in astronomical sources and the earth’s atmosphere.<sup>9</sup> For example, the famous “green line” (5777 Å) of atomic oxygen seen in atmospheric aurora is due to the  $^1S_0 - ^1D_2$  quadrupole transition ( $E2$ ), and the “red line” at 6300 Å is the  $^1D_2 - ^3P_2$  magnetic dipole transition ( $M1$ ).

## 5.8 Zeeman Effect

Associated with the various angular momenta of atoms are magnetic moments. The theory begins with orbital angular momentum since there is a classical analogy. The orbit of a negatively charged electron around the nucleus is equivalent to a small current loop that creates a magnetic moment. If the constant of proportionality between the orbital angular momentum  $\hat{\mathbf{L}}$  and the magnetic moment  $\hat{\boldsymbol{\mu}}_L$  is  $\gamma$ , the magnetogyric ratio, so that

$$\hat{\boldsymbol{\mu}}_L = \gamma \hat{\mathbf{L}}, \quad (5.147)$$

then  $\gamma$  can be shown to be given by  $-e/(2m_e)$  in SI units or  $-e/(2m_e c)$  in cgs units. The magnitude of  $\hat{\mathbf{L}}$  is

$$|\hat{\mathbf{L}}| = \sqrt{L(L+1)}\hbar, \quad (5.148)$$

and consequently the magnitude of the associated orbital magnetic moment is given by

$$|\hat{\mu}_L| = - \left( \frac{e\hbar}{2m_e} \right) \sqrt{L(L+1)} = -\mu_B \sqrt{L(L+1)}. \quad (5.149)$$

It is convenient to define a quantity  $\mu_B$  referred to as the *Bohr magneton*, as

$$\mu_B = \frac{e\hbar}{2m_e} = -\hbar\gamma \quad (5.150)$$

in SI units. The Bohr magneton is the magnitude of the magnetic moment that one unit of orbital angular momentum produces in an atom.

Other angular momenta such as spin also have associated magnetic moments, but there are no classical analogies to draw upon. In the case of the electron it is possible to write, by analogy with the orbital angular momentum, the expression

$$\hat{\mu}_S = g_e \gamma \hat{S} = g_e \left( \frac{-\mu_B}{\hbar} \right) \hat{S}, \quad (5.151)$$

in which the numerical factor  $g_e$  is defined via this equation. The  $g_e$  value for a single free electron is found to be 2.0023. In the absence of quantum electrodynamics, the value of  $g_e$  is precisely 2. The negative sign of the magnetogyric ratio indicates that  $\hat{\mu}_S$  and  $\hat{S}$  point in opposite directions (notice that  $\hat{\mu}_L$  and  $\hat{L}$  also point in opposite directions). Recent custom<sup>10</sup> (Appendix A) is to include the negative sign in equation (5.151) in the  $g_e$  value, i.e., to take  $g_e$  as  $-2.0023$ .

Similarly, for nuclear spin a nuclear moment  $\hat{\mu}_I$  can be described by the equation

$$\hat{\mu}_I = \gamma_I \hat{I} = g_I \frac{\mu_N}{\hbar} \hat{I}, \quad (5.152)$$

in which  $\gamma_I$  is the nuclear magnetogyric ratio and  $\mu_N$  is the basic unit for nuclear magnetic moments, called the *nuclear magneton*. For nuclei, the  $g_I$  values can be either positive or negative. The nuclear magneton

$$\mu_N = \frac{e\hbar}{2m_p} \quad (5.153)$$

is defined by analogy with the Bohr magneton, but with the mass of the proton ( $m_p$ ) rather than the mass of the electron. This means that (Appendix A)

$$\frac{\mu_N}{\mu_B} = \frac{m_e}{m_p} \approx \frac{1}{1837}, \quad (5.154)$$

so that nuclear magnetic moments are typically two to three orders of magnitude smaller than are electronic moments, as  $g_I \sim 1$  (e.g.,  $g_I = 5.585$  for  $^1\text{H}$ ).

When an atom is placed in a magnetic field, its magnetic moments interact with the field and, as a consequence, an interaction energy term

$$\hat{H}' = -\hat{\mu} \cdot \mathbf{B} \quad (5.155)$$

must be included in the Hamiltonian operator. This is called the Zeeman interaction Hamiltonian operator and leads to the *Zeeman effect*. When  $S = 0$ , the Zeeman effect is called “normal” and when  $S \neq 0$  it is called “anomalous” since, historically, the Zeeman effect was discovered before electron spin was known.

The normal Zeeman effect applies to singlet states, for which only  $\hat{\mu}_L$  is present. If  $\mathbf{B}$  is aligned along the laboratory  $z$ -axis, then the Zeeman Hamiltonian operator takes the form

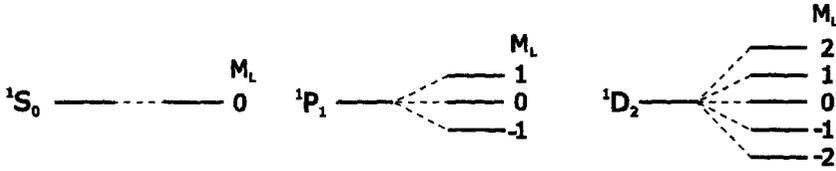


Figure 5.19: The Zeeman pattern for  $^1S$ ,  $^1P$ , and  $^1D$  states.

$$\hat{H}' = -\hat{\mu}_L \cdot \mathbf{B} = -\gamma \hat{L}_z B_z. \tag{5.156}$$

Simple perturbation theory gives the Zeeman energy as

$$\begin{aligned} E &= \langle LM_L | -\gamma \hat{L}_z B_z | LM_L \rangle \\ &= -\gamma \hbar M_L B = \mu_B B M_L. \end{aligned} \tag{5.157}$$

The energy levels are thus split into the  $2L + 1$  states (Figure 5.19) characterized by individual  $M_L$  values. The energy of each state varies linearly with the strength of the magnetic field, and hence the interval between adjacent  $M_L$  values is  $\mu_L B$ , which is independent of  $L$ . The selection rules for electronic transitions are  $\Delta M_L = 0$  for light polarized along  $z$  and parallel to  $\mathbf{B}$ ,  $\Delta M_L = \pm 1$  for light polarized perpendicular to  $\mathbf{B}$ .

When a magnetic field is applied to atomic transitions between singlet ( $S = 0$ ) states, the atomic line is split into three components (Figure 5.20). The line coinciding with the zero-field position is a  $\Delta M = 0$  transition, the line shifted to higher frequency is a  $\Delta M = +1$  transition, while the line shifted to lower frequency is a  $\Delta M = -1$  transition. Classically, one can view the orbital magnetic moment as precessing (Figure 5.21) at a frequency called the *Larmor frequency* around the direction of the applied field. The Larmor frequency is given by  $h\nu_L = \mu_B B$  or

$$\nu_L = \frac{\mu_B B}{h}. \tag{5.158}$$

The case with  $S \neq 0$  is more complex, since  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  couple first to give  $\hat{\mathbf{J}}$ , while  $\hat{\mu}_S$  and  $\hat{\mu}_L$  interact to give the corresponding  $\hat{\mu}_J$ . By analogy with  $\hat{\mu}_L$  one writes

$$\hat{\mu}_S = g_S \gamma \hat{\mathbf{S}} = -g_S \frac{\mu_B}{\hbar} \hat{\mathbf{S}} \tag{5.159}$$

and

$$\hat{\mu}_J = g_J \gamma \hat{\mathbf{J}} = -g_J \frac{\mu_B}{\hbar} \hat{\mathbf{J}}, \tag{5.160}$$

so that the problem reduces to finding an expression for  $g_J$ . According to the classical vector coupling picture  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  precess rapidly about  $\hat{\mathbf{J}}$ , so that only those components of  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  parallel to  $\hat{\mathbf{J}}$  contribute to the total magnetic moment (Figure 5.22).

The total magnetic moment  $\hat{\mu}_J$  can be written, according to what has been previously said, as

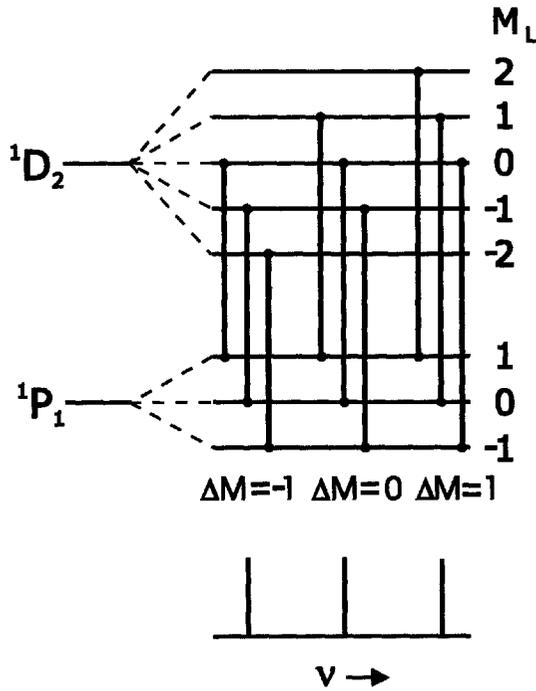


Figure 5.20: The three-line pattern of the “normal” Zeeman effect.

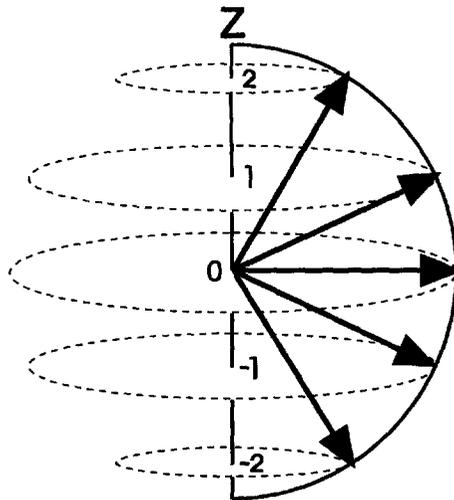


Figure 5.21: The precessional motion of  $L = 2$  with the possible  $M_L = 2, 1, 0, -1,$  and  $-2$  values.

$$\begin{aligned}
 \hat{\mu}_J &= \left( \hat{\mu}_S \cdot \frac{\hat{\mathbf{J}}}{|\hat{\mathbf{J}}|^2} + \hat{\mu}_L \cdot \frac{\hat{\mathbf{J}}}{|\hat{\mathbf{J}}|^2} \right) \hat{\mathbf{J}} \\
 &= \left( \gamma(g_S \hat{\mathbf{S}} + \hat{\mathbf{L}}) \cdot \frac{\hat{\mathbf{J}}}{|\hat{\mathbf{J}}|^2} \right) \hat{\mathbf{J}},
 \end{aligned}
 \tag{5.161}$$

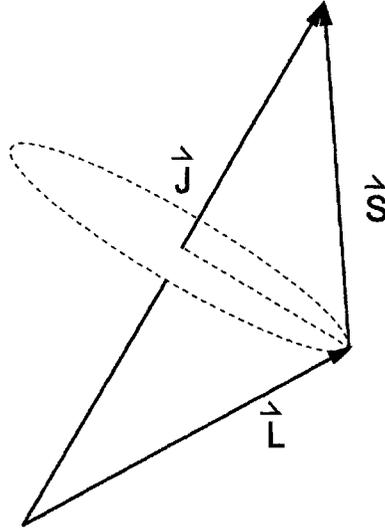


Figure 5.22: The rapid precessional motion of  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  about  $\hat{\mathbf{J}}$ .

from which we see that the appropriate  $g_J$ -factor is

$$\begin{aligned} \hat{g}_J &= (g_S \hat{\mathbf{S}} + \hat{\mathbf{L}}) \cdot \frac{\hat{\mathbf{J}}}{|\hat{\mathbf{J}}|^2} = \frac{(g_S \hat{\mathbf{S}} + \hat{\mathbf{L}}) \cdot (\hat{\mathbf{L}} + \hat{\mathbf{S}})}{|\hat{\mathbf{J}}|^2} \\ &= \frac{g_S(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} + |\hat{\mathbf{S}}|^2) + \hat{\mathbf{L}} \cdot \hat{\mathbf{S}} + |\hat{\mathbf{L}}|^2}{|\hat{\mathbf{J}}|^2}. \end{aligned} \tag{5.162}$$

Notice that  $\hat{g}_J$  is not a simple number but is technically a scalar operator which will give rise to different numerical values for different levels. Now since

$$\hat{J}^2 = (\hat{\mathbf{L}} + \hat{\mathbf{S}}) \cdot (\hat{\mathbf{L}} + \hat{\mathbf{S}}) = \hat{L}^2 + \hat{S}^2 + 2\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} \tag{5.163}$$

we can solve for  $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$  in terms of  $\hat{J}^2$ ,  $\hat{L}^2$ , and  $\hat{S}^2$  as

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \frac{\hat{J}^2 - \hat{L}^2 - \hat{S}^2}{2} \tag{5.164}$$

to obtain for  $\hat{g}_J$  the expression

$$\hat{g}_J = \frac{g_S(\hat{J}^2 - \hat{L}^2 + \hat{S}^2) + (\hat{J}^2 - \hat{S}^2 + \hat{L}^2)}{2\hat{J}^2}. \tag{5.165}$$

Because  $g_S \approx 2$ , the expression for  $g_J$  can also be written in the form

$$\hat{g}_J \approx \frac{(3\hat{J}^2 + \hat{S}^2 - \hat{L}^2)}{2\hat{J}^2}. \tag{5.166}$$

In the corresponding energy expression (analogous to equation (5.157)), we would replace  $\hat{J}^2$ ,  $\hat{L}^2$ , and  $\hat{S}^2$  by their magnitudes  $J(J + 1)$ ,  $L(L + 1)$ , and  $S(S + 1)$  to obtain for  $g_J$  (now a number),

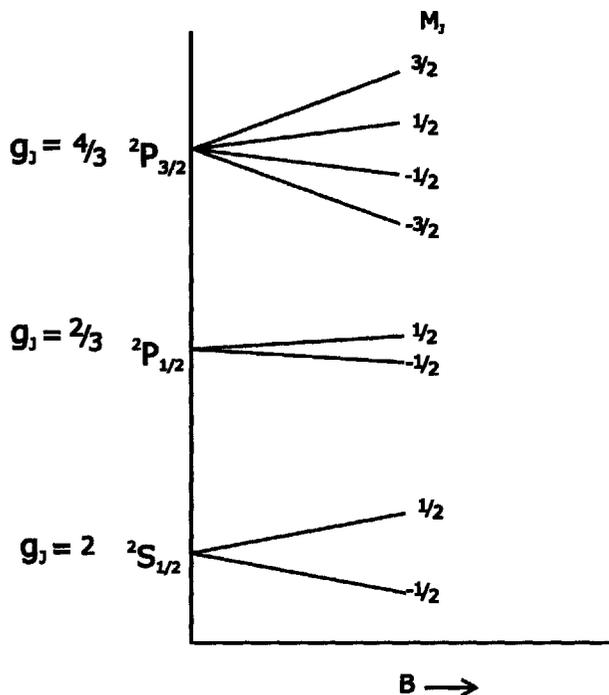


Figure 5.23: Zeeman energy-level pattern for the first three states of an alkali atom.

$$g_J = \frac{3J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}. \quad (5.167)$$

By applying a magnetic field  $\mathbf{B}$  along the  $z$ -axis of an atom, one obtains an energy splitting of

$$E_{M_J} = \langle \hat{H}' \rangle = -\langle \hat{\mu}_J \cdot \mathbf{B} \rangle = g_J \frac{\mu_B}{\hbar} \langle \hat{\mathbf{J}} \cdot \mathbf{B} \rangle = g_J \frac{\mu_B B}{\hbar} \langle \hat{J}_z \rangle = g_J \mu_B M_J B, \quad (5.168)$$

with  $g_J$  given by equation (5.167). This “anomalous” Zeeman effect has proved to be of great value in atomic spectroscopy, since the number of components is related to the  $J$  value, and the measured  $g_J$  value allows  $L$ ,  $S$ , and  $J$  to be determined (Figure 5.23).

### *Paschen-Back Effect*

As the applied magnetic field becomes very large, the splitting between the  $M_J$  components becomes larger than the splitting between the spin-orbit components. The coupling of  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  with the magnetic field is then stronger than the spin-orbit coupling. Spin-orbit coupling breakdown then occurs because of the decoupling of  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  from  $\hat{\mathbf{J}}$  by the magnetic field. As the magnetic field increases, the complex pattern of the anomalous effect is replaced by the simple three-line pattern of the normal Zeeman effect. This is called the *Paschen-Back effect*.

## 5.9 Stark Effect

The application of an electric field to an atom is called the *Stark effect*. In this case the atomic levels split into their  $(L + 1)$  values of  $|M_J|$  and the  $(2J + 1)$ -degeneracy is only partly removed. While the Stark effect is widely used in molecular spectroscopy to obtain dipole moments, it has rarely been used in atomic spectroscopy. The laser cooling and trapping of atoms<sup>11</sup> and the spectroscopy of Rydberg states,<sup>12</sup> however, often make use of the Stark effect and the Zeeman effect. The application of an electric field to the  $n = 2$  states of the hydrogen atom results in a space-fixed dipole moment in the laboratory coordinate system (see Problem 17).

### Problems

1. Positronium is an atom-like system formed from an electron and a positron. Predict the energy-level pattern and the wavelengths of some of the electronic transitions of positronium.
2. Show that if the parity operator is given in spherical polar coordinates by

$$E^* Y_{L, M_L} = Y_{L, M_L}(\pi - \theta, \phi + \pi),$$

then the spherical harmonics obey

$$E^* Y_{L, M_L} = (-1)^L Y_{L, M_L}.$$

Note that  $E^*$  inverts all spatial coordinates through the origin.

3. (a) Plot the  $R_{10}$ ,  $R_{20}$ , and  $R_{21}$  radial functions for the hydrogen atom.  
(b) Plot the angular part of the real form of the hydrogen function for the  $2s$ ,  $2p_x$ ,  $2p_y$ , and  $2p_z$  orbitals. What are the shapes of the orbitals?
4. Derive the spin matrices for  $\hat{S}^2$ ,  $\hat{S}_x$ ,  $\hat{S}_y$ , and  $\hat{S}_z$  for the case  $S = 3/2$ .
5. Verify the relationship

$$(\hat{\sigma} \cdot \mathbf{a})(\hat{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\hat{\sigma} \cdot \mathbf{a} \times \mathbf{b}$$

where the components of  $\hat{\sigma}$  are the Pauli matrices, and  $\mathbf{a}$  and  $\mathbf{b}$  are any two vector operators that commute with  $\hat{\sigma}$ .

6. (a) Construct the table of microstates and derive the terms for the atomic configurations  $p^3$  and  $d^2$ .  
(b) For  $p^3$  and  $d^2$  atomic configurations, what are the lowest energy terms and energy levels?
7. Without using microstates, derive the ground-state terms and energy levels for the transition elements of the third row (Sc through Zn) of the periodic table. (Remember Cr and Cu are exceptions to the regular *Aufbau* filling of electrons into orbitals.)
8. For the three configurations  $ns n's$ ,  $ns n'd$ , and  $np n'p n''p$ ,

- (a) What are the possible terms?  
 (b) For each term, what are the possible energy levels?
9. For the  $\text{He}^+$  atom calculate the vacuum wavelength and wavenumber of the transition corresponding to the red Balmer  $\text{H}_\alpha$  transition in the H atom.
10. The spectrum of  $\text{He}^+$  contains transitions at 303.780 Å, 256.317 Å, 243.027 Å, and 237.331 Å. Assign principal quantum numbers to these transitions.
11. (a) What are  $\langle r \rangle$  and  $\langle 1/r \rangle$  for the 1s orbital of hydrogen?  
 (b) What is the transition dipole moment in debye for the  $2p_z \leftarrow 1s$  transition of hydrogen?
12. The air wavelengths for the Balmer series are 6562.72 Å and 6562.852 Å ( $\text{H}_\alpha$ ), 4861.33 Å ( $\text{H}_\beta$ ), 4340.47 Å ( $\text{H}_\gamma$ ), and 4101.74 Å ( $\text{H}_\delta$ ). Derive a value for the Rydberg constant  $\tilde{R}_\infty$ . Why are two wavelengths listed for  $\text{H}_\alpha$ ?
13. Plot the angular dependence of the square magnitude of the spherical harmonics  $|Y_{l,m_l}|^2$  for  $l = 0, 1$ , and 2.
14. (a) For the hydrogen atom in  $n = 2$  evaluate the  $\zeta_{2p}$  integral.  
 (b) Calculate the splitting in  $\text{cm}^{-1}$  for the  ${}^2P_{3/2} - {}^2P_{1/2}$  interval for  $n = 2$  of H.
15. In the atomic spectrum of neutral Ca there is a normal multiplet of six lines at 0, 14, 36, 106, 120, and 158  $\text{cm}^{-1}$  above the lowest frequency line of the multiplet. What are the quantum numbers of the states involved in the transition?
16. For the two Na  $D$  lines calculate the spectral patterns for emission lines at a magnetic field strength of 0.25 tesla (T). What are the Zeeman splittings of the lines in  $\text{cm}^{-1}$ ?
17. An electric field along the laboratory  $z$ -axis is applied to the hydrogen atom.
- (a) If the interaction energy is represented by  $\hat{H}' = eEz = eEr \cos \theta$ , evaluate the Hamiltonian matrix for  $n = 2$  using the complex form of the hydrogen wavefunctions. (Ignore the effects of the Lamb shift, fine, and hyperfine structure.) *Hint:* Parity considerations will simplify the problem.  
 (b) What is the energy-level pattern?  
 (c) For an applied electric field of 1000 V/m, what are the energy splittings in  $\text{cm}^{-1}$ ?
18. The following wavenumbers are listed in Moore's tables for the  $n^2P^\circ - 3^2S$  transitions of Na.
- (a) Correct the line positions for the effects of spin-orbit coupling and determine  $\zeta$  for the excited  $n^2P$  terms of Na.  
 (b) Devise an extrapolation procedure to determine the ionization potential and the quantum defect for this Rydberg series.

$n$	$J$	Wavenumber/cm <sup>-1</sup>
5	0.5	35 040.27
5	1.5	35 042.79
6	0.5	37 296.51
6	1.5	37 297.76
7	0.5	38 540.40
7	1.5	38 541.14
8	0.5	39 298.54
8	1.5	39 299.01
9	0.5	39 794.53
9	1.5	39 795.00
10	0.5 and 1.5	40 137.23

19. On the basis of first-order perturbation theory, the hyperfine structure of the ground electronic state of the H atom involves the interaction of the spins of the electron and proton with one another, and with any applied magnetic fields. It is possible to integrate out the spatial coordinates and to consider the system as two spins  $S = I = 1/2$  governed by the spin Hamiltonian operator

$$\hat{H}_{\text{spin}} = \frac{b_F}{\hbar^2} \hat{I} \cdot \hat{S} + \frac{k_S}{\hbar} \hat{S}_z + \frac{k_I}{\hbar} \hat{I}_z \equiv \hat{H}_{\text{hfs}} + \hat{H}_{\text{Zeeman}},$$

in which  $b_F$ ,  $k_S$ , and  $k_I$  are given by

$$\begin{aligned} b_F &= \frac{2\mu_0}{3} g_e \mu_B g_I \mu_N |\psi_{1s}(0)|^2 \\ k_S &= g_e \mu_B B_0 \\ k_I &= -g_I \mu_N B_0 \end{aligned}$$

and  $g_e$ ,  $g_I$ ,  $\mu_B$ ,  $\mu_N$  are the  $g$ -factors and magnetons for the electron and the proton. The spin Hamiltonian operator can be split into two parts,  $b_F \hat{I} \cdot \hat{S} / \hbar^2$  (referred to as the hyperfine structure (hfs) Hamiltonian operator), and  $(k_S \hat{S}_z + k_I \hat{I}_z) / \hbar$  (referred to as the Zeeman Hamiltonian operator). SI units are used and  $\mu_0 = 4\pi \times 10^{-7} \text{ N A}^{-2}$  is the permeability of vacuum.

- Calculate the values of  $b_F$ ,  $k_S$ , and  $k_I$  (the latter two as multiples of the field strength  $B_0$ ) for the hydrogen  $1s$  state.
- Now consider an isolated H atom (with no applied magnetic field). Show that the matrix of  $\hat{H}_{\text{hfs}}$  with respect to the  $|m_S m_I\rangle$  basis is

$$\hat{H}_{\text{hfs}} = \frac{b_F}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Find the energies and eigenstates in this basis and construct the matrix  $\mathbf{X}$  that diagonalizes  $\hat{\mathbf{H}}_{\text{hfs}}$ . What will the eigenstates  $|FM_F\rangle$  of  $\hat{\mathbf{H}}_{\text{hfs}}$  be in terms of the  $|m_S m_I\rangle$  states? Give a discussion of this in terms of vector coupling.

- (c) Determine (in terms of  $b_F$ ,  $k_S$ ,  $k_I$ ) the matrices with elements  $\langle m'_S m'_I | \hat{H}_{\text{spin}} | m''_S m''_I \rangle$  and  $\langle F' M'_F | \hat{H}_{\text{spin}} | F'' M''_F \rangle$  in the general case when an applied field  $B_0$  is present.
- (d) From the results of part (c) show how the zero-field  $|FM_F\rangle$  levels split in a weak magnetic field. In this case it is necessary to treat the magnetic field as a perturbation, namely

$$\hat{H}^{(0)} = \frac{b_F}{\hbar^2} \hat{I} \cdot \hat{S}, \quad \hat{H}^{(1)} = \frac{k_S}{\hbar} \hat{S}_z + \frac{k_I}{\hbar} \hat{I}_z.$$

Give a plot of the splitting of these levels as calculated earlier for fields  $B_0$  from 0 to 0.2 T (put your energy scale in MHz).

- (e) Determine the energy levels in a strong magnetic field of 1 T, regarding the hyperfine interaction as a small perturbation, that is,

$$\hat{H}^{(0)} = \frac{k_S}{\hbar} \hat{S}_z + \frac{k_I}{\hbar} \hat{I}_z, \quad \hat{H}^{(1)} = \frac{b_F}{\hbar^2} \hat{I} \cdot \hat{S}.$$

In this case show explicitly that the first-order perturbation spin functions are

$$\begin{aligned} \psi_1^{(1)} &= \phi_1^{(0)}, & \psi_4^{(1)} &= \phi_4^{(0)}, \\ \psi_2^{(1)} &= \phi_2^{(0)} + \frac{b_F}{2(g_e \mu_B B_0 + g_I \mu_N B_0)} \phi_3^{(0)}, \\ \psi_3^{(1)} &= \phi_3^{(0)} - \frac{b_F}{2(g_e \mu_B B_0 + g_I \mu_N B_0)} \phi_2^{(0)}, \end{aligned}$$

while the energies to second order corresponding to these four functions are

$$\begin{aligned} E_1 &= \frac{1}{2} g_e \mu_B B_0 - \frac{1}{2} g_I \mu_N B_0 + \frac{1}{4} b_F, \\ E_2 &= \frac{1}{2} g_e \mu_B B_0 + \frac{1}{2} g_I \mu_N B_0 - \frac{1}{4} b_F + \frac{b_F^2}{4(g_e \mu_B B_0 + g_I \mu_N B_0)}, \\ E_3 &= -\frac{1}{2} g_e \mu_B B_0 - \frac{1}{2} g_I \mu_N B_0 - \frac{1}{4} b_F - \frac{b_F^2}{4(g_e \mu_B B_0 + g_I \mu_N B_0)}, \\ E_4 &= -\frac{1}{2} g_e \mu_B B_0 + \frac{1}{2} g_I \mu_N B_0 + \frac{1}{4} b_F. \end{aligned}$$

The electron spin resonance (ESR) spectrum for hydrogen atoms has only *two* equally intense lines, because the magnetic moment of the proton is too small to contribute to the intensity, and because the mixing of the  $|m_S m_I\rangle$  states in the strong field is small. Show explicitly with numerical results that this is indeed the case for the problem that you are considering. Calculate the splitting of the two ESR lines in MHz, and compare your result with the experimentally observed value of 1420.4 MHz. What is the corresponding wavelength? How could you use this calculation to substantiate the existence of interstellar clouds of atomic hydrogen?

20. Consider the  $2p^1 3d^1$  electron configuration of an atom.
- What terms are possible?
  - Construct wavefunctions for one  $M_L$  component of the total  $L = 2$  states for all possible total  $S$  values.
21. The lifetime of the  $2p$  level of hydrogen is 1.6 ns. Estimate the lifetime of the  $2p$  level of the hydrogenic ion,  $O^{+7}$ .
22. A heated sodium vapor cell of 10 cm length is prepared. When a laser beam is tuned to the center of the  $3^2P_{3/2} - 3^2S_{1/2}$  D line at 5890 Å, the cell transmission is 50%. Assume that the line is Doppler broadened at a temperature  $T$  and ignore hyperfine structure. The vapor pressure of Na is given by the equation  $\log_{10} p = -5652/T - 1.11 \log_{10} T + 8.217$  with  $p$  in bar (1 bar =  $10^5$  Pa) and  $T$  in K. What is the number density of Na atoms and the temperature of the cell?
23. The Lyman  $L_\alpha$  transition ( $2^2P_{3/2} - 1^2S_{1/2}$ ) at 1215.338 Å has a  $gf$  value of 0.5549.
- Calculate the atomic line strength in debye<sup>2</sup>.
  - What are the Einstein  $A$  and  $B$  coefficients (integrated over the lineshape function)?
  - What is the lifetime of the transition?
  - What is the absorption cross section at line center assuming only lifetime broadening?
24. Verify the numerical values in equations (5.141) to (5.145).
25. The Hubble Space Telescope observed interstellar absorption of starlight by O atoms towards the star HD 75309 using the STIS spectrograph. The observed equivalent width of the forbidden oxygen line ( $^5S_2^o - ^3P_2$ ) at 1356 Å was 9.3 mÅ. The equivalent width  $W_\lambda$  is a measure of the area under an absorption line. It is the width that the line would have if the line had 100% absorption, i.e., if the lineshape was a rectangle with  $I/I_0 = 0$  for  $\lambda$  between line center and  $\pm W_\lambda/2$ , and  $I/I_0 = 1$  elsewhere. The definition of equivalent width,  $W_\lambda$  is thus

$$W_\lambda = \int \left(1 - \frac{I}{I_0}\right) d\lambda. \quad (5.169)$$

- Convert the equivalent width,  $W_\lambda$ , in mÅ into SI units ( $W_\nu$  in  $s^{-1}$ ) for the oxygen line.
- For weak absorption (i.e., when  $I/I_0$  is small) derive a relationship between  $W_\nu$  and the oscillator strength,  $f_{\text{abs}}$ .
- If  $f_{\text{abs}} = 1.161 \times 10^{-6}$  for the O transition at 1356 Å, what is the column density (atoms/cm<sup>2</sup>) of interstellar O atoms towards HD 75309 (assuming that the line is only weakly absorbing)?
- In the astronomical literature, the relationship between column density,  $Nl$ , and equivalent width is given in SI units as

$$Nl = \frac{4\epsilon_0 m_e c^2}{e^2} \frac{W_\lambda}{\lambda^2 f_{\text{abs}}}.$$

Show that this equation is correct for a weak line (and assuming no population in the excited state).

26. The potential  $V(r)$  for the hydrogen atom is not given exactly by Coulomb's law because the proton has a finite size. Assume that the nuclear charge of the proton is distributed uniformly in a sphere of radius  $10^{-13}$  cm. Use first order perturbation theory to find the shift in the hydrogen  $1s$  energy due to the finite nuclear size. Assume that the potential energy of the electron inside the nucleus at a distance  $r$  is given by  $V(r) = -eQ/(4\pi\epsilon_0 r)$  with  $Q$  equal to the amount of charge enclosed by the sphere of radius  $r$ . Notice also that  $\psi_{1s}$  has a nearly constant value inside the nucleus.

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# Chapter 6

## Rotational Spectroscopy

### 6.1 Rotation of Rigid Bodies

The classical mechanics of rotational motion of a rigid body remains a relatively mysterious subject compared to that for linear motion. In order to dispel some of the mystery it is useful to note the extensive correspondence between linear motion of a point particle of mass  $m$  and rotational motion of the same particle (Figure 6.1 and Table 6.1). For simplicity the vector natures of most of the quantities are suppressed. The correspondences between the analogous linear and angular quantities in Table 6.1 are quite striking. The linear and angular variables are related by various equations,

$$\begin{aligned}\theta &= \frac{x}{r} \\ \omega &= \frac{v}{r} & (\boldsymbol{\omega} \times \mathbf{r} = \mathbf{v}) \\ a &= \frac{v^2}{r} & (\text{constant } \omega) \\ L &= r p & (\mathbf{L} = \mathbf{r} \times \mathbf{p}) \\ T &= r F & (\mathbf{T} = \mathbf{r} \times \mathbf{F})\end{aligned}$$

in which the full vector forms are listed in parentheses. For a single particle, the angular velocity  $\boldsymbol{\omega}$  and the angular momentum  $\mathbf{L}$  are vectors that point out of the plane of the rotation. In this case, the  $\boldsymbol{\omega}$  and  $\mathbf{L}$  vectors point in the same direction (Figure 6.2). If an extended object is rotating, then  $\mathbf{L}$  and  $\boldsymbol{\omega}$  need not point in the same direction (Figure 6.3). This behavior is represented mathematically by the matrix product

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}, \tag{6.1}$$

where  $\mathbf{I}$  is represented by a symmetric  $3 \times 3$  matrix with equation (6.1) written explicitly as

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \tag{6.2}$$

The matrix  $\mathbf{I}$  is called the *moment of inertia tensor* in classical mechanics.

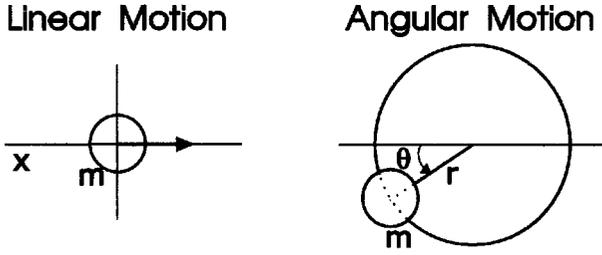


Figure 6.1: Linear and circular motion of a particle of mass  $m$ .

Table 6.1: The Correspondence between Linear and Angular Motion

Linear Motion	“Property”	Angular Motion
Distance, $x$	<i>Position</i>	Angle, $\theta$
Velocity, $v = \dot{x} = dx/dt$	<i>Velocity</i>	Angular velocity, $\omega = \dot{\theta} = d\theta/dt$
Acceleration, $a = \ddot{x} = d^2x/dt^2$	<i>Acceleration</i>	Angular acceleration, $\alpha = \ddot{\theta} = d^2\theta/dt^2$
Mass, $m$	<i>Mass</i>	Moment of inertia, $I = mr^2$
Linear momentum, $p = mv$	<i>Momentum</i>	Angular momentum, $L = I\omega$
$E_k = \frac{1}{2}mv^2 = p^2/2m$	<i>Kinetic energy</i>	$E_k = \frac{1}{2}I\omega^2 = L^2/2I$
Force, $F$	<i>Force</i>	Torque, $T$
$F = ma = dp/dt$	<i>Newton's 2nd law</i>	$T = I\alpha = dL/dt$

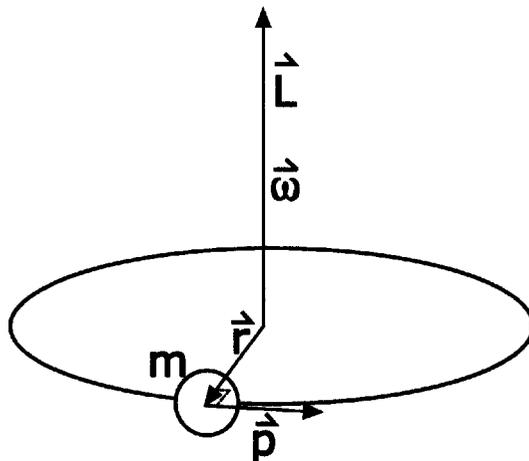


Figure 6.2: The circular motion of a particle of mass  $m$ .

The derivation of the form of the moment of inertia tensor for a collection of nuclei rotating together requires the use of some vector identities and the definition of angular momentum. Consider a collection of nuclei of mass  $m_\alpha$  located at positions  $\mathbf{r}_\alpha$  relative

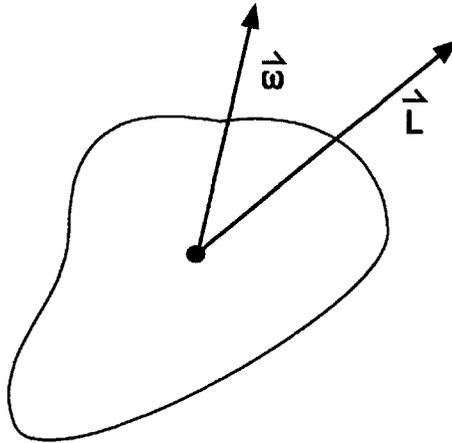


Figure 6.3: For an extended object  $\omega$  and  $L$  can point in different directions.

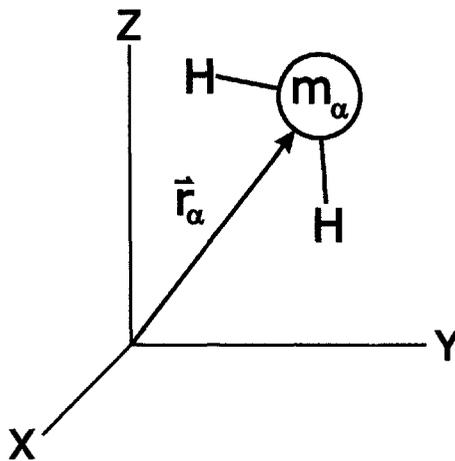


Figure 6.4: A typical molecule with nuclei located by  $r_\alpha$  vectors.

to the origin in a Cartesian coordinate system (Figure 6.4) and all rotating with angular velocity  $\omega$ , so that the angular momentum is given by

$$L = \sum_{\alpha} r_{\alpha} \times p_{\alpha} = \sum_{\alpha} m_{\alpha} r_{\alpha} \times (\omega \times r_{\alpha}), \tag{6.3}$$

in which

$$\omega_{\alpha} = \omega \quad \text{and} \quad p_{\alpha} = m_{\alpha} v = m_{\alpha} \omega \times r.$$

The cross product identity

$$P \times (Q \times R) = Q(P \cdot R) - R(P \cdot Q) \tag{6.4}$$

gives

$$\begin{aligned}
\mathbf{L} &= \sum_{\alpha} m_{\alpha} (\boldsymbol{\omega}(\mathbf{r}_{\alpha} \cdot \mathbf{r}_{\alpha}) - \mathbf{r}_{\alpha}(\mathbf{r}_{\alpha} \cdot \boldsymbol{\omega})) \\
&= \sum_{\alpha} m_{\alpha} (\boldsymbol{\omega}(x_{\alpha}^2 + y_{\alpha}^2 + z_{\alpha}^2) - \mathbf{r}_{\alpha}(x_{\alpha}\omega_x + y_{\alpha}\omega_y + z_{\alpha}\omega_z)). \quad (6.5)
\end{aligned}$$

Writing out the vector components gives

$$\begin{aligned}
\mathbf{L} &= \sum_{\alpha} m_{\alpha} (\omega_x(x_{\alpha}^2 + y_{\alpha}^2 + z_{\alpha}^2)\hat{\mathbf{e}}_1 + \omega_y(x_{\alpha}^2 + y_{\alpha}^2 + z_{\alpha}^2)\hat{\mathbf{e}}_2 + \omega_z(x_{\alpha}^2 + y_{\alpha}^2 + z_{\alpha}^2)\hat{\mathbf{e}}_3 \\
&\quad - x_{\alpha}^2\omega_x\hat{\mathbf{e}}_1 - x_{\alpha}y_{\alpha}\omega_y\hat{\mathbf{e}}_1 - x_{\alpha}z_{\alpha}\omega_z\hat{\mathbf{e}}_1 \\
&\quad - y_{\alpha}x_{\alpha}\omega_x\hat{\mathbf{e}}_2 - y_{\alpha}^2\omega_y\hat{\mathbf{e}}_2 - y_{\alpha}z_{\alpha}\omega_z\hat{\mathbf{e}}_2 \\
&\quad - z_{\alpha}x_{\alpha}\omega_x\hat{\mathbf{e}}_3 - z_{\alpha}y_{\alpha}\omega_y\hat{\mathbf{e}}_3 - z_{\alpha}^2\omega_z\hat{\mathbf{e}}_3), \quad (6.6)
\end{aligned}$$

which can also be expressed in matrix form as

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \sum m_{\alpha}(y_{\alpha}^2 + z_{\alpha}^2) & -\sum m_{\alpha}x_{\alpha}y_{\alpha} & -\sum m_{\alpha}x_{\alpha}z_{\alpha} \\ -\sum m_{\alpha}y_{\alpha}x_{\alpha} & \sum m_{\alpha}(x_{\alpha}^2 + z_{\alpha}^2) & -\sum m_{\alpha}y_{\alpha}z_{\alpha} \\ -\sum m_{\alpha}z_{\alpha}x_{\alpha} & -\sum m_{\alpha}z_{\alpha}y_{\alpha} & \sum m_{\alpha}(x_{\alpha}^2 + y_{\alpha}^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \quad (6.7)$$

Let us now identify the diagonal matrix elements of the matrix  $\mathbf{I}$  as

$$I_{xx} = \sum_{\alpha} m_{\alpha}(y_{\alpha}^2 + z_{\alpha}^2) = \sum_{\alpha} m_{\alpha}r_{x,\perp}^2, \quad (6.8a)$$

$$I_{yy} = \sum_{\alpha} m_{\alpha}(x_{\alpha}^2 + z_{\alpha}^2) = \sum_{\alpha} m_{\alpha}r_{y,\perp}^2, \quad (6.8b)$$

$$I_{zz} = \sum_{\alpha} m_{\alpha}(x_{\alpha}^2 + y_{\alpha}^2) = \sum_{\alpha} m_{\alpha}r_{z,\perp}^2. \quad (6.8c)$$

These elements are referred to as the *moments of inertia*. Similarly, let us identify the nondiagonal matrix elements as

$$I_{xy} = -\sum_{\alpha} m_{\alpha}x_{\alpha}y_{\alpha}, \quad (6.9a)$$

$$I_{xz} = -\sum_{\alpha} m_{\alpha}x_{\alpha}z_{\alpha}, \quad (6.9b)$$

$$I_{yz} = -\sum_{\alpha} m_{\alpha}y_{\alpha}z_{\alpha}. \quad (6.9c)$$

These elements are referred to as *products of inertia*. Notice that the moment of inertia with respect to an axis involves the squares of the perpendicular distances of the masses from that axis, for example,  $r_{x,\perp}^2$  from the  $x$ -axis.

In classical mechanics the motion of a collection of objects can be broken into the center of mass translational motion (see below) and the rotational motion about the center of mass. If a rigid rotor is assumed, the  $3N - 6$  internal vibrations are ignored.

The natural origin for the molecular coordinate system is the center of mass of the molecule.

The location of the center of mass (given by a vector  $\mathbf{R}$ ) for a system of total mass

$$M = \sum_{\alpha} m_{\alpha} \quad (6.10)$$

made up of a collection of particles is given by

$$M\mathbf{R} = \sum_{\alpha} m_{\alpha}\mathbf{r}_{\alpha}. \quad (6.11)$$

If the origin of the coordinate system is at the center of mass, then  $\mathbf{R} = 0$  and

$$\sum_{\alpha} m_{\alpha}\mathbf{r}_{\alpha} = 0. \quad (6.12)$$

The moment of inertia tensor is a real symmetric matrix, so it is always possible to find an orthogonal transformation matrix  $\mathbf{X}$  that transforms the moment of inertia tensor  $\mathbf{I}$  (in equation (6.7)) into diagonal form. The matrix  $\mathbf{X}$  represents a rotation of the coordinate system, which can be written as

$$\mathbf{r}' = \mathbf{X}^{-1}\mathbf{r} \quad \text{or} \quad \mathbf{r} = \mathbf{X}\mathbf{r}'. \quad (6.13)$$

The columns of the matrix  $\mathbf{X}$  are made up of the normalized eigenvectors of  $\mathbf{I}$ .

As discussed in Chapter 3, the diagonalized matrix  $\mathbf{I}'$  is related to the original matrix  $\mathbf{I}$  by a similarity transformation—that is,

$$\mathbf{I}\mathbf{X} = \mathbf{X}\mathbf{I}' \quad (6.14)$$

or

$$\mathbf{X}^{-1}\mathbf{I}\mathbf{X} = \mathbf{I}'. \quad (6.15)$$

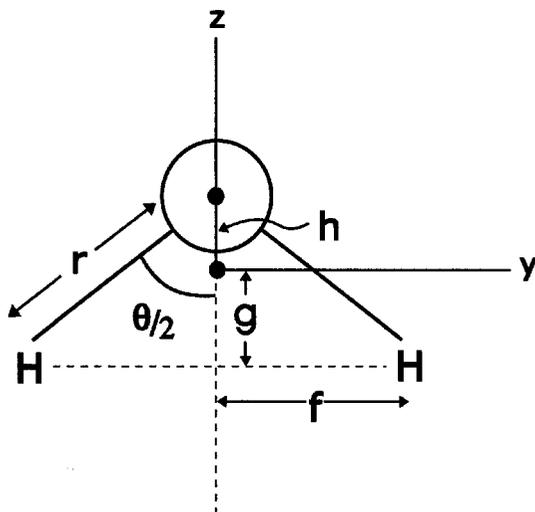
The  $\mathbf{I}'$  matrix is a diagonal matrix whose elements are the eigenvalues of  $\mathbf{I}$ . This new coordinate system is called the *principal axis system* and  $\mathbf{I}'$  has the form

$$\mathbf{I}' = \begin{pmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{pmatrix}. \quad (6.16)$$

In most work the use of the principal axis system is assumed so that the primes will be dropped and  $I_x = I_{x'x'}$ ,  $I_y = I_{y'y'}$ , and  $I_z = I_{z'z'}$ . In the principal axis system we write

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad (6.17)$$

or  $L_x = I_x\omega_x$ ,  $L_y = I_y\omega_y$ , and  $L_z = I_z\omega_z$ . The kinetic energy expression also has the very simple form

Figure 6.5: The H<sub>2</sub>O molecule.

$$\begin{aligned}
 E_k &= T = \frac{1}{2} \boldsymbol{\omega}^t \mathbf{I} \boldsymbol{\omega} \\
 &= \frac{1}{2} (\omega_x \ \omega_y \ \omega_z) \begin{pmatrix} I_x \omega_x \\ I_y \omega_y \\ I_z \omega_z \end{pmatrix} \\
 &= \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2 \\
 &= \frac{L_x^2}{2I_x} + \frac{L_y^2}{2I_y} + \frac{L_z^2}{2I_z}.
 \end{aligned} \tag{6.18}$$

The  $x$ -,  $y$ -, and  $z$ -axes are chosen by some set of geometrical conventions. For example, the  $z$ -axis of a molecule is always chosen to be the highest order axis of rotational symmetry, and the  $x$ -axis is out of the plane for a planar molecule. For example, the moments of inertia for the H<sub>2</sub>O molecule (Figure 6.5) are

$$I_z = 2m_H f^2, \tag{6.19}$$

$$I_y = m_O h^2 + 2m_H g^2, \tag{6.20}$$

and

$$I_x = I_z + I_y = m_O h^2 + 2m_H (g^2 + f^2). \tag{6.21}$$

For any planar molecule the out-of-plane moment of inertia is equal to the sum of the two in-plane moments of inertia. There is another labeling scheme for the axes in a molecule based upon the magnitude of the moments of inertia. In this case, the axes are labeled  $a$ ,  $b$ , and  $c$  with

$$I_A \leq I_B \leq I_C \tag{6.22}$$

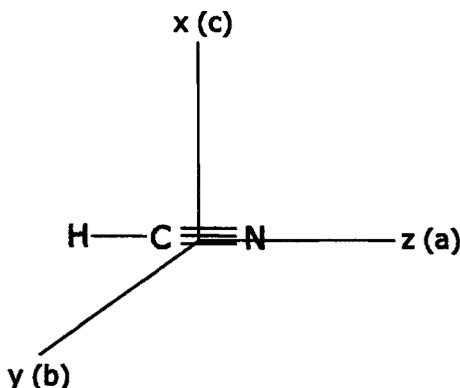


Figure 6.6: Linear molecule.

so that  $I_C$  is always the largest moment of inertia and  $I_A$  is the smallest. The  $a$ -,  $b$ -, and  $c$ -axes are chosen in order to ensure that this inequality holds.

For example, using  $r = 0.958 \text{ \AA}$ ,  $\theta = 104.5^\circ$ ,  $m_H = 1.00$  atomic mass unit (u), and  $m_O = 16.00$  u for  $\text{H}_2\text{O}$  results in  $f = 0.7575 \text{ \AA}$ ,  $g = 0.5213 \text{ \AA}$ , and  $h = 0.0652 \text{ \AA}$  using the center of mass definition (6.12). Thus from equations (6.19), (6.20), and (6.21) we obtain

$$\begin{aligned} I_z &= 1.148 \text{ u \AA}^2 (= I_B) \\ I_y &= 0.6115 \text{ u \AA}^2 (= I_A) \\ I_x &= 1.760 \text{ u \AA}^2 (= I_C) \end{aligned}$$

and  $z = b$ ,  $y = a$ , and  $x = c$ . There are six possible ways that  $(x, y, z)$  can be mapped into  $(a, b, c)$  depending on the particular values of the moments of inertia. The  $x$ -,  $y$ -,  $z$ -axes are picked by a customary set of rules, such as  $z$  is along the highest axis of rotational symmetry, but  $a$ ,  $b$ , and  $c$  are chosen to make equation (6.22) true.

Molecules can be classified on the basis of the values of the three moments of inertia. The five cases are as follows:

1. Linear molecules,  $I_B = I_C, I_A = 0$ ; for example, HCN (Figure 6.6).
2. Spherical tops,  $I_A = I_B = I_C$ ; for example,  $\text{SF}_6$  and  $\text{CH}_4$  (Figure 6.7).
3. Prolate symmetric tops,  $I_A < I_B = I_C$ ; for example,  $\text{CH}_3\text{Cl}$  (Figure 6.8).
4. Oblate symmetric tops,  $I_A = I_B < I_C$ ; for example,  $\text{BF}_3$  (Figure 6.9).
5. Asymmetric tops,  $I_A < I_B < I_C$ ; for example,  $\text{H}_2\text{O}$  (Figure 6.10).

Group theory can be used to classify the rotational properties of molecules. The spherical tops ( $O_h, T_d$ , and  $I_h$  point groups) and linear molecules ( $C_{\infty v}$  and  $D_{\infty h}$ ) are readily recognized. All symmetric tops have a  $C_n$ -axis, with  $n$  greater than 2. For example, the symmetric tops  $\text{CH}_3\text{Cl}$  and benzene have  $C_3$ - and  $C_6$ -axes, while the asymmetric top  $\text{H}_2\text{O}$  has only a  $C_2$ -axis. But what about allene (Figure 6.11)? By symmetry allene has  $I_B = I_C$ , and hence it must be a prolate symmetric top. Allene

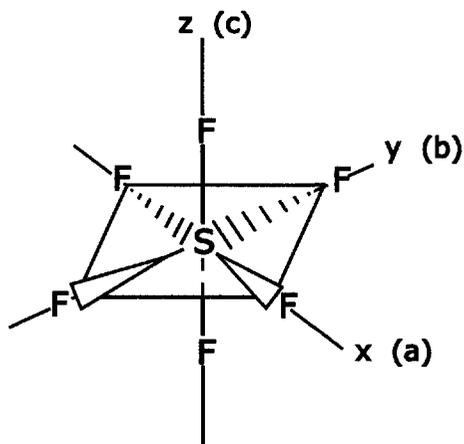


Figure 6.7: Spherical top.

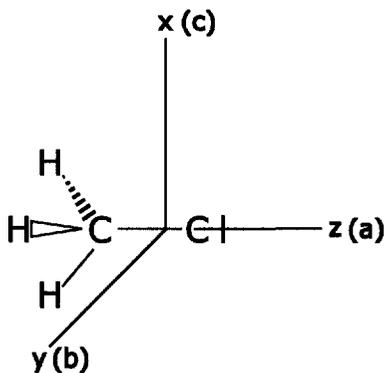


Figure 6.8: Prolate symmetric top.

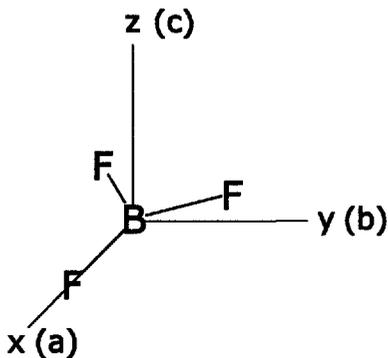


Figure 6.9: Oblate symmetric top.

has only a  $C_2$ -axis, but it does have an  $S_4$ -axis. The complete rule is, therefore, all molecules with a  $C_n$ -( $n > 2$ ) or an  $S_4$ -axis are symmetric tops. Note that the presence

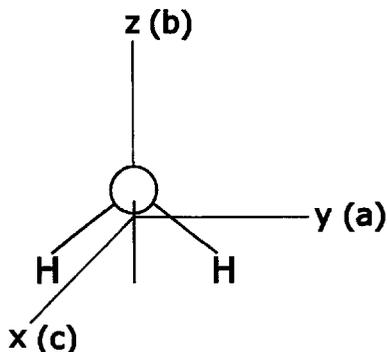


Figure 6.10: Asymmetric top.

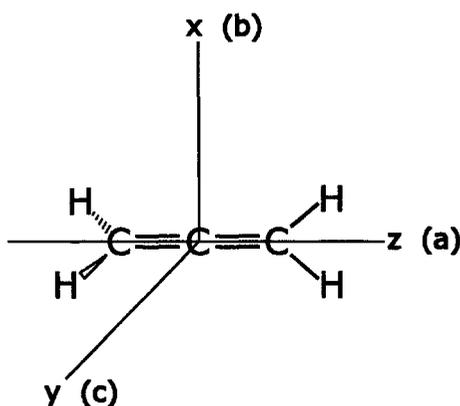


Figure 6.11: Allene, a symmetric top.

of an  $S_n$ -axis with  $n > 4$  implies the presence of a  $C_n$ -axis,  $n > 2$ , so this case need not be explicitly stated.

The symmetry properties of a molecule are also helpful in locating the principal axes. For example, if there is a  $C_n$ -axis with  $n > 1$ , then one of the principal axes lies along it (e.g.,  $\text{H}_2\text{O}$ ). Any molecule with a plane of symmetry has one of the principal inertial axes perpendicular to the plane (e.g.,  $\text{H}_2\text{O}$ ).

## 6.2 Diatomic and Linear Molecules

For a rigid linear molecule with no net orbital and spin angular momentum the classical expression for the rotational kinetic energy is, from equation (6.18),

$$\begin{aligned}
 E_k &= T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 \\
 &= \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 \\
 &= \frac{J_x^2}{2I} + \frac{J_y^2}{2I} = \frac{J^2}{2I}
 \end{aligned}
 \tag{6.23}$$

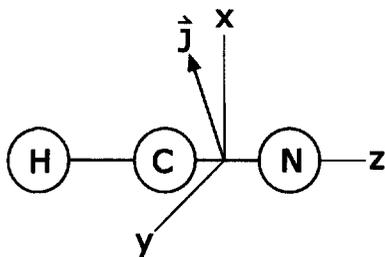


Figure 6.12: Rotational angular momentum in a linear molecule.

since  $I_z = 0$ ,  $I_x = I_y = I$  for a linear molecule, and the customary symbol  $J$  is used to represent the total angular momentum (exclusive of nuclear spin) (Figure 6.12). For a rigid rotor in isotropic (field-free) space the rotational Hamiltonian operator for a linear molecule is

$$\hat{H} = \frac{\hat{J}^2}{2I}. \quad (6.24)$$

The Schrödinger equation can be solved immediately, since  $\psi$  must be one of the spherical harmonics,  $\psi_{LM} = Y_{JM}$ . The specific Schrödinger equation for this case is

$$\frac{\hat{J}^2\psi}{2I} = E\psi, \quad (6.25)$$

so that

$$\frac{\hat{J}^2\psi}{2I} = \frac{J(J+1)\hbar^2\psi}{2I} = BJ(J+1)\psi. \quad (6.26)$$

Thus we see that the energy eigenvalue  $F(J)$  is

$$F(J) = BJ(J+1), \quad (6.27)$$

in which

$$B = \frac{\hbar^2}{2I} = \frac{h^2}{8\pi^2I}, \quad (6.28)$$

with  $B$  in the SI units of joules. In spectroscopy it is customary to use  $F(J)$  to represent the rotational energy-level expression and the value of  $B$  is usually given in MHz or  $\text{cm}^{-1}$  rather than in joules. Since  $E = h\nu = hc/\lambda = 10^2hc\tilde{\nu}$ , the value of  $B$  in Hz is

$$B[\text{Hz}] = \frac{h}{8\pi^2I}, \quad (6.29a)$$

or in MHz,

$$B[\text{MHz}] = \frac{h}{8\pi^2I} \times 10^{-6}, \quad (6.29b)$$

or in  $\text{cm}^{-1}$ ,

$$B[\text{cm}^{-1}] = \frac{h}{8\pi^2 c I} \times 10^{-2}. \quad (6.30)$$

Convenient explicit expressions for  $B$  are

$$B/\text{cm}^{-1} = \frac{16.857\,629\,08}{I/(\text{u } \text{Å}^2)}, \quad (6.31)$$

and

$$B/\text{MHz} = \frac{505\,379.006}{I/(\text{u } \text{Å}^2)}. \quad (6.32)$$

For a diatomic molecule A—B we have

$$I = \mu r^2, \quad (6.33)$$

with  $\mu$  the reduced mass

$$\mu = \frac{m_A m_B}{m_A + m_B}. \quad (6.34)$$

The use of a single symbol  $B$  for the rotational constant to represent a number which may be in units of joules, MHz, or  $\text{cm}^{-1}$  is an unfortunate but common practice. This convention will nonetheless be followed in this book.

### Selection Rules

The intensity of a pure rotational transition is determined by the transition dipole moment

$$\mathbf{M} = \int \psi_{J'M'} \boldsymbol{\mu} \psi_{JM} d\tau. \quad (6.35)$$

For a linear molecule the wavefunction  $\psi_{JM}$  can be written explicitly as

$$\psi_{JM} = Y_{JM}(\theta, \phi) = \Theta_{JM}(\theta) e^{iM\phi} / \sqrt{2\pi} \quad (6.36)$$

and the dipole moment is oriented along the internuclear axis of the molecule, so that its components in the laboratory axis system can be expressed in the form

$$\begin{aligned} \boldsymbol{\mu} &= \mu_x \hat{\mathbf{e}}_1 + \mu_y \hat{\mathbf{e}}_2 + \mu_z \hat{\mathbf{e}}_3 \\ &= \mu_0 (\sin \theta \cos \phi \hat{\mathbf{e}}_1 + \sin \theta \sin \phi \hat{\mathbf{e}}_2 + \cos \theta \hat{\mathbf{e}}_3), \end{aligned} \quad (6.37)$$

so that equation (6.35) becomes

$$\begin{aligned} \mathbf{M} &= \frac{\mu_0}{2\pi} \left( \hat{\mathbf{e}}_1 \int_0^{2\pi} \int_0^\pi \Theta_{J'M'} e^{-iM'\phi} \sin \theta \cos \phi \Theta_{JM} e^{iM\phi} \sin \theta d\theta d\phi \right. \\ &\quad + \hat{\mathbf{e}}_2 \int_0^{2\pi} \int_0^\pi \Theta_{J'M'} e^{-iM'\phi} \sin \theta \sin \phi \Theta_{JM} e^{iM\phi} \sin \theta d\theta d\phi \\ &\quad \left. + \hat{\mathbf{e}}_3 \int_0^{2\pi} \int_0^\pi \Theta_{J'M'} e^{-iM'\phi} \cos \theta \Theta_{JM} e^{iM\phi} \sin \theta d\theta d\phi \right). \end{aligned} \quad (6.38)$$

If we now employ the relationships  $\cos \phi = (e^{i\phi} + e^{-i\phi})/2$ ,  $\sin \phi = (e^{i\phi} - e^{-i\phi})/2i$  and a recursion relationship for the associated Legendre polynomials, namely

$$(2l + 1)zP_l^m(z) = (l + m)P_{l-1}^m(z) + (l - m + 1)P_{l+1}^m(z), \quad (6.39)$$

in which  $z = \cos \theta$  and  $\Theta_{JM}(\theta) = NP_l^m(\cos \theta)$ , the selection rules  $\Delta M = 0, \pm 1$  and  $\Delta J = \pm 1$  are obtained.

In addition, if the molecule has no permanent dipole moment ( $\mu_0 = 0$ ), then there are no allowed transitions. Thus, symmetric molecules  $\text{O}=\text{C}=\text{O}$ ,  $\text{Cl}-\text{Cl}$ ,  $\text{H}-\text{C}\equiv\text{C}-\text{H}$  have no pure rotational transitions, if only one-photon electric-dipole selection rules are considered. Molecules such as  $\text{O}_2$  ( $X^3\Sigma_g^-$ ) undergo weakly allowed magnetic-dipole pure rotational transitions. Molecules such as  $\text{H}-\text{C}\equiv\text{C}-\text{D}$  or  $\text{H}-\text{D}$ , for which the center of mass is displaced from the center of charge when the molecule is vibrating, possess a small dipole moment ( $8 \times 10^{-4}$  D for  $\text{HD}^1$ ) and also undergo weak rotational transitions.

The above derivation of selection rules has also assumed that there is no additional vibrational, orbital, or spin angular momentum present (i.e.,  $\Sigma$  vibronic states are assumed). If there is additional angular momentum, then  $Q$  branch ( $\Delta J = 0$ ) rotational transitions are possible, such as for a  $\Pi$  vibrational or electronic state, in which case the  $Q$  transitions are between the two nearly degenerate levels with the same  $J$  value but opposite total parity. The energy levels of  $\Pi$  vibrational states of linear molecules are considered later in this chapter.

The selection rule  $\Delta J = \pm 1$  for a linear molecule results in transitions with frequencies

$$\begin{aligned} \nu_{J+1 \leftarrow J} &= F(J') - F(J'') \\ &= B(J+1)(J+2) - BJ(J+1) \\ &= 2B(J+1). \end{aligned} \quad (6.40)$$

Customarily, transitions are written with the upper state, indicated by primes ( $J'$ ), first and the lower state, indicated by double primes ( $J''$ ), second with an arrow to indicate absorption  $J' \leftarrow J''$  or emission  $J' \rightarrow J''$ . The first transition  $J = 1 \leftarrow 0$  occurs at  $2B$ , and the other transitions are spaced by multiples of  $2B$  from one another (Figure 6.13). This is illustrated by the pure rotational transitions of hot HF (Figure 6.14) and the far-infrared absorption spectrum of CO (Figure 6.15).

The intensity of a rotational transition is determined both by the dipole moment and the population difference between the two levels (Chapter 1). The rotational populations can be calculated from statistical thermodynamics. If the total concentration of molecules is  $N$ , then the concentration of molecules  $N_J$  with the rotational quantum number  $J$  is

$$N_J = N(2J + 1) \frac{e^{-BJ(J+1)/kT}}{q_r} = NP_J, \quad (6.41)$$

where  $q_r$  is the rotational partition function

$$q_r = \sum_J (2J + 1) e^{-BJ(J+1)/kT} \approx \frac{kT}{\sigma B}, \quad (6.42)$$

with  $\sigma$ , the symmetry number, equal to 2 or 1 for a symmetric or nonsymmetric molecule, respectively. The expression (6.41) assumes that only the ground vibrational and

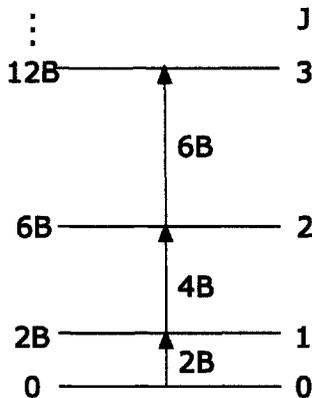
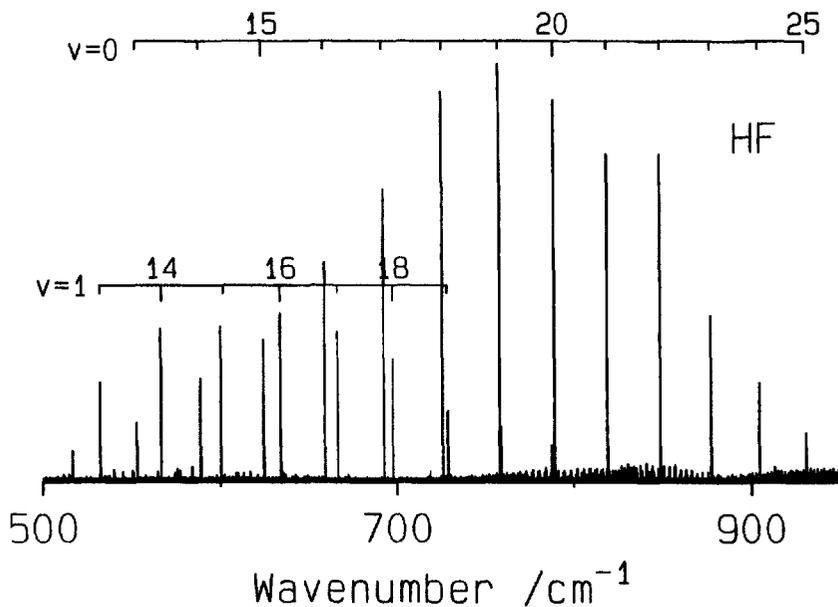


Figure 6.13: Transitions of a linear molecule.

Figure 6.14: Pure rotational emission of hot HF molecules. The spectrum also contains weaker lines due to H<sub>2</sub>O and LiF molecules.

electronic states are populated at temperature  $T$ . This distribution is plotted in Figure 6.16 for CO ( $B = 1.9225 \text{ cm}^{-1}$ ) at room temperature (298 K). The rotational state with maximum population  $J_{\text{max}}$  is determined by setting  $dN_J/dJ = 0$  and solving for  $J$ . This gives

$$J_{\text{max}} = \left( \frac{kT}{2B} \right)^{1/2} - \frac{1}{2}. \quad (6.43)$$

For CO at room temperature the state with maximum population has a  $J$  value of 7.

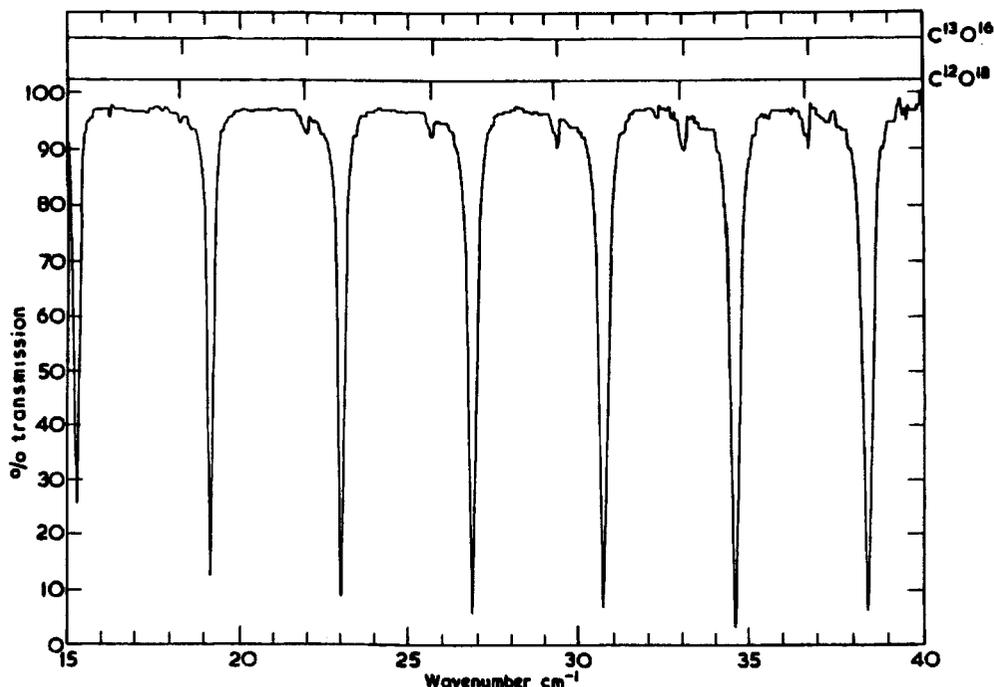


Figure 6.15: Far-infrared absorption spectrum of CO showing transitions  $J = 4 \leftarrow 3$  at  $15.38 \text{ cm}^{-1}$  to  $J = 10 \leftarrow 9$  at  $38.41 \text{ cm}^{-1}$ .

### Centrifugal Distortion

A molecule is not strictly a rigid rotor. As a molecule rotates, the atoms experience a centrifugal force in the rotating molecular reference frame that distorts the internuclear positions (Figure 6.17). For a diatomic molecule one can obtain an expression for the stretching of the internuclear separation  $r$  by allowing the bond to stretch from  $r_e$  to  $r_c$  under the action of the centrifugal force

$$F_c = \frac{\mu v^2}{r} = \mu \omega^2 r = \frac{J^2}{\mu r^3}. \quad (6.44)$$

The centrifugal force is balanced by the Hooke's law restoring force

$$F_r = k(r_e - r_c) \quad (6.45)$$

in the bond, and after some algebra (Problem 14) one finds that

$$F(J) = BJ(J+1) - D(J(J+1))^2 = (B - DJ(J+1))J(J+1). \quad (6.46)$$

The constant  $D$  is called the *centrifugal distortion constant* and, in fact, there are additional higher-order distortion corrections that lead to the rotational energy expression

$$F(J) = BJ(J+1) - D(J(J+1))^2 + H(J(J+1))^3 + L(J(J+1))^4 + M(J(J+1))^5 + \dots \quad (6.47)$$

A useful expression for  $D$  is given by the Kratzer relationship (Problem 14)

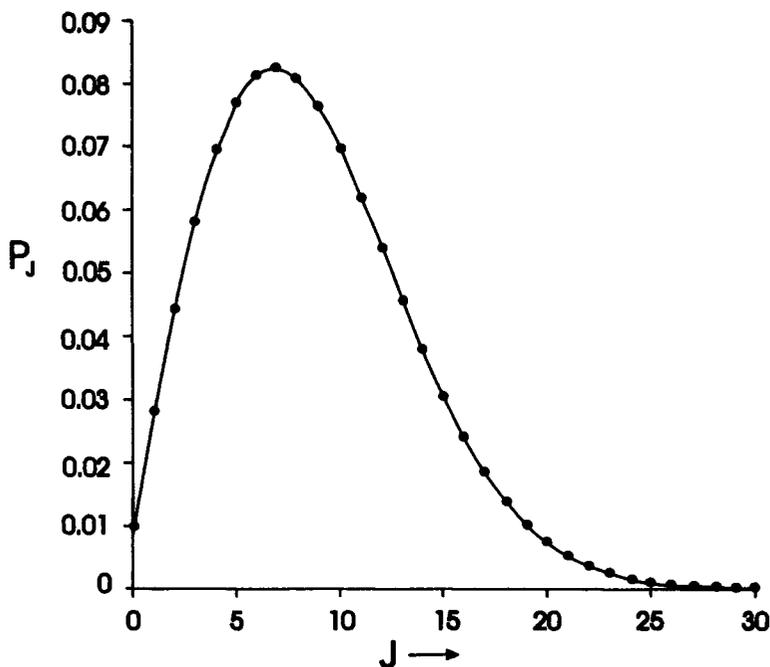


Figure 6.16: Distribution of population among rotational states of CO at room temperature.

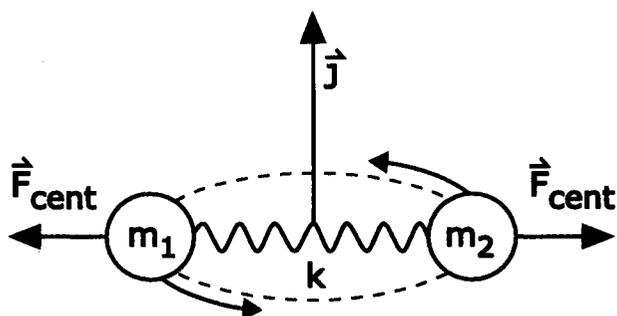


Figure 6.17: Nonrigid diatomic rotor with  $m_1$  and  $m_2$  connected by a spring.

$$D = \frac{4B_e^3}{\omega_e^2}, \quad (6.48)$$

in which  $\omega_e$  is the equilibrium vibration frequency. The negative sign in front of  $D$  in (6.46) and (6.47) has been introduced in order to make  $D$  a positive number. Equation (6.47) applies to both diatomic and linear polyatomic molecules.

Centrifugal distortion increases the internuclear separation  $r$ , which decreases the effective rotation constant " $B_{\text{eff}} = B - DJ(J + 1)$ " of a pure rotational transition, so that the transition frequency can be written as

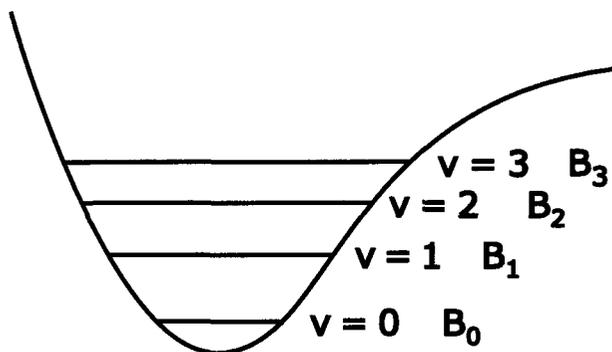


Figure 6.18: Each vibrational level of a diatomic molecule has its own rotational constant  $B_v$ .

$$\nu_{J+1 \leftarrow J} = F(J+1) - F(J) = 2B(J+1) - 4D(J+1)^3 = 2(B - 2D(J+1)^2)(J+1). \quad (6.49)$$

For example, the values of  $B$  and  $D$  for CO in the vibrational ground state are  $B(v=0) = 57.635\,968\,3$  GHz,  $D(v=0) = 0.183\,505\,5$  MHz,  $H(v=0) = 1.725 \times 10^{-7}$  MHz, and  $L(v=0) = 3.1 \times 10^{-13}$  MHz.<sup>2</sup>

The rotational constant also depends on the vibrational and electronic state (Figure 6.18). For a diatomic molecule, as  $v$  increases the molecule spends more of its time at large  $r$  where the potential energy curve is flatter (Figure 7.5). Thus, the average internuclear separation  $\langle r \rangle$  increases with  $v$  while

$$B_v = \frac{\hbar^2}{8\pi^2\mu} \left\langle \frac{1}{r^2} \right\rangle \quad (6.50)$$

decreases. This vibrational dependence is customarily parameterized<sup>3</sup> by the equations

$$B_v = B_e - \alpha_e(v + \frac{1}{2}) + \gamma_e(v + \frac{1}{2})^2 + \dots \quad (6.51)$$

and

$$D_v = D_e + \beta_e(v + \frac{1}{2}) + \dots \quad (6.52)$$

The rotational energy level expression also becomes dependent on  $v$ , namely

$$F_v(J) = B_v J(J+1) - D_v (J(J+1))^2 + \dots \quad (6.53)$$

At room temperature the pure rotational spectrum of a small molecule will not usually display the effects of vibration because the excited vibrational energy levels have little population. For a more floppy molecule with low-frequency vibrations, “vibrational satellites” appear in the pure rotational spectrum (Figure 6.19) since each vibrational level has its own set of rotational constants. Including the effects of centrifugal distortion and the vibrational dependence of the rotational constants results in transition frequencies given by

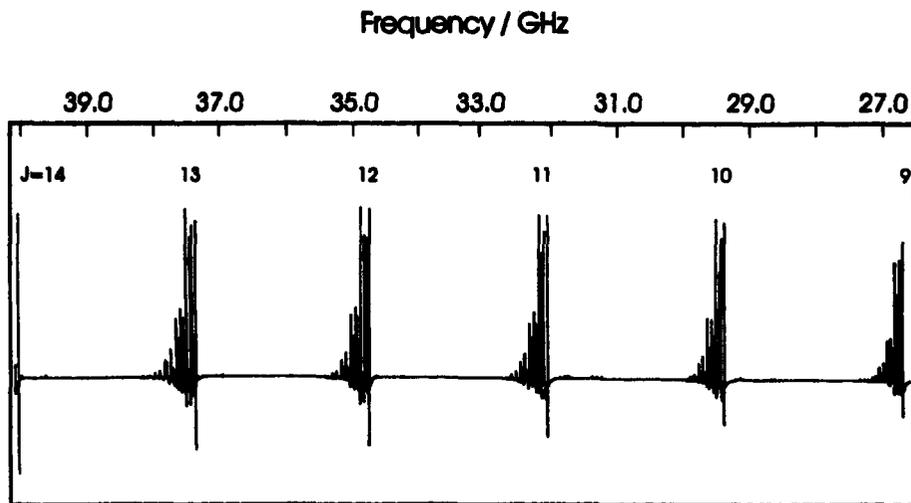


Figure 6.19: The microwave spectrum of  $\text{H}-\text{C}\equiv\text{C}-\text{C}\equiv\text{C}-\text{C}\equiv\text{N}$  showing vibrational satellites.

$$\begin{aligned}
 \nu_{J+1 \leftarrow J} &= B_v(J+1)(J+2) - D_v((J+1)(J+2))^2 \\
 &\quad - B_v J(J+1) + D_v(J(J+1))^2 \\
 &= 2B_v(J+1) - 4D_v(J+1)^3.
 \end{aligned} \tag{6.54}$$

### Vibrational Angular Momentum

The total angular momentum  $\hat{\mathbf{J}}$  in a linear molecule is given by

$$\hat{\mathbf{J}} = \hat{\mathbf{R}} + \hat{\mathbf{L}} + \hat{\mathbf{S}} + \hat{\mathbf{I}} \tag{6.55}$$

in which  $\hat{\mathbf{R}}$ ,  $\hat{\mathbf{L}}$ ,  $\hat{\mathbf{S}}$ , and  $\hat{\mathbf{I}}$  are the rotational, electronic orbital, spin, and vibrational angular momenta, respectively. In spectroscopy it is customary to associate different standard symbols<sup>4,5</sup> with different types of angular momenta. Most common molecules ( $\text{O}_2(X^3\Sigma_g^-)$  and  $\text{NO}(X^2\Pi)$  are exceptions) have no unpaired spins or electronic orbital angular momenta ( $\hat{\mathbf{L}} = \hat{\mathbf{S}} = 0$ ) and only  $\hat{\mathbf{I}}$  needs to be considered in addition to  $\hat{\mathbf{R}}$  for linear polyatomic molecules. In recent years the sensitivity of pure rotational spectroscopy has improved, particularly with the development of submillimeter wave technology, so that microwave spectroscopy of free radicals and ions,<sup>6</sup> often with  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  not equal to zero, is now an important area of research. However, it is beyond the scope of this book.

Vibrationally-excited linear polyatomic molecules can display the effects of vibrational angular momentum. A molecule like  $\text{H}-\text{C}\equiv\text{N}$  or  $\text{H}-\text{C}\equiv\text{C}-\text{Cl}$  has doubly degenerate bending modes, since the molecule could bend in plane or out of plane (Figure 6.20). For example,  $\text{HCN}$  has  $3N - 6 = 4$  vibrational modes with

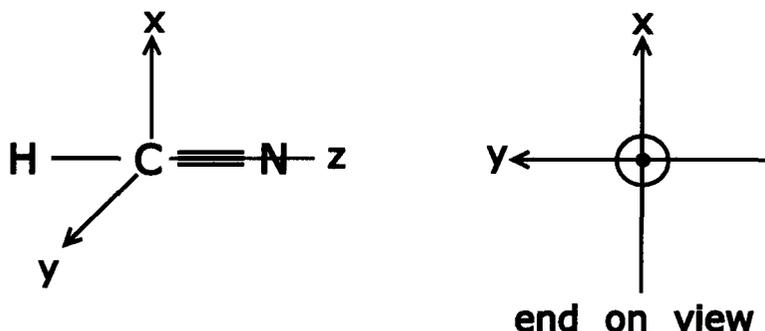


Figure 6.20: The doubly degenerate bending mode of a linear molecule.

$$\nu_1(\sigma^+) \quad 3311 \text{ cm}^{-1} \quad \text{for the H—C stretching mode,}$$

$$\nu_2(\pi) \quad 713 \text{ cm}^{-1} \quad \text{for the H—C}\equiv\text{N bending mode,}$$

$$\nu_3(\sigma^+) \quad 2097 \text{ cm}^{-1} \quad \text{for the C}\equiv\text{N stretching mode,}$$

with  $\nu_2$  being a doubly degenerate bending mode.

The degenerate bending mode  $\nu_2$  is modeled by a two-dimensional harmonic oscillator<sup>7</sup> with a Hamiltonian operator given by

$$\hat{H} = \frac{-\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2}k(x^2 + y^2), \quad (6.56)$$

in which  $\mu$  and  $k$  are the effective mass and force constant, respectively. The  $x$  and  $y$  parts are separable, so the Schrödinger equation is solved by writing the wavefunctions as

$$\psi(x, y) = \psi_{\text{HO}}(x)\psi_{\text{HO}}(y) \quad (6.57)$$

and splitting the total energy into two parts as

$$\begin{aligned} E &= h\nu(v_x + \frac{1}{2}) + h\nu(v_y + \frac{1}{2}) \\ &= h\nu(v + 1), \quad v = 0, 1, 2, \dots, \end{aligned} \quad (6.58)$$

with  $v = v_x + v_y$ , and each level  $v$  has a degeneracy of  $v + 1$ . In general for the  $d$ -dimensional harmonic oscillator

$$E = h\nu(v + \frac{d}{2}), \quad v = 0, 1, 2, \dots, \quad (6.59)$$

with  $d = 1, 2, 3, \dots$ , depending on the number of degenerate oscillators, each contributing  $h\nu/2$  of zero-point energy.

The two-dimensional harmonic oscillator Hamiltonian operator<sup>7</sup> can be converted to plane polar coordinates in which  $\rho = (x^2 + y^2)^{1/2}$  and  $\phi = \tan^{-1}(y/x)$  (Figure 6.21). The Hamiltonian operator becomes

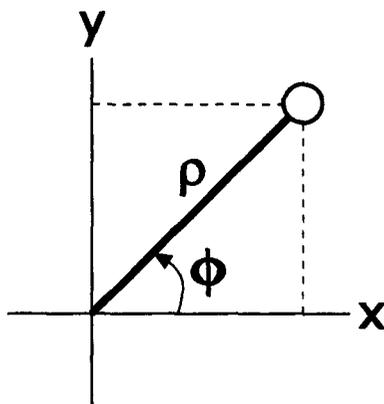


Figure 6.21: Plane polar coordinates.

$$\hat{H} = \frac{-\hbar^2}{2\mu} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) + \frac{1}{2} k \rho^2. \quad (6.60)$$

In this coordinate system the problem is also separable and results in a wavefunction

$$\psi_{vl} = R_{vl}(\rho) e^{il\phi}, \quad (6.61)$$

in which  $l$  is a new quantum number associated with vibrational angular momentum of  $\pm |l|\hbar$ . The operator for vibrational angular momentum about  $z$  is

$$\hat{p}_z = -i\hbar \frac{\partial}{\partial \phi} \quad (6.62)$$

since

$$\hat{p}_z \psi_{vl} = l\hbar \psi_{vl}. \quad (6.63)$$

The possible values<sup>7</sup> of  $|l|$  are  $v, v-2, \dots, 0$  or  $1$ .

Following the usual custom in spectroscopy, a single positive value of  $|l|$  is used although  $\pm |l|$  are possible. The double degeneracy for each value of  $l$  is associated with clockwise or counterclockwise motion of the nuclei in a linear molecule (Figure 6.22). As before, the total degeneracy for the level  $v$  is  $v+1$ . Classically, the two oscillators in the  $x$  and  $y$  directions can be phased such that the nuclei execute circular motion of small amplitude about the  $z$ -axis. In quantum mechanics this motion is quantized and only  $\pm l\hbar$  units of angular momentum are possible. Sometimes Greek letters are used to designate vibrational angular momentum (in analogy to the use of  $\Sigma, \Pi, \Delta$ , and so forth, to represent  $\Lambda = 0, 1, 2, \dots$  for the component of the orbital angular momentum about the internuclear axis of a diatomic molecule, see Chapter 9) and  $l$  is often written as a superscript,  $v_2^l$  (Figure 6.23).

Although the different  $l$  values for a given  $v$  are degenerate for the two-dimensional harmonic oscillator, they become split if the oscillator is anharmonic. Since real molecules are always anharmonic oscillators, the different  $|l|$  values are split by typically a few  $\text{cm}^{-1}$ . The twofold degeneracy for each  $l$  value ( $\pm |l|$ ) remains in the nonrotating molecule (Figure 6.23).

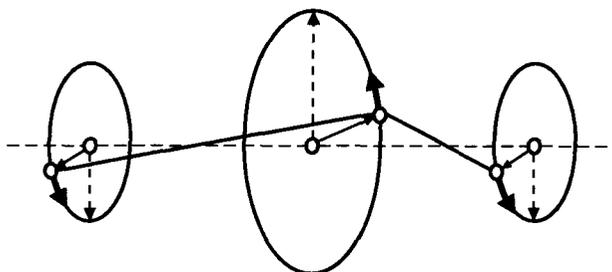


Figure 6.22: Classical picture of vibrational angular momentum.

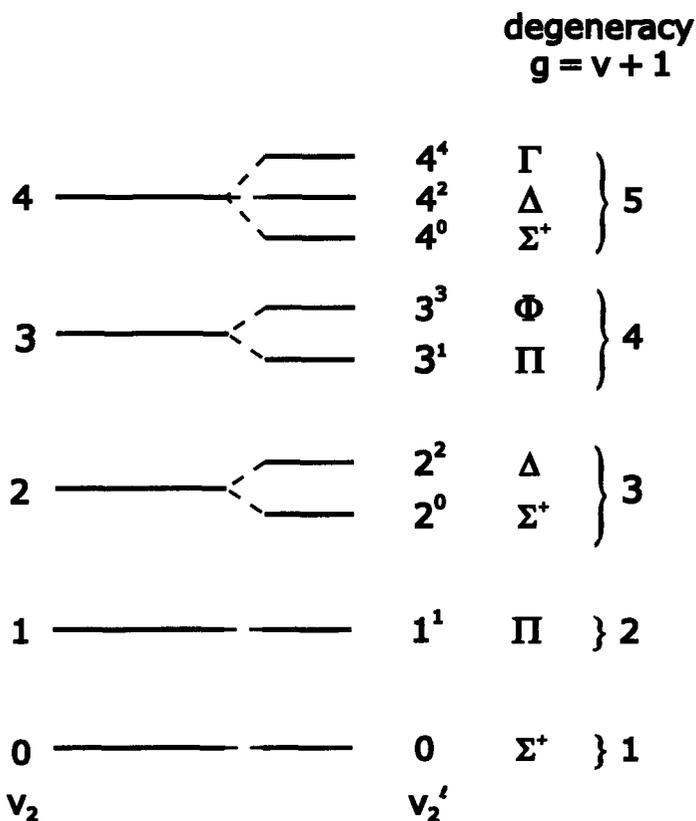


Figure 6.23: Vibrational energy-level pattern for the bending mode of a linear molecule.

When only vibrational and rotational angular momentum (Figure 6.24) are present, we have

$$\hat{\mathbf{J}} = \hat{\mathbf{R}} + \hat{\mathbf{I}}. \quad (6.64)$$

The possible values of the quantum number  $J$  are  $|l|, |l| + 1, \dots$ , since a vector cannot be shorter than its projection on the  $z$ -axis (Figure 6.25).

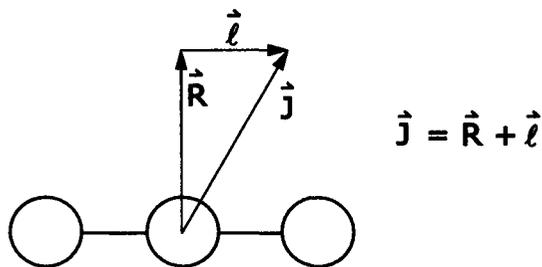
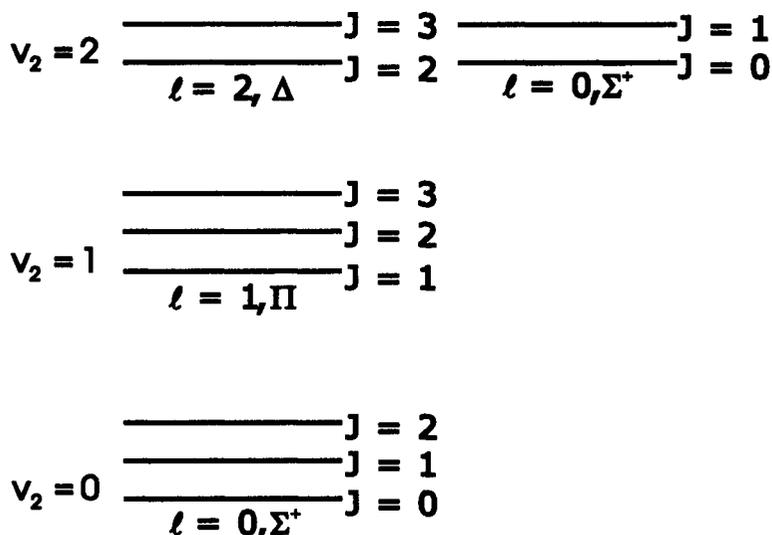
Figure 6.24: The rotational  $\hat{R}$  and vibrational  $\hat{l}$  angular momenta couple to give  $\hat{J}$ .

Figure 6.25: Rotational structure (not to scale) of the first few bending vibrational and rotational energy levels of a linear triatomic molecule.

There is a different rotational constant for each vibrational level, which is customarily expressed as

$$B_v = B_e - \sum \alpha_i \left( v_i + \frac{d_i}{2} \right), \quad (6.65)$$

with  $d_i$  the degeneracy of the  $i$ th mode. For example, the vibrational dependence of the rotational constant for  $\text{BeF}_2$  is<sup>8</sup>

$$B_{v_1 v_2 v_3} = 0.235\,356 - 0.000\,794(v_1 + \frac{1}{2}) + 0.001\,254(v_2 + 1) - 0.002\,446(v_3 + \frac{1}{2})\text{cm}^{-1}.$$

This  $B_e$  value gives an  $r_e = 1.374\,971 \text{ \AA}$  for the Be–F bond length while the  $B_{000}$  value of  $0.234\,990 \text{ cm}^{-1}$  gives an  $r_0$  value of  $1.374\,042 \text{ \AA}$  (see section 6.6).

### 6.3 Line Intensities for Diatomic and Linear Molecules

In quantitative applications of spectroscopy for astronomy, remote sensing, or analytical chemistry, one seeks to determine the amount of material by application of the equations governing absorption or emission of radiation presented in Chapters 1 and 5. For the case of pure rotational emission of diatomic and linear molecules, the value of  $\mu_{10}$  in equation (1.53) for the Einstein  $A$  coefficient needs to be found. For the transition  $|J'M'\rangle \leftrightarrow |J''M''\rangle$  between two quantum states, the transition dipole moment (6.35) is

$$\mathbf{M} = \langle J'M' | \boldsymbol{\mu} | J''M'' \rangle \quad (6.66)$$

or equation (6.38) in spherical polar coordinates. The integrals in (6.38) for the  $x$ ,  $y$ , and  $z$  ( $\hat{\mathbf{e}}_1$ ,  $\hat{\mathbf{e}}_2$ , and  $\hat{\mathbf{e}}_3$ ) directions will be evaluated separately to obtain an explicit expression for  $\mathbf{M}$ .

For the  $z$  component, the selection rule on  $M$  is  $\Delta M = 0$  and the  $\phi$  part of the integral is  $2\pi$ , leaving only the  $\theta$  part to be evaluated as

$$M_z = \mu_0 \int_0^\pi \Theta_{J'M} \cos \theta \Theta_{J''M} \sin \theta d\theta. \quad (6.67)$$

From the definition of the spherical harmonics,  $Y_{JM}$ , and the associated Legendre functions,  $P_J^M$ , in Table 5.1, one finds

$$\Theta_{JM} = (-1)^M \left( \frac{(2J+1)(J-M)!}{2(J+M)!} \right)^{1/2} P_J^M(\cos \theta) \quad (6.68)$$

and the recursion relationship (6.39) becomes

$$(2J+1) \cos \theta P_J^M(\cos \theta) = (J+M)P_{J-1}^M + (J-M+1)P_{J+1}^M \quad (6.69)$$

or

$$\cos \theta \Theta_{JM} = \left( \frac{J^2 - M^2}{(2J+1)(2J-1)} \right)^{1/2} \Theta_{J-1,M} + \left( \frac{(J+1)^2 - M^2}{(2J+1)(2J+3)} \right)^{1/2} \Theta_{J+1,M}. \quad (6.70)$$

Substituting equation (6.70) into equation (6.67) yields

$$\begin{aligned} M_z = & \mu_0 \left( \left( \frac{(J')^2 - M^2}{(2J'+1)(2J'-1)} \right)^{1/2} \int_0^\pi \Theta_{J'-1,M} \Theta_{J'',M} \sin \theta d\theta \right. \\ & \left. + \left( \frac{(J'+1)^2 - M^2}{(2J'+1)(2J'+3)} \right)^{1/2} \int_0^\pi \Theta_{J'+1,M} \Theta_{J'',M} \sin \theta d\theta \right) \quad (6.71) \end{aligned}$$

For the case of  $|J+1, M\rangle \leftarrow |J, M\rangle$ ,  $\Delta J = +1$  or  $J' - 1 = J''$  the integral becomes

$$M_z = \mu_0 \left( \frac{(J')^2 - M^2}{(2J'+1)(2J'-1)} \right)^{1/2} = \mu_0 \left( \frac{(J''+1) - M^2}{(2J''+3)(2J''+1)} \right)^{1/2} \quad (6.72)$$

and for the  $|J-1, M\rangle \leftarrow |J, M\rangle$ ,  $\Delta J = -1$  transition,

$$M_z = \mu_0 \left( \frac{(J' + 1)^2 - M^2}{(2J' + 1)(2J' + 3)} \right)^{1/2} = \mu_0 \left( \frac{(J'')^2 - M^2}{(2J'' - 1)(2J'' + 1)} \right)^{1/2} \quad (6.73)$$

Similarly using equation (5.108) yields

$$M_x = -iM_y = \frac{-\mu_0}{2} \left( \frac{(J + M + 2)(J + M + 1)}{(2J + 1)(2J + 3)} \right)^{1/2} \quad (6.74)$$

for the  $|J + 1, M + 1\rangle \leftarrow |J, M\rangle$  transition and

$$M_x = iM_y = \frac{\mu_0}{2} \left( \frac{(J - M + 1)(J - M + 2)}{(2J + 1)(2J + 3)} \right)^{1/2} \quad (6.75)$$

for the  $|J + 1, M - 1\rangle \leftarrow |J, M\rangle$  transition.

Starting from  $|JM\rangle$  there are thus three possible transitions to  $|J + 1\rangle$ , i.e., to  $|J + 1, M + 1\rangle$ ,  $|J + 1, M\rangle$  and  $|J + 1, M - 1\rangle$ —so the square magnitude of the transition dipole moment from  $|J, M\rangle$  to  $|J + 1\rangle$  is

$$|\mathbf{M}|_{J+1 \leftarrow J, M}^2 = \sum_{M'} M_x^2 + M_y^2 + M_z^2 = \frac{\mu_0^2(J + 1)}{2J + 1}. \quad (6.76)$$

However, there are  $2J + 1$  values of  $M$  to be counted for the lower state so the final expression is

$$|\mathbf{M}|_{J+1 \leftarrow J}^2 = \mu_0^2(J + 1). \quad (6.77)$$

For emission from state  $|J + 1, M\rangle$  to state  $|J\rangle$ , a similar calculation gives

$$|\mathbf{M}|_{J+1, M \rightarrow J}^2 = \frac{\mu_0^2(J + 1)}{2J + 3}, \quad (6.78)$$

but now there are  $2J + 3$  values of  $M$  in the upper state so the final expression (6.77) is the same as for emission or absorption.

The upper state has a total population  $N_{J+1}$  and a population density  $N_{J+1}/(2J+3)$  per  $M'$  state so the rate of emission, equation (1.17) (which applies to a single  $|J'M'\rangle$  state) becomes

$$\frac{dN_{J+1}}{dt} = - \sum_{M'} \sum_{M''} A_{J+1, M' \rightarrow J, M''} \frac{N_{J+1}}{2J + 3} \quad (6.79)$$

and

$$\begin{aligned} \frac{dN_{J+1}}{dt} &= - \frac{16\pi^3\nu^3}{3\varepsilon_0hc^3} \frac{N_{J+1}}{2J + 3} \sum_{M'} \sum_{M''} |\langle J + 1, M' | \boldsymbol{\mu} | JM'' \rangle|^2 \\ &= - \frac{16\pi^3\nu^3\mu_0^2(J + 1)}{3\varepsilon_0hc^3(2J + 3)} N_{J+1} \end{aligned} \quad (6.80)$$

so

$$A_{J+1 \rightarrow J} = \frac{16\pi^3\nu^3\mu_0^2(J + 1)}{3\varepsilon_0hc^3(2J + 3)}. \quad (6.81)$$

If a lineshape function  $g(\nu - \nu_{10})$  is included, then the Einstein  $A$  coefficient becomes

$$(A_{J+1 \rightarrow J})_{\nu} = \frac{16\pi^3 \nu^3 \mu_0^2 (J+1)}{3\epsilon_0 h c^3 (2J+3)} g(\nu - \nu_{10}). \quad (6.82)$$

As noted earlier, the same symbols are often used with (6.82) and without (6.81) the lineshape function so care with units is required. All of the equations for  $A$ ,  $B$ ,  $\sigma$ , and  $f$  derived for atoms in section 5.7 apply with  $2J' + 1 = 2J + 3$  and  $S_{J', J''}$  replaced by  $\mu_0^2 (J+1)$ . The  $(J+1)$ -part of the square of the transition dipole moment is an example of a rotational line strength factor, commonly called a *Hönl-London factor* (see Chapter 9).

The absorption cross section for the transition  $J+1 \leftarrow J$  for a linear molecule is thus

$$\sigma = \frac{2\pi^2 \nu \mu_0^2 (J+1)}{3\epsilon_0 h c (2J+1)} g(\nu - \nu_{10}) \quad (6.83)$$

and Beer's law, equation (1.62), including the stimulated emission correction, is

$$I = I_0 e^{-\sigma(N_0 - N_1 \frac{2J+1}{2J+3})l} = I_0 e^{-\alpha l} \quad (6.84)$$

with the absorption coefficient  $\alpha$  (units of  $\text{m}^{-1}$ ) given as

$$\alpha = \sigma \left( N_0 - N_1 \frac{2J+1}{2J+3} \right). \quad (6.85)$$

In equation (6.85) it is convenient to replace  $N_0$  and  $N_1$  by the total concentration,  $N$ .

For a system at temperature  $T$ , the absorption coefficient  $\alpha$  becomes

$$\alpha = \frac{2\pi^2 \nu \mu_0^2 (J+1) N e^{-E_J/kT}}{3\epsilon_0 h c q} \left( 1 - e^{-h\nu/kT} \right) g(\nu - \nu_{10}) \quad (6.86)$$

with  $q = q_{\text{el}} q_{\text{vib}} q_{\text{rot}}$  as the partition function and assuming that

$$N_0 = P_J N = \frac{N(2J+1)e^{-E_J/kT}}{q}, \quad (6.87)$$

i.e., that the state  $J$  has  $(2J+1)$ -fold rotational degeneracy, but no additional vibrational or electronic degeneracy.

At low frequencies, Doppler broadening is generally negligible relative to pressure broadening so the molecular line shape  $g(\nu - \nu_{10})$  is typically given by the Lorentzian function (1.78). Interestingly, for high precision work at low frequency, the "antiresonant" term containing  $\omega + \omega_{10}$  neglected in going from (1.71) to (1.72) needs to be included and the lineshape function is then approximately

$$g(\nu - \nu_{10}) = \frac{\nu}{\pi \nu_{10}} \left( \frac{\Delta\nu/2}{(\Delta\nu/2)^2 + (\nu - \nu_{10})^2} + \frac{\Delta\nu/2}{(\Delta\nu/2)^2 + (\nu + \nu_{10})^2} \right), \quad (6.88)$$

which is called the Van Vleck-Weisskopf lineshape function.<sup>9</sup> The  $\Delta\nu$  parameter is given as  $(\pi T_2)^{-1}$ , equation (1.81), with  $T_2$  the average time between collisions. For large  $\nu_{10}$  (in the infrared and optical regions) the second term on the right-hand side of equation (6.88) can be neglected and the usual Lorentzian line shape is recovered with  $\Delta\nu = \Delta\nu_{1/2}$ , the full width at half maximum. The Van Vleck-Weisskopf line shape agrees well with experimental observations of pure rotational transitions in the microwave and millimeter wave spectral region.

## 6.4 Symmetric Tops

The classical energy-level expression for a rigid symmetric top is

$$\begin{aligned} E &= \frac{J_a^2}{2I_A} + \frac{J_b^2}{2I_B} + \frac{J_c^2}{2I_C} \\ &= \frac{J_a^2}{2I_A} + \frac{1}{2I_B}(J_b^2 + J_c^2) \quad (\text{prolate top}) \end{aligned} \quad (6.89)$$

or

$$E = \frac{1}{2I_B}(J_a^2 + J_b^2) + \frac{J_c^2}{2I_C} \quad (\text{oblate top}). \quad (6.90)$$

For simplicity the treatment will be limited to a prolate top, but the results also apply to an oblate top by interchanging the labels  $a$  and  $c$ . Since

$$J_a^2 + J_b^2 + J_c^2 = J^2 \quad (6.91)$$

or

$$J_b^2 + J_c^2 = J^2 - J_a^2 \quad (6.92)$$

then

$$E = \frac{1}{2I_B}J^2 + \left( \frac{1}{2I_A} - \frac{1}{2I_B} \right) J_a^2. \quad (6.93)$$

The corresponding quantum mechanical Hamiltonian operator to equation (6.93) is

$$\hat{H} = \frac{1}{2I_B}\hat{J}^2 + \left( \frac{1}{2I_A} - \frac{1}{2I_B} \right) \hat{J}_a^2. \quad (6.94)$$

The solution of the symmetric top Schrödinger equation requires a small digression into quantum mechanics.

### *Molecule and Space-Fixed Angular Momenta*

The symmetric top molecule is described in two coordinate systems—the space-fixed laboratory system,  $X, Y, Z$ , and the molecular coordinate system,  $x, y, z$  (or  $a, b, c$ )—both with origins at the center of mass (Figure 6.26). The orientation of the molecular system relative to the laboratory system is described by three Euler angles,  $\theta$ ,  $\phi$ , and  $\chi$ , defined in different ways by various authors. Our convention<sup>10</sup> is illustrated in Figure 6.27. The angles  $\theta$  and  $\phi$  correspond to the usual polar and azimuthal angles of the molecular  $z$ -axis in the  $X, Y, Z$  frame, while  $\chi$  describes the internal orientation of the molecule relative to the molecular  $z$ -axis. In the example of  $\text{CH}_3\text{Cl}$ ,  $\chi$  is the angle which describes the rotation of the  $\text{CH}_3$  group around the molecular  $z$ -axis (Figure 6.26).

The laboratory and molecular coordinate systems are related by the transformation matrix  $\mathbf{S}$ , that is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{S} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \Phi_{xX} & \Phi_{xY} & \Phi_{xZ} \\ \Phi_{yX} & \Phi_{yY} & \Phi_{yZ} \\ \Phi_{zX} & \Phi_{zY} & \Phi_{zZ} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \quad (6.95)$$

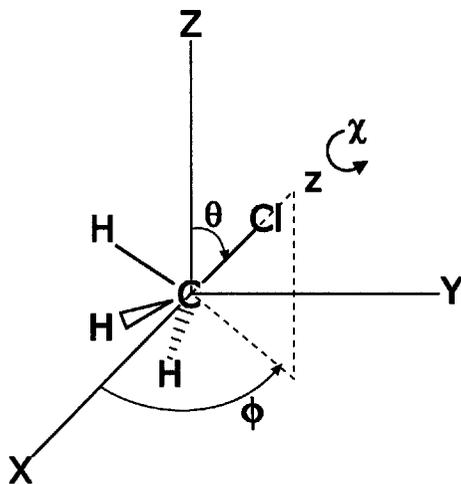


Figure 6.26: The laboratory and molecular coordinate systems for  $\text{CH}_3\text{Cl}$ .

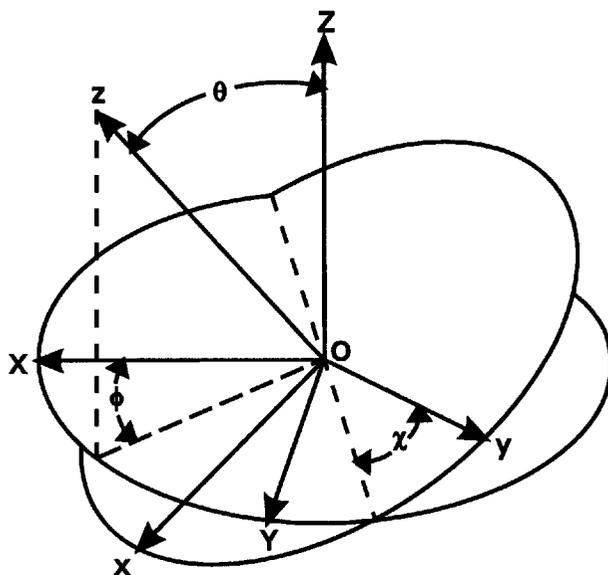


Figure 6.27: The Euler angles  $\theta$ ,  $\phi$ , and  $\chi$  that relate the space-fixed coordinate system  $(X, Y, Z)$  to the molecular coordinate system  $(x, y, z)$ .

The elements of  $\mathbf{S}$  are just the direction cosines of vector algebra (Figure 6.28) with

$$\begin{aligned}
 x &= \Phi_{xX}X + \Phi_{xY}Y + \Phi_{xZ}Z \\
 &= \cos \alpha_1 X + \cos \beta_1 Y + \cos \gamma_1 Z \\
 &= \hat{\mathbf{x}} \cdot \hat{\mathbf{X}}X + \hat{\mathbf{x}} \cdot \hat{\mathbf{Y}}Y + \hat{\mathbf{x}} \cdot \hat{\mathbf{Z}}Z
 \end{aligned} \tag{6.96}$$

and so forth, where  $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ , and  $\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}$  are sets of unit vectors for the molecular and laboratory coordinate systems, respectively.

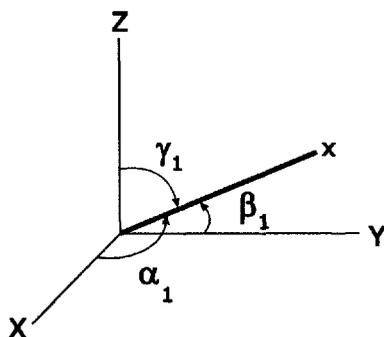


Figure 6.28: The direction cosine angles for the molecular  $x$ -axis in the laboratory frame.

The  $\mathbf{S}$  matrix can be derived using the description of the Euler angles as rotations about axes (Figure 6.27):

1. Rotate  $X$  and  $Y$  by an angle  $\phi$  about  $Z$  into  $X'$  and  $Y'$ ;
2. Rotate  $X'$  and  $Z$  by an angle  $\theta$  about  $Y'$  into  $X''$  and  $z$ ;
3. Rotate  $X''$  and  $Y'$  by an angle  $\chi$  about  $z$  into  $x$  and  $y$ .

Thus,

$$\begin{aligned} \mathbf{S} &= \begin{pmatrix} \cos \chi & \sin \chi & 0 \\ -\sin \chi & \cos \chi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta \cos \phi \cos \chi - \sin \phi \sin \chi & \cos \theta \sin \phi \cos \chi + \cos \phi \sin \chi & -\sin \theta \cos \chi \\ -\cos \theta \cos \phi \sin \chi - \sin \phi \cos \chi & -\cos \theta \sin \phi \sin \chi + \cos \phi \cos \chi & \sin \theta \sin \chi \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix}, \end{aligned} \quad (6.97)$$

in which  $\mathbf{S}$  is an orthogonal matrix ( $\mathbf{S}^{-1} = \mathbf{S}^t$ ), since  $\mathbf{S}$  represents a rotational transformation of the coordinate system. The angular momenta can be measured in the laboratory frame ( $\hat{J}_X, \hat{J}_Y, \hat{J}_Z$ ) or in the molecular frame ( $\hat{J}_x, \hat{J}_y, \hat{J}_z$ ) with

$$\hat{J}_x = \Phi_{xX} \hat{J}_X + \Phi_{xY} \hat{J}_Y + \Phi_{xZ} \hat{J}_Z, \quad (6.98)$$

$$\hat{J}_y = \Phi_{yX} \hat{J}_X + \Phi_{yY} \hat{J}_Y + \Phi_{yZ} \hat{J}_Z, \quad (6.99)$$

$$\hat{J}_z = \Phi_{zX} \hat{J}_X + \Phi_{zY} \hat{J}_Y + \Phi_{zZ} \hat{J}_Z. \quad (6.100)$$

Using the matrix elements of  $\mathbf{S}$  and the operator expressions for  $\hat{J}_X, \hat{J}_Y,$  and  $\hat{J}_Z$  one obtains<sup>10</sup> a set of expressions for  $\hat{J}_x, \hat{J}_y,$  and  $\hat{J}_z$ , namely

$$\hat{J}_x = -i\hbar \left( \frac{-\cos\chi}{\sin\theta} \frac{\partial}{\partial\phi} + \frac{\cos\chi \cos\theta}{\sin\theta} \frac{\partial}{\partial\chi} + \sin\chi \frac{\partial}{\partial\theta} \right) \quad (6.101)$$

$$\hat{J}_y = -i\hbar \left( \frac{\sin\chi}{\sin\theta} \frac{\partial}{\partial\phi} - \frac{\sin\chi \cos\theta}{\sin\theta} \frac{\partial}{\partial\chi} + \cos\chi \frac{\partial}{\partial\theta} \right) \quad (6.102)$$

$$\hat{J}_z = -i\hbar \frac{\partial}{\partial\chi}. \quad (6.103)$$

The corresponding expressions in the laboratory frame<sup>10</sup> are

$$J_X = -i\hbar \left( \frac{-\cos\phi \cos\theta}{\sin\theta} \frac{\partial}{\partial\phi} + \frac{\cos\phi}{\sin\theta} \frac{\partial}{\partial\chi} - \sin\phi \frac{\partial}{\partial\theta} \right) \quad (6.104)$$

$$J_Y = -i\hbar \left( \frac{-\sin\phi \cos\theta}{\sin\theta} \frac{\partial}{\partial\phi} + \frac{\sin\phi}{\sin\theta} \frac{\partial}{\partial\chi} + \cos\phi \frac{\partial}{\partial\theta} \right) \quad (6.105)$$

$$J_Z = -i\hbar \frac{\partial}{\partial\phi}. \quad (6.106)$$

Checking the commutation relationships gives the surprising result

$$[\hat{J}_x, \hat{J}_y] = -i\hbar \hat{J}_z, \quad (6.107)$$

in contrast to the usual commutation relation

$$[\hat{J}_X, \hat{J}_Y] = i\hbar \hat{J}_Z. \quad (6.108)$$

All of the commutation relationships have a minus sign in the molecular frame when compared to the corresponding equation in the laboratory frame. The “anomalous” commutation relationships in the molecular frame, equation (6.107), are due entirely to the direction cosine terms. Note that the molecular  $\hat{J}_z$  operator commutes with  $\hat{J}^2$  and that the space-fixed and molecular frame operators commute with each other, that is,

$$[\hat{J}^2, \hat{J}_z] = [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \hat{J}_z] = 0 \quad (6.109)$$

and

$$[\hat{J}_\alpha, \hat{J}_i] = 0; \quad \alpha = X, Y, Z, \quad i = x, y, z. \quad (6.110)$$

Consider now the rigid rotor symmetric top Hamiltonian operator

$$\hat{H} = \left( \frac{1}{2I_B} \right) \hat{J}^2 + \left( \frac{1}{2I_A} - \frac{1}{2I_B} \right) \hat{J}_z \quad (6.111)$$

in the molecular frame. Clearly  $\hat{J}^2$ ,  $\hat{J}_z$ , and  $\hat{H}$  all commute with  $\hat{H}$  so that a set of simultaneous eigenfunctions can be found, namely

$$\hat{H}|JKM\rangle = E|JKM\rangle, \quad (6.112)$$

$$\hat{J}^2|JKM\rangle = J(J+1)\hbar^2|JKM\rangle, \quad (6.113)$$

$$\hat{J}_z|JKM\rangle = M_J\hbar|JKM\rangle, \quad (6.114)$$

$$\hat{J}_z|JKM\rangle = K\hbar|JKM\rangle, \quad (6.115)$$

in which  $K\hbar$  is defined as the projection of  $\hat{\mathbf{J}}$  along the molecular  $z$ -axis. Furthermore, since

$$\hat{J}_Z = -i\hbar \frac{\partial}{\partial \phi}, \quad (6.116)$$

and

$$\hat{J}_z = -i\hbar \frac{\partial}{\partial \chi}, \quad (6.117)$$

the symmetric top wavefunctions  $|JKM\rangle$  must have the form<sup>11</sup>

$$|JKM\rangle = \left(\frac{2J+1}{8\pi^2}\right)^{1/2} e^{iM\phi} d_{MK}^{(J)}(\theta) e^{iK\chi}. \quad (6.118)$$

The  $d_{MK}^{(J)}$  functions are hypergeometric functions of  $\sin^2(\theta/2)$  and are also related to the rotation matrices of angular momentum theory. The symmetric top functions are rarely listed since the explicit functions are not needed for calculations.

The anomalous commutation relationships in the molecular frame mean that  $\hat{J}^+ = \hat{J}_x + i\hat{J}_y$  is a lowering operator and  $\hat{J}^- = \hat{J}_x - i\hat{J}_y$  is a raising operator. (Note that  $\hat{J}^+$  and  $\hat{J}^-$  are in the molecular frame, but  $\hat{J}_-$  and  $\hat{J}_+$  are in the laboratory frame.) The effects of the raising and lowering operators on the symmetric top eigenfunctions are given by the equations

$$\hat{J}_+|JKM\rangle = \hbar\sqrt{J(J+1) - M(M+1)}|JKM+1\rangle, \quad (6.119)$$

$$\hat{J}_-|JKM\rangle = \hbar\sqrt{J(J+1) - M(M-1)}|JKM-1\rangle, \quad (6.120)$$

$$\hat{J}^+|JKM\rangle = \hbar\sqrt{J(J+1) - K(K-1)}|JK-1M\rangle, \quad (6.121)$$

$$\hat{J}^-|JKM\rangle = \hbar\sqrt{J(J+1) - K(K+1)}|JK+1M\rangle. \quad (6.122)$$

Returning to the symmetric top Hamiltonian operator, equation (6.111), one can solve the Schrödinger equation using the symmetric top wavefunctions as

$$\begin{aligned} \hat{H}\psi &= \left(\frac{\hat{J}^2}{2I_B} + \left(\frac{1}{2I_A} - \frac{1}{2I_B}\right)\hat{J}_a^2\right)|JKM\rangle \\ &= \left(\frac{\hbar^2}{2I_B}J(J+1) + \left(\frac{\hbar^2}{2I_A} - \frac{\hbar^2}{2I_B}\right)K^2\right)|JKM\rangle \quad (\text{prolate top}) \end{aligned} \quad (6.123a)$$

$$\begin{aligned} \hat{H}\psi &= \left(\frac{\hat{J}^2}{2I_B} + \left(\frac{1}{2I_C} - \frac{1}{2I_B}\right)\hat{J}_c^2\right)|JKM\rangle \\ &= \left(\frac{\hbar^2}{2I_B}J(J+1) + \left(\frac{\hbar^2}{2I_C} - \frac{\hbar^2}{2I_B}\right)K^2\right)|JKM\rangle \quad (\text{oblate top}) \end{aligned} \quad (6.123b)$$

so

$$E_{JK_a} = BJ(J+1) + (A-B)K_a^2 \quad (\text{prolate top}) \quad (6.124)$$

or

$$E_{JK_c} = BJ(J+1) + (C-B)K_c^2 \quad (\text{oblate top}). \quad (6.125)$$

The rotational constants in energy units (joules) are defined by

$$A = \frac{h^2}{8\pi^2 I_A}, \quad (6.126a)$$

$$B = \frac{h^2}{8\pi^2 I_B}, \quad (6.126b)$$

$$C = \frac{h^2}{8\pi^2 I_C}, \quad (6.126c)$$

and equations analogous to equations (6.29) to (6.32) are valid for nonlinear molecules. As expected, the energy-level expression is not a function of  $M$  so that the  $M_J$  degeneracy of  $2J + 1$  remains. The degeneracy of states with the  $2J + 1$  possible values of  $K$  is partially lifted; however, states with  $\pm K$  still have the same energy. The energy of the symmetric top is the same for clockwise and counterclockwise rotation around the molecular  $z$ -axis so that a twofold  $K$  degeneracy remains.

It is convenient to classify the energy levels of symmetric tops by the  $K$  quantum number. For a given  $K$ , we have  $J \geq K$  and the energies have a simple linear molecule structure apart from a  $(A - B)K^2 (> 0)$  or  $(C - B)K^2 (< 0)$  offset (Figure 6.29). Note that levels of a given  $J$  value increase in energy with increasing  $K$  for a prolate top while they decrease in energy for an oblate top. For example,  $\text{CH}_3\text{I}$  has  $A = 5.11 \text{ cm}^{-1}$  and  $B = 0.250 \text{ cm}^{-1}$  so<sup>12</sup>

$$E_{JK}/\text{cm}^{-1} = 0.250J(J + 1) + 4.86K^2. \quad (6.127)$$

### Rotational Spectra

The pure rotational spectra of symmetric tops are determined by the application of selection rules to the energy-level pattern in Figure 6.29. The derivation of selection rules for the symmetric top is somewhat involved since the transformation from the laboratory frame to the molecular frame needs to be considered. The selection rules are  $\Delta J = \pm 1$ ,  $\Delta M = 0, \pm 1$  and  $\Delta K = 0$ , and they result in very simple pure rotational spectra (Figures 6.30 and 6.31). The transitions are confined to lie within a  $K$ -stack so the transition frequencies are given by the diatomic expression, that is,

$$\nu_{J+1, K \leftarrow J, K} = 2B(J + 1) \quad (6.128)$$

### Centrifugal Distortion

As a molecule rotates, it also distorts under the effects of centrifugal forces resulting in an energy-level expression

$$F(J, K) = BJ(J + 1) - D_J(J(J + 1))^2 + (A - B)K^2 - D_K K^4 - D_{JK}J(J + 1)K^2, \quad (6.129)$$

in which there are now three centrifugal distortion constants  $D_J$ ,  $D_K$ , and  $D_{JK}$ . The transition frequencies are then given by

$$\begin{aligned} \nu_{J+1, K \leftarrow J, K} &= F(J + 1, K) - F(J, K) \\ &= 2B(J + 1) - 4D_J(J + 1)^3 - 2D_{JK}(J + 1)K^2. \end{aligned} \quad (6.130)$$

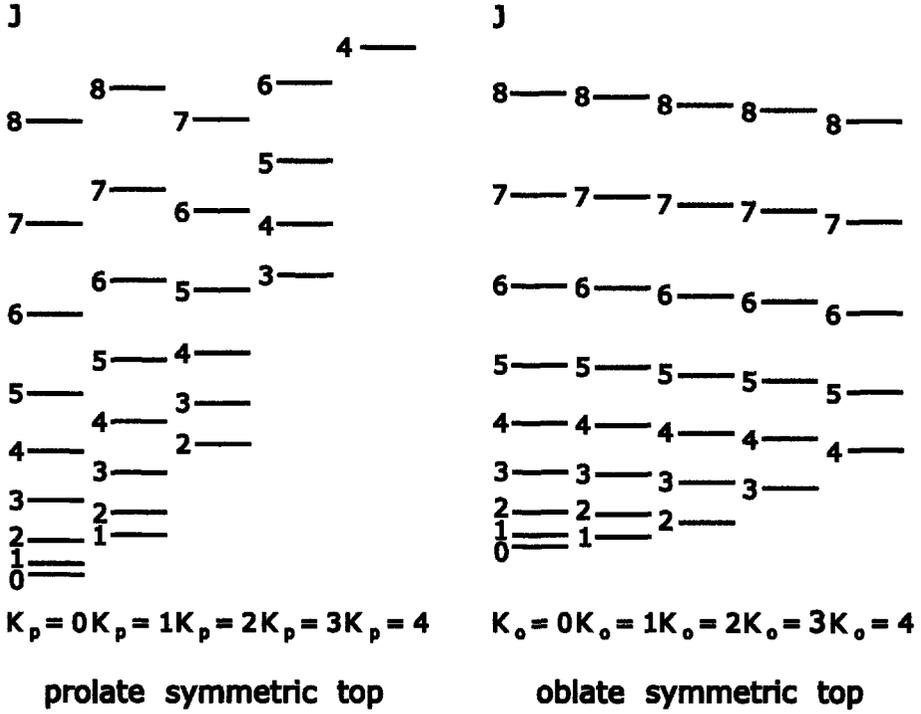


Figure 6.29: Energy levels of a prolate and an oblate symmetric top.

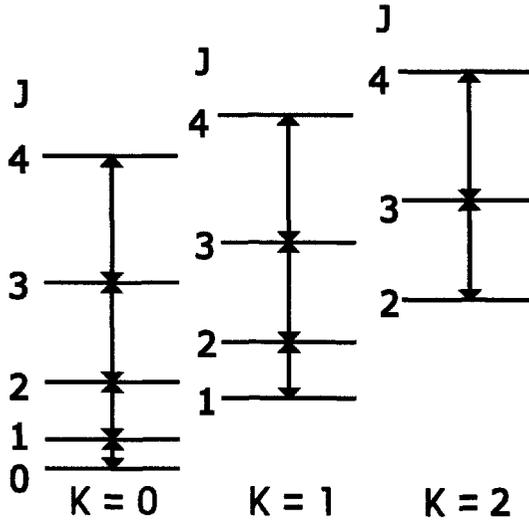


Figure 6.30: The allowed electric dipole transitions of a prolate symmetric top.

The constant  $D_{JK}$  splits out the transitions with different  $K$  for a given  $J + 1 \leftarrow J$  transition, as shown in Figure 6.32.

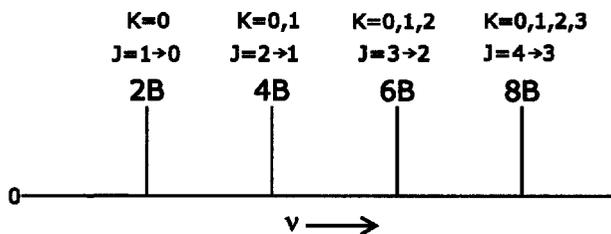
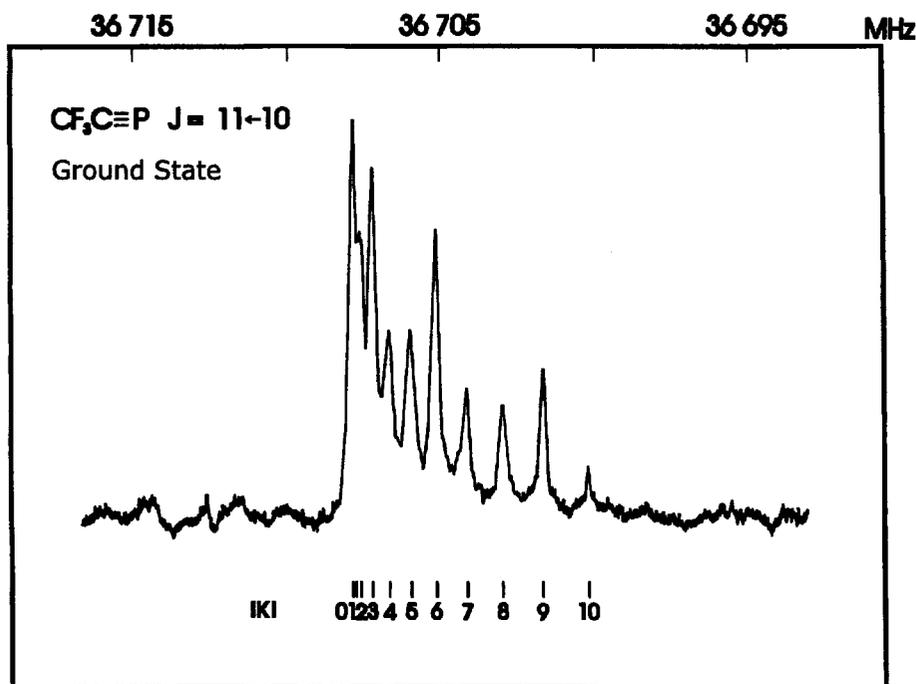


Figure 6.31: Pure rotational emission spectrum of a symmetric top.

Figure 6.32: The  $J = 11 \leftarrow 10$  transition of a  $\text{CF}_3\text{C}\equiv\text{P}$  showing  $K$ -structure. The intensity pattern is affected by nuclear spin statistics.

### Line Intensity

All of the intensity equations for atoms in Section 5.7 apply with the atomic line strength factor  $S_{J',J''}$  replaced by the appropriate expression for a symmetric top. The derivation for a symmetric top is more involved than for an atom or a linear molecule because the wavefunctions  $\psi_{JKM}(\theta, \phi, \chi)$  more complicated. The final result is

$$S_{J+1, K \leftarrow J, K} = \mu_0^2 \frac{(J+1)^2 - K^2}{J+1} \quad (6.131)$$

and

$$S_{J-1, K \leftarrow J, K} = \mu_0^2 \frac{J^2 - K^2}{J} \quad (6.132)$$

for a symmetric top with dipole moment  $\mu_0$  and the selection rules are  $\Delta J = \pm 1$ ,  $\Delta K = 0$ . These factors (6.131 and 6.132), apart from  $\mu_0^2$ , are given as Hönl-London factors in Table 9.4 with  $K$  replacing  $\Lambda$ . The primary complication is the effect of nuclear spin statistics<sup>13</sup> on populations as displayed in Figure 6.32.

## 6.5 Asymmetric Tops

For an asymmetric top,  $I_A \neq I_B \neq I_C$ , the classical energy for a rigid rotor is given by

$$E = \frac{J_a^2}{2I_A} + \frac{J_b^2}{2I_B} + \frac{J_c^2}{2I_C}. \quad (6.133)$$

This results in the rigid asymmetric rotor Hamiltonian operator

$$\hat{H} = \frac{\hat{J}_a^2}{2I_A} + \frac{\hat{J}_b^2}{2I_B} + \frac{\hat{J}_c^2}{2I_C}. \quad (6.134)$$

The Schrödinger equation for the asymmetric top has no general analytical solutions and therefore must be solved numerically with the help of a computer. For certain special values of  $J$ , however, analytical solutions are available for the rigid rotor.

The asymmetric top Schrödinger equation can be solved using a symmetric top basis set, changing the form of the terms in the Hamiltonian operator for convenience. Let

$$A = \frac{\hbar^2}{2I_A}, \quad (6.135)$$

$$B = \frac{\hbar^2}{2I_B}, \quad (6.136)$$

and

$$C = \frac{\hbar^2}{2I_C} \quad (6.137)$$

so that

$$\begin{aligned} \hbar^2 \hat{H} &= A\hat{J}_a^2 + B\hat{J}_b^2 + C\hat{J}_c^2 \\ &= \left(\frac{A+B}{2}\right)(\hat{J}_a^2 + \hat{J}_b^2) + C\hat{J}_c^2 + \frac{(A-B)}{2}(\hat{J}_a^2 - \hat{J}_b^2) \\ &= \left(\frac{A+B}{2}\right)\hat{J}^2 + \left(C - \frac{A+B}{2}\right)\hat{J}_c^2 + \left(\frac{A-B}{4}\right)((\hat{J}^+)^2 + (\hat{J}^-)^2). \end{aligned} \quad (6.138)$$

The following symmetric top matrix elements<sup>14</sup> are useful

$$\langle JK | \hat{J}^2 | JK \rangle = \hbar^2 J(J+1), \quad (6.139)$$

$$\langle JK | \hat{J}_c^2 | JK \rangle = \hbar^2 K^2, \quad (6.140)$$

$$\langle JK + 2 | (\hat{J}^-)^2 | JK \rangle = \hbar^2 ((J-K)(J+K+1)(J-K-1)(J+K+2))^{1/2}, \quad (6.141)$$

$$\langle JK - 2 | (\hat{J}^+)^2 | JK \rangle = \hbar^2 ((J+K)(J-K+1)(J+K-1)(J-K+2))^{1/2}, \quad (6.142)$$

remembering that  $\hat{J}^+$  is a lowering operator and  $\hat{J}^-$  is a raising operator. With the symmetric top basis functions the asymmetric top Hamiltonian operator has matrix elements with  $\Delta K = 0$  and  $\Delta K = \pm 2$ .

For example, the basis set for  $J = 1$  has three members  $|J = 1, K = 1\rangle$ ,  $|1, 0\rangle$ , and  $|1, -1\rangle$ , and the Hamiltonian is the  $3 \times 3$  matrix

$$\hat{\mathbf{H}} = \begin{array}{c} \langle 1, 1| \\ \langle 1, 0| \\ \langle 1, -1| \end{array} \begin{array}{ccc} |1, 1\rangle & |1, 0\rangle & |1, -1\rangle \\ \left( \begin{array}{ccc} C + \frac{A+B}{2} & 0 & \frac{A-B}{2} \\ 0 & A+B & 0 \\ \frac{A-B}{2} & 0 & C + \frac{A+B}{2} \end{array} \right) \end{array}. \quad (6.143)$$

The eigenvalues of this equation are easily determined by first exchanging the second and third rows and columns to give

$$\hat{\mathbf{H}} = \begin{array}{c} \langle 1, 1| \\ \langle 1, -1| \\ \langle 1, 0| \end{array} \begin{array}{ccc} |1, 1\rangle & |1, -1\rangle & |1, 0\rangle \\ \left( \begin{array}{ccc} C + \frac{A+B}{2} & \frac{A-B}{2} & 0 \\ \frac{A-B}{2} & C + \frac{A+B}{2} & 0 \\ 0 & 0 & A+B \end{array} \right) \end{array}. \quad (6.144)$$

and then solving the secular equation (Chapter 4) for the  $2 \times 2$  block

$$\left| \begin{array}{cc} C + \frac{A+B}{2} - \lambda & \frac{A-B}{2} \\ \frac{A-B}{2} & C + \frac{A+B}{2} - \lambda \end{array} \right| = 0 \quad (6.145)$$

to give

$$\lambda = C + A \quad (6.146)$$

or

$$\lambda = C + B. \quad (6.147)$$

The three solutions for  $J = 1$  are thus  $A + B$ ,  $A + C$ , and  $B + C$ . Labeling of the energy levels is carried out by considering the correlation diagram that connects the energy levels of a prolate top with those of an oblate top in Figure 6.33 and the requirement that  $I_A \leq I_B \leq I_C$ .

The energy levels of prolate and oblate symmetric tops are

$$E_p = (A - B)K_a^2 + BJ(J + 1); \quad E_o = (C - B)K_c^2 + BJ(J + 1). \quad (6.148)$$

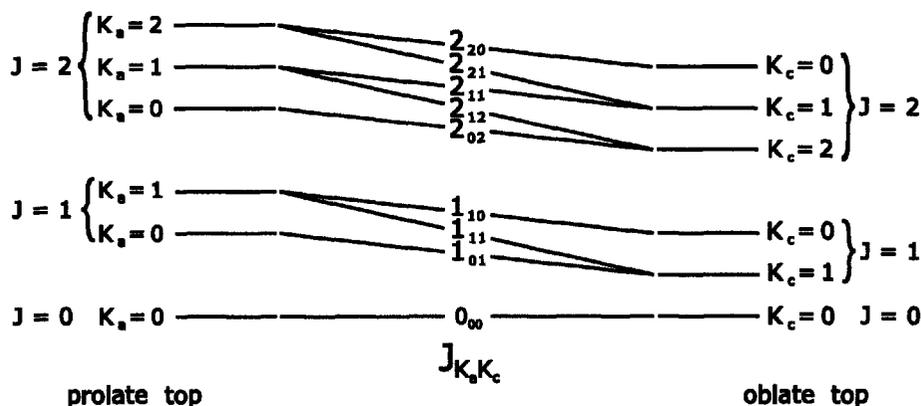


Figure 6.33: Prolate-oblate correlation diagram useful for labeling asymmetric top levels.

Notice that by definition  $(A - B) > 0$  and  $(C - B) < 0$ , so that for a given value of  $J$  the  $K_a$  levels increase in energy as  $K_a$  increases for a prolate top, while the  $K_c$  levels decrease in energy as  $K_c$  increases for an oblate top. As is typical in correlation diagrams, the energy levels are not to scale and the lines connecting the prolate levels with the oblate levels correspond to a hypothetical distortion of a molecule from a prolate to an oblate top. The levels are labeled by  $J_{K_a K_c}$ , where  $J$  is a good quantum number, but  $K_a$  and  $K_c$  are just labels for the asymmetric top. Clearly,  $K_a$  and  $K_c$  become good quantum numbers only in the prolate or oblate symmetric top limits. Note that the sum of  $K_a$  and  $K_c$  is  $J$  or  $J + 1$ .

It is sometimes convenient to define a label  $\tau = K_a - K_c$ , which runs from  $\tau = +J$  to  $\tau = -J$  in order of descending energy. The label  $\tau$  emphasizes that for the asymmetric top there are  $2J + 1$  distinct energy levels corresponding to the  $2J + 1$  different possible values of  $\tau$  or “K” for every  $J$ . The degree of asymmetry can be quantified by an asymmetry parameter (“Ray’s asymmetry parameter”)  $\kappa$ , which runs from  $-1$  for a prolate top to  $+1$  for an oblate top. The asymmetry parameter is defined as

$$\kappa = \frac{2B - A - C}{A - C}. \quad (6.149)$$

The asymmetric top labels  $K_a = K_p$  and  $K_c = K_o$  are sometimes called  $K_{-1}$  and  $K_{+1}$  because of the values of the asymmetry parameter for the prolate and oblate symmetric top limits. The notation  $J_{K_p K_o}$  allows the three energy levels associated with  $J = 1$  to be labeled  $E(1_{10}) = A + B$ ,  $E(1_{11}) = A + C$ , and  $E(1_{01}) = B + C$ , since  $I_A \leq I_B \leq I_C$  means  $A \geq B \geq C$ . The explicit energy level expressions for a rigid asymmetric rotor are provided in Table 6.2 for  $J = 0, 1, 2$ , and 3.

### Selection Rules

The asymmetric top selection rules are more complicated than those of a linear molecule or a symmetric top. In general, an arbitrary molecule has three dipole moment components  $\mu_a$ ,  $\mu_b$ , and  $\mu_c$  along the principal axes (Figure 6.34). Each nonvanishing dipole moment component makes a certain set of transitions possible and leads to a set of selection rules. The selection rules on  $J$  and  $M$  are  $\Delta J = 0, \pm 1$  and  $\Delta M = 0, \pm 1$ .

Table 6.2: Rigid Asymmetric Rotor Energy Levels for  $J = 0, 1, 2, 3$ 

$J_{K_a K_c}$	$J_\tau$	$F(J_\tau)$
0 <sub>00</sub>	0 <sub>0</sub>	0
1 <sub>10</sub>	1 <sub>1</sub>	$A + B$
1 <sub>11</sub>	1 <sub>0</sub>	$A + C$
1 <sub>01</sub>	1 <sub>-1</sub>	$B + C$
2 <sub>20</sub>	2 <sub>2</sub>	$2A + 2B + 2C + 2((B - C)^2 + (A - C)(A - B))^{1/2}$
2 <sub>21</sub>	2 <sub>1</sub>	$4A + B + C$
2 <sub>11</sub>	2 <sub>0</sub>	$A + 4B + C$
2 <sub>12</sub>	2 <sub>-1</sub>	$A + B + 4C$
2 <sub>02</sub>	2 <sub>-2</sub>	$2A + 2B + 2C - 2((B - C)^2 + (A - C)(A - B))^{1/2}$
3 <sub>30</sub>	3 <sub>3</sub>	$5A + 5B + 2C + 2(4(A - B)^2 + (A - C)(B - C))^{1/2}$
3 <sub>31</sub>	3 <sub>2</sub>	$5A + 2B + 5C + 2(4(A - C)^2 - (A - B)(B - C))^{1/2}$
3 <sub>21</sub>	3 <sub>1</sub>	$2A + 5B + 5C + 2(4(B - C)^2 + (A - B)(A - C))^{1/2}$
3 <sub>22</sub>	3 <sub>0</sub>	$4A + 4B + 4C$
3 <sub>12</sub>	3 <sub>-1</sub>	$5A + 5B + 2C - 2(4(A - B)^2 + (A - C)(B - C))^{1/2}$
3 <sub>13</sub>	3 <sub>-2</sub>	$5A + 2B + 5C - 2(4(A - C)^2 - (A - B)(B - C))^{1/2}$
3 <sub>03</sub>	3 <sub>-3</sub>	$2A + 5B + 5C - 2(4(B - C)^2 + (A - B)(A - C))^{1/2}$

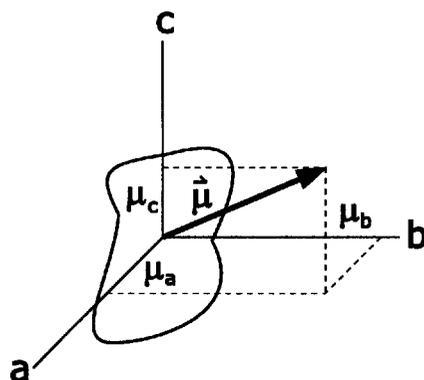
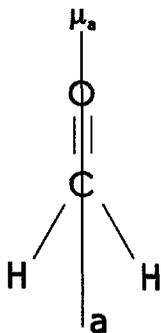
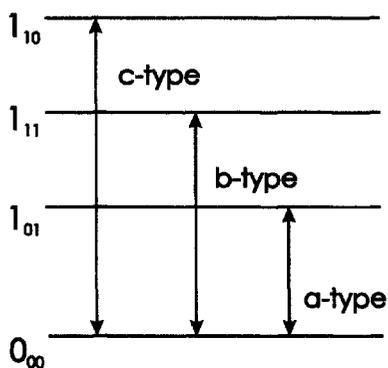


Figure 6.34: An arbitrary molecule has three components of the dipole moment in the principal axis system of the molecule.

### $a$ -Type Transitions

If  $\mu_a \neq 0$  and  $\mu_b = \mu_c = 0$ , then a molecule such as  $\text{H}_2\text{CO}$  is said to obey  $a$ -type selection rules (Figure 6.35),  $\Delta K_a = 0(\pm 2, \pm 4 \dots)$  and  $\Delta K_c = \pm 1(\pm 3, \pm 5 \dots)$ ; the transitions in parentheses are much weaker than the main ones. Thus, for example, the  $1_{01} - 0_{00}$  transition of formaldehyde is allowed, but the transitions  $1_{10} - 0_{00}$  and  $1_{11} - 0_{00}$  are forbidden because they require  $\mu_c \neq 0$  and  $\mu_b \neq 0$ , respectively.

Figure 6.35: Formaldehyde has  $\mu_a \neq 0$ .Figure 6.36: *a*-, *b*-, and *c*-type transitions.

### *b*-Type Transitions

If  $\mu_b \neq 0$ , then transitions with the selection rules

$$\Delta K_a = \pm 1(\pm 3, \pm 5 \dots) \quad (6.150)$$

$$\Delta K_c = \pm 1(\pm 3, \pm 5 \dots) \quad (6.151)$$

are allowed.

### *c*-Type Transitions

If  $\mu_c \neq 0$ , then transitions with the selection rules

$$\Delta K_a = \pm 1(\pm 3, \pm 5, \dots) \quad (6.152)$$

$$\Delta K_c = 0(\pm 2, \pm 4, \dots) \quad (6.153)$$

are allowed. The three possible types of transitions are illustrated in Figure 6.36. Note that for molecules of low symmetry, all three types could occur together.

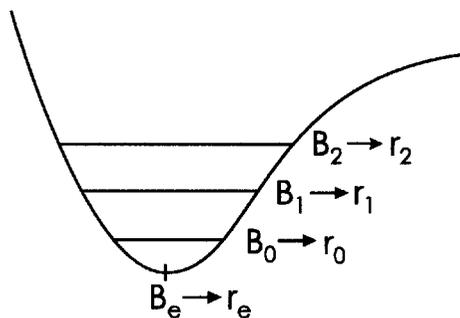


Figure 6.37:  $B_v$  and  $r_v$  values for a diatomic molecule.

### Line Intensity

Again the basic intensity equations in section 5.7 apply, but the line strength factor  $S_{J',J''}$  is difficult to relate back to the dipole moment components and the rotational quantum numbers. The problem of intensity of lines for an asymmetric rotor is complicated by the lack of analytical expressions. In practice, the dipole moments  $\mu_a$ ,  $\mu_b$ , and  $\mu_c$  along the principal axes are used with an asymmetric top computer program to calculate numerically the line strengths  $S_{J',J''}$ . The eigenvectors needed to diagonalize the rotational Hamiltonian matrix, e.g., equation (6.143), give the amount of mixing of the symmetric top basis functions for each asymmetric rotor state. Analytical expressions for the rotational line strength factors for transitions between symmetric top levels are known (the Hönl-London factors of Table 9.4 with  $K$  replacing  $\Lambda$ ). When combined with the mixing coefficients and the selection rules (previous section), these symmetric top expressions give the needed asymmetric top line strengths as numerical values. Various computer programs<sup>15,16</sup> are freely available for the calculation of asymmetric rotor line positions and line intensities using the standard Watson<sup>17</sup> Hamiltonian operator.

## 6.6 Structure Determination

One of the main applications of molecular spectroscopy is the determination of molecular structures. The moments of inertia are related to bond lengths and bond angles. For a diatomic molecule the determination of  $r$  from  $B$  is simple, but each vibrational level has a different  $B_v$  value so that there are numerous corresponding  $r_v$  values (Figure 6.37). Each  $B_v = \langle \hbar^2/2\mu \rangle \langle v|1/r^2|v \rangle$  corresponds to an average over a different vibrational wavefunction. Perhaps the “best” value of  $r$  is  $r_e$ , which is computed by extrapolating  $B_v$  down to the bottom of the potential well, that is,

$$B_v = B_e - \alpha_e(v + \frac{1}{2}) + \gamma_e(v + \frac{1}{2})^2 + \dots \quad (6.154)$$

Given at least  $B_1$  and  $B_0$ ,  $B_e$  is easily computed as  $B_0 - B_1 = \alpha_e$  and  $B_e = B_0 + \alpha_e/2$ , and hence  $r_e$  is determined. This is usually possible for diatomic molecules, so that  $r_e$  values are customarily reported.

The determination of molecular structures in polyatomic molecules is much more difficult. The first problem is that there are now  $3N - 6$  (or  $5$ )  $\alpha$ 's for each  $A$ ,  $B$ , and  $C$ :

Table 6.3: Rotational Constants<sup>19</sup> for H<sub>2</sub><sup>12</sup>CO and H<sub>2</sub><sup>13</sup>CO (in MHz)

	H <sub>2</sub> <sup>12</sup> CO	H <sub>2</sub> <sup>13</sup> CO
<i>A</i>	281 970.572	281 993.257
<i>B</i>	38 836.0455	37 811.0886
<i>C</i>	34 002.2034	33 213.9789

$$A_v = A_e - \sum_{i=1}^{3N-6} \alpha_i^A \left( v_i + \frac{d_i}{2} \right)$$

$$B_v = B_e - \sum_{i=1}^{3N-6} \alpha_i^B \left( v_i + \frac{d_i}{2} \right)$$

$$C_v = C_e - \sum_{i=1}^{3N-6} \alpha_i^C \left( v_i + \frac{d_i}{2} \right),$$

where  $d_i$  is the degeneracy of the  $i$ th mode. The determination of so many  $\alpha$ 's is a tedious task even for a triatomic molecule, so very few  $r_e$  structures are known. For most polyatomic molecules  $r_0$  structures are computed from  $A_0$ ,  $B_0$ , and  $C_0$ .

Another difficulty is that there are at most three moments of inertia (even for the largest molecules!), but usually more than three structural parameters need to be determined. Consider the case of formaldehyde (H<sub>2</sub>CO) for which the three structural parameters are  $r_{\text{CO}}$ ,  $r_{\text{CH}}$ , and  $\theta_{\text{HCH}}$ . At first sight it seems that there is no difficulty, since  $A$ ,  $B$ , and  $C$  allow  $I_A$ ,  $I_B$ , and  $I_C$  to be determined. Formaldehyde is a planar molecule, however, so that  $I_C = I_A + I_B$  and three structural parameters need to be determined from only two independent moments of inertia.

In fact  $I_C$  is not exactly equal to  $I_A + I_B$  because of several effects including vibrations.<sup>18</sup> It is useful to define the inertial defect  $\Delta = I_C - I_A - I_B$ , which generally has a small positive value (e.g.,  $\Delta = 0.05767 \text{ u } \text{\AA}^2$  for formaldehyde<sup>19</sup>) for planar molecules. Any deviation from the empirically expected value of  $\Delta$  is taken as evidence of nonplanarity, or fluxional behavior, or the presence of low frequency out-of-plane vibrations<sup>18</sup> in the molecule.

A solution to the structural problem in formaldehyde lies in making use of data from isotopic molecules (Table 6.3). It is necessary to assume that all isotopic variants of formaldehyde have the same  $r_{\text{CO}}$ ,  $r_{\text{CH}}$ , and  $\theta_{\text{HCH}}$  values. This is a good approximation because the potential surface is independent of nuclear mass within the Born–Oppenheimer approximation. From the pure rotational spectrum of, for example, <sup>13</sup>C-substituted H<sub>2</sub>CO a set of two additional independent moments of inertia can be derived. Now four independent moments of inertia (from the six total) are available to determine the three structural parameters by least squares fitting<sup>20</sup> (Problem 8).

An  $r_0$  structure is very useful, but some of the geometrical parameters, particularly C–H bond lengths, are not very reliable in that they lie far from equilibrium  $r_e$  val-

ues. A “better” structure can be derived by isotopic substitution using Kraitchman’s equations<sup>20</sup> or other, more sophisticated, techniques. Kraitchman’s equations allow the distance between the center of mass and the isotopically substituted atom to be calculated (see Gordy and Cook<sup>20</sup> or Domenicano and Hargittai<sup>21</sup>). Structures determined by substituting a different isotope for every atom in a molecule (in turn, one at a time) are called  $r_s$  structures<sup>20</sup> ( $s$  stands for substitution). A complete  $r_s$  structure is very tedious to determine because a large number of isotopic forms of a molecule must be synthesized and rotationally analyzed. For example, for formaldehyde, HDCO,  $\text{H}_2^{13}\text{CO}$ ,  $\text{H}_2\text{C}^{18}\text{O}$ , and  $\text{H}_2\text{CO}$  are needed. In the case of formaldehyde an equilibrium or  $r_e$  structure has been determined<sup>19</sup> with  $r_e(\text{CH}) = 1.100 \text{ \AA}$ ,  $r_e(\text{CO}) = 1.203 \text{ \AA}$ , and  $\theta_e(\text{HCH}) = 116^\circ 8'$ .

### Problems

1. Classify each of the following molecules as spherical, symmetric, or asymmetric top molecules:

- (a)  $\text{CH}_4$
- (b)  $\text{CH}_3\text{F}$
- (c)  $\text{CH}_3\text{D}$
- (d)  $\text{SF}_6$
- (e)  $\text{SF}_5\text{Br}$
- (f) *trans*- $\text{SF}_4\text{Br}_2$
- (g) *cis*- $\text{SF}_4\text{Br}_2$
- (h)  $\text{HCN}$
- (i)  $\text{H}_2\text{S}$
- (j)  $\text{C}_5\text{H}_8$ , spiropentane.

2. (a) Show for a linear triatomic molecule made of atoms with masses  $m_1$ ,  $m_2$ , and  $m_3$  that

$$I = (1/M)(m_1m_2r_{12}^2 + m_1m_3r_{13}^2 + m_2m_3r_{23}^2) \quad (6.155)$$

with  $M$  the total mass of the molecule.

- (b) The lowest frequency microwave transitions of  $^1\text{H}^{12}\text{C}^{14}\text{N}$  and  $^2\text{H}^{12}\text{C}^{14}\text{N}$  occur at 88631 and 72415 MHz, respectively. (These are for the ground vibrational state.) Calculate the bond distances in  $\text{HCN}$ .
3. A triatomic molecule has the formula  $\text{A}_2\text{B}$ . Its microwave spectrum shows strong lines at 15, 30, 45, ... GHz, and no other lines. Which of the following structures is (are) compatible with this spectrum?
- (a) linear AAB
  - (b) linear ABA
  - (c) bent AAB
  - (d) bent ABA

4. For the  $^{12}\text{C}^{32}\text{S}$  molecule the following millimeter wave pure rotational transitions have been observed (in MHz):

Transition	$v = 0$	$v = 1$	$v = 2$
$J = 1 - 0$	48 990.978	48 635.977	48 280.887
$J = 2 - 1$	97 980.950	97 270.980	96 560.800
$J = 3 - 2$	146 969.033	145 904.167	144 838.826
$J = 4 - 3$	195 954.226	194 534.321	193 113.957

- (a) For each vibrational level derive a set of rotational constants by fitting the data.
- (b) From results of (a) derive an expression (by fitting) for the vibrational dependence of  $B$ .
- (c) From  $B_0$  calculate  $r_0$ ; from  $B_e$  calculate  $r_e$ .
5. The  $\text{F}_2\text{O}$  molecule of  $C_{2v}$  symmetry has an O—F bond length of 1.405 Å and a FOF bond angle of 103.0°.
- (a) Calculate  $A$ ,  $B$ , and  $C$  for  $\text{F}_2\text{O}$ .
- (b) Will the microwave spectrum of  $\text{F}_2\text{O}$  show  $a$ -,  $b$ -, or  $c$ -type transitions?
- (c) Predict the frequency of the  $J = 1 - 0$  microwave transition.
6. For the  $\text{BF}_3$  molecule of  $D_{3h}$  symmetry the B—F bond length is 1.310 Å. Calculate  $A$ ,  $B$ , and  $C$ . What is the rotational energy-level expression?
7. The  $J = 2 \leftarrow 1$  microwave absorption is observed near 42 723 MHz for  $^{14}\text{NF}_3$  and 42 517 MHz for  $^{15}\text{NF}_3$ .
- (a) Derive the rotation constants for  $^{14}\text{NF}_3$  and  $^{15}\text{NF}_3$ .
- (b) Determine the N—F bond length and the F—N—F bond angle.
8. The following is a complete list of observed transitions involving levels  $J = 0, 1,$  and 2 for two isotopic forms of formaldehyde in their vibrational ground states:

$\text{H}_2^{12}\text{C}^{16}\text{O}$ (MHz)	$\text{H}_2^{13}\text{C}^{16}\text{O}$ (MHz)
71.14	—
4 829.66	4 593.09
14 488.65	13 778.86
72 837.97	71 024.80
140 839.54	137 449.97
145 602.98	141 983.75
150 498.36	146 635.69

- (a) Assign these microwave transitions for both isotopologue. Assume that  $\text{H}_2\text{CO}$  belongs to the  $C_{2v}$  point group and estimate a molecular geometry using bond-length tables. Assign the spectrum by prediction of the expected rotational spectrum.
- (b) What are  $A$ ,  $B$ , and  $C$  for the two isotopic species? Since we have neglected centrifugal distortion, it will not be possible to fit all transitions exactly with only three rotational constants. Devise a procedure that gives a “best fit” to all lines.
- (c) Explain why the inertial defect

$$\Delta = I_C - I_A - I_B \quad (6.156)$$

is a good test for planarity. Why does  $\text{H}_2\text{CO}$  not have  $\Delta = 0$ ?

- (d) Obtain a best possible geometry for  $\text{H}_2\text{CO}$  using your  $A$ ,  $B$ ,  $C$  values for the two isotopologues.
9. For  $\text{CF}_3\text{I}$ ,  $r_{\text{CF}} = 1.332 \text{ \AA}$ ,  $r_{\text{CI}} = 2.134 \text{ \AA}$ ,  $\theta_{\text{FCF}} = 108^\circ$ .
- (a) Calculate the rotational constants.
- (b) Predict the pattern (be quantitative) of the microwave spectrum.
10. The  $\text{S}_2\text{O}$  molecule is a bent triatomic molecule isovalent with ozone. The S—S bond length is  $1.884 \text{ \AA}$ , the S—O bond length is  $1.465 \text{ \AA}$ , and the SSO bond angle is  $118.0^\circ$ .
- (a) Locate the center of mass and set up the moment of inertia tensor. Pick the  $z$ -axis out of the plane and the  $x$ -axis parallel to the S—S bond.
- (b) Diagonalize the moment of inertia tensor to find  $I_A$ ,  $I_B$ , and  $I_C$ .
- (c) Predict frequencies of the possible transitions from the  $0_{00}$  rotational state.
11. The  $\text{HCl}$  molecule has a  $B_0$  value of  $10.4 \text{ cm}^{-1}$ .
- (a) What are the  $J$  values of the levels with maximum population at 300 K and 2000 K?
- (b) Graph the populations of the  $J$  levels as a function of  $J$  for 300 K and 2000 K.
12. What is the moment of inertia of a cube of uniform density  $\rho$  and sides of length  $a$ ?
13. For the  $\text{HCl}$  molecule with a  $B_0$  value of  $10.4 \text{ cm}^{-1}$  and  $J = 1$ , treat the rotational motion classically.
- (a) What is the period of rotation?
- (b) What is the linear velocity of the H atom?
- (c) What is the angular momentum?

14. Derive equation (6.46) by showing that the total energy of the distorted molecule

$$E = \frac{J^2}{2\mu r_c^2} + \frac{1}{2}k(r_c - r_e)^2 \quad (6.157)$$

gives rise to a centrifugal distortion term of approximately  $E_{cd} = -DJ^4 = -\hbar^4 J^4 / (2\mu^2 r_e^6 k)$  for a nonrigid rotor. Show that this term gives the Kratzer formula (6.48) for  $D$ . (*Hint*: Remember to replace the classical  $J$  by  $\hbar J$  at the end.)

15. Check the commutation relationships (6.107) and (6.108) using the differential form of the operators.
16. The rotational constants for the ground vibrational state of  $\text{CH}_3\text{I}$  are found to be  $B = 0.25022 \text{ cm}^{-1}$ ,  $A = 5.1739 \text{ cm}^{-1}$ ,  $D_J = 2.09 \times 10^{-7} \text{ cm}^{-1}$ ,  $D_{JK} = 3.29 \times 10^{-6} \text{ cm}^{-1}$ , and  $D_K = 87.6 \times 10^{-6} \text{ cm}^{-1}$ .
- (a) Predict the microwave spectrum of the  $J = 1 \leftarrow 0$  and  $J = 4 \leftarrow 3$  transitions. (Ignore the possibility of hyperfine structure.)
17. (a) Show that the rigid rotor Hamiltonian operator (6.138) is equivalent to (6.134).
- (b) Derive the matrix elements (6.141) and (6.142).
- (c) Construct the Hamiltonian matrix for  $J = 2$  and derive the equations (for  $J = 2$ ) in Table 6.2. It is helpful to transform the Hamiltonian matrix to a new basis set defined by  $|JK\pm\rangle = (|JK\rangle \pm |J-K\rangle)/\sqrt{2}$  via the Wang transformation matrix

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (6.158)$$

18. Consider the HOD (partially deuterated water) molecule with bond length  $r = 0.958 \text{ \AA}$  and bond angle  $\theta(\text{HOD}) = 104.5^\circ$ .
- (a) Find the moment of inertia tensor in  $\text{u \AA}^2$  in any convenient coordinate system with the origin at the center of mass.
- (b) Find the principal axis moments of inertia, rotational constants (MHz), and transformation from the original axis system to the principal axis system.
19. For the symmetric top wavefunction  $|J, K, M\rangle = C e^{i\chi} e^{-2i\phi} \sin\theta (3\sin^2\theta + 2\cos\theta - 2)$ , find  $J$ ,  $K$ , and  $M$ .
20. The application of an electric field to a molecular system partially lifts the  $M_J$  degeneracy. This Stark effect may be treated as a perturbation of the rotational energies. The perturbation Hamiltonian operator  $\hat{H}' = -\mu_z E_z$ , where  $z$  is a laboratory frame coordinate and  $E_z$  is the electric field along the laboratory  $z$ -axis.
- (a) Show that there will be no first-order Stark effect for a linear molecule.

- (b) Develop a formula for the second-order Stark effect of a linear molecule.
21. Fill in the missing steps between equations (6.74) and (6.75), and between (6.75) and (6.76).
22. The typical volume mixing ratio for CO in the earth's atmosphere is 0.12 ppm. For the  $J = 1 \rightarrow 0$  transition, what is absorption coefficient,  $\alpha$ , per kilometer at line center? The dipole moment is 0.1101 D and the pressure broadening coefficient for air is 3.14 MHz/Torr. Assume 1 atm pressure and 20°C. Repeat the calculation for pure CO at the same concentration and temperature, but in the absence of air. In this case, the pressure broadening coefficient is 3.43 MHz/Torr for pure CO. (You will need to check whether Doppler or pressure broadening predominates.)
23. A convenient catalog of pure rotational line positions and line intensities is maintained by H. Pickett and coworkers (<http://spec.jpl.nasa.gov/>). The line intensities are given as an effective integrated absorption cross sections (i.e., to remove  $g(\nu - \nu_{10})$ ) in units of  $\text{nm}^2 \text{ MHz}$  at 300 K obtained by dividing equation (6.86) by  $N$ , and reported as a logarithm to the base ten.
- (a) For the CO molecule ( $\mu_0 = 0.1101 \text{ D}$ ) verify that the tabulated value of  $-4.1197$  for the  $J = 2 \leftarrow 1$  transition is correct.
- (b) Why does the value obtained in (a) differ from the integrated cross section obtained using equation (6.83)? What is the Einstein  $A_{2 \rightarrow 1}$  coefficient and the oscillator strength  $f_{\text{abs}}$  for the  $J = 2 \rightarrow 1$  transition?
- (c) Derive the conversion factor from "JPL units" of  $\text{nm}^2 \text{ MHz}$  to "HITRAN units" (see Chapter 7) of  $\text{cm}/\text{molecule}$  for line intensities.
24. Radio astronomers measure the intensity of their lines by the "antenna temperature,"  $T_a$ . At long wavelengths the Rayleigh-Jeans approximation applies and the power detected is directly proportional to the source temperature. Assume that the source emitting the line is optically thin, i.e., negligible reabsorption of the emitted radiation occurs.
- (a) Show that the radiance  $L_\nu$  (see Chapter 1, equation (1.44)) detected by a radio telescope can be converted into an equivalent temperature of a blackbody by the equation

$$T_a = \frac{c^2}{2k\nu^2} L_\nu.$$

- (b) If the antenna temperature is integrated over frequency to obtain the area of the line, show that

$$\int T_a d\nu = \frac{hc^2 A_{1 \rightarrow 0} (N_1 l)}{8\pi k \nu_{10}}$$

with  $A_{1 \rightarrow 0}$  the Einstein  $A$  factor for emission from the upper state  $|1\rangle$  to  $|0\rangle$  and  $N_1 l$  is the upper state column density ( $\text{molecules}/\text{m}^2$ ).

- (c) Assuming that the upper state  $|1\rangle$  is in thermodynamic equilibrium with an excitation temperature  $T_{\text{ex}}$ , derive the expressions for the total column density  $Nl$ ,

$$Nl = \frac{8\pi k\nu_{10}q}{c^2 h g_1 A_{1\rightarrow 0}} e^{E_1/kT_{\text{ex}}} \int T_a d\nu$$

or converting from frequency units  $\nu$  to velocity units  $v$  using the Doppler formula,

$$Nl = \frac{8\pi k\nu_{10}^2 q}{c^3 h g_1 A_{1\rightarrow 0}} e^{E_1/kT_{\text{ex}}} \int T_a dv$$

with  $E_1$  the energy of the upper state,  $g_1$  the upper state degeneracy, and  $q$  the partition function. Radio astronomers like to measure line widths and line shifts in velocity units. Thus spectra are usually displayed with units of  $T_a$  for the  $y$ -axis and velocity,  $v$ , rather than frequency,  $\nu$ , for the  $x$ -axis, with the zero of the  $x$ -axis set at the laboratory rest frequency of the line.

- (d) Emission from the  $\text{C}^{18}\text{O } J = 1 \rightarrow 0$  transition is observed at a rest frequency of 109 782.173 MHz. This emission was detected in a molecular cloud DR21 with an integrated line intensity of  $12 \text{ K km s}^{-1}$ . Assuming thermodynamic equilibrium with an excitation temperature of 25 K, what is the column density of  $\text{C}^{18}\text{O}$  in the source? (See question 23.)

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# Chapter 7

## Vibrational Spectroscopy

### 7.1 Diatomic Molecules

The solution of the Schrödinger equation for a diatomic molecule plays an important role in spectroscopy. The study of the vibration-rotation spectra of diatomic molecules is an area of spectroscopy with many practical applications. In addition the vibrational spectra of diatomics serve as models for polyatomic molecules.

Consider a diatomic molecule A—B rotating and vibrating in the laboratory coordinate system  $X, Y, Z$  (Figure 7.1). The motion of the two nuclei can always be exactly separated into a center-of-mass part and an internal part by using the internal coordinates

$$\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A \quad (7.1a)$$

and the definition for the center-of-mass position,

$$\mathbf{R} = \frac{m_A \mathbf{r}_A + m_B \mathbf{r}_B}{m_A + m_B} = \frac{m_A \mathbf{r}_A + m_B \mathbf{r}_B}{M} \quad (7.1b)$$

with

$$M = m_A + m_B. \quad (7.2)$$

These two equations can be solved for  $\mathbf{r}_A$  and  $\mathbf{r}_B$  in terms of  $\mathbf{R}$  and  $\mathbf{r}$  to give

$$\mathbf{r}_A = \mathbf{R} - \frac{m_B}{M} \mathbf{r} \quad (7.3)$$

and

$$\mathbf{r}_B = \mathbf{R} + \frac{m_A}{M} \mathbf{r}. \quad (7.4)$$

The corresponding velocities are given by the time derivatives of  $\mathbf{r}_A$  and  $\mathbf{r}_B$ , that is,  $\dot{\mathbf{r}}_A$  and  $\dot{\mathbf{r}}_B$ . If the velocities obtained from equations (7.3) and (7.4) are substituted into the kinetic energy expression, it becomes

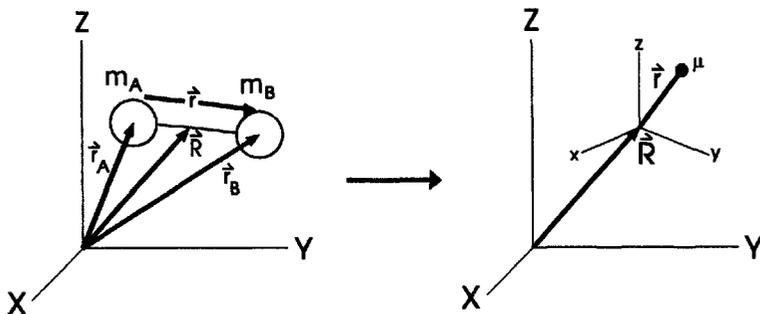


Figure 7.1: Center-of-mass transformation of a two-particle system.

$$\begin{aligned}
 T &= \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 \\
 &= \frac{1}{2}m_A \left( \dot{\mathbf{R}} - \frac{m_B \dot{\mathbf{r}}}{M} \right) \cdot \left( \dot{\mathbf{R}} - \frac{m_B \dot{\mathbf{r}}}{M} \right) \\
 &\quad + \frac{1}{2}m_B \left( \dot{\mathbf{R}} + \frac{m_A \dot{\mathbf{r}}}{M} \right) \cdot \left( \dot{\mathbf{R}} + \frac{m_A \dot{\mathbf{r}}}{M} \right), \tag{7.5}
 \end{aligned}$$

which simplifies to

$$T = \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{1}{2}\mu|\dot{\mathbf{r}}|^2, \tag{7.6}$$

with  $\mu$  the reduced mass given by

$$\mu = \frac{m_A m_B}{m_A + m_B}. \tag{7.7}$$

The kinetic energy has thus been split in equation (7.6) into an overall center-of-mass term and an equivalent one-particle (mass  $\mu$ ) term (Figure 7.1). Note that the usual atomic masses (not the masses of bare nuclei) are used in equation (7.7) because the electrons are considered to be bound to the nuclei during vibrational motion.

By expressing the kinetic energy in terms of the momentum rather than velocity, one obtains the classical Hamiltonian for the two-particle system,

$$\begin{aligned}
 H &= \frac{p_A^2}{2m_A} + \frac{p_B^2}{2m_B} + V(r) \\
 &= \frac{p_R^2}{2M} + \frac{p_r^2}{2\mu} + V(r), \tag{7.8}
 \end{aligned}$$

in which the potential energy depends only upon the distance  $r$  between the atoms. The center-of-mass contribution to the kinetic energy is ignored, since it only represents a shift in the total energy of the system. The quantum mechanical Schrödinger equation for a vibrating rotor is therefore

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V(r)\psi = E\psi. \tag{7.9}$$

Upon replacing the Cartesian coordinates  $x$ ,  $y$ , and  $z$  for the location of the equivalent mass in equation (7.9) by the spherical polar coordinates  $r$ ,  $\theta$ , and  $\phi$ , one obtains

$$\frac{-\hbar^2}{2\mu} \left( \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial \psi}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right) + V(r)\psi = E\psi, \quad (7.10)$$

or

$$\frac{-\hbar^2}{2\mu} \left( \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{2\mu r^2} \hat{J}^2 \psi + V(r)\psi = E\psi, \quad (7.11)$$

in which  $\hat{J}^2$  is the operator representing the square of the angular momentum. Substitution of

$$\psi = R(r)Y_{JM}(\theta, \phi), \quad (7.12)$$

in which  $Y_{JM}$  is a spherical harmonic, into equation (7.11) yields the one-dimensional radial Schrödinger equation,

$$\frac{-\hbar^2}{2\mu r^2} \frac{d}{dr} r^2 \frac{dR}{dr} + \left( \frac{\hbar^2 J(J+1)}{2\mu r^2} + V(r) \right) R = ER. \quad (7.13)$$

Let us define

$$\frac{\hbar^2 J(J+1)}{2\mu r^2} = V_{\text{cent}} \quad (7.14)$$

as the centrifugal potential, and the sum

$$V(r) + V_{\text{cent}} = V_{\text{eff}} \quad (7.15)$$

as the effective potential. Only a specific functional form of  $V(r)$  is needed in order to obtain the energy levels and wavefunctions of the vibrating rotor by solving equation (7.13). The substitution

$$S(r) = rR(r) \quad (7.16)$$

into equation (7.13) leads to the equation

$$\frac{-\hbar^2}{2\mu} \frac{d^2 S}{dr^2} + \left( \frac{\hbar^2 J(J+1)}{2\mu r^2} + V(r) \right) S = ES. \quad (7.17)$$

In general  $V(r) = E_{\text{el}}(r) + V_{NN}$  (Chapter 4) where  $E_{\text{el}}$  is obtained by solving the electronic Schrödinger equation

$$\hat{H}_{\text{el}}\psi = E_{\text{el}}\psi. \quad (7.18)$$

For the electronic Schrödinger equation (7.18) the energy depends on the particular value of  $r$  chosen for the calculation. As a result,  $E_{\text{el}}$  is a parametric function of  $r$ , so that no simple analytical form for  $E_{\text{el}}(r)$  exists in general. Instead, considerable effort has been devoted to developing empirical forms for  $V(r)$ , the typical shape of which is shown in Figure 7.2. One of the most general forms, often denoted as the Dunham potential,<sup>1</sup> is a Taylor series expansion about  $r_e$ , namely

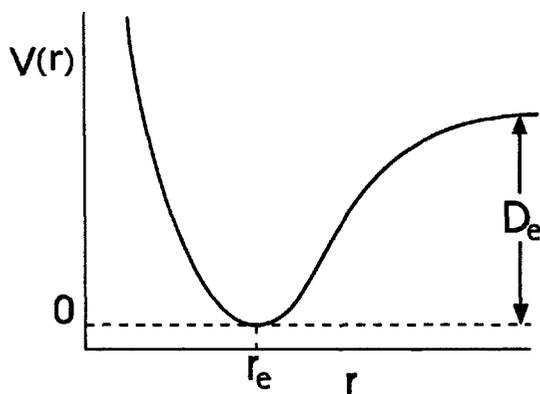


Figure 7.2: Potential energy curve for a diatomic molecule.

$$V(r) = V(r_e) + \left. \frac{dV}{dr} \right|_{r_e} (r - r_e) + \frac{1}{2} \left. \frac{d^2V}{dr^2} \right|_{r_e} (r - r_e)^2 + \dots \quad (7.19)$$

By setting  $V(r_e) = 0$ , the bottom of the well is arbitrarily chosen as the point of zero energy. In the expansion of  $V(r)$  about its minimum at  $r_e$ , the first derivative is zero,

$$\left. \frac{dV}{dr} \right|_{r_e} = 0 \quad (7.20)$$

and therefore

$$V(r) = \frac{1}{2}k(r - r_e)^2 + \frac{1}{6}k_3(r - r_e)^3 + \frac{1}{24}k_4(r - r_e)^4 + \dots \quad (7.21)$$

with

$$k = \left. \frac{d^2V}{dr^2} \right|_{r_e} \quad (7.22)$$

and

$$k_n = \left. \frac{d^n V}{dr^n} \right|_{r_e} \quad (7.23)$$

By retaining only the leading term  $\frac{1}{2}k(r - r_e)^2$  in this expansion for the nonrotating molecule ( $J = 0$ ), one obtains the harmonic oscillator solutions

$$S = N_v H_v(\sqrt{\alpha}x) e^{-\alpha x^2/2}, \quad (7.24)$$

with

$$x = r - r_e, \quad \alpha = \frac{\mu\omega}{\hbar}, \quad N_v = \left( \frac{1}{2^v v!} \left( \frac{\alpha}{\pi} \right)^{1/2} \right)^{1/2}.$$

The functions  $H_v(\sqrt{\alpha}x)$  are the Hermite polynomials listed in Table 7.1. The corresponding eigenvalues for the nonrotating harmonic oscillator are

Table 7.1: The Hermite Polynomials  $H_v(x)$ 


---

$H_0 = 1$	$H_4 = 16x^4 - 48x^2 + 12$
$H_1 = 2x$	$H_5 = 32x^5 - 160x^3 + 120x$
$H_2 = 4x^2 - 2$	$H_6 = 64x^6 - 480x^4 + 720x^2 - 120$
$H_3 = 8x^3 - 12x$	$H_v(x) = (-1)^v e^{x^2} d^v/dx^v e^{-x^2}$

---

$$\begin{aligned}
 E_v &= h\nu(v + \frac{1}{2}), & v = 0, 1, 2, \dots \\
 &= \hbar\omega(v + \frac{1}{2})
 \end{aligned}
 \tag{7.25}$$

with

$$\omega = \left(\frac{k}{\mu}\right)^{1/2}, \quad \nu = \frac{1}{2\pi} \left(\frac{k}{\mu}\right)^{1/2}.
 \tag{7.26}$$

Another popular choice for a simple form for the potential function is the Morse potential<sup>2</sup>

$$V(r) = D(1 - e^{-\beta(r-r_e)})^2.
 \tag{7.27}$$

The Morse potential, unlike the harmonic oscillator, asymptotically approaches a dissociation limit  $V(r) = D$  as  $r \rightarrow \infty$ . Moreover, the Schrödinger equation can be solved analytically for the Morse potential. Specifically, one can show<sup>2</sup> that the eigenvalues for the Morse potential (plus the centrifugal term) can be written as (customarily with  $E$  in  $\text{cm}^{-1}$ , rather than joules):

$$\begin{aligned}
 E(\text{cm}^{-1}) &= \omega_e(v + \frac{1}{2}) - \omega_e x_e(v + \frac{1}{2})^2 + B_e J(J+1) - D_e(J(J+1))^2 \\
 &\quad - \alpha_e(v + \frac{1}{2})J(J+1)
 \end{aligned}
 \tag{7.28}$$

with

$$\omega_e = \beta \left(\frac{Dh \times 10^2}{2\pi^2 c \mu}\right)^{1/2},
 \tag{7.29}$$

$$\omega_e x_e = \frac{h\beta^2 \times 10^2}{8\pi^2 \mu c},
 \tag{7.30}$$

$$B_e = \frac{h \times 10^{-2}}{8\pi^2 \mu r_e^2 c},
 \tag{7.31}$$

$$D_e = \frac{4B_e^3}{\omega_e^2}
 \tag{7.32}$$

and

$$\alpha_e = \frac{6(\omega_e x_e B_e^3)^{1/2}}{\omega_e} - \frac{6B_e^2}{\omega_e}. \quad (7.33)$$

When using equations (7.28) to (7.33), some care with units is required since all spectroscopic constants and the Morse potential parameter  $\beta$  (equation (7.27)) are in  $\text{cm}^{-1}$ , while SI units are used for the physical constants. Note that in these equations  $D_e$  denotes the centrifugal distortion constant (equation (7.28)) as opposed to  $D$ , which denotes the dissociation energy, equation (7.27). The equation  $D_e = 4B_e^3/\omega_e^2$ , equation (7.32), applies to all realistic diatomic potentials and is known as the *Kratzer relationship*. The equation for  $\alpha_e$ , equation (7.33), applies only to the Morse potential and is often referred to as the *Pekeris relationship*. Notice that the vibrational energy expression

$$G(v) = \omega_e(v + \frac{1}{2}) - \omega_e x_e(v + \frac{1}{2})^2 \quad (7.34)$$

for the Morse oscillator has exactly two terms, and  $G(v)$  is the customary symbol for the vibrational energy levels. In contrast, the rotational parts of the Morse oscillator energy-level equation (7.28) are only the leading terms of a series solution.

An even more general form than the Morse potential is the Dunham potential<sup>1</sup>

$$V(\xi) = a_0 \xi^2 (1 + a_1 \xi + a_2 \xi^2 + \dots) \quad (7.35)$$

with

$$\xi = \frac{r - r_e}{r_e}. \quad (7.36)$$

The Dunham potential is just the Taylor series expansion (7.21) with some minor changes in notation such as

$$a_0 = \frac{k r_e^2}{2} = \frac{\omega_e^2}{4B_e}. \quad (7.37)$$

Although exact analytical forms for the wavefunctions and eigenvalues are impossible to derive for the Dunham potential, approximate analytical forms are relatively easy to obtain.

Dunham obtained an analytical expression for the energy levels of the vibrating rotor by using the first-order semiclassical quantization condition<sup>3</sup> from WKB (Wentzel–Kramers–Brillouin) theory, specifically

$$\left(\frac{2\mu}{\hbar^2}\right)^{1/2} \int_{r_-}^{r_+} \sqrt{E - V(r)} dr = (v + \frac{1}{2})\pi, \quad (7.38)$$

in which  $r_-$  and  $r_+$  are the classical inner and outer turning points for  $V(r)$  at the energy  $E$ . The approximate wavefunctions are given by

$$\psi = A \exp\left(\pm i \left(\frac{2\mu}{\hbar^2}\right)^{1/2} \int (E - V(r))^{1/2} dr\right), \quad (7.39)$$

and the energy levels are given by

$$E_{vJ} = \sum_{jk} Y_{jk} (v + \frac{1}{2})^j (J(J + 1))^k. \quad (7.40)$$

Dunham<sup>1</sup> was able to relate the coefficients  $Y_{jk}$  back to the potential energy parameters  $a_i$  by a series of equations listed, for example, in Townes and Schawlow.<sup>4</sup> The customary energy-level expressions<sup>5</sup>

$$F_v(J) = B_v J(J+1) - D_v (J(J+1))^2 + H_v (J(J+1))^3 + \dots \quad (7.41)$$

$$G(v) = \omega_e (v + \frac{1}{2}) - \omega_e x_e (v + \frac{1}{2})^2 + \omega_e y_e (v + \frac{1}{2})^3 + \omega_e z_e (v + \frac{1}{2})^4 + \dots \quad (7.42)$$

$$B_v = B_e - \alpha_e (v + \frac{1}{2}) + \gamma_e (v + \frac{1}{2})^2 + \dots \quad (7.43)$$

$$D_v = D_e + \beta_e (v + \frac{1}{2}) + \dots \quad (7.44)$$

allow the relationships<sup>4</sup> between the Dunham  $Y_{jk}$  parameters and the conventional spectroscopic constants to be derived:

$$\begin{aligned} Y_{10} &\approx \omega_e & Y_{20} &\approx -\omega_e x_e & Y_{30} &\approx \omega_e y_e \\ Y_{01} &\approx B_e & Y_{11} &\approx -\alpha_e & Y_{21} &\approx \gamma_e \\ Y_{02} &\approx -D_e & Y_{12} &\approx -\beta_e & Y_{40} &\approx \omega_e z_e \\ Y_{03} &\approx H_e & & & & \end{aligned}$$

The various isotopic forms of a molecule have different spectroscopic constants because the reduced mass is different. The pattern of isotopic mass dependence for a few of the spectroscopic constants can be seen in equations (7.29) to (7.33)—that is,  $\omega_e \propto \mu^{-1/2}$ ,  $B_e \propto \mu^{-1}$ ,  $\omega_e x_e \propto \mu^{-1}$ ,  $D_e \propto \mu^{-2}$ , and  $\alpha_e \propto \mu^{-3/2}$ . In general the isotopic dependence of the Dunham  $Y_{jk}$  constants is given by

$$Y_{jk} \propto (\mu^{-j/2})(\mu^{-k}) = \mu^{-(j+2k)/2}. \quad (7.45)$$

Defining a set of mass-independent constants  $U_{jk}$  using the relationship

$$U_{jk} = \mu^{(j+2k)/2} Y_{jk}, \quad (7.46)$$

enables one to combine spectroscopic data from different isotopic forms of a molecule using the single equation

$$E_{vJ} = \sum_{jk} \mu^{-(j+2k)/2} U_{jk} (v + \frac{1}{2})^j (J(J+1))^k. \quad (7.47)$$

When the Born–Oppenheimer approximation breaks down and the first-order WKB condition of equation (7.38) is inadequate, small correction terms<sup>6,7</sup> must be added to equation (7.47).

Although the Dunham energy level formula (7.40) is widely used to represent energy levels, the Dunham relationships<sup>4</sup> between the  $Y$ 's and the potential parameters ( $a$ 's) are used more rarely. For diatomic molecules,  $V(r)$  potentials are typically derived from the  $G(v)$ , equation (7.42), and  $B_v$ , equation (7.43), polynomials by application of the Rydberg–Klein–Rees (RKR) method using readily available computer programs.<sup>8</sup> The RKR method starts with the WKB quantization, equation (7.38), which is manipulated extensively<sup>8,9</sup> to give the two Klein integrals (in SI units),

$$r_+ - r_- = \sqrt{2\hbar^2/\mu} \int_{v_{\min}}^v \frac{dv'}{\sqrt{G(v) - G(v')}} \quad (7.48)$$

$$\frac{1}{r_-} - \frac{1}{r_+} = \sqrt{8\mu/\hbar^2} \int_{v_{\min}}^v \frac{B_v' dv'}{\sqrt{G(v) - G(v')}} \quad (7.49)$$

in which the vibrational quantum number  $v$  is taken as a continuous variable, as is possible in the semiclassical world. The Klein integrals are evaluated numerically<sup>8</sup> from the semiclassical  $v_{\min} = -1/2$  at the bottom of the well to the  $v$  of interest and give the difference between the classical turning points, equation (7.48), or the difference of the reciprocals of the classical turning points, equation (7.49). The two equations, (7.48) and (7.49), can be solved to give the two unknown classical turning points  $r_+$  and  $r_-$  at integer values of  $v$ , and additional points can be generated by using noninteger  $v$ 's.

The RKR procedure generates the diatomic potential  $V(r)$  as pairs of classical turning points that can be interpolated and used to solve the one-dimensional vibration-rotation Schrödinger equation (7.17). Rather than use the semiclassical quantization condition (7.38) to solve equation (7.17) analytically, the differential equation (7.17) is numerically integrated<sup>10</sup> to obtain the vibration-rotation energy levels  $E_{vJ}$  and the corresponding wavefunctions  $\psi_{vJ}(r)$ , represented as points on a grid. These numerical energy levels and wavefunctions can be used for a variety of purposes such as the computation of rotational constants, centrifugal distortion constants, and Franck-Condon factors (see Chapter 9).<sup>10</sup>

### *Wavefunctions for Harmonic and Anharmonic Oscillators*

The harmonic oscillator wavefunctions are given in Table 7.2 and are plotted in Figure 7.3. There are several notable features of these wavefunctions, including a finite probability density outside the walls of the confining potential. As the vibrational quantum number  $v$  increases, the probability for the oscillator being found near a classical turning point increases.

A diatomic molecule behaves like an anharmonic oscillator because the inner wall of a realistic potential is steeper than the harmonic oscillator potential, while the outer wall is much less steep than the harmonic oscillator (Figure 7.4). For small values of  $v$ , the harmonic oscillator model provides a reasonable approximation and the differences between the harmonic and anharmonic oscillator wavefunctions are small. As  $v$  increases, however, the amplitude of the wavefunction for the anharmonic oscillator increases at the outer turning point relative to its value at the inner turning point because the system spends most of its time at large  $r$  (Figure 7.5). The harmonic oscillator approximation is then no longer realistic.

Table 7.2: The Harmonic Oscillator Wavefunctions

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$$\begin{aligned} \psi_0 &= \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2} \\ \psi_1 &= \sqrt{2} \left(\frac{\alpha}{\pi}\right)^{1/4} \alpha^{1/2} x e^{-\alpha x^2/2} \\ \psi_2 &= \frac{1}{\sqrt{2}} \left(\frac{\alpha}{\pi}\right)^{1/4} (2\alpha x^2 - 1) e^{-\alpha x^2/2} \\ \psi_3 &= \sqrt{3} \left(\frac{\alpha}{\pi}\right)^{1/4} (2\alpha^{3/2} x^3/3 - \alpha^{1/2} x) e^{-\alpha x^2/2} \\ \psi_4 &= \frac{1}{\sqrt{6}} \left(\frac{\alpha}{\pi}\right)^{1/4} (2\alpha^2 x^4 - 6\alpha x^2 + 3/2) e^{-\alpha x^2/2} \\ \psi_5 &= \frac{1}{\sqrt{15}} \left(\frac{\alpha}{\pi}\right)^{1/4} (2\alpha^{5/2} x^5 - 10\alpha^{3/2} x^3 + 15\alpha^{1/2} x/2) e^{-\alpha x^2/2} \\ \psi_v &= \left(\frac{1}{2^v v!}\right)^{1/2} \left(\frac{\alpha}{\pi}\right)^{1/4} H_v(\alpha^{1/2} x) e^{-\alpha x^2/2}, \quad \alpha = \mu\omega/\hbar \end{aligned}$$


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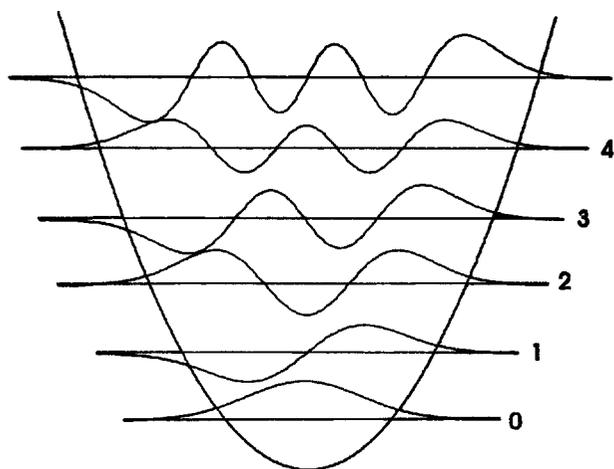


Figure 7.3: The harmonic oscillator wavefunctions.

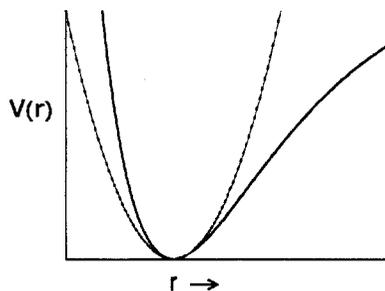


Figure 7.4: A harmonic oscillator potential (dots) as compared to a realistic diatomic potential (solid).

### *Vibrational Selection Rules for Diatomics*

To predict a spectrum from the energy levels, selection rules are required. The intensity of a vibrational transition is governed by the transition dipole moment integral,

$$\mathbf{M}_{v'v''} = \int \psi_{\text{vib}}'^* \boldsymbol{\mu}(r) \psi_{\text{vib}}'' dr, \quad (7.50)$$

in which single primes refer to the upper level of a transition and double primes to the lower. The dipole moment of a diatomic molecule is a function of  $r$  and the functional dependence of  $\boldsymbol{\mu}$  on  $r$  can be determined from Stark effect measurements, from the intensities of infrared bands, or from *ab initio* calculations. As an example, the dipole moment function<sup>11</sup> calculated for the  $X^2\Pi$  ground state of OH is illustrated in Figure 7.6.

Since any well-behaved function can be expanded as a Taylor series, let us expand  $\boldsymbol{\mu}(r)$  about  $r = r_e$  as

$$\boldsymbol{\mu} = \boldsymbol{\mu}_e + \left. \frac{d\boldsymbol{\mu}}{dr} \right|_{r_e} (r - r_e) + \frac{1}{2} \left. \frac{d^2\boldsymbol{\mu}}{dr^2} \right|_{r_e} (r - r_e)^2 + \dots \quad (7.51)$$

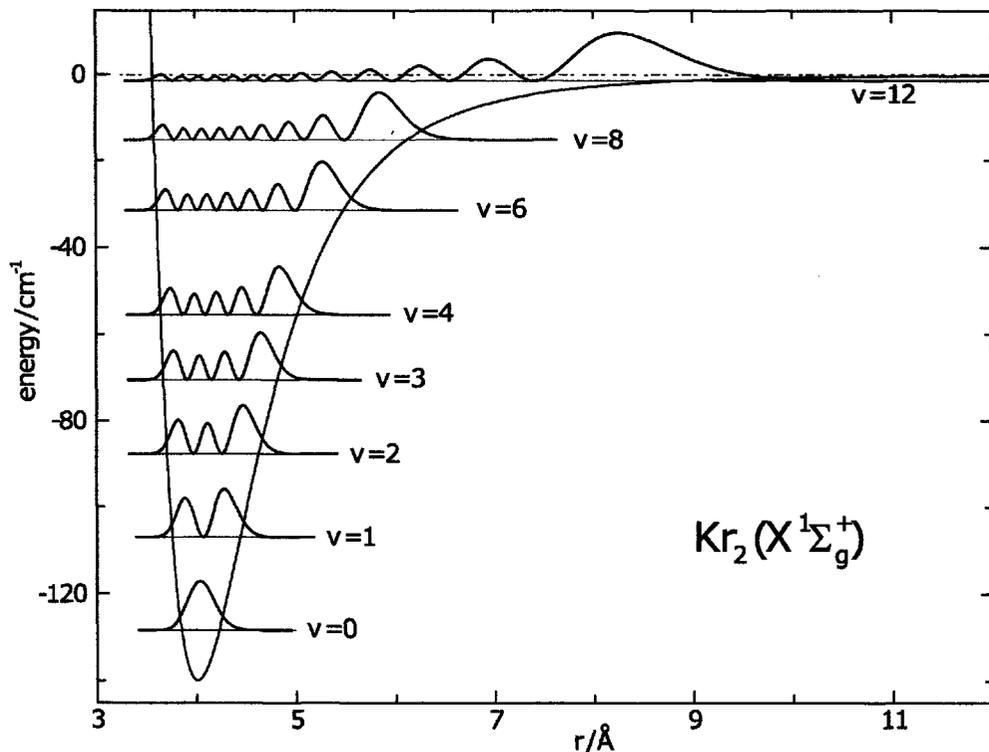


Figure 7.5: The intermolecular potential and the square of the vibrational wavefunctions for  $\text{Kr}_2$ .

so that

$$\mathbf{M}_{v'v''} = \mu_e \int \psi_{\text{vib}}'^* \psi_{\text{vib}}'' dr + \left. \frac{d\mu}{dr} \right|_{r_e} \int \psi_{\text{vib}}'^* (r - r_e) \psi_{\text{vib}}'' dr + \dots \quad (7.52)$$

The first term on the right-hand side of equation (7.52) is exactly zero because different vibrational wavefunctions of the same potential curve are orthogonal to each other. The second term makes the dominant contribution to the intensity of most infrared fundamental transitions and it depends on the value of the dipole moment derivative at the equilibrium distance,  $d\mu/dr|_{r_e}$ . More precisely, the intensity of a vibrational emission or absorption transition is given by

$$I \propto |\mathbf{M}_{v'v''}|^2 \propto \left. \frac{d\mu}{dr} \right|_{r_e}^2. \quad (7.53)$$

This approximation neglects quadratic and higher power terms in equation (7.51) and assumes that the electrical dipole moment function is a linear function of  $r$  in the region close to  $r = r_e$ .

According to equation (6.83), the intensity of pure rotational transitions depend on  $|\mu|^2$ , rather than on the square of the derivative given in equation (7.53), as is the case for vibrational transitions. Since homonuclear diatomic molecules such as  $\text{Cl}_2$  have  $\mu = 0$  and  $d\mu/dr = 0$ , they do not have electric dipole-allowed pure rotational

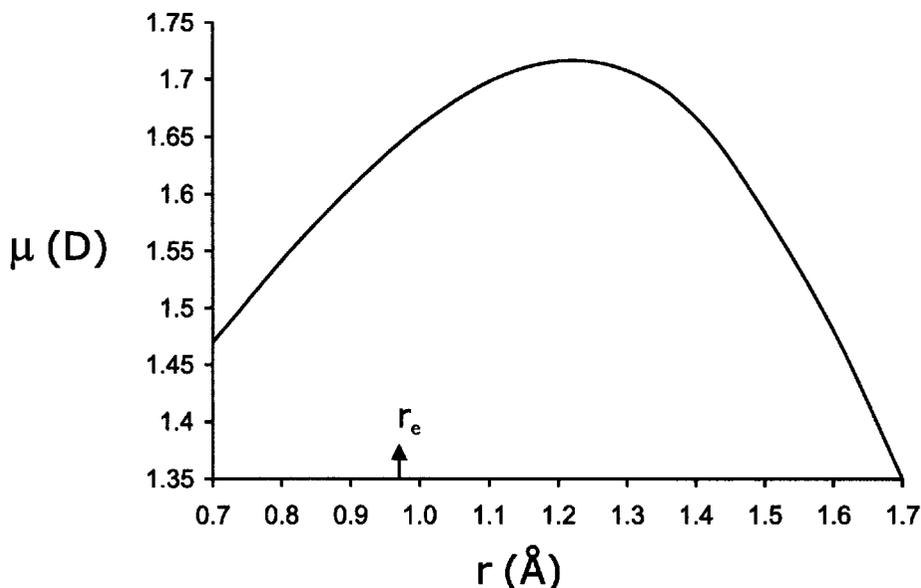


Figure 7.6: The *ab initio* dipole moment function of the  $X^2\Pi$  state of OH.

or vibrational spectra. However, homonuclear diatomic molecules do have very weak electric quadrupole vibrational transitions that can be detected with very long path lengths.<sup>12</sup> These electric quadrupole transitions are about a factor of  $10^{-6}$  weaker than typical electric dipole-allowed infrared transitions.

The intensity of an infrared vibrational transition also depends upon the value of the integral

$$\langle v' | x | v'' \rangle = \int \psi'_{\text{vib}}^*(r - r_e) \psi''_{\text{vib}} dr, \quad (7.54)$$

with  $x = r - r_e$ . If harmonic oscillator wavefunctions (Table 7.2) are used to represent the wavefunctions in equation (7.54), and if the recursion relationship

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x) \quad (7.55)$$

between Hermite polynomials is employed, the result

$$\langle v' | x | v \rangle = \left( \frac{\hbar}{2m\omega} \right)^{1/2} (\sqrt{v+1}\delta_{v',v+1} + \sqrt{v}\delta_{v',v-1}) \quad (7.56)$$

is obtained. The vibrational selection rule is thus  $\Delta v = \pm 1$  for harmonic oscillator wavefunctions since  $v' = v + 1$  or  $v - 1$  in the Kronecker  $\delta$  of equation (7.56).

If anharmonic wavefunctions are used, then transitions with  $\Delta v = \pm 2, \pm 3, \dots$  also become allowed because each anharmonic wavefunction can be represented by an expansion of harmonic oscillator wavefunctions,  $\psi_{i,\text{HO}}$ :

$$\psi_{\text{vib}} = \sum c_i \psi_{i,\text{HO}}. \quad (7.57)$$

Although this mechanical anharmonicity allows overtone transitions to occur, the intensities of such transitions drop with increasing  $\Delta v$ . Typically an increase in  $\Delta v$  by

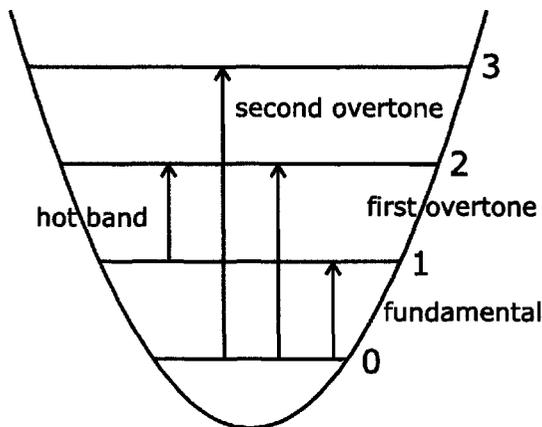


Figure 7.7: Names for infrared vibrational transitions.

one unit diminishes the intensity of an overtone band by a factor of 10 or 20 in infrared absorption spectroscopy. If the dipole moment function, equation (7.51), is not truncated after the linear term, then integrals of the type  $\langle v' | (r - r_e)^2 | v \rangle$  and  $\langle v' | (r - r_e)^3 | v \rangle$  are also present in the transition moment expression, equation (7.52). These terms give rise to matrix elements with  $\Delta v = \pm 2, \pm 3, \dots$  so that they also give overtones of appreciable intensity. The oscillator is said to be “electrically anharmonic” if terms higher than linear are used to represent  $\mu(r)$ . Thus both electrical (equation (7.51)) and mechanical (equation (7.21)) anharmonic terms contribute to the appearance of overtones in a spectrum.

The various types of infrared transitions have specific names associated with them (Figure 7.7). The  $v = 1 \leftarrow 0$  transition is called the *fundamental*, while any transition with  $v'' \neq 0$  is called a *hot band*. The name hot band originates from the experimental observation that the intensities of these bands increase upon heating the sample. The first overtone is the  $v = 2 \leftarrow 0$  transition, the second overtone has  $v = 3 \leftarrow 0$ , and so on.

The mechanical anharmonicity of a diatomic oscillator results in an energy-level expression (7.42)

$$G(v) = \omega_e(v + \frac{1}{2}) - \omega_e x_e(v + \frac{1}{2})^2 + \omega_e y_e(v + \frac{1}{2})^3 + \omega_e z_e(v + \frac{1}{2})^4 + \dots$$

so that a transition between vibrational levels characterized by  $v + 1$  and  $v$  has an associated energy given by

$$\Delta G_{v+1/2} = G(v+1) - G(v) = \omega_e - 2\omega_e x_e(v + \frac{1}{2}) + \omega_e y_e(3v^2 + 6v + 13/4) + \dots \quad (7.58)$$

As an example, for  $\text{H}^{35}\text{Cl}$ , the vibrational energy expression is<sup>13</sup>

$$G(v) = 2990.946(v + \frac{1}{2}) - 52.8186(v + \frac{1}{2})^2 + 0.2244(v + \frac{1}{2})^3 - 0.0122(v + \frac{1}{2})^4 \text{ cm}^{-1}, \quad (7.59a)$$

while for  $\text{D}^{35}\text{Cl}$  the expression is<sup>13</sup>

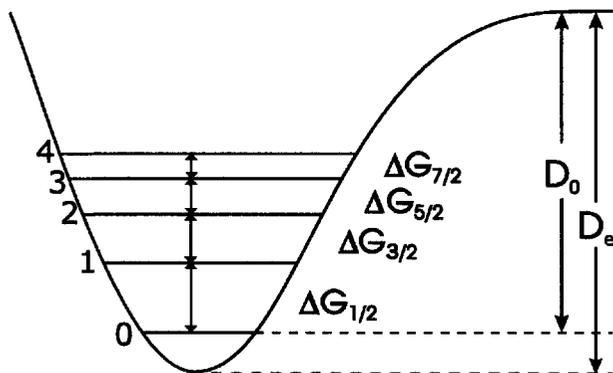


Figure 7.8: The vibrational intervals of a diatomic molecule.

$$G(v) = 2145.163(v + \frac{1}{2}) - 27.1825(v + \frac{1}{2})^2 + 0.08649(v + \frac{1}{2})^3 - 0.00355(v + \frac{1}{2})^4 \text{ cm}^{-1}. \quad (7.59b)$$

Notice that the anharmonicity constants decrease rapidly in magnitude with  $|\omega_e x_e| \gg |\omega_e y_e| \gg |\omega_e z_e|$  in the case of HCl and DCl.

### *Dissociation Energies from Spectroscopic Data*

Under favorable circumstances it is possible to deduce the dissociation energy from spectroscopic data. Indeed, this is usually the most accurate of all methods for determining dissociation energies for diatomic molecules. In principle, if all of the vibrational intervals  $\Delta G_{v+1/2}$  are available, then the dissociation energy  $D_0$  (measured from  $v = 0$ ) is given by the sum of the intervals

$$D_0 = \sum_v \Delta G_{v+1/2} \quad (7.60)$$

as illustrated in Figure 7.8. Graphically this can be represented by a Birge–Sponer plot<sup>14</sup> of  $\Delta G_{v+1/2}$  versus  $v + \frac{1}{2}$  with the dissociation energy given by the area under the curve (see Figure 7.9). If the vibrational energy expression has only two terms  $G(v) = \omega_e(v + \frac{1}{2}) - \omega_e x_e(v + \frac{1}{2})^2$  (e.g., for the Morse oscillator), then  $\Delta G_{v+1/2} = \omega_e - 2\omega_e x_e - 2\omega_e x_e v$ . Thus the Birge–Sponer plot is linear over the entire range of  $v$  and the equilibrium dissociation energy  $D_e$  (Figure 7.8) for a Morse oscillator is

$$D_e = \frac{\omega_e^2}{4\omega_e x_e}. \quad (7.61)$$

This result is derived from equations (7.29) and (7.30). Note that the same symbol  $D_e$  is customarily used for the equilibrium centrifugal distortion constant (equation (7.32)) and for the equilibrium dissociation energy (e.g., equation (7.61)) and needs to be distinguished by the context.

If all of the vibrational levels of a molecule are known, then the simple application of equation (7.60) gives the dissociation energy. Only rarely, however, are all of the vibrational levels associated with a particular electronic state of a molecule known experimentally (e.g.,  $\text{H}_2$ ).<sup>15</sup> In practice an extrapolation from the last few observed levels

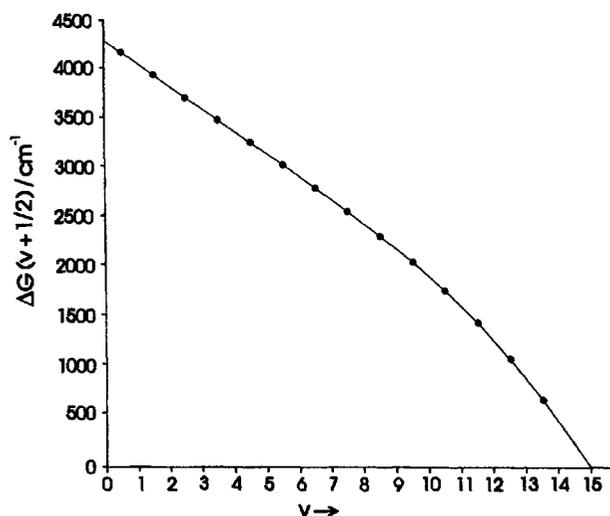


Figure 7.9: A Birge-Sponer plot for the ground state of  $\text{H}_2$ . Notice the curvature at high vibrational levels.

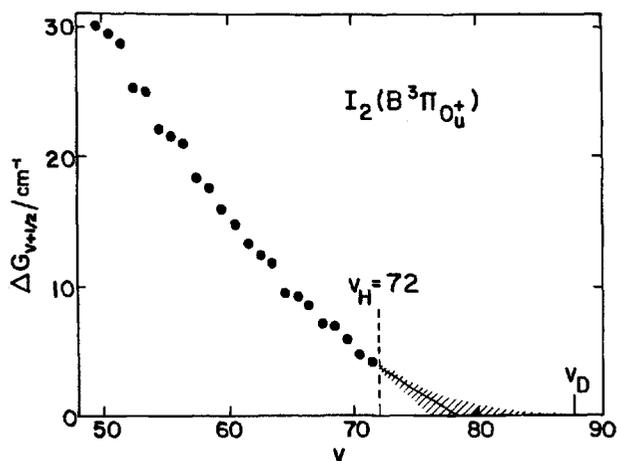


Figure 7.10: A Birge-Sponer plot for the  $B$  state of  $\text{I}_2$ . The highest observed vibrational level is  $v_H = 72$ .

to the unobserved dissociation limit ( $v_D$ ) is necessary. A simple linear extrapolation has often been used, but this typically introduces considerable uncertainty into the exact location of the dissociation limit even when the extrapolation is short (Figure 7.10). The number  $v_D$  is the effective vibrational “quantum number” at dissociation and can be noninteger. It corresponds to the intercept of the Birge-Sponer curve (Figure 7.9) with the  $v$ -axis of the plot.

A more reliable procedure makes use of a Le Roy-Bernstein<sup>16</sup> plot in which the extrapolation is based on the theoretical long-range behavior of the potential. It has been shown that the vibrational spacings and other properties of levels lying near dissociation depend mainly on the long-range part of the potential, which is known to

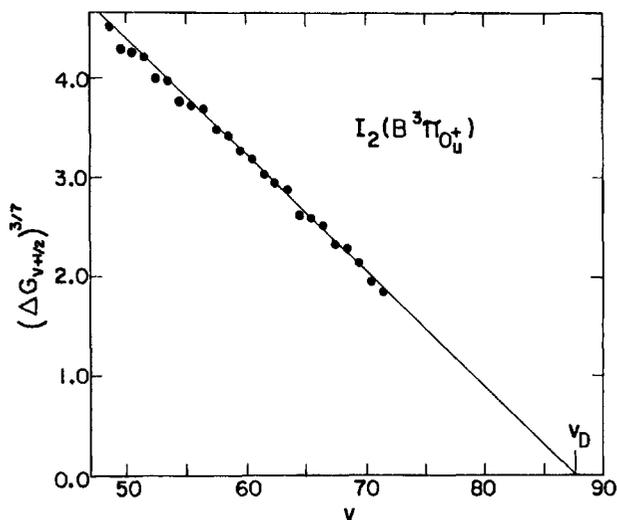


Figure 7.11: A Le Roy-Bernstein plot for the  $B$  state of  $I_2$ .

have the form

$$V(r) = D - \frac{C_n}{r^n} + \dots, \quad (7.62)$$

in which  $D$  is the dissociation energy,  $n$  is an integer (typically 5 or 6 for a neutral molecule), and  $C_n$  is a constant. Substitution of equation (7.62) for  $V(r)$  into the semiclassical quantization condition (equation (7.38)) followed by mathematical manipulation,<sup>17</sup> yields the approximate equation

$$(\Delta G_{v+1/2})^{(n-2)/(n+2)} = (v_D - v - \frac{1}{2})L(n, C_n), \quad (7.63)$$

in which  $L(n, C_n)$  is a constant. A Le Roy-Bernstein plot of  $(\Delta G_{v+1/2})^{(n-2)/(n+2)}$  versus  $v$  is a straight line at long range, so that linear extrapolation gives the dissociation limit marked by  $v_D$  in Figure 7.11. In essence the Le Roy-Bernstein procedure corrects for the curvature of the Birge-Sponer extrapolation (Figure 7.10) by making use of the known form (7.62) of the long-range interaction of two atoms.<sup>17</sup> For the case of the  $B$  state of  $I_2$  this plot shows that the last bound vibrational level is  $v' = 87$ , which contrasts markedly with the uncertainty of the intercept on the conventional Birge-Sponer plot of Figure 7.10.

The leading long-range interaction term  $C_n/r^n$  depends upon the nature of the two interacting atoms. All atom pairs have at least a  $C_6/r^6$  term from the fluctuating induced dipole-induced dipole interaction. For  $I_2$ , however, the leading long range term is  $C_5/r^5$  (this is associated with the quadrupole-quadrupole interaction<sup>17</sup> between the two open-shell  $^2P_{3/2}$  atoms). Indeed, the leading long-range interaction terms are known for all possible combinations of atoms.<sup>17</sup> For the  $B$  state of  $I_2$  the Le Roy-Bernstein plot of  $(\Delta G_{v+1/2})^{3/7}$  versus  $v$  predicts  $v_D = 87.7$  from the old vibrational data of Brown<sup>18</sup> measured in 1931. More recent data,<sup>19</sup> including observations of levels up to  $v = 82$ , have confirmed this value of  $v_D$ .

### Vibration-Rotation Transitions of Diatomics

Molecules vibrate and rotate at the same time, thus giving rise to vibration-rotation spectra. The selection rules for a diatomic molecule are obtained simply by combining the pure rotational selection rules  $\Delta J = \pm 1$  with the vibrational selection rules ( $\Delta v = \pm 1$ ). The selection rules  $\Delta J = \pm 1$  apply to molecules with no net spin or orbital angular momentum (i.e.,  $^1\Sigma^+$  states). For molecules such as NO ( $X^2\Pi$ ), and free radicals in which  $\hat{L}$  or  $\hat{S}$  are nonzero, weak  $Q$  branches ( $\Delta J = 0$ ) are also possible (Chapter 9).

Transitions are organized into branches on the basis of the change in  $J$  value. For one-photon, electric-dipole-allowed transitions only  $\Delta J = 0, \pm 1$  are possible, but for Raman transitions (Chapter 8), multiphoton transitions, or electric quadrupole transitions, there are additional possibilities. Magnetic dipole transitions like electric dipole transitions have only  $\Delta J = 0, \pm 1$ . Transitions with  $\Delta J = -3, -2, -1, 0, 1, 2, 3$  are labeled  $N, O, P, Q, R, S$ , and  $T$ , respectively.

For a molecule such as HCl ( $X^1\Sigma^+$ ) the spectrum contains only  $P$  and  $R$  branches. The energy of a given  $v, J$  level is

$$\begin{aligned} E_{vJ} &= G(v) + F(J) \\ &= \omega_e(v + \frac{1}{2}) - \omega_e x_e(v + \frac{1}{2})^2 + \omega_e y_e(v + \frac{1}{2})^3 + \omega_e z_e(v + \frac{1}{2})^4 \dots \\ &\quad + B_v J(J+1) - D_v(J(J+1))^2 + \dots \end{aligned} \quad (7.64)$$

so that (ignoring the effect of centrifugal distortion) the line positions for  $R$  and  $P$  branch transitions are given by

$$\begin{aligned} \tilde{\nu}_R(v', J+1 \leftarrow v'', J) &= \tilde{\nu}_0 + 2B' + (3B' - B'')J + (B' - B'')J^2, \\ &= \tilde{\nu}_0 + (B' + B'')(J+1) + (B' - B'')(J+1)^2, \end{aligned} \quad (7.65)$$

$$\tilde{\nu}_P(v', J-1 \leftarrow v'', J) = \tilde{\nu}_0 - (B' + B'')J + (B' - B'')J^2, \quad (7.66)$$

in which  $\tilde{\nu}_0$ , the band origin, is  $G(v') - G(v'')$ . The  $P$  and  $R$  branch formulas can be combined into the single expression,

$$\tilde{\nu} = \tilde{\nu}_0 + (B' + B'')m + (B' - B'')m^2, \quad (7.67)$$

by defining  $m = J+1$  for the  $R$  branch and  $m = -J$  for the  $P$  branch. By convention upper state quantum numbers and spectroscopic constants are labeled by single primes, while lower state quantum numbers and constants are labeled by double primes.

The fundamental band of HCl is the  $v = 1 \leftarrow 0$  transition and from equation (7.58) the band origin is given by

$$\tilde{\nu}_0 = \omega_e - 2\omega_e x_e + \frac{13}{4}\omega_e y_e + 5\omega_e z_e. \quad (7.68)$$

From expressions (7.59a) and (7.59b) we can obtain the band origin for  $\text{H}^{35}\text{Cl}$  as  $2885.977 \text{ cm}^{-1}$ , while the band origin for  $\text{D}^{35}\text{Cl}$  is  $2091.061 \text{ cm}^{-1}$ . The vibration-rotation transitions are illustrated in Figure 7.12 for the DCl infrared spectrum. The labeled peaks in Figure 7.12 are due to the more abundant  $\text{D}^{35}\text{Cl}$  isotopologue, while the weaker satellite features are due to transitions of  $\text{D}^{37}\text{Cl}$ . The relative intensities of the  $\text{D}^{35}\text{Cl}$  and  $\text{D}^{37}\text{Cl}$  lines with the same  $J$  value seem to change with  $J$  because the

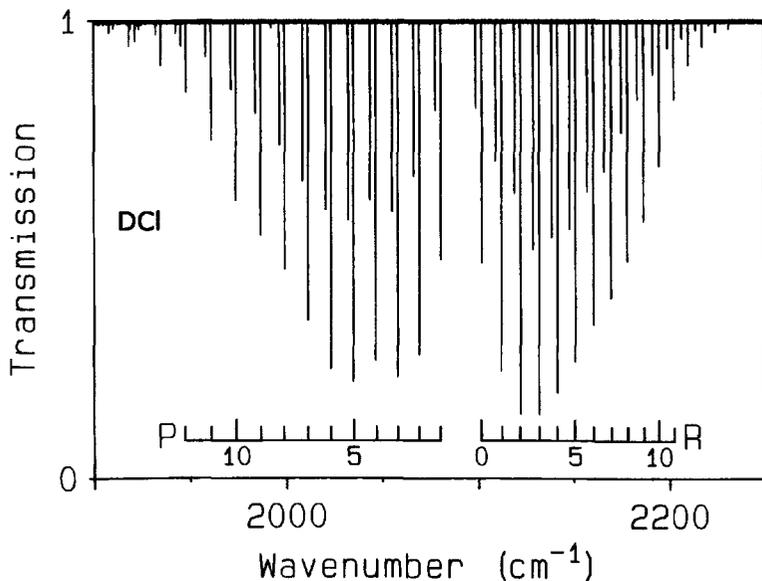


Figure 7.12: The fundamental vibration-rotation band of  $D^{35}\text{Cl}$  and  $D^{37}\text{Cl}$ .

stronger  $D^{35}\text{Cl}$  lines saturate before the weaker  $D^{37}\text{Cl}$  lines. For DCl the vibrational dependence of  $B$  is given by<sup>13</sup>

$$B_v = 5.448\,794 - 0.113\,291(v + \frac{1}{2}) + 0.000\,458\,9(v + \frac{1}{2})^2 \text{cm}^{-1}, \quad (7.69)$$

so that  $B_0 = 5.392\,263 \text{ cm}^{-1}$  and  $B_1 = 5.279\,890 \text{ cm}^{-1}$ . Thus, even for a light hydride  $B_0 \approx B_1$ . One can use equation (7.67) to show that for lines near the band origin, the spacing between consecutive lines is approximately  $B' + B'' = 2\bar{B}$ , with the average rotational constant given by  $\bar{B} = (B' + B'')/2$ . Notice that there is a gap at the band origin where a  $Q$  branch would be present if  $\Delta J = 0$  transitions were allowed. This “band gap” between the first lines  $R(0)$  and  $P(1)$  of the two branches is approximately  $4\bar{B}$ .

### Combination Differences

In general, a transition energy depends on the constants of both the upper and lower states as shown in equation (7.67) so that the two sets of rotational constants cannot be treated independently. The differences between lines that share a common upper or lower level are known as *combination differences* (Figure 7.13). These differences of line positions are very useful because they depend only on upper or lower state spectroscopic constants. The lower state combination differences (Figure 7.13) are

$$\begin{aligned} \Delta_2 F''(J) &= \tilde{\nu}(R(J-1)) - \tilde{\nu}(P(J+1)) \\ &= B''(J+1)(J+2) - B''(J-1)J \\ &= 4B''(J + \frac{1}{2}), \end{aligned} \quad (7.70)$$

while the upper state combination differences are

$$\Delta_2 F'(J) = \tilde{\nu}(R(J)) - \tilde{\nu}(P(J)) = 4B'(J + \frac{1}{2}). \quad (7.71)$$

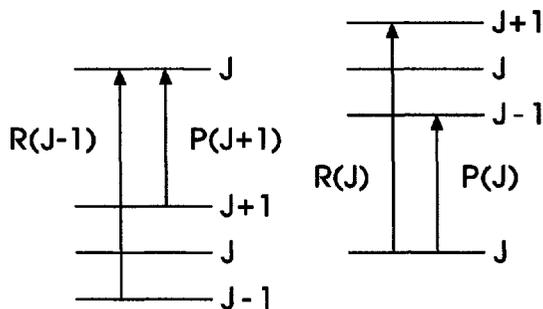
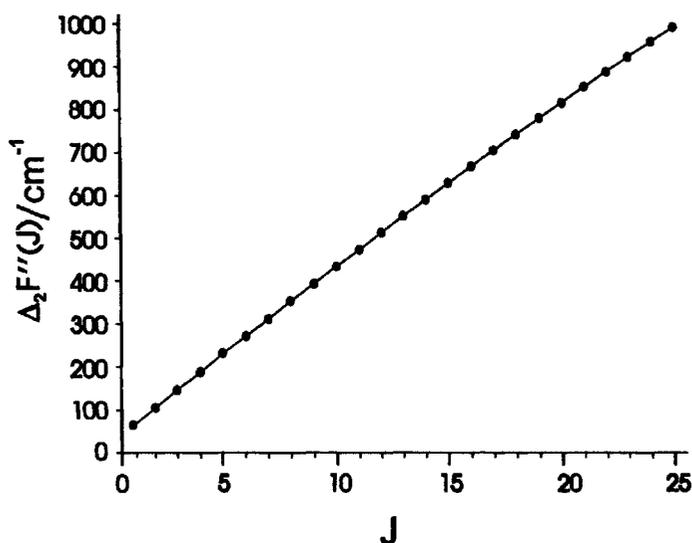


Figure 7.13: Ground-state and excited-state combination differences.

Figure 7.14: Ground-state combination differences  $\bar{\nu}(R(J-1)) - \bar{\nu}(P(J+1))$  for the fundamental band  $\nu = 1 - 0$  of HCl.

In equations (7.70) and (7.71) the  $\Delta$  indicates a difference between line positions represented by the standard  $F(J)$  formulas, and the subscript 2 signifies that  $\Delta J = 2$  for the differences. A plot of  $\Delta_2 F''(J)$  versus  $J$  yields approximately a straight line with a slope of  $4B''$  as shown in Figure 7.14 for HCl.<sup>20</sup> The slight curvature is due to the neglect of centrifugal distortion, which gives

$$\Delta_2 F(J) = (4B - 6D)(J + \frac{1}{2}) - 8D(J + \frac{1}{2})^3 \quad (7.72)$$

when it is included. The combination differences thus allow the rotational constants of the upper and lower states to be determined independently.

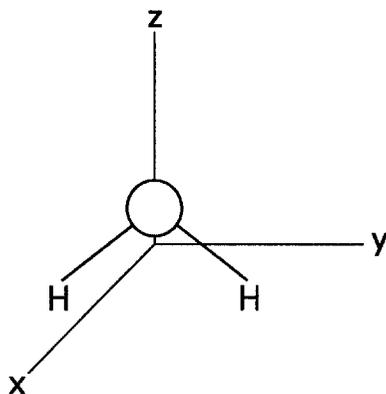


Figure 7.15: Coordinate system for a molecule with  $N (=3)$  atoms.

## 7.2 Vibrational Motion of Polyatomic Molecules

### *Classical Mechanical Description*

The classical Hamiltonian for the vibrational motion of a nonrotating molecule (Figure 7.15) with  $N$  atoms is given by  $H = T + V$  where the kinetic energy  $T$  is

$$\begin{aligned} T &= \frac{1}{2} \sum_{i=1}^N m_i |\mathbf{v}_i|^2 \\ &= \frac{1}{2} \sum_{i=1}^N m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2), \end{aligned} \quad (7.73)$$

in which the dot notation has been used for derivatives with respect to time, as for example,  $\dot{x}_i \equiv dx_i/dt$ .

This expression can be rewritten by introducing mass-weighted Cartesian displacement coordinates. Let

$$q_1 = (m_1)^{1/2}(x_1 - x_{1e}) \quad (7.74a)$$

$$q_2 = (m_1)^{1/2}(y_1 - y_{1e}) \quad (7.74b)$$

$$q_3 = (m_1)^{1/2}(z_1 - z_{1e}) \quad (7.74c)$$

$$q_4 = (m_2)^{1/2}(x_2 - x_{2e}) \quad (7.74d)$$

⋮

$$q_{3N} = (m_N)^{1/2}(z_N - z_{Ne}), \quad (7.74e)$$

in which the  $q_i$  coordinate is proportional to the displacement from the equilibrium value. The set of equilibrium nuclear coordinates,  $\{\mathbf{r}_{ie}\}$ , describes the location of the nuclei for the absolute minimum in the potential energy. In terms of the  $q_i$  coordinates the kinetic energy of nuclear motion takes the particularly simple form

$$T = \frac{1}{2} \sum_{i=1}^{3N} \dot{q}_i^2. \quad (7.75)$$

In general the potential energy  $V(r_i)$  is a complicated function of the Cartesian coordinates of the atoms. Expanding the potential  $V$  in a Taylor series about the equilibrium nuclear positions using the mass-weighted Cartesian displacement coordinates gives

$$V = V(q_i = 0) + \sum_i \left. \frac{\partial V}{\partial q_i} \right|_{q_i=0} q_i + \frac{1}{2} \sum_i \sum_j \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{q_i=0, q_j=0} q_i q_j + \dots \quad (7.76)$$

The potential energy is arbitrarily chosen to be zero at equilibrium, that is,  $V(q_i = 0) = 0$ . At the same time we also have by definition  $\left. \partial V / \partial q_i \right|_{q_i=0} = 0$  at equilibrium.

The present discussion is based on the harmonic approximation for the potential energy, according to which terms in the expansion (7.76) with order greater than two are neglected. The second derivatives of the potential are force constants  $f_{ij}$  defined by

$$f_{ij} = \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{q_i=0, q_j=0} \quad (7.77)$$

so that

$$2V = \sum_{i,j} f_{ij} q_i q_j. \quad (7.78)$$

The  $q_i$  form a set of generalized coordinates for which Newton's laws of motion are best formulated using Lagrange's equations. Lagrange's equations are equivalent to

$$\sum \mathbf{F}_i = m_i \mathbf{a}_i \quad (7.79)$$

but are valid for any coordinates, not just the Cartesian coordinates implicit in equation (7.79). Lagrange's formulation of the classical equations of motion is based on the construction of the Lagrangian  $L$ ,

$$L = T(\dot{q}_i) - V(q_i), \quad (7.80)$$

which is a function of the generalized coordinates and velocities  $q_i$  and  $\dot{q}_i$ . Newton's second law of motion, equation (7.79), is equivalent to Lagrange's equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right)_{q_i} - \left( \frac{\partial L}{\partial q_i} \right)_{\dot{q}_i} = 0. \quad (7.81)$$

It is easy to verify this equivalence for a single particle moving in one dimension  $x$ , for a conservative system in which the potential is independent of time ( $V \neq V(t)$ ). In this case the Lagrangian is

$$L = \frac{1}{2} m \dot{x}^2 - V(x), \quad (7.82)$$

and Lagrange's equation (7.81) becomes

$$\frac{d}{dt} \frac{\partial}{\partial \dot{x}} \left( \frac{1}{2} m \dot{x}^2 - V \right) - \frac{\partial}{\partial x} \left( \frac{1}{2} m \dot{x}^2 - V \right) = 0. \quad (7.83)$$

Taking the derivatives with respect to  $\dot{x}$  and  $x$  gives

$$\frac{d}{dt}m\dot{x} + \frac{\partial V}{\partial x} = 0, \quad (7.84)$$

or

$$-\frac{\partial V}{\partial x} = \frac{d(m\dot{x})}{dt} = \frac{dp}{dt} = m\ddot{x} = ma. \quad (7.85)$$

However, since force is also related to the potential function by

$$\mathbf{F} = -\nabla V \quad (7.86a)$$

we also have, in this one dimensional case,

$$F = -\frac{\partial V}{\partial x} = ma. \quad (7.86b)$$

As expected Lagrange's equation is equivalent to  $F = ma = \dot{p}$ . In Lagrangian mechanics the generalized force is  $(\partial L/\partial q_i)_{q_i}$  and the generalized momentum is

$$\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q_i} = p_i. \quad (7.87)$$

Applying Lagrange's equation to the vibrations of polyatomic molecules gives

$$L = T - V = \frac{1}{2} \sum_{i=1}^{3N} \dot{q}_i^2 - \frac{1}{2} \sum_{i,j}^{3N} f_{ij} q_i q_j, \quad (7.88)$$

but since

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q_i} - \left(\frac{\partial L}{\partial q_i}\right)_{q_i} = 0, \quad i = 1, \dots, 3N, \quad (7.89)$$

the equations of motion become

$$\ddot{q}_i + f_{ii} q_i + \sum_{j \neq i} f_{ij} q_j = 0, \quad (7.90)$$

or

$$\ddot{q}_i + \sum_j f_{ij} q_j = 0, \quad i = 1, \dots, 3N. \quad (7.91)$$

This is a set of  $3N$  coupled second-order differential equations (7.91) with constant coefficients. Such a system of equations can be solved by assuming a solution of the form

$$q_i = A_i \cos(\lambda^{1/2} t + \phi), \quad (7.92)$$

in which  $\sqrt{\lambda}$  is an angular frequency associated with the vibration of the nuclei about their equilibrium positions. Substitution of equation (7.92) into equation (7.91) converts the set of second-order differential equations into a set of  $3N$  homogeneous linear equations:

$$-\lambda A_i \cos(\lambda^{1/2} t + \phi) + \sum_j f_{ij} A_j \cos(\lambda^{1/2} t + \phi) = 0, \quad (7.93)$$

or

$$-\lambda A_i + \sum_j f_{ij} A_j = 0. \quad (7.94)$$

The set of  $3N$  equations has only the trivial solution for the amplitudes  $A_1 = A_2 = \dots A_{3N} = 0$  unless the determinant of the coefficients is zero or

$$\begin{vmatrix} f_{11} - \lambda & f_{12} & \cdots & f_{1,3N} \\ f_{21} & f_{22} - \lambda & \cdots & \\ \vdots & & & \vdots \\ f_{3N,1} & \cdots & & f_{3N,3N} - \lambda \end{vmatrix} = 0. \quad (7.95)$$

The secular equation (7.95) is a polynomial of order  $3N$  so that there exist  $3N$  values of  $\lambda$  for which equation (7.95) is satisfied. The  $3N \times 3N$  force constants can be arranged in a force constant matrix  $\mathbf{f}$ , and the  $3N$  values of  $\lambda$  are the eigenvalues of  $\mathbf{f}$ . It turns out that six of the eigenvalues are zero for a nonlinear molecule, and five are zero for a linear molecule. Three degrees of freedom are associated with the translation ( $x, y, z$ ) of the center of mass and three (or two for a linear molecule) with rotational motion of the molecule as a whole ( $\theta, \phi, \chi$ ). Since there is no restoring force acting on these degrees of freedom, their frequencies are zero.

Associated with each eigenvalue  $\lambda_i$  is a coordinate called a *normal mode coordinate*,  $Q_i$ . The normal modes represent a new set of coordinates related to the old  $q_j$  by a linear transformation,

$$Q_k = \sum_j l_{kj} q_j \quad (7.96)$$

or

$$\mathbf{Q} = \mathbf{lq} \quad (7.97)$$

with  $\mathbf{l}$  being a real orthogonal matrix ( $\mathbf{l}^{-1} = \mathbf{l}^t$ ) so that

$$\mathbf{q} = \mathbf{l}^t \mathbf{Q}. \quad (7.98)$$

In matrix notation the original  $3N$  differential equations (7.91) are written as

$$\ddot{\mathbf{q}} + \mathbf{fq} = 0. \quad (7.99)$$

Substituting equation (7.98) gives

$$\mathbf{l}^t \ddot{\mathbf{Q}} + \mathbf{fl}^t \mathbf{Q} = 0, \quad (7.100)$$

so that multiplication by  $\mathbf{l}$  from the left gives

$$\ddot{\mathbf{Q}} + (\mathbf{fl}^t) \mathbf{Q} = 0. \quad (7.101)$$

The transformation matrix  $\mathbf{l}$  is chosen to diagonalize  $\mathbf{f}$ , that is,

$$\mathbf{fl}^t = \mathbf{\Lambda} \quad (7.102)$$

so that the eigenvalues of  $\mathbf{f}$  are the diagonal elements of  $\mathbf{\Lambda}$ . Since  $\mathbf{f}$  is a real symmetric matrix, there are  $3N$  real eigenvalues. Furthermore, an orthogonal matrix can

always be constructed from the orthonormal eigenvectors of  $\mathbf{f}$ . We see that the linear transformation has uncoupled the  $3N$  equations so that now

$$\ddot{\mathbf{Q}} + \mathbf{\Lambda}\mathbf{Q} = 0 \quad (7.103)$$

or written out as

$$\begin{pmatrix} \ddot{Q}_1 \\ \ddot{Q}_2 \\ \vdots \\ \ddot{Q}_{3N} \end{pmatrix} + \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_{3N} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{3N} \end{pmatrix} = 0. \quad (7.104)$$

Expanding the matrix equation (7.104) gives

$$\ddot{Q}_1 + \lambda_1 Q_1 = 0 \quad (7.105a)$$

$$\ddot{Q}_2 + \lambda_2 Q_2 = 0 \quad (7.105b)$$

$$\vdots$$

$$\ddot{Q}_{3N} + \lambda_{3N} Q_{3N} = 0. \quad (7.105c)$$

Applying the same normal coordinate transformation to  $T$  and  $V$  gives

$$2T = \dot{\mathbf{q}}^t \dot{\mathbf{q}} = (\mathbf{l}^t \dot{\mathbf{Q}})^t (\mathbf{l}^t \dot{\mathbf{Q}}) = (\dot{\mathbf{Q}})^t \mathbf{l} \mathbf{l}^t (\dot{\mathbf{Q}}) = \dot{\mathbf{Q}}^t \dot{\mathbf{Q}} \quad (7.106)$$

or

$$T = \frac{1}{2} \sum \dot{Q}_i^2 \quad (7.107)$$

and

$$2V = \mathbf{q}^t \mathbf{f} \mathbf{q} = \mathbf{Q}^t \mathbf{l} \mathbf{f} \mathbf{l}^t \mathbf{Q} = \mathbf{Q}^t \mathbf{\Lambda} \mathbf{Q} \quad (7.108)$$

or

$$V = \frac{1}{2} \sum \lambda_i Q_i^2. \quad (7.109)$$

Thus, both the kinetic and potential energy terms of the Hamiltonian have no cross-terms that connect different coordinates. The system therefore behaves like a set of  $3N-6$  (or  $3N-5$ ) independent harmonic oscillators, each oscillating without interaction with the others. Naturally, a real system has cubic and quartic terms (and higher!) in the potential energy expansion. For a real molecule, the  $\mathbf{l}$  matrix and the normal coordinates are still defined in the way outlined above, using only the harmonic terms, but the  $\mathbf{l}$  matrix transformation no longer completely uncouples the  $3N$  differential equations. The anharmonic terms in the potential energy expansion are then said to couple the normal modes, so that the normal mode approximation is not completely valid.

### Quantum Mechanical Description

With the classical Hamiltonian available, the transition to quantum mechanics is (deceptively) simple. First the classical Hamiltonian is written in terms of the generalized coordinates, the normal modes  $Q_i$ , and the associated generalized momenta  $P_i$  with

$$P_i = \left( \frac{\partial L}{\partial \dot{Q}_i} \right)_{Q_i} = \dot{Q}_i. \quad (7.110)$$

The resulting classical Hamiltonian is

$$H = \frac{1}{2} \sum P_i^2 + \frac{1}{2} \sum \lambda_i Q_i^2, \quad (7.111)$$

with the summations running from 1 to  $3N - 6$  (or  $3N - 5$ ). The classical Hamiltonian is converted to a quantum mechanical operator by making the usual substitutions  $Q_i \rightarrow \hat{Q}_i$  and

$$P_i \rightarrow \hat{P}_i = -i\hbar \frac{\partial}{\partial Q_i},$$

which gives

$$\begin{aligned} \hat{H} &= -\frac{\hbar^2}{2} \sum_i \frac{\partial^2}{\partial Q_i^2} + \frac{1}{2} \sum_i \lambda_i \hat{Q}_i^2 \\ &= \sum \left( -\frac{\hbar^2}{2} \frac{\partial^2}{\partial Q_i^2} + \frac{1}{2} \lambda_i \hat{Q}_i^2 \right) \\ &= \sum \hat{H}_i. \end{aligned} \quad (7.112)$$

In terms of the normal coordinates the Hamiltonian operator equation (7.112) is just a sum of  $3N - 6$  (or  $3N - 5$ ) independent harmonic oscillator Hamiltonian operators. Consequently, the total wavefunction  $\psi$  for the Schrödinger equation  $\hat{H}\psi = E\psi$  is just a product of  $3N - 6$  (or  $3N - 5$ ) harmonic oscillator wavefunctions

$$\psi = \psi_1(Q_1)\psi_2(Q_2) \cdots \psi_{3N-6}(Q_{3N-6}) \quad (7.113)$$

with

$$\psi_i(Q_i) = N_{v_i} H_{v_i}(\xi_i) e^{-\xi_i^2/2} \quad \text{and} \quad \xi_i = Q_i \left( \frac{\lambda_i^{1/2}}{\hbar} \right)^{1/2} \quad (7.114)$$

The  $3N - 6$  (or  $3N - 5$ ) fictitious harmonic oscillators all have unit mass since the actual atomic masses were already used to mass-weight the Cartesian displacement coordinates. The square of the angular frequency ( $\omega = 2\pi\nu$ ) is  $\omega^2 = k/m = \lambda$  so that  $\omega = \lambda^{1/2}$ . The total energy is thus the sum of the energies of  $3N - 6$  (or  $3N - 5$  for a linear molecule) harmonic oscillator energies, namely

$$\begin{aligned} E &= \hbar\omega_1(v_1 + \frac{1}{2}) + \cdots + \hbar\omega_{3N-6}(v_{3N-6} + \frac{1}{2}) \\ &= \sum \hbar\omega_i(v_i + \frac{1}{2}) \\ &= \sum \hbar\lambda_i^{1/2}(v_i + \frac{1}{2}). \end{aligned} \quad (7.115)$$

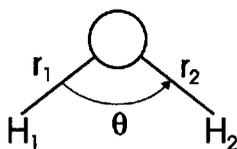


Figure 7.16: Internal coordinates of the  $\text{H}_2\text{O}$  molecule.

The preceding treatment is valid within the harmonic oscillator approximation and assumes that the molecule is not rotating. Real molecules, however, are rotating, anharmonic oscillators. The approximate vibrational wavefunction (7.113) is nevertheless a very good starting point for the description of true molecular vibrations. Thus, the vibrational wavefunctions of the true vibrational Hamiltonian operator, which contains anharmonic terms in  $V$ , can be represented as linear combinations of harmonic oscillator functions. The anharmonic terms in the potential cause the true wavefunctions to be mixtures of harmonic oscillator wavefunctions.

The effect of rotation is much more difficult to handle because the vibrational and rotational motions are not separable. The use of a noninertial (accelerated) coordinate system (internal molecular coordinates) rotating with the molecule introduces Coriolis terms into the classical Hamiltonian. Special techniques are then required to transform the classical Hamiltonian to an appropriate quantum mechanical form and, of course, there are additional terms in the Hamiltonian operator. The coupling of vibration with rotation introduces vibrational angular momentum and prevents the exact separation of vibrational and rotational motion. These additional terms can mix the normal-mode wavefunctions by Coriolis coupling and can mix the vibrational and rotational wavefunctions. Despite these problems the simple normal mode picture is a remarkably successful model. Only for highly excited modes, such as the fifth overtone of the OH stretching motion of  $\text{H}_2\text{O}$ , is a different, non-normal mode picture (the local mode approximation) commonly used.

### Internal Coordinates

The use of force constants  $f_{ij}$  associated with mass-weighted Cartesian coordinates is very convenient mathematically, but they are difficult to associate with specific internal motions such as bond stretching. Modern *ab initio* quantum chemistry programs do, in fact, use Cartesian force constants because they can be computed easily by displacing an atom and computing the change in energy, or by using analytical derivatives, equation (7.77). To obtain physical insight, however, it is preferable to describe the vibrational motion of a molecule in terms of readily recognizable structural features—namely, bond lengths and angles. For example, in the water molecule three internal coordinates ( $r_1$ ,  $r_2$ , and  $\theta$ ) are required to describe the relative positions of the atoms (i.e., Figure 7.16). There are also  $3N - 6 = 3$  vibrational modes which must be related in some manner to changes in  $r_1$ ,  $r_2$ , and  $\theta$ . It is convenient to define the internal displacement coordinates  $\Delta r_1$ ,  $\Delta r_2$ , and  $\Delta \theta$  to correspond to bond-stretching and bond-bending motions. It is also convenient to use  $(r_1 r_2)^{1/2} \Delta \theta = r \Delta \theta$  as the bending coordinate so that all internal coordinates have the same dimensions.

### Symmetry Coordinates

Even more preferable than internal coordinates are symmetry coordinates, since the vibrational Hamiltonian operator is unchanged by the symmetry operations associated with the molecule. This also means that the vibrational wavefunctions can be labeled with the irreducible representations of the molecular point group. The best plan is to utilize as much symmetry information as possible.

Symmetry coordinates can be obtained by inspection or more systematically with the aid of projection operators (Chapter 4). From the character table all possible projection operators are constructed and applied to the internal coordinates until the required  $3N - 6$  symmetry coordinates are generated. For example, using the  $\text{H}_2\text{O}$  molecule, the totally symmetric projection operator

$$\hat{P}^{A_1} = \sum_{\hat{R}} (\chi_R^{A_1})^* \hat{O}_R = \hat{O}_E + \hat{O}_{C_2} + \hat{O}_{\sigma_v} + \hat{O}_{\sigma_v'} \quad (7.116)$$

when applied to  $\Delta r_1$  gives

$$\begin{aligned} \hat{P}^{A_1} \Delta r_1 &= \Delta r_1 + \Delta r_2 + \Delta r_1 + \Delta r_2 \\ &= 2(\Delta r_1 + \Delta r_2). \end{aligned} \quad (7.117)$$

Similarly, the application of  $\hat{P}^{B_2}$  to  $\Delta r_1$  and  $\hat{P}^{A_1}$  to  $r\Delta\theta$  gives

$$\hat{P}^{B_2} \Delta r_1 = 2(\Delta r_1 - \Delta r_2), \quad (7.118)$$

$$\hat{P}^{A_1}(r\Delta\theta) = r\Delta\theta. \quad (7.119)$$

Let the three symmetry coordinates  $s_1$ ,  $s_2$ , and  $s_3$  be defined by

$$s_1 = \frac{\Delta r_1 + \Delta r_2}{\sqrt{2}} \quad (A_1 \text{ symmetry}) \quad (7.120)$$

$$s_2 = r\Delta\theta \quad (A_1 \text{ symmetry}) \quad (7.121)$$

$$s_3 = \frac{\Delta r_1 - \Delta r_2}{\sqrt{2}} \quad (B_2 \text{ symmetry}). \quad (7.122)$$

The  $x$ -axis is defined to be out of plane (i.e., perpendicular to the plane of the water molecule). The three symmetry coordinates ( $s_1$ ,  $s_2$ , and  $s_3$ ) have symmetry appropriate to a molecule belonging to the  $C_{2v}$  point group.

Using internal coordinates the harmonic potential energy function of  $\text{H}_2\text{O}$  is

$$\begin{aligned} V &= \frac{1}{2}f_{11}(\Delta r_1)^2 + \frac{1}{2}f_{22}(\Delta r_2)^2 + \frac{1}{2}f_{33}(r\Delta\theta)^2 + f_{12}(\Delta r_1)(\Delta r_2) \\ &\quad + f_{13}(\Delta r_1)(r\Delta\theta) + f_{23}(\Delta r_2)(r\Delta\theta) \end{aligned} \quad (7.123)$$

and  $f_{11} = f_{22}$ ,  $f_{13} = f_{23}$  by symmetry. This potential energy function (7.123) is derived by considering the three possible quadratic terms ( $f_{11}$ ,  $f_{22}$ , and  $f_{33}$ ) and the three possible cross-terms ( $f_{12}$ ,  $f_{13}$ , and  $f_{23}$ ) obtained from the three internal coordinates. Converting to symmetry coordinates by using

$$\Delta r_1 = \frac{s_1 + s_3}{\sqrt{2}}, \quad (7.124)$$

$$r\Delta\theta = s_2, \quad (7.125)$$

and

$$\Delta r_2 = \frac{s_1 - s_3}{\sqrt{2}} \quad (7.126)$$

gives

$$V(s_1, s_2, s_3) = \frac{(f_{11} + f_{12})s_1^2}{2} + \frac{(f_{11} - f_{12})s_3^2}{2} + \frac{f_{33}s_2^2}{2} + \frac{\sqrt{2}f_{13}s_1s_2}{\text{stretch-bend interaction}}. \quad (7.127)$$

Notice that there are no  $s_1s_3$  terms present in equation (7.127) because  $s_1$  and  $s_3$  have different symmetry, so that no terms in the Hamiltonian operator can connect them (see Chapter 4). If a term such as  $s_1s_3$  were present, it would have  $A_1 \otimes B_2 = B_2$  symmetry and thus would be changed by a symmetry operation such as  $\hat{C}_2$  or  $\hat{\sigma}_v$ , contradicting the principle that the Hamiltonian operator is unchanged by the symmetry operations of the molecule.

The transformation to symmetry coordinates, equations (7.120) to (7.122), can be written in matrix form,

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \Delta r_1 \\ \Delta r_2 \\ r\Delta\theta \end{pmatrix} \quad (7.128)$$

or

$$\mathbf{s} = \mathbf{U}\mathbf{r}. \quad (7.129)$$

Upon inversion equation (7.128) gives

$$\begin{pmatrix} \Delta r_1 \\ \Delta r_2 \\ r\Delta\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \quad (7.130)$$

or

$$\mathbf{r} = \mathbf{U}^{-1}\mathbf{s} = \mathbf{U}^t\mathbf{s}, \quad (7.131)$$

in which  $\mathbf{s}$  is the vector of symmetry coordinates,  $\mathbf{r}$  is the vector of internal coordinates, and  $\mathbf{U}$  is the orthogonal transformation matrix defined in equation (7.129).

In matrix notation the potential energy  $V$  for the  $\text{H}_2\text{O}$  molecule is

$$\begin{aligned} V(\Delta r_1, \Delta r_2, r\Delta\theta) &= \frac{1}{2}(\Delta r_1, \Delta r_2, r\Delta\theta) \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{11} & f_{13} \\ f_{13} & f_{13} & f_{33} \end{pmatrix} \begin{pmatrix} \Delta r_1 \\ \Delta r_2 \\ r\Delta\theta \end{pmatrix} \\ &= \frac{1}{2}\mathbf{r}^t\mathbf{F}_r\mathbf{r}. \end{aligned} \quad (7.132)$$

In terms of the symmetry coordinates, the potential energy is

$$\begin{aligned} V(s_1, s_2, s_3) &= \frac{1}{2}(s_1, s_2, s_3) \begin{pmatrix} f_{11} + f_{12} & \sqrt{2}f_{13} & 0 \\ \sqrt{2}f_{13} & f_{33} & 0 \\ 0 & 0 & f_{11} - f_{12} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \\ &= \frac{1}{2}\mathbf{s}^t \mathbf{F}_s \mathbf{s}. \end{aligned} \quad (7.133)$$

Equations (7.132) and (7.133) are two different ways of representing the same quantity so we must have

$$\begin{aligned} V &= \frac{1}{2}\mathbf{r}^t \mathbf{F}_r \mathbf{r} = \frac{1}{2}(\mathbf{U}^t \mathbf{s})^t \mathbf{F}_r \mathbf{U}^t \mathbf{s} \\ &= \frac{1}{2}\mathbf{s}^t (\mathbf{U} \mathbf{F}_r \mathbf{U}^t) \mathbf{s}. \end{aligned} \quad (7.134)$$

Thus, the internal coordinate force constant matrix  $\mathbf{F}_r$  is related to the symmetry coordinate force constant matrix  $\mathbf{F}_s$  by the equation

$$\mathbf{F}_s = \mathbf{U} \mathbf{F}_r \mathbf{U}^t. \quad (7.135)$$

The change of basis affects the force constant matrix via a similarity transformation. Let the symmetry-adapted force constants of  $\text{H}_2\text{O}$  be defined by the equation

$$V = \frac{1}{2}(s_1, s_2, s_3) \begin{pmatrix} F_{11} & F_{12} & 0 \\ F_{12} & F_{22} & 0 \\ 0 & 0 & F_{33} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \quad (7.136)$$

with  $F_{11} = f_{11} + f_{12}$ ,  $F_{12} = \sqrt{2}f_{13}$ , and  $F_{33} = f_{11} - f_{12}$ . The force constant matrix  $\mathbf{F}_s$  is thus block factored into a  $2 \times 2$   $A_1$  block and a  $1 \times 1$   $B_2$  block. The uppercase  $F_{ij}$ 's are symmetry-adapted force constants, while lowercase  $f_{ij}$ 's are internal coordinate force constants.

The water molecule is described at the harmonic oscillator level by four force constants and three normal modes. There is not enough information in an infrared spectrum of  $\text{H}_2\text{O}$  to determine the force field since there are four unknowns to be derived from three vibrational frequencies. This is a general problem that gets worse as the molecule becomes larger and less symmetric. One experimental solution is to use vibrational frequencies from isotopic molecules such as  $\text{D}_2\text{O}$ . Although the nuclear masses are different, it is generally assumed that the equilibrium geometries and force constants are unaffected by isotopic substitution. For larger molecules, modern *ab initio* calculations are often used to compute some or all of the force constants.

The major problem with using internal coordinates or symmetry coordinates is that the kinetic energy operator becomes more complicated<sup>21</sup> than when Cartesian coordinates are used. In terms of the mass-weighted Cartesian coordinates  $T$  has the simple form given previously:

$$T = \frac{1}{2} \sum_{i=1}^{3N} \dot{q}_i^2, \quad (7.137)$$

When Cartesian coordinates are used, the classical kinetic energy operator contains no cross-terms between coordinates. When internal or symmetry coordinates are used,  $T$  has the more general form

$$T = \frac{1}{2} \sum_{i=1}^{3N-6} \sum_{j=1}^{3N-6} (G^{-1})_{ij} \dot{s}_i \dot{s}_j, \quad (7.138)$$

where  $\mathbf{G}^{-1}$  is the inverse of the matrix  $\mathbf{G}$ , the elements of which can be derived for a given molecular geometry using the methods described in Wilson, Decius, and Cross.<sup>21</sup> There are cross-terms  $\dot{s}_i(G^{-1})_{ij}\dot{s}_j$  connecting the various coordinates. Of course, if symmetry coordinates are used, there can be no terms with  $\dot{s}_i$  and  $\dot{s}_j$  of different symmetry. This means that the matrix forms of  $T$  and  $V$  have the same block structure and are factored into blocks belonging to the same irreducible representations.

The classical normal modes of vibration can be derived using symmetry coordinates in just the same way as for mass-weighted Cartesian displacement coordinates. The kinetic energy  $T$  is

$$T = \frac{1}{2} \dot{\mathbf{s}}^t \mathbf{G}^{-1} \dot{\mathbf{s}}, \quad (7.139)$$

which is just equation (7.138) written in matrix notation. The classical Hamiltonian can be written as

$$H = \frac{1}{2} \dot{\mathbf{s}}^t \mathbf{G}^{-1} \dot{\mathbf{s}} + \frac{1}{2} \mathbf{s}^t \mathbf{F} \mathbf{s}. \quad (7.140)$$

A solution of the form

$$s_j = A_j \cos(\lambda^{1/2} t + \phi) \quad (7.141)$$

will be assumed. The Lagrangian is

$$L = T - V \quad (7.142)$$

and Lagrange's equations are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{s}_i} \right)_{s_i} - \left( \frac{\partial L}{\partial s_i} \right)_{\dot{s}_i} = 0. \quad (7.143)$$

Using equations (7.133), (7.136), and (7.139) in equation (7.142) results in

$$\frac{d}{dt} \sum_j (G^{-1})_{ij} \dot{s}_j - \left( - \sum_j F_{ij} s_j \right) = 0, \quad i = 1, \dots, 3N - 6 \quad (7.144)$$

or

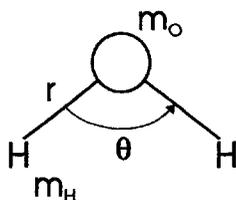
$$\sum_j (G^{-1})_{ij} \ddot{s}_j + \sum_j F_{ij} s_j = 0. \quad (7.145)$$

Substitution of equation (7.141) into (7.145) gives

$$\sum_j (G^{-1})_{ij} (-\lambda) A_j \cos(\lambda^{1/2} t + \phi) + \sum_j F_{ij} A_j \cos(\lambda^{1/2} t + \phi) = 0 \quad (7.146)$$

or

$$\sum_j (-\lambda(G^{-1})_{ij} + F_{ij}) A_j = 0, \quad i = 1, \dots, 3N - 6. \quad (7.147)$$

Figure 7.17: The H<sub>2</sub>O molecule.

In matrix form equation (7.147) is written as

$$(-\lambda \mathbf{G}^{-1} + \mathbf{F})\mathbf{A} = 0. \quad (7.148)$$

Multiplying by  $\mathbf{G}$  from the left gives

$$(-\lambda \mathbf{I} + \mathbf{GF})\mathbf{A} = 0. \quad (7.149)$$

Thus equation (7.149) is a set of  $3N - 6$  homogeneous linear equations that has a nontrivial solution only if the determinant of the coefficients is zero, that is,

$$|\mathbf{GF} - \lambda \mathbf{I}| = 0. \quad (7.150)$$

This is the celebrated  $\mathbf{GF}$  matrix solution for the vibrational modes of a polyatomic molecule.

The  $\mathbf{G}$  matrix is easier to derive than  $\mathbf{G}^{-1}$ , but still requires some work. The technique of deriving  $\mathbf{G}$  matrix elements is given in the classic book by Wilson, Decius, and Cross.<sup>21</sup> For example, the  $\mathbf{G}$  matrix for H<sub>2</sub>O is<sup>22</sup>

$$\begin{aligned} \mathbf{G} &= \begin{pmatrix} \mu_H + \mu_O(1 + \cos \theta) & -\sqrt{2}\mu_O \sin \theta & 0 \\ -\sqrt{2}\mu_O \sin \theta & 2(\mu_H + \mu_O - \mu_O \cos \theta) & 0 \\ 0 & 0 & \mu_H + \mu_O(1 - \cos \theta) \end{pmatrix} \\ &= \begin{pmatrix} 1.039 & -0.0858 & 0 \\ -0.0858 & 2.139 & 0 \\ 0 & 0 & 1.070 \end{pmatrix} \end{aligned} \quad (7.151)$$

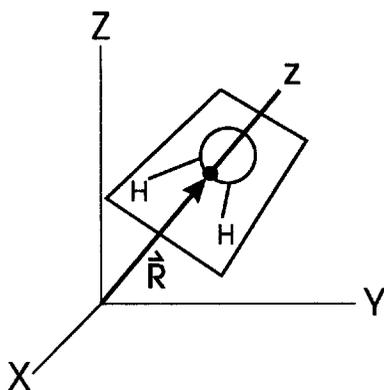
with  $\mu_H = 1/m_H$ ,  $\mu_O = 1/m_O$  for  $m_H = 1.008$  u,  $m_O = 16$  u,  $\theta = 104^\circ$ , and  $r = 0.958$  Å (Figure 7.17).

A typical force constant analysis proceeds by selecting initial values for the force constants  $F_{11}$ ,  $F_{12}$ ,  $F_{22}$ , and  $F_{33}$ , and then calculating  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$ , the three eigenvalues of the  $\mathbf{GF}$  matrix for both H<sub>2</sub>O and D<sub>2</sub>O. The  $\mathbf{G}$  matrix is different for H<sub>2</sub>O and D<sub>2</sub>O, but the  $\mathbf{F}$  matrix is the same. The six calculated vibrational frequencies are then compared with the experimentally measured frequencies, and the values of the four force constants are adjusted through iterative refinement to improve the agreement between observed and calculated frequencies.<sup>22</sup> The iterative refinement involves a nonlinear least squares fitting procedure to minimize the sum of the squared deviations between the observed and calculated vibrational frequencies.

Force constants<sup>23</sup> for some bent XY<sub>2</sub> molecules are provided in Table 7.3 in traditional units of millidynes/Å (1 millidyne/Å = 100 N/m).

Table 7.3: Force Constants (in millidyne/Å) for Some  $XY_2$  Molecules

$V$ Term	Interaction	H <sub>2</sub> O	F <sub>2</sub> O	O <sub>3</sub>	SO <sub>2</sub>
$\frac{1}{2}f_{11}(\Delta r_1)^2$	$f_{11}$ (X – Y stretch)	7.684	3.97	5.74	10.33
$f_{12}(\Delta r_1 \Delta r_2)$	$f_{12}$ (stretch-stretch interaction)	-0.082	0.83	1.57	0.08
$\frac{1}{2}f_{33}(r\Delta\theta)^2$	$f_{33}$ (bend)	0.707	0.70	1.26	0.82
$f_{13}\Delta r_1(r\Delta\theta)$	$f_{13}$ (bend-stretch interaction)	0.169	0.15	0.39	0.23

Figure 7.18: The H<sub>2</sub>O molecule oriented in space relative to the center-of-mass vector  $\mathbf{R}$ .

### Symmetry of Normal Modes

As an example, consider the H<sub>2</sub>O molecule. Three Cartesian coordinates are required to specify the position of each atom in space (Figure 7.18). Of the resulting nine degrees of freedom, however, three coordinates are required to locate the center of mass and three additional coordinates—for example, the Euler angles  $\theta$ ,  $\phi$ , and  $\chi$ —specify the orientation of the molecule in space. The spherical polar coordinates  $\theta$  and  $\phi$  specify how the molecular  $z$ -axis is oriented relative to the laboratory  $Z$ -axis. The Euler angle  $\chi$  specifies the relative angular position of the plane containing the two hydrogen atoms and the O atom with respect to the molecular  $z$ -axis (Figure 7.19).

For a linear molecule the angle  $\chi$  is replaced by another vibrational coordinate. Only two angles,  $\theta$  and  $\phi$ , are required to specify the orientation of a linear molecule in space. Therefore  $3N - 5$  coordinates are necessary to describe the relative internal positions of the atoms in a linear molecule (Figure 7.20), but  $3N - 6$  are needed for a nonlinear molecule.

Following the methods outlined in Chapter 3, the set of  $3N$  mass-weighted Cartesian displacement coordinates (Figure 7.21) can be used to construct a reducible representation for the group. For example  $\mathbf{D}(E)$  is given by

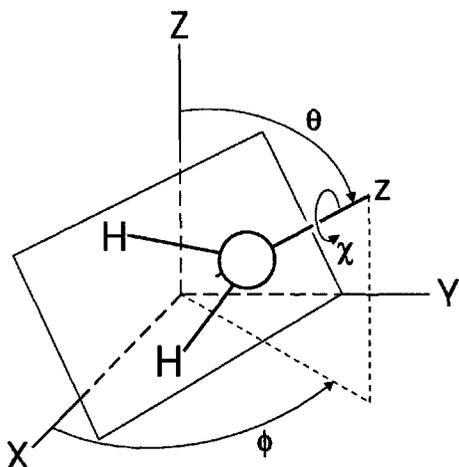


Figure 7.19: The orientation of the  $\text{H}_2\text{O}$  molecule in space, specified by the three Euler angles  $\theta$ ,  $\phi$ , and  $\chi$ .

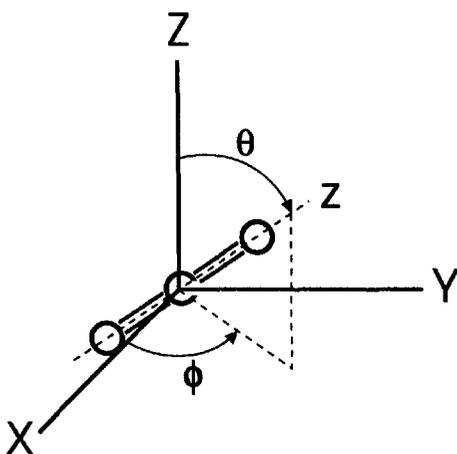


Figure 7.20: The orientation of a linear molecule in space, specified by the two angles  $\theta$  and  $\phi$ .

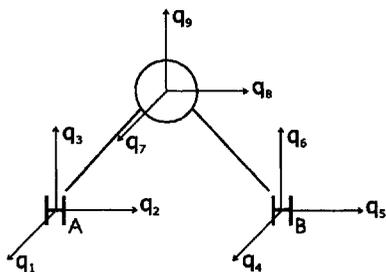
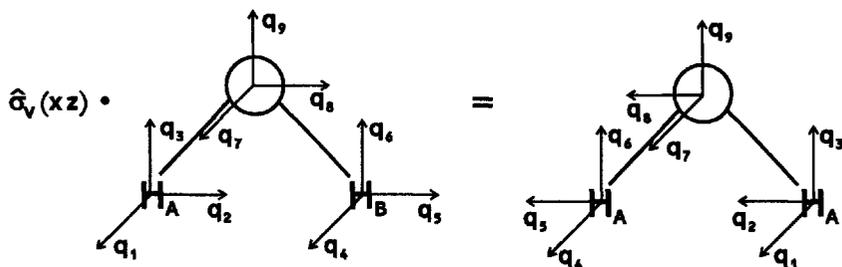
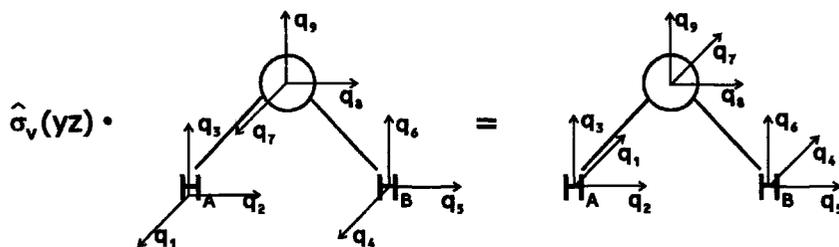


Figure 7.21: The nine mass-weighted Cartesian displacement coordinates for  $\text{H}_2\text{O}$ .



Figure 7.23: The effect of  $\hat{\sigma}_v(xz)$  on  $\text{H}_2\text{O}$ .Figure 7.24: The effect of  $\hat{\sigma}_v(yz)$  on  $\text{H}_2\text{O}$ .

From Figure 7.24, the matrix representation of  $\hat{\sigma}_v(yz)$  is

$$D(\hat{\sigma}_v(yz)) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (7.155)$$

The characters for the  $9 \times 9$  representations are

$$\chi^{3N}(\hat{E}) = 9, \quad \chi^{3N}(\hat{C}_2) = -1, \quad \chi^{3N}(\hat{\sigma}_v(xz)) = 1, \quad \chi^{3N}(\hat{\sigma}_v(yz)) = 3.$$

These characters can be easily generated by inspection without writing down the complete matrices since only the diagonal elements of a matrix are needed to determine the character. For the  $\hat{E}$  operation the contribution by each atom to  $\chi(\hat{E})$  is 3 so that  $\chi^{3N}(\hat{E}) = 3N$ . If any displacement vectors are moved from one atom to another atom by a symmetry operation, then they contribute zero to the total character of that symmetry operation. The total character for any operation is the sum of the contributions from each atom. Any atom for which the displacement vectors are rotated by  $\theta$  contributes  $1 + 2 \cos \theta$  to the total character  $\chi(\hat{C}_n)$  since the rotation matrix is

$$D(\hat{C}_\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (7.156)$$

and the trace is  $1 + 2 \cos \theta$ . Any  $\hat{\sigma}$  operation contributes +1 for each atom on the symmetry plane to the total character since the trace of the reflection matrix

$$\mathbf{D}(\hat{\sigma}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (7.157)$$

is one. Because  $\hat{S}_\theta = \hat{C}_\theta \hat{\sigma}_h = \hat{\sigma}_h \hat{C}_\theta$ , each atom not displaced contributes  $-1 + 2 \cos \theta$  to the character for an improper rotation by  $\theta$ . Finally the application of an inversion operation to an atom inverts the displacement vectors so that  $\chi(\hat{i}) = -3$  for any atom at the center of symmetry.

The nine-dimensional representation can be reduced as

$$\Gamma^{3N} = \oplus \sum a_i \Gamma^i \quad (7.158)$$

using the relationship

$$a_i = \frac{1}{g} \sum_{\hat{R}} \chi^{3N}(\hat{R}) \chi^i(\hat{R})^* \quad (7.159)$$

to give

$$\begin{aligned} a_{A_1} &= \frac{1}{4}(9 - 1 + 1 + 3) = 3, \\ a_{A_2} &= \frac{1}{4}(9 - 1 - 1 - 3) = 1, \\ a_{B_1} &= \frac{1}{4}(9 + 1 + 1 - 3) = 2, \\ a_{B_2} &= \frac{1}{4}(9 + 1 - 1 + 3) = 3, \end{aligned}$$

so that the reducible representation  $\Gamma^{3N}$  can be written as the direct sum of irreducible representations as

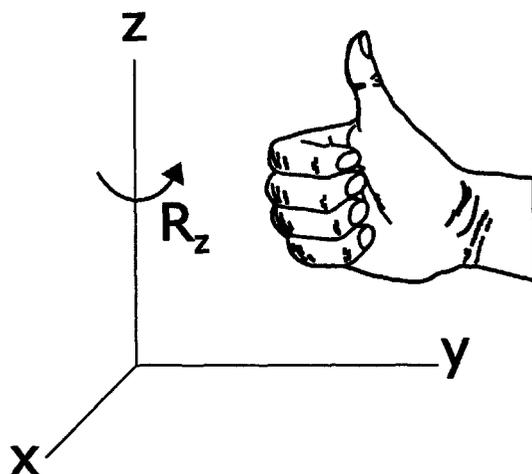
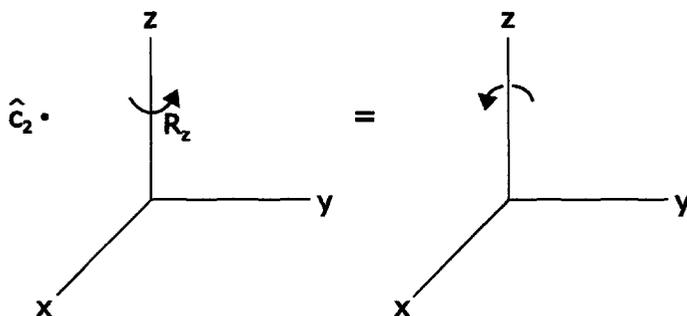
$$\Gamma^{3N} = 3\Gamma^{A_1} \oplus \Gamma^{A_2} \oplus 2\Gamma^{B_1} \oplus 3\Gamma^{B_2} \quad (7.160)$$

for the  $\text{H}_2\text{O}$  molecule. The  $3N$  representation, however, still contains three translations and three rotations; these must be removed from the full representation to leave the symmetry representation of the pure vibrational motions.

The symmetry of the translational coordinates can be determined by considering the effect of the symmetry operations on the three Cartesian basis vectors,  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$ ,  $\hat{\mathbf{k}}$ , or on a point  $\mathbf{r}$ . The reasoning behind this is that the motion of the center of mass is equivalent to the translation of a point through space. For the  $C_{2v}$  case, translational motion in the  $x$  direction behaves like the  $B_1$  line of the character table (Appendix B) with respect to the symmetry operations, translational motion along  $y$  behaves like  $B_2$ , and along  $z$  behaves like  $A_1$ , that is,

$$\Gamma^{\text{tr}} = \Gamma^{A_1} \oplus \Gamma^{B_1} \oplus \Gamma^{B_2}. \quad (7.161)$$

The symmetry of the three rotational motions is more difficult to ascertain. Any arbitrary rotation can be expressed in terms of rotations about the  $x$ -,  $y$ -, and  $z$ -axes, which are designated as  $R_x$ ,  $R_y$ , and  $R_z$ . The effect of symmetry operations on an arbitrary rotation can be determined by representing the rotation by a curved arrow to represent the fingers of the right hand, using the right-hand rule (Figure 7.25). The rotation is represented by the counterclockwise movement of the right-hand fingers.

Figure 7.25: A rotation about the  $z$ -axis,  $R_z$ .Figure 7.26: The effect of  $\hat{C}_2$  on  $R_z$ .

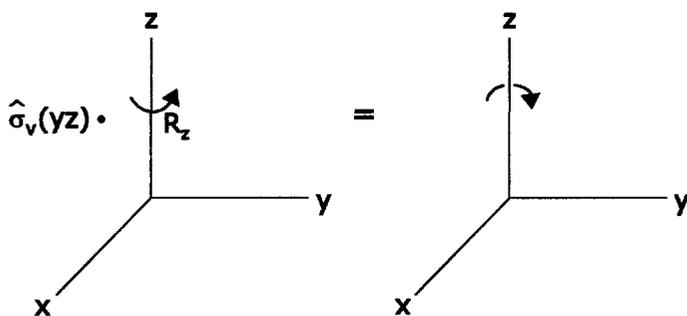
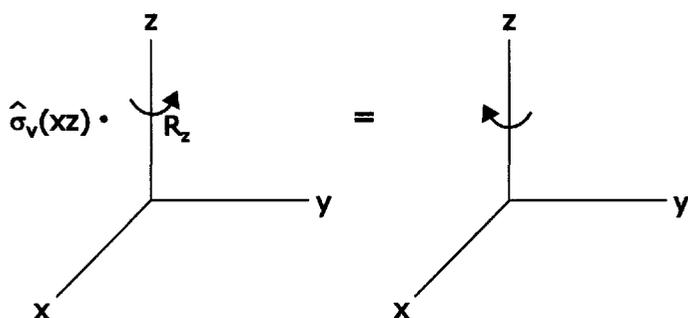
The effect of the point group symmetry operations on the movement of the right hand yields the symmetry. The sign of the rotation is given by the right-hand rule: that is, positive for counterclockwise rotation and negative for clockwise rotation. The effect of applying  $\hat{C}_2$ ,  $\hat{\sigma}_v(yz)$ , and  $\hat{\sigma}_v(xz)$  operations on  $R_z$  is illustrated in Figures 7.26, 7.27, and 7.28. From these figures, we find  $\hat{C}_2 R_z = R_z$ ,  $\hat{\sigma}_v(xz) R_z = -R_z$ ,  $\hat{\sigma}_v(yz) R_z = -R_z$  and, of course,  $\hat{E} R_z = R_z$ . We find from the  $C_{2v}$  character table that  $R_z$  behaves like the  $A_2$  line (i.e., has  $A_2$  symmetry). Similarly,  $R_x$  and  $R_y$  are found to have  $B_2$  and  $B_1$  symmetry, respectively, so that

$$\Gamma^{\text{rot}} = \Gamma^{A_2} \oplus \Gamma^{B_1} \oplus \Gamma^{B_2}. \quad (7.162)$$

Removing the translational and rotational degrees of freedom from  $\Gamma^{3N}$  leaves

$$\Gamma^{\text{vib}} = 2\Gamma^{A_1} \oplus \Gamma^{B_2}. \quad (7.163)$$

At first sight this procedure might appear to be flawed because the problem was set up using Cartesian displacement coordinates instead of normal coordinates. The mass-weighted Cartesian displacement coordinates  $q_i$  are related, however, to the normal coordinates  $Q_i$  by an orthogonal transformation

Figure 7.27: The effect of  $\hat{\sigma}_v(yz)$  on  $R_z$ .Figure 7.28: The effect of  $\hat{\sigma}_v(xz)$  on  $R_z$ .

$$\mathbf{Q} = \mathbf{l}\mathbf{q} \quad \text{or} \quad \mathbf{q} = \mathbf{l}^t\mathbf{Q}, \quad (7.164)$$

in which the 6 (or 5) rotations and translations are included in the set of  $Q_i$ .

The  $3N$ -dimensional matrix representation is generated from the equation

$$(\mathbf{q}')^t = \mathbf{q}^t \mathbf{D}^{3N}(\hat{R}) \quad (7.165)$$

and substituting equation (7.164) for  $\mathbf{q}$  into equation (7.165) gives

$$(\mathbf{l}^t\mathbf{Q}')^t = (\mathbf{l}^t\mathbf{Q})^t \mathbf{D}^{3N}(\hat{R}) \quad (7.166)$$

or

$$(\mathbf{Q}')^t \mathbf{l} = \mathbf{Q}^t \mathbf{I} \mathbf{D}^{3N}(\hat{R}). \quad (7.167)$$

Upon multiplying (7.167) from the right by the inverse  $\mathbf{l}^t$  we obtain

$$(\mathbf{Q}')^t = \mathbf{Q}^t \mathbf{I} \mathbf{D}^{3N}(\hat{R}) \mathbf{l}^t = \mathbf{Q}^t \mathbf{D}^Q(\hat{R}) \quad (7.168)$$

so that

$$\mathbf{D}^Q(\hat{R}) = \mathbf{I} \mathbf{D}^{3N}(\hat{R}) \mathbf{l}^t. \quad (7.169)$$

The matrix representation generated by using normal modes  $D^Q(\hat{R})$  is thus related by a similarity transformation to that generated using mass-weighted Cartesian displacement coordinates. This means that the two representations are equivalent and have the same characters. Thus the symmetries of the normal modes of vibration  $Q_i$  are correctly generated by using the  $q_i$ .

For the  $H_2O$  molecule the three modes correspond to the<sup>24</sup>

symmetric stretching mode	$\nu_1(a_1)$	$3\,657\text{ cm}^{-1}$ ,
bending mode	$\nu_2(a_1)$	$1\,595\text{ cm}^{-1}$ ,
antisymmetric stretching mode	$\nu_3(b_2)$	$3\,756\text{ cm}^{-1}$ ,

where the rules for labeling normal modes of vibration have been used. Normal modes are labeled in numerical order as  $\nu_1, \nu_2, \nu_3$ , and so on. The order used for labeling modes follows the order listed in Herzberg's character tables<sup>24</sup> (sometimes called the Herzberg order) which proceeds as follows:  $A_1$  before  $A_2$  before  $B_1$  before  $B_2$ , using  $C_{2v}$  as an example. For a given symmetry type the frequencies of the modes are arranged in descending order. Finally, lowercase letters are used to describe individual modes similar to the use of lowercase letters for the individual molecular orbitals (Chapters 9 and 10). The use of capital letters for irreducible representations is restricted to the total vibrational or electronic symmetry of a molecule. In the case of  $H_2O$  the two  $a_1$  modes precede the  $b_2$  mode, and among the two  $a_1$  modes  $\nu_1$  is chosen to be the higher frequency symmetric stretching mode. Degenerate modes are given only a single label so that for  $NH_3$ ,  $3N - 6 = 6$ , but the modes are  $\nu_1(a_1), \nu_2(a_1), \nu_3(e),$  and  $\nu_4(e)$ . In this case there are two modes for each frequency,  $\nu_3$  and  $\nu_4$ . For linear triatomics,  $\nu_2$  is always the bending mode.

### *Selection Rules for Vibrational Transitions*

Within the harmonic approximation, the vibrational wavefunction for the ground state is given by the totally symmetric product

$$\begin{aligned}\psi_0 &= \phi_0(\xi_1)\phi_0(\xi_2)\cdots\phi_0(\xi_{3N-6}) \\ &= N_0 e^{-\xi_1^2/2} e^{-\xi_2^2/2} \dots e^{-\xi_{3N-6}^2/2},\end{aligned}\quad (7.170)$$

in which

$$\xi_i = Q_i \left( \frac{\omega_i}{\hbar} \right)^{1/2},$$

and  $N_0$  is a normalization constant. The ground state wavefunction belongs to the  $A_1$  irreducible representation, since all of the group operations leave  $\psi_0$  unchanged. If the  $j$ th vibrational mode is excited by one quantum, then the wavefunction becomes

$$\begin{aligned}\psi_1 &= \phi_0(\xi_1)\cdots\phi_1(\xi_j)\cdots\phi_0(\xi_{3N-6}) \\ &= N_1 e^{-\xi_1^2/2} \dots H_1(\xi_j) e^{-\xi_j^2/2} \dots e^{-\xi_{3N-6}^2/2},\end{aligned}\quad (7.171)$$

in which the harmonic oscillator wavefunction for  $v_j = 1$  has replaced that for  $v_j = 0$ . The Hermite polynomial part of the wavefunction is  $H_1(\xi_j) = 2\xi_j$  for  $v_j = 1$  (Table

7.1). The fact that  $\xi_j$  is proportional to  $Q_j$  means that  $\psi_1$  is of the same symmetry as  $Q_j$ .

The intensity of an infrared transition is given by the absolute square of the transition moment integral

$$\mathbf{M} = \int \psi_f^* \boldsymbol{\mu} \psi_i d\tau = \int \psi_f^*(Q) \boldsymbol{\mu}(Q) \psi_i(Q) dQ, \quad (7.172)$$

in which  $\psi_f$  and  $\psi_i$  are final and initial vibrational wavefunctions within the same electronic state,  $\boldsymbol{\mu}$  is the dipole moment function, and the integral is over all vibrational coordinates ( $dQ = dQ_1 dQ_2 \cdots dQ_{3N-6}$ ). The dipole moment depends on the positions of the nuclei and hence on the set of  $Q_i$ . By expressing the dipole moment as a Taylor series expansion

$$\boldsymbol{\mu} = \boldsymbol{\mu}_0 + \sum_{k=1}^{3N-6} \left( \frac{\partial \boldsymbol{\mu}}{\partial Q_k} \right)_0 Q_k + \cdots, \quad (7.173)$$

equation (7.172) becomes

$$\mathbf{M} = \boldsymbol{\mu}_0 \int \psi_f^* \psi_i dQ + \sum_{k=1}^{3N-6} \left( \frac{\partial \boldsymbol{\mu}}{\partial Q_k} \right)_0 \int \psi_f^* Q_k \psi_i dQ + \cdots. \quad (7.174)$$

The first term on the right-hand side of this expression is zero because the vibrational wavefunctions are orthogonal. For a fundamental vibrational transition

$$\psi_i = \phi_0(\xi_1) \cdots \phi_0(\xi_{3N-6}) \quad (7.175)$$

and

$$\psi_f = \phi_0(\xi_1) \cdots \phi_1(\xi_j) \cdots \phi_0(\xi_{3N-6}), \quad (7.176)$$

in which  $\psi_f$  differs from  $\psi_i$  only in the  $j$ th normal mode, and the intensity of a vibrational mode is

$$I_{f \leftarrow i} \propto \left| \left( \frac{\partial \boldsymbol{\mu}}{\partial Q_j} \right)_0 \right|^2 \left| \int \phi_1(\xi_j) Q_j \phi_0(\xi_j) dQ_j \right|^2. \quad (7.177)$$

All of the terms but the  $j$ th one in the sum in (7.174) vanish due to the orthogonality of the Hermite polynomials and the neglect of higher-order terms. In fact  $I_{f \leftarrow i} = 0$  unless a single vibrational mode changes its vibrational quantum number by one unit, leading to the selection rule  $\Delta v_i = \pm 1$  (arising from the properties of Hermite polynomials, equation (7.55)). This selection rule directly follows from the use of the harmonic oscillator wavefunctions for  $\phi_i$  and from the truncation of the expansion of the dipole moment (7.173) at terms linear in  $Q$ . Consequently, to a first-order approximation, the intensity of an infrared transition is proportional to the square of the dipole moment derivative (7.177).

The intensity of electric dipole-allowed vibrational transitions is given by the square of equation (7.172). The integrand  $\psi_f^* \boldsymbol{\mu} \psi_i$  must be totally symmetric, and  $\Gamma(\psi_f^*) \otimes \Gamma(\boldsymbol{\mu}) \otimes \Gamma(\psi_i)$  must therefore contain the  $A_1$  irreducible representation. For fundamental transitions,  $\psi_i$  has  $A_1$  symmetry while  $\psi_f^*$  belongs to the irreducible representation of the  $j$ th mode, which is excited up to  $v_j = 1$ . The dipole moment operator is a vector  $\boldsymbol{\mu} = \mu_x \hat{\mathbf{e}}_1 + \mu_y \hat{\mathbf{e}}_2 + \mu_z \hat{\mathbf{e}}_3$  that behaves like the point  $\mathbf{r} = x \hat{\mathbf{e}}_1 + y \hat{\mathbf{e}}_2 + z \hat{\mathbf{e}}_3$  when the symmetry operations of the group are applied. This implies that

$$\Gamma(\psi_f^*) \otimes \Gamma(\mu) \otimes \Gamma(\psi_i) = \Gamma(\psi_f^*) \otimes \Gamma(\mathbf{r}), \quad (7.178)$$

and  $\psi_f$  must have the same symmetry as  $x$  or  $y$  or  $z$  to make the direct product symmetric. For convenience the symmetry of the Cartesian  $x$ ,  $y$ , and  $z$  components are listed to the right of the character table (Table 7.4) as are the rotations  $R_x$ ,  $R_y$ , and  $R_z$  about the  $x$ -,  $y$ -, and  $z$ -axes. All three normal modes of  $\text{H}_2\text{O}$  are infrared active since  $\nu_1(a_1)$ ,  $\nu_2(a_1)$ , and  $\nu_3(b_2)$  have the same symmetry as  $z$  or  $y$ .

Table 7.4: The  $C_{2v}$  Character Table

$C_{2v}$	$\hat{E}$	$\hat{C}_2$	$\hat{\sigma}_v(xz)$	$\hat{\sigma}_v(yz)$	
$A_1$	1	1	1	1	$z$
$A_2$	1	1	-1	-1	$R_z$
$B_1$	1	-1	1	-1	$x, R_y$
$B_2$	1	-1	-1	1	$y, R_x$

Considerable information about the vibrational modes of molecules can be predicted with the use of character tables and a table of characteristic vibrational frequencies (Table 7.5). For example, the chloroform molecule  $\text{HCCl}_3$  molecule of  $C_{3v}$  symmetry has nine normal modes that reduce as

$$\Gamma^{\text{vib}} = 3\Gamma^{a_1} \oplus 3\Gamma^e.$$

There are six distinct fundamental vibrational frequencies and all modes are infrared (and Raman) active. Since there are three C—Cl bonds and one C—H bond, there must be four stretching modes and  $9 - 4 = 5$  bending modes. If the three C—Cl stretching modes are represented by three bond-stretching coordinates  $\Delta r_1$ ,  $\Delta r_2$ , and  $\Delta r_3$ , then the three-dimensional representation reduces to  $a_1 \oplus e$ . Consulting the group frequency table (Table 7.5), one therefore predicts a C—H stretching mode of  $2960 \text{ cm}^{-1}$  ( $a_1$ ) and two C—Cl stretching frequencies ( $a_1$  and  $e$ ) at  $650 \text{ cm}^{-1}$ .

The symmetry of the bending modes can be predicted by removing two  $a_1$  modes and one  $e$  mode from the total of three  $a_1$  and three  $e$  modes. The bending modes must have  $a_1$ ,  $e$ , and  $e$  symmetry. For the C—Cl bonds, three bond-bending coordinates can be defined ( $\Delta\theta_1$ ,  $\Delta\theta_2$ , and  $\Delta\theta_3$ ) and they form a reducible three-dimensional representation. The three C—Cl bending modes therefore reduce to an  $a_1$  and an  $e$  mode. This means that the remaining bending mode must be a C—H bend of  $e$  symmetry. Predicting the frequency of bending modes is very difficult, but a bending C—H mode in  $\text{CHCl}_3$  should have a somewhat lower frequency than in  $\text{H}_2\text{O}$  (say  $\sim 1000 \text{ cm}^{-1}$ ), while the C—Cl bends might be near  $300 \text{ cm}^{-1}$ . The frequencies of bending modes are typically half of the bond-stretching frequencies. The predictions and observations for  $\text{CHCl}_3$  are summarized in Table 7.6. The numbering of the frequencies follows the order in the character table, and within a given symmetry type the modes are numbered in decreasing frequency order.

### ***Vibration-Rotation Transitions of Linear Molecules***

In many respects, the vibration-rotation transitions of linear polyatomic molecules closely resemble those of diatomic molecules. The molecular symmetry of linear poly-

Table 7.5: Infrared Group Wavenumber Table

Group	$\tilde{\nu}/\text{cm}^{-1}$	Group	$\tilde{\nu}/\text{cm}^{-1}$
$\equiv\text{C}-\text{H}$	3300	$-\text{O}-\text{H}$	3600
$=\text{C}<\text{H}$	3020	$>\text{N}-\text{H}$	3350
$>\text{C}-\text{H}$	2960	$-\text{S}-\text{H}$	2500
$-\text{C}\equiv\text{C}-$	2050	$>\text{P}=\text{O}$	1295
$>\text{C}=\text{C}<$	1650	$>\text{S}=\text{O}$	1310
$>\text{C}-\text{C}<$	900	$=\text{C}-\text{H}$	700
$-\text{N}=\text{N}-$	1600	$=\text{C}<\text{H}$	1100
$>\text{C}=\text{O}$	1700	$-\text{C}<\text{H}$	1000
$-\text{C}\equiv\text{N}$	2100	$>\text{C}<\text{H}$	1450
$>\text{C}-\text{F}$	1100	$\text{C}\equiv\text{C}-\text{C}$	300
$>\text{C}-\text{Cl}$	650		
$>\text{C}-\text{Br}$	560		
$>\text{C}-\text{I}$	500		

Table 7.6: Vibrational Modes of  $\text{CHCl}_3$ 

Mode	Symmetry	Type of Mode	Prediction	Observation <sup>25</sup>
$\nu_1$	$a_1$	C—H stretch	2960 $\text{cm}^{-1}$	3033 $\text{cm}^{-1}$
$\nu_2$	$a_1$	symmetric C—Cl stretch	650 $\text{cm}^{-1}$	667 $\text{cm}^{-1}$
$\nu_3$	$a_1$	symmetric C—Cl bend	300 $\text{cm}^{-1}$	364 $\text{cm}^{-1}$
$\nu_4$	$e$	C—H bend	1000 $\text{cm}^{-1}$	1205 $\text{cm}^{-1}$
$\nu_5$	$e$	C—Cl stretch	650 $\text{cm}^{-1}$	760 $\text{cm}^{-1}$
$\nu_6$	$e$	C—Cl bend	300 $\text{cm}^{-1}$	260 $\text{cm}^{-1}$

atomic molecules is either  $D_{\infty h}$  or  $C_{\infty v}$  and there are  $3N - 5$  modes of vibration. Let us consider the  $\text{H}-\text{C}\equiv\text{N}$  and  $\text{H}-\text{C}\equiv\text{C}-\text{H}$  molecules as examples. The  $\text{HCN}$  molecule<sup>24</sup> has three fundamental modes of vibration: the C—H stretching mode  $\nu_1(\sigma^+)$  at 3311  $\text{cm}^{-1}$ , the bending mode  $\nu_2(\pi)$  at 713  $\text{cm}^{-1}$ , and the  $\text{C}\equiv\text{N}$  stretching mode  $\nu_3(\sigma^+)$  at

2097  $\text{cm}^{-1}$ . Notice that linear triatomics are an exception to the frequency numbering scheme because  $\nu_2(\pi)$  is reserved for the bending mode. Certain functional groups such as C—H and  $\text{C}\equiv\text{N}$  have characteristic vibrational frequencies. Some of these group frequencies are listed in Table 7.5. The bending mode is doubly degenerate because of the possibility of bending in two mutually orthogonal planes. All modes of HCN are infrared active although  $\nu_3$  is very weak.

The acetylene molecule  $\text{H}-\text{C}\equiv\text{C}-\text{H}$  has  $D_{\infty h} = C_{\infty v} \otimes C_i$  symmetry. There are  $3N - 5 = 7$  modes with three stretching modes (the number of stretching modes equals the number of bonds) and  $7 - 3 = 4$  bending modes. The fundamental modes<sup>24</sup> are:

the symmetric C—H stretch  $\nu_1(\sigma_g^+)$  at 3373  $\text{cm}^{-1}$ ,

the  $\text{C}\equiv\text{C}$  stretch  $\nu_2(\sigma_g^+)$  at 1974  $\text{cm}^{-1}$ ,

the antisymmetric C—H stretch  $\nu_3(\sigma_u^+)$  at 3295  $\text{cm}^{-1}$ ,

the *trans* bend  $\text{H}-\text{C}\equiv\text{C}-\overset{\uparrow}{\text{H}} \nu_4(\pi_g)$  at 612  $\text{cm}^{-1}$ ,

and the *cis* bend  $\text{H}-\text{C}\equiv\text{C}-\overset{\downarrow}{\text{H}} \nu_5(\pi_u)$  at 729  $\text{cm}^{-1}$ .

The numbering of the modes is determined by the conventional order of the irreducible representations in the  $D_{\infty h}$  character table of Herzberg. Also, notice that the  $\nu_4$  and  $\nu_5$  bending modes are doubly degenerate with two modes associated with each frequency. The ground vibrational state has  $\sigma_g^+$  symmetry, and because  $z$  belongs to  $\sigma_u^+$  and  $x, y$  belong to  $\pi_u$ , only the  $\sigma_u^+$  and  $\pi_u$  modes ( $\nu_3$  and  $\nu_5$ ) are infrared active.

The number and types of normal modes can be quickly determined for all linear molecules. If there are  $N$  atoms, then there will be  $N - 1$  stretching frequencies and  $((3N - 5) - (N - 1))/2 = N - 2$  bending frequencies. In the case of symmetric molecules of  $D_{\infty h}$  symmetry, the  $g$  or  $u$  labels need to be added by symmetrizing the stretching of bonds or the bending of the molecule. For example, for acetylene there is a symmetric C—H stretching mode of  $\sigma_g^+$  symmetry ( $\nu_1 = 3373 \text{ cm}^{-1}$ ) and an antisymmetric C—H stretching mode of  $\sigma_u^+$  symmetry ( $\nu_3 = 3295 \text{ cm}^{-1}$ ).

The fundamental vibrational transitions of linear molecules are either of the  $\Sigma - \Sigma$  (parallel) type for stretching modes or of the  $\Pi - \Sigma$  (perpendicular) type for bending modes. For symmetric linear molecules, which belong to the  $D_{\infty h}$  point group,  $g$  and  $u$  subscripts are needed. The terms parallel and perpendicular are used because the transition dipole moment is either parallel ( $\mu_z$ ) or perpendicular ( $\mu_x$  and  $\mu_y$ ) to the molecular  $z$ -axis. Allowed parallel transitions arise from the  $\mu_z$  component of the transition dipole moment with  $\sigma_u^+$  symmetry,

$$\Gamma\psi_f^* \otimes \Gamma\mu_z \otimes \Gamma\psi_i = \sigma_u^+ \otimes \sigma_u^+ \otimes \sigma_g^+ = \sigma_g^+, \quad (7.179)$$

while allowed perpendicular transitions arise from the  $\mu_x$  and  $\mu_y$  components,

$$\Gamma\psi_f^* \otimes \Gamma\mu_{x,y} \otimes \Gamma\psi_i = \pi_u \otimes \pi_u \otimes \sigma_g^+ = \delta_g \oplus \sigma_g^+ \oplus \sigma_g^-. \quad (7.180)$$

The  $\Sigma - \Sigma$  transitions can have  $P$  and  $R$  branches only, so that the appearance of the spectrum closely resembles that of the infrared spectrum of a diatomic molecule (Figure 7.29).

The  $\Pi - \Sigma$  transitions have  $P$ ,  $Q$ , and  $R$  branches as shown in Figure 7.30. The rotational energy levels associated with the  $\Pi$  state are doubly degenerate because

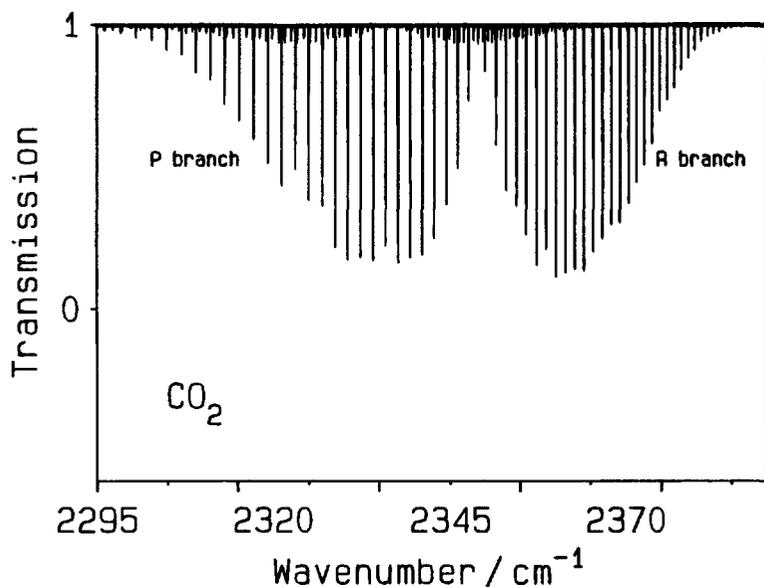


Figure 7.29: The  $\nu_3(\sigma_u^+)$  antisymmetric stretching fundamental band of  $\text{CO}_2$ . Notice the weaker bending hot band ( $01^11 - 01^10$ ) that is also present.

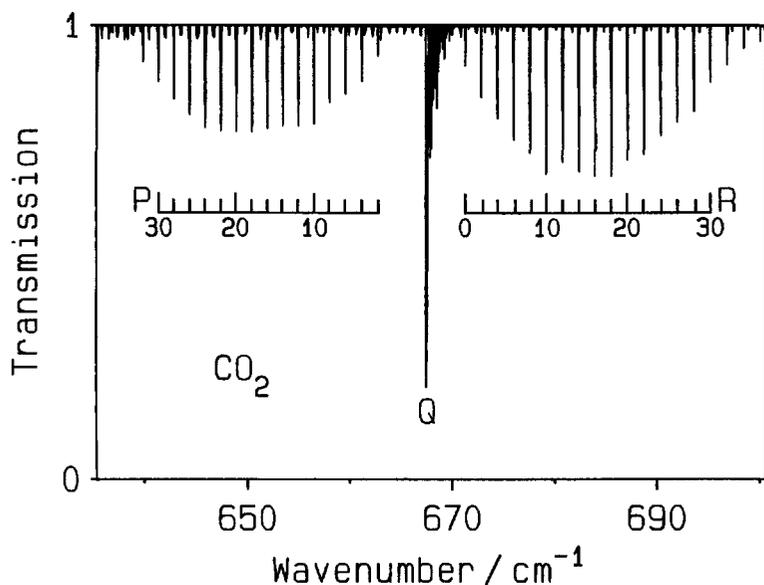


Figure 7.30: The  $\nu_2(\pi_u)$  bending fundamental band of  $\text{CO}_2$ .

$l = \pm 1$ , where  $l$  is the quantum number of vibrational angular momentum (Chapter 6). As the molecule begins to rotate, the two components for a given  $J$  begin to split slightly because of the interaction of rotational angular momentum ( $\hat{J}$ ) with vibrational angular momentum ( $\hat{l}$ ). The splitting  $\Delta\nu$  is proportional to<sup>24</sup>

$$\Delta\nu = qJ(J + 1) \quad (7.181)$$

and  $q$  is called the  $l$ -type doubling constant. It is useful to use parity labels to distinguish the two nearly degenerate levels for each  $J$ . There are many different types of parity, but the two most common varieties are total parity and  $e/f$  parity. (A more detailed description of parity is provided in Chapter 9.) Total parity considers the effect of inversion of all coordinates *in the laboratory frame* of the total wavefunction  $\psi = \psi_{\text{el}}\psi_{\text{vib}}\psi_{\text{rot}}$ . This inversion operation  $\hat{E}^*$  inverts the laboratory coordinates of all atoms in a molecule,

$$\hat{E}^*\psi(X_i, Y_i, Z_i) = \psi(-X_i, -Y_i, -Z_i) = (\pm 1)\psi, \quad (7.182)$$

and leaves the wavefunction unchanged, except possibly for a change in sign. Total parity can be either positive + (upper sign) or negative - (lower sign). Total parity is commonly used to label the energy levels of atoms as well as the rotational energy levels of diatomic and linear molecules.

This laboratory symmetry operator  $\hat{E}^*$  is different from the geometric molecular symmetry operator  $\hat{i}$  discussed previously. Only  $D_{\infty h}$  molecules have  $\hat{i}$  as a symmetry operator, while all molecules have  $\hat{E}^*$  as a symmetry operator. Note that  $\hat{E}^*$  is a very peculiar operator because it inverts the entire molecular coordinate system as well as the location of the nuclei. It is therefore a permutation-inversion operator rather than a molecular symmetry operator of the type discussed in Chapters 2 and 3.

The wavefunction can be written as  $\psi = \psi_{\text{el}}\psi_{\text{vib}}\psi_{\text{rot}}$ , and the effect of  $\hat{E}^*$  on each part must be considered. The operation  $\hat{E}^*$  leaves the relative positions of the nuclei unchanged, so  $\hat{E}^*\psi_{\text{vib}} = \psi_{\text{vib}}$  for nondegenerate  $\sigma^+$  vibrations. The rotational part of the wavefunction,  $\psi_{\text{rot}} = \psi_{JM}(\theta, \phi)$ , changes sign for odd  $J$  under the operation  $\hat{E}^*$ , since

$$\hat{E}^*Y_{JM} = (-1)^J Y_{JM}. \quad (7.183)$$

The effect of  $\hat{E}^*$  on  $\psi_{\text{el}}$  is much more difficult to ascertain because  $\psi_{\text{el}}$  is a function of internal molecular coordinates. It is possible to show the surprising result<sup>26</sup> that  $\hat{E}^*$  in the laboratory frame is equivalent to  $\hat{\sigma}_v$  (chosen to be  $\hat{\sigma}_v(xz)$ , for convenience) in the molecular frame. Thus for the totally symmetric electronic ground state  $X^1\Sigma^+$ ,

$$\hat{E}^*\psi_{\text{el}}(x_i, y_i, z_i) = \hat{\sigma}_v\psi_{\text{el}}(x_i, y_i, z_i) = +1\psi_{\text{el}}. \quad (7.184)$$

The total parity of a linear molecule wavefunction alternates with  $J$  as shown in Figure 7.31 for a  $^1\Sigma^+$  state. Since this alternation of total parity with  $J$  occurs for all electronic states, it is convenient to factor out the  $J$  dependence and designate those rotational levels with a total parity of  $+(-1)^J$  as  $e$  parity and those with a total parity of  $-(-1)^J$  as  $f$  parity (for half-integer  $J$  a total parity of  $+(-1)^{J-1/2}$  corresponds to  $e$ ,  $-(-1)^{J-1/2}$  to  $f$ ).<sup>27</sup> The  $e/f$  parity is thus a  $J$  independent parity labeling scheme for rovibronic wavefunctions (rovibronic = rotational  $\times$  vibrational  $\times$  electronic). All  $^1\Sigma^+$  rotational energy levels, therefore, have  $e$  parity (Figure 7.31). The  $e/f$  parity labels correspond to the residual intrinsic parity of a rotational level after the  $(-1)^J$  part has been removed. Note that Figures 7.31, 7.32, and 7.33 also apply to electronic transitions (Chapter 9).

The one-photon, electric-dipole selection rule  $+\leftrightarrow-$  is derived by recognizing that the parity of  $\mu$  is  $-1$  (i.e.,  $E^*\mu = -1\mu$ ), while the parity of the transition moment integral

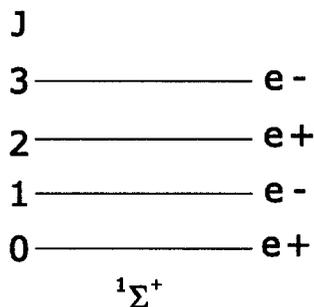


Figure 7.31: Parities of the rotational levels in a  ${}^1\Sigma^+$  state.

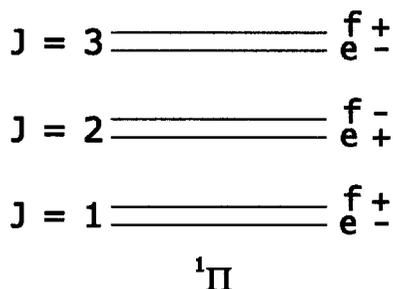


Figure 7.32: Parity labels for the rotational levels of a  $\Pi$  state with a negative value for the parameter  $q$ , that places  $f$  above  $e$  for each  $J$ .

$$\int \psi_i^* \mu \psi_f d\tau \quad (7.185)$$

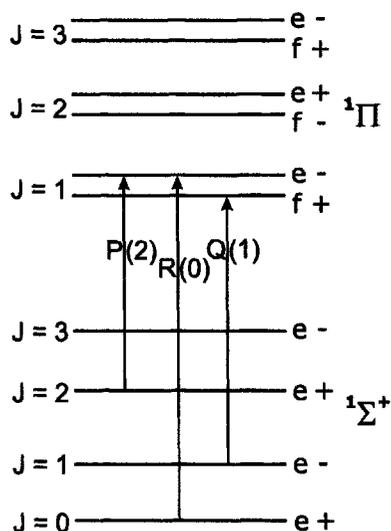
must be  $+1$ . This  $+/-$  selection rule becomes  $e \leftrightarrow e$ ,  $f \leftrightarrow f$  for  $P$  and  $R$  branches, and  $e \leftrightarrow f$  for  $Q$  branches in the  $e/f$  parity labeling scheme.

Parity labeling is essential when nearly degenerate energy levels are present, as for example, in  $\Pi$  vibrational states of linear molecules. For a  $\Pi$  state  $l = \pm 1$ , and it is possible to form linear combinations of the vibrational wavefunctions from the two-dimensional harmonic oscillator wavefunctions (Chapter 6), so that they are eigenfunctions of  $\hat{\sigma}_v$ :

$$\psi_{\text{vib}} = R(\rho)(e^{i\phi} \pm e^{-i\phi}) = R(\rho)\Phi(\phi). \quad (7.186)$$

Note that  $\hat{\sigma}_v\Phi(\phi) = \Phi(\pi - \phi)$  so that the upper sign in equation (7.186) corresponds to  $f$  parity, while the lower sign corresponds to  $e$  parity. The total parity still changes with  $J$ , as shown in Figure 7.32, while the  $e/f$  ordering is determined by the sign of  $q$ , the  $l$ -doubling constant. Therefore,  $e/f$  parity labels are convenient for differentiating between the two near-degenerate levels associated with  $l$ -type doubling (Figure 7.32).

In a  $\Pi - \Sigma$  transition (Figure 7.33) the  $Q$  branch lines terminate on rotational levels of the *opposite* parity ( $f \leftrightarrow e$ ), as opposed to  $P$  and  $R$  branch lines that terminate on rotational levels of the *same* parity ( $e \leftrightarrow e$ ). Thus the usual combination differences involving  $P$  and  $R$  branches provide rotational constants for the upper and lower levels

Figure 7.33: Energy-level diagram for a  $\Pi - \Sigma$  transition.

of  $e$  parity only, while analysis of the Q branch yields constants for the upper levels of  $f$  parity.

The  $+/-$  total parity labels or  $e/f$  parity labels are unrelated to the  $g/u$  vibrational labels used in  $D_{\infty h}$  molecules in which

$$\hat{i}\psi_{\text{vib}} = \pm\psi_{\text{vib}}, \quad (7.187)$$

where  $+$  corresponds to  $g$  and  $-$  corresponds to  $u$ . Since  $D_{\infty h} = C_{\infty v} \otimes C_i$ , the  $g$  and  $u$  labels are required when the molecule has a center of symmetry.

### Nuclear Spin Statistics

An additional symmetry requirement is associated with the constraint placed on molecular wavefunctions by the Pauli exclusion principle. Because identical nuclei are indistinguishable, their exchange can, at most, change the sign of the total wavefunction  $\psi_{\text{total}}$  that includes nuclear spin. If  $P_{12}$  is the operator which exchanges identical nuclei, then the Pauli exclusion principle requires that

$$\hat{P}_{12}\psi_{\text{total}} = \pm\psi_{\text{total}}. \quad (7.188)$$

For particles with integer nuclear spin ( $I = 0, 1, 2, \dots$ ), called bosons, the sign in equation (7.188) is found to be positive ( $+1$ ), while for fermions with half-integer nuclear spin ( $I = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ ) the negative sign ( $-1$ ) applies.

The total wavefunction can be written as a product of a nuclear spin part  $\psi_{\text{spin}}$  and a space part,  $\psi_{\text{space}} \equiv \psi$ ,

$$\psi_{\text{total}} = \psi_{\text{space}}\psi_{\text{spin}} = \psi\psi_{\text{spin}}, \quad (7.189)$$

so that the effect of  $\hat{P}_{12}$  on either part can be examined separately. When  $\hat{P}_{12}$  operates on the “normal” space part of the total wavefunction of a symmetric linear molecule

$$\hat{P}_{12}\psi = \pm\psi, \quad (7.190)$$

the + levels are labeled as symmetric or *s*, and - as antisymmetric or *a*.

For a diatomic molecule, the  $\hat{P}_{12}$  permutation operator in the laboratory frame is equivalent to the  $\hat{C}_2(y)$  symmetry operator in the molecular frame of a diatomic molecule.<sup>26</sup> This surprising equivalence can be rationalized if one remembers that the  $\hat{C}_2(y)$  operation does not, in fact, exchange the nuclei. The  $\hat{C}_2$  operator changes the electrons, the nuclear *displacement* vectors, and the rotational variables  $\theta$  and  $\phi$ , but leaves the positions of the nuclei unaltered. The location of the nuclei define the molecular *z*-axis, which is not affected by a symmetry operation such as  $\hat{C}_2(y)$ . Clearly this is physically equivalent to just interchanging the nuclei while leaving the positions of all of the other particles fixed.

The nature of the nuclear spin part of  $\psi_{\text{total}}$  depends on the particular nuclei under consideration. For example in the F—Be—F or H<sub>2</sub> molecules of  $D_{\infty h}$  symmetry, the nuclear spins of F and H are  $\frac{1}{2}$ . The symmetric and antisymmetric nuclear spin wave functions can be constructed for nuclei A and B with  $\alpha = |m_I = +\frac{1}{2}\rangle$ ,  $\beta = |m_I = -\frac{1}{2}\rangle$ , as

$$\psi_{\text{spin}} = \begin{cases} \alpha(A)\alpha(B) \\ \frac{\alpha(A)\beta(B) + \beta(A)\alpha(B)}{\sqrt{2}} & \text{(symmetric)} \\ \beta(A)\beta(B) \end{cases} \quad (7.191)$$

and

$$\psi_{\text{spin}} = \frac{\alpha(A)\beta(B) - \beta(A)\alpha(B)}{\sqrt{2}} \quad \text{(antisymmetric)}. \quad (7.192)$$

The total wavefunctions must obey the equation

$$\hat{P}_{12}\psi_{\text{total}} = -\psi_{\text{total}}, \quad (7.193)$$

because H and F nuclei are both fermions. This means that *s* symmetry spatial wavefunctions must be combined with antisymmetric spin functions, while *a* symmetry spatial wavefunctions are combined with symmetric spin functions. Since there are three symmetric nuclear spin wavefunctions but only one antisymmetric function, the energy levels with *a* symmetry have statistical weights three times those of the *s* levels. This means that, all other things being equal, the transitions from *a* levels are three times as intense as are those from *s* levels. Note that *s* and *a* labels describe the wavefunction exclusive of nuclear spin.

The  $\hat{P}_{12}$  permutation operator in the laboratory frame, or the  $\hat{C}_2(y)$  symmetry operator in the molecular frame, needs to be applied to the total wavefunction

$$\psi = \psi_{\text{el}}\psi_{\text{vib}}\psi_{\text{rot}}. \quad (7.194)$$

Once again one can conclude that for a symmetric vibration and for a symmetric  ${}^1\Sigma_g^+$  electronic state,  $\hat{C}_2(y)$  has no effect. The operation of  $\hat{C}_2(y)$  on  $\psi_{\text{rot}}$  replaces  $\theta$  by  $\pi - \theta$  and  $\phi$  by  $\pi + \phi$  in  $Y_{JM}(\theta, \phi)$  or

$$\hat{C}_2(y)Y_{JM}(\theta, \phi) = Y_{JM}(\pi - \theta, \pi + \phi) = (-1)^J Y_{JM}(\theta, \phi). \quad (7.195)$$

J	nuclear spin weight, g
3 ————— e - a	3
2 ————— e + s	1
1 ————— e - a	3
0 ————— e + s	1

${}^1\Sigma_g^+$

Figure 7.34: Nuclear spin weights and labels for a  ${}^1\Sigma_g^+$  state with two equivalent spin- $\frac{1}{2}$  nuclei.

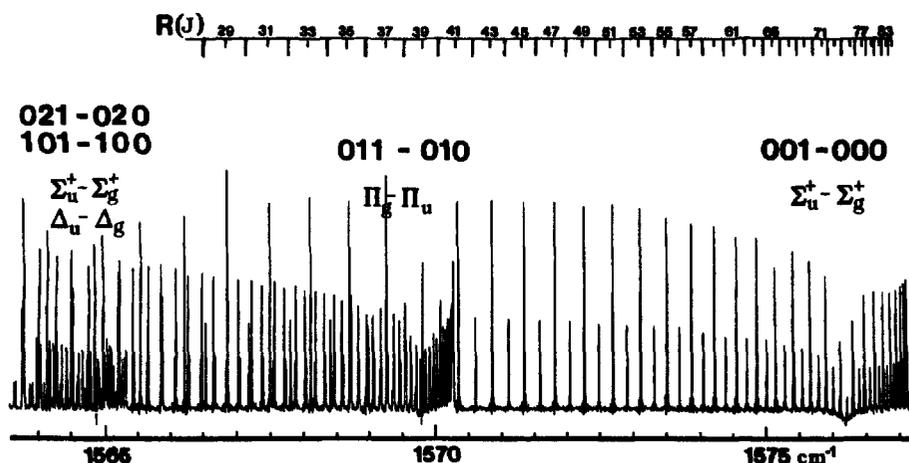


Figure 7.35: Infrared emission spectrum of the  $\nu_3(\sigma_u^+)$  antisymmetric stretching mode of  $\text{BeF}_2$ .

The energy-level diagram for  $\text{H}_2$  or  $\text{BeF}_2$  is illustrated in Figure 7.34. The selection rules on  $s$  and  $a$  are  $s \leftrightarrow s$  and  $a \leftrightarrow a$  for transitions. The levels with the larger nuclear spin weighting are designated *ortho*, while the levels with the smaller weighting are designated *para*. The effect of nuclear spin statistics can clearly be seen in the spectrum<sup>28</sup> of  $\text{BeF}_2$  (Figure 7.35).

The  $g$  and  $u$  symmetry labels for an electronic or vibrational state are determined by the  $\hat{i}$  symmetry operator acting in the molecular frame or by the  $\hat{P}_{12}\hat{E}^*$  product of permutation-inversion operators in the laboratory frame. This is because the  $\hat{E}^*$  operator first inverts all coordinates of all particles and then  $\hat{P}_{12}$  places the nuclei back into their original position. The net effect of  $\hat{P}_{12}\hat{E}^*$  is thus to invert the electronic coordinates through the origin (i.e., the operator  $\hat{i}$ ). This also means that the  $s$  or  $a$  symmetry (associated with  $\hat{P}_{12}$ ) of a rovibronic wavefunction operator is determined by the  $g$  or  $u$  symmetry (associated with  $\hat{i}$  or  $\hat{P}_{12}\hat{E}^*$ ) of the vibronic part of the wavefunction (vibronic = vibrational  $\times$  electronic; rovibronic = rotational  $\times$  vibrational  $\times$  electronic) and by the total parity (associated with  $\hat{E}^*$ ). The various possibilities are illustrated in Figure 7.36.

The exchange of the two equivalent nuclei in a homonuclear diatomic molecule ( $\hat{P}_{12}$  operator) can be carried out in other ways equivalent to the operator  $\hat{C}_2(y)$ . For

J 3 — e - a	J 3 — e - s	J 3 — f + s	J 3 — f + a
2 — e + s	2 — e + a	2 — f - a	2 — f - s
1 — e - a	1 — e - s	1 — f + s	1 — f + a
0 — e + s	0 — e + a	0 — f - a	0 — f - s
${}^1\Sigma_g^+$	${}^1\Sigma_u^+$	${}^1\Sigma_g^-$	${}^1\Sigma_u^-$

Figure 7.36: Parities for the rotational levels of  ${}^1\Sigma$  states of  $D_{\infty h}$  molecules.

example, all of the particles are inverted through the origin by the  $\hat{E}^*$  operator in the laboratory frame and then the electrons (and the nuclear displacement vectors) are inverted back by the  $\hat{i}$  operator in the molecular frame. This leaves the two nuclei exchanged; mathematically  $\hat{i}\hat{E}^* = \hat{P}_{12}\hat{E}^*\hat{E}^* = \hat{P}_{12}$ , as required. Thus, the  $s$  or  $a$  parity is determined by the total parity ( $\hat{E}^*$  operator) and the  $g$  or  $u$  parity of the electronic state ( $\hat{i}$  operator) as shown in Figure 7.36.

Finally, the superscript  $+$  or  $-$  on the  ${}^1\Sigma^+$  and  ${}^1\Sigma^-$  symbols is necessary in order to distinguish between the effect of  $\hat{\sigma}_v$  on the electronic (or vibronic) wavefunctions,

$$\hat{\sigma}_v\psi_{el} = \pm\psi_{el}. \quad (7.196)$$

For doubly degenerate vibronic wavefunctions  $\Pi$ ,  $\Delta$ ,  $\Phi$ , and so forth, one component can always be labeled  $+$  while the other can be labeled  $-$ . However, writing  $\Pi^\pm$ ,  $\Delta^\pm$ ,  $\Phi^\pm$  generally serves no useful purpose, so that only for  $\Sigma$  states is the superscript  $+$  or  $-$  used.

The effect of nuclear spin statistics is most apparent for molecules such as  $\text{CO}_2$  for which the nuclear spins of equivalent nuclei are zero. In this case the equivalent nuclei are bosons so only the  $s$  levels are present. The  $a$  levels have no antisymmetric nuclear spin functions to combine with and are therefore absent. This means that all of the odd  $J$  lines are missing in the infrared spectrum of the  $\nu_3$  mode of  $\text{CO}_2$  and the spacing between the lines is approximately  $4B$ . In general the relative nuclear spin weights<sup>5</sup> for two equivalent nuclei are  $I/(I+1)$ .

### Excited Vibrational States of Linear Molecules

The symmetry of excited vibrational states of linear molecules is obtained by taking direct products of the symmetry species. For example for a doubly excited state ( $v_2 = 2$ ) of  $\Pi$  symmetry one obtains  $\Pi \otimes \Pi = \Sigma^+ \oplus [\Sigma^-] \oplus \Delta$ . This product, however, can be reduced to a symmetric part ( $\Sigma^+ \oplus \Delta$ ) and an antisymmetric part ( $\Sigma^-$ ), but only the symmetric part is allowed by symmetry (cf. the Pauli exclusion principle). In the direct product tables of Appendix C, the antisymmetric part of the product is in square brackets. Notice that  $\Pi \otimes \Pi$  is the same as coupling two  $l = \pm 1$  vibrational states to make a  $l = \pm 2(\Delta)$  and two  $l = 0$  states ( $\Sigma^+$  and  $\Sigma^-$ ). However, the  $l_1 = +1, l_2 = -1$  and  $l_1 = -1, l_2 = +1$  states are indistinguishable, so that only a  $\Sigma^+$  results. Note that if the  $\Pi \otimes \Pi$  product resulted from a vibronic product where a  $\Pi$  electronic state was coupling to a  $\pi$  bending mode or if each  $\pi$  was from a different vibrational mode, then both  $\Sigma^+$  and  $\Sigma^-$  states would be present. The stack of energy levels for the bending mode of a linear triatomic molecule, such as  $\text{HCN}$ , is shown in Figure 7.37. For a two-dimensional harmonic oscillator, energy levels with the same  $v$  are degenerate, but this degeneracy is removed when anharmonicity is taken into account.

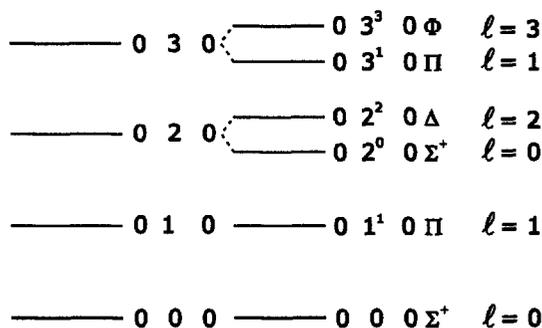


Figure 7.37: Bending energy levels of a linear triatomic molecule. Notice that  $l$  is often written as a superscript on  $v_2$ , i.e., a vibrational state is labeled by  $v_1 v_2^l v_3$ .

The energy levels for a collection of  $3N - 5$  harmonic oscillators ( $3N - 6$  for nonlinear molecules) is

$$G(v_1 v_2 \cdots v_{3N-5}) = \sum_{r=1}^{3N-5} \omega_r \left( v_r + \frac{d_r}{2} \right), \quad (7.197)$$

where  $d_r$  is the degeneracy of the  $r$ th vibrational mode. For the anharmonic oscillator the modes are no longer independent and cross-terms are present, so that<sup>24</sup>

$$G(v_1 v_2 \cdots v_{3N-5}) = \sum_r \omega_r \left( v_r + \frac{d_r}{2} \right) + \sum_{r \leq s} x_{rs} \left( v_r + \frac{d_r}{2} \right) \left( v_s + \frac{d_s}{2} \right) + \sum_{t \leq t'} g_{tt'} l_t l_{t'}, \quad (7.198)$$

in which the index  $t$  applies to degenerate modes with vibrational angular momentum  $l_t$ . As an example, the vibrational energy-level expression for HCN is

$$G(v_1 v_2 v_3) = \omega_1 \left( v_1 + \frac{1}{2} \right) + \omega_2 (v_2 + 1) + \omega_3 \left( v_3 + \frac{1}{2} \right) + x_{11} \left( v_1 + \frac{1}{2} \right)^2 + x_{22} (v_2 + 1)^2 + x_{33} \left( v_3 + \frac{1}{2} \right)^2 + x_{12} \left( v_1 + \frac{1}{2} \right) (v_2 + 1) + x_{13} \left( v_1 + \frac{1}{2} \right) \left( v_3 + \frac{1}{2} \right) + x_{23} (v_2 + 1) \left( v_3 + \frac{1}{2} \right) + g l^2. \quad (7.199)$$

For most large molecules the constants  $x_{rs}$  and  $g_{tt'}$  are not known. The vibrational energy levels (in  $\text{cm}^{-1}$ ) for HCN are given by<sup>29</sup>

$$G(v_1 v_2^l v_3) = 3441.221 \left( v_1 + \frac{1}{2} \right) + 726.995 (v_2 + 1) + 2119.864 \left( v_3 + \frac{1}{2} \right) - 52.490 \left( v_1 + \frac{1}{2} \right)^2 - 2.653 (v_2 + 1)^2 - 7.074 \left( v_3 + \frac{1}{2} \right)^2 - 19.006 \left( v_1 + \frac{1}{2} \right) (v_2 + 1) - 10.443 \left( v_1 + \frac{1}{2} \right) \left( v_3 + \frac{1}{2} \right) - 2.527 (v_2 + 1) \left( v_3 + \frac{1}{2} \right) + 5.160 l^2, \quad (7.200)$$

in which higher-order terms are dropped.

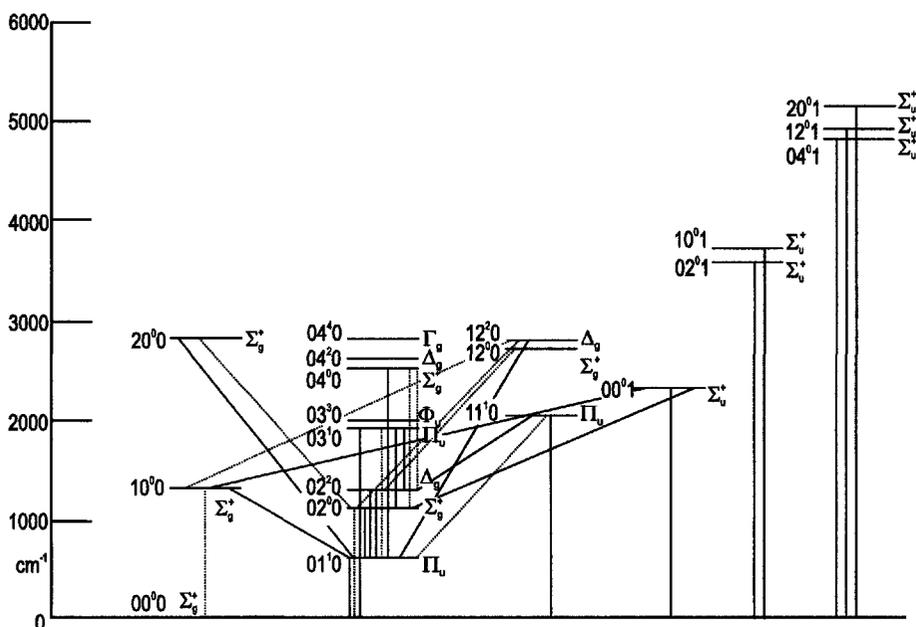


Figure 7.38: Energy-level diagram for some of the vibrational levels of  $\text{CO}_2$ . The solid lines correspond to infrared transitions, while the dotted transitions are observed by Raman spectroscopy (Chapter 8).

The energy level diagram is quite complicated even for a triatomic molecule. For example the energy levels of some of the known states of  $\text{CO}_2$  are shown in Figure 7.38. The selection rules for transitions among the excited energy levels are derivable, as usual, from the transition dipole moment integral. The general selection rules can be summarized as  $\Delta l = 0, \pm 1$ ,  $g \leftrightarrow u$ , and  $\Sigma^+ \leftrightarrow \Sigma^-$ .

The various possibilities are:

1.  $\Delta l = 0$  with  $l = 0$ . This is a parallel transition of the  $\Sigma^+ - \Sigma^+$  type with  $P$  and  $R$  branches ( $\Delta J = \pm 1$ ).
2.  $\Delta l = \pm 1$ . This is a perpendicular transition such as  $\Pi - \Sigma$ ,  $\Delta - \Pi$ , and so forth, with  $P$  and  $R$  ( $\Delta J = \pm 1$ ) branches and a strong  $Q$  branch ( $\Delta J = 0$ ).
3.  $\Delta l = 0$  with  $l \neq 0$ . Transitions of the type  $\Pi - \Pi$ ,  $\Delta - \Delta$ , and so forth, with  $P$  and  $R$  branches and weak  $Q$  branches. The  $Q$  branch lines are not always observed. The relative intensities of the lines in the various branches are given by the rotational populations and the Hönl-London factors<sup>5</sup> (from Table 9.4 with  $\Lambda$  replaced by  $l$ ).

Some of the possible transitions are displayed in the energy-level diagram for  $\text{CO}_2$  (Figure 7.38). Notice that in addition to fundamentals, overtones, and hot bands, transitions such as  $01^11 \leftarrow 000$  are possible. These transitions, in which the quantum numbers for two or more modes change, are called *combination bands*. For example, all possible allowed transitions among the first four bending energy levels of  $\text{HCN}$  are illustrated in Figure 7.39.

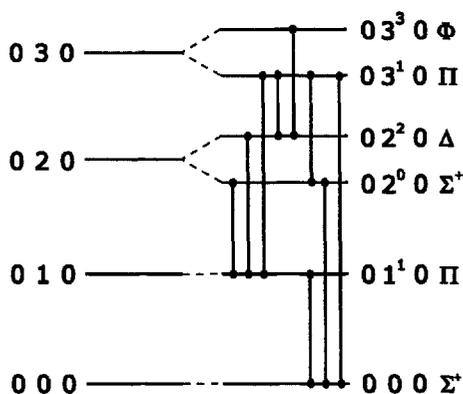


Figure 7.39: All possible allowed transitions among the first four bending energy levels of a triatomic molecule such as HCN.

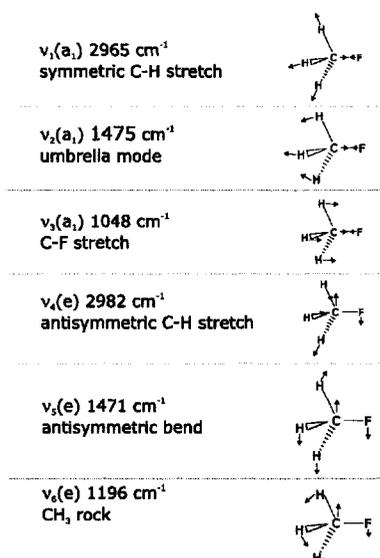


Figure 7.40: Vibrational modes<sup>25</sup> of CH<sub>3</sub>F with only one member of each degenerate pair of modes displayed.

### 7.3 Vibrational Spectra of Symmetric Tops

Consider a molecule such as CH<sub>3</sub>F with  $3N - 6 = 9$  modes of vibration. The application of group theory indicates that there are three  $a_1$  modes and three  $e$  modes<sup>25</sup> of vibration using the  $C_{3v}$  character table. The four bonds in the molecule give rise to four stretching modes: the three C—H stretches and a C—F stretch. The symmetry of the C—H stretching modes are  $a_1$  and  $e$ , while the C—F stretch has  $a_1$  symmetry. The remaining five of the nine possible modes must be bending modes. The H—C bending modes can be reduced to a symmetric CH<sub>3</sub> bending mode (umbrella mode) and an antisymmetric CH<sub>3</sub> bending mode of  $e$  symmetry. The symmetry of the —C—F bending mode (or CH<sub>3</sub> rock) is  $e$  (Figure 7.40).

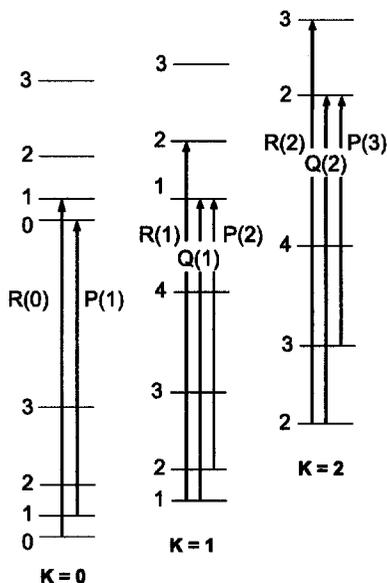


Figure 7.41: A parallel  $A_1 - A_1$  transition of a symmetric top molecule.

As for linear molecules, transitions in symmetric tops can have transition dipole moments parallel to the  $z$ -axis (symmetry axis) or perpendicular to the  $z$ -axis. Parallel transitions are of the  $A_1 - A_1$  type, while perpendicular transitions are of the  $E - A_1$  type. The  $A_1 - A_1$  energy level diagram is given in Figure 7.41. The transitions obey the parallel selection rules  $\Delta K = 0$  with  $\Delta J = \pm 1$  for the  $K' = 0 \leftarrow K'' = 0$  transition and  $\Delta J = 0, \pm 1$  transitions for  $K \neq 0$ . It is useful to note that these are exactly the same selection rules as obtained for the linear molecule, with  $K$  playing the role formerly played by  $l$ . Indeed the intensity expressions are given by the same Hönl-London factors given in Table 9.4 with  $K$  replacing  $\Lambda$ . The transitions associated with each  $K$  are called sub-bands. The observed spectrum can be viewed as a superposition of sub-bands as shown in Figure 7.42. At low resolution the band exhibits the characteristic  $PQR$  pattern, Figure 7.43. When examined at higher resolution, the  $K$  splittings for each rotational line are resolved (Figure 7.44).

### Coriolis Interactions in Molecules

The  $E - A_1$  type transitions of a symmetric top molecule require the addition of Coriolis terms to the vibrational Hamiltonian operator. Coriolis forces are very important for the doubly degenerate  $E$  level.

Consider a molecular reference frame  $xyz$  rotating in space relative to the laboratory coordinate system  $XYZ$ . This means that the molecular frame of reference is an accelerated coordinate system, which will have "fictitious" centrifugal and Coriolis forces. These forces are not present when the molecule is viewed in the inertial laboratory coordinate system. It is more convenient, however, to work in the molecular frame and to live with the presence of centrifugal and Coriolis forces.

The origin of centrifugal forces are best explained by considering a particle of mass  $\mu$  rotating at a constant angular velocity  $\omega$  (Figure 7.45). This models a rotating diatomic

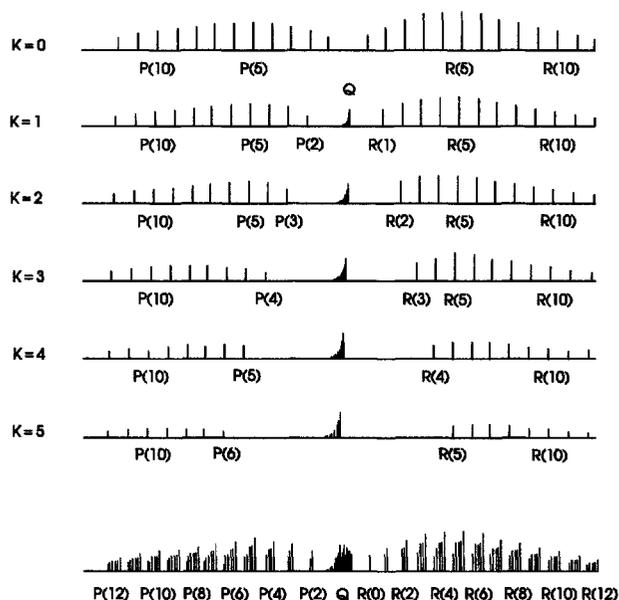


Figure 7.42: Sub-bands of a parallel transition of a symmetric top. On the top the sub-bands are shown separately, and they are combined to simulate a real spectrum in the bottom panel.

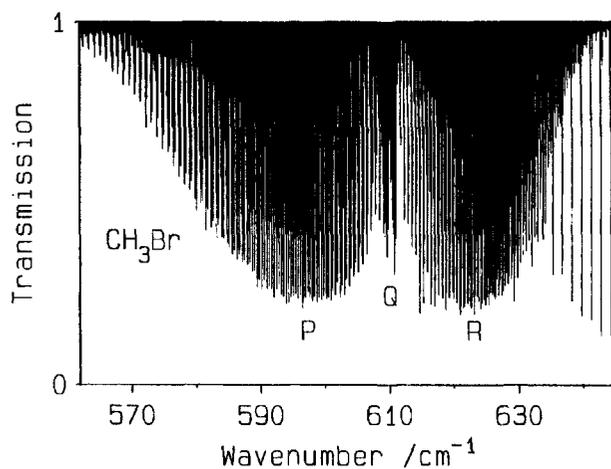


Figure 7.43: The infrared spectrum of the  $\text{CH}_3\text{Br}$   $\nu_3$  mode. Notice the presence of two  $Q$  branches, one due to  $\text{CH}_3^{79}\text{Br}$  and the other to  $\text{CH}_3^{81}\text{Br}$ .

molecule of reduced mass  $\mu$  as seen from the laboratory frame. Although the magnitude of the velocity of the particle is constant, the direction of the velocity is constantly changing (Figure 7.45). In the laboratory frame the particle is constrained to move in a circle by application of a force of magnitude

$$|\mathbf{T}| = \frac{mv^2}{r} = m\omega^2 r. \quad (7.201)$$

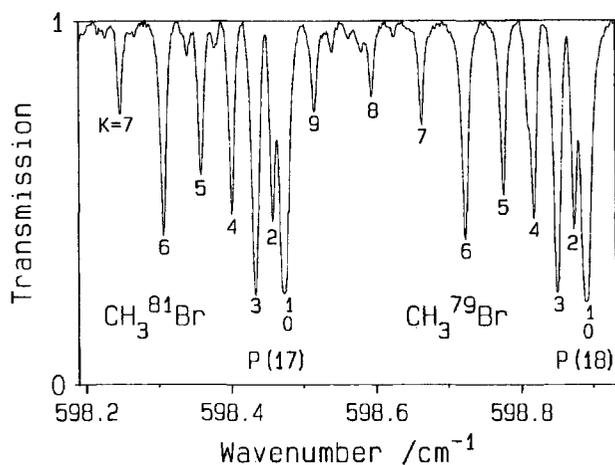


Figure 7.44: The  $K$  structure of two rotational lines of the  $\text{CH}_3\text{Br } \nu_3$  mode. The intensity alternation is caused by nuclear spin statistics.

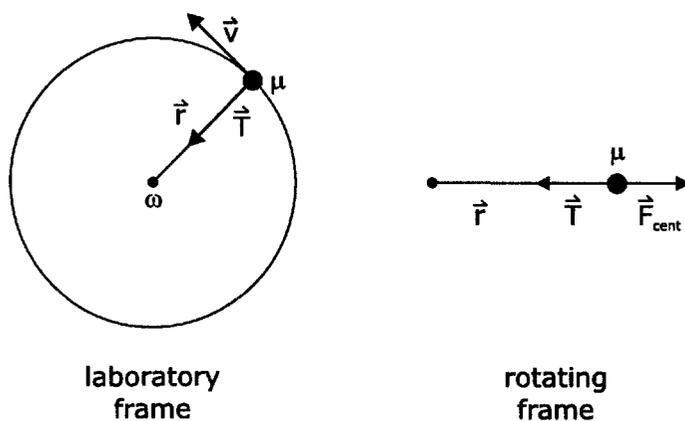


Figure 7.45: A rotating particle of mass  $\mu$  viewed in the laboratory frame and in the rotating (molecular) frame.

The particle is therefore undergoing acceleration, since Newton's second law is

$$\mathbf{T} = \mathbf{F} = \mu \mathbf{a}. \quad (7.202)$$

In the rotating molecular frame (Figure 7.45) the particle is stationary since the angular velocity  $\omega$  of the particle and of the molecular frame is the same. A new centrifugal force has appeared that exactly balances the force  $\mathbf{T}$ . As anyone who has been in a car that corners sharply can attest, these "fictitious" forces are very real in an accelerated frame of reference.

A Coriolis force is the second type of fictitious force that can appear in an accelerated coordinate system. Consider a particle of mass  $\mu$  initially moving at a constant angular velocity  $\omega$  (Figure 7.46). Some time later (position 5 in Figure 7.46) the particle is released and proceeds to move in a straight line at constant velocity because there are no applied forces (Newton's first law). After the particle has been released, the

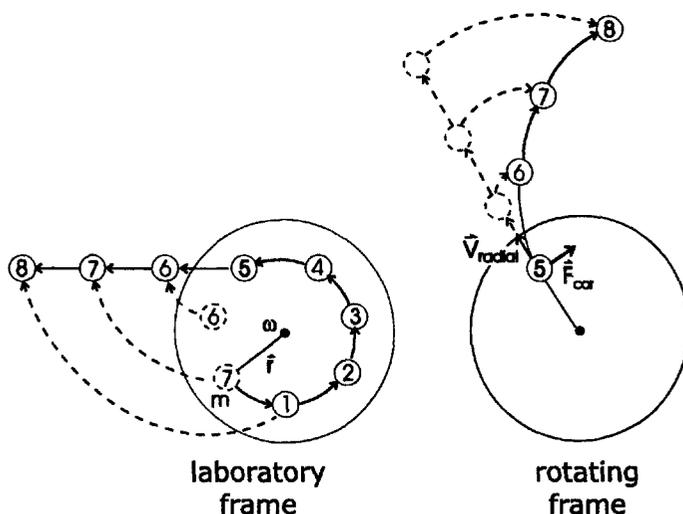


Figure 7.46: A particle is constrained to move in a circle at a constant angular velocity  $\omega$  (positions 1 to 5). At 5 the particle is released and continues its motion unconstrained by external forces (positions 6 to 8).

motion in the laboratory frame is simple, but when viewed in the rotating frame the motion appears peculiar. In the frame rotating with an angular velocity  $\omega$  the particle moves both radially at a constant velocity  $v_{\text{radial}}$  and veers to the right (Figure 7.46). The particle seems to veer to the right because the rotating frame is rotating out from underneath the particle moving at a constant velocity in the laboratory frame. The motion of the particle to the right is caused by a “fictitious” Coriolis force. The Coriolis force is

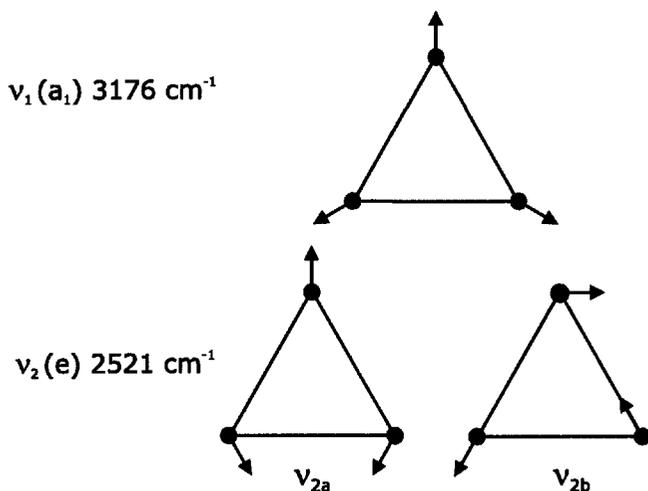
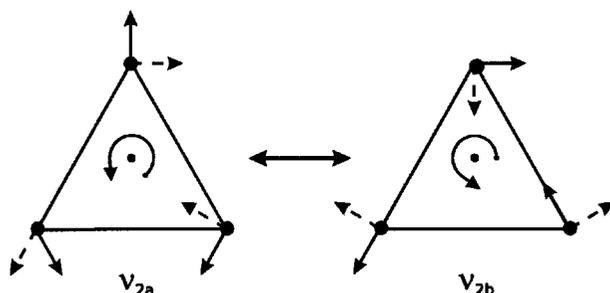
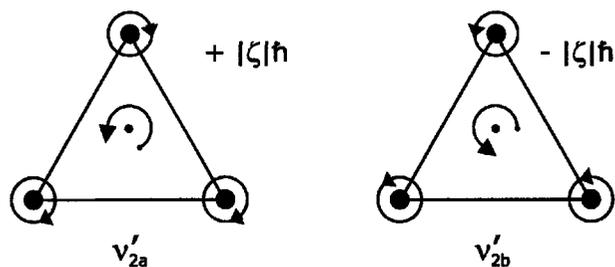
$$\mathbf{F}_{\text{cor}} = -2\mu(\boldsymbol{\omega} \times \mathbf{v}') \quad (7.203)$$

where  $\mathbf{v}'$  is the velocity in the rotating frame. The magnitude of the Coriolis force is

$$|\mathbf{F}_{\text{cor}}| = 2\mu\omega v_{\text{radial}}. \quad (7.204)$$

The Coriolis force is responsible for the counterclockwise rotation of tornados and hurricanes in the northern hemisphere. This is because the Earth is a rotating reference frame that has a Coriolis force that makes the winds veer to the right in the northern hemisphere.

Coriolis forces are also important in molecules. Consider the  $\text{H}_3^+$  molecule that has the structure of an equilateral triangle with  $D_{3h}$  symmetry.<sup>30</sup> The  $3N - 6 = 3$  modes of vibration are shown in Figure 7.47. The degenerate vibration  $\nu_2$  at  $2521 \text{ cm}^{-1}$  has two orthogonal modes of vibration  $\nu_{2a}$  and  $\nu_{2b}$  that can be chosen as shown in Figure 7.47. If the vibrational mode of the molecule is  $\nu_{2a}$ , then the Coriolis forces (7.203) act as shown by dashed lines in Figure 7.48. The Coriolis forces acting on  $\nu_{2a}$  lead to the excitation of  $\nu_{2b}$ ; when the molecule is in  $\nu_{2b}$ , the Coriolis forces excite  $\nu_{2a}$ . This is analogous to the strong coupling of two pendula of the same frequency. The molecule will therefore rapidly convert back and forth between  $\nu_{2a}$  and  $\nu_{2b}$ . The  $\nu_{2a}$  and  $\nu_{2b}$  modes are thus coupled via a first-order Coriolis effect.

Figure 7.47: The vibrational modes of  $\text{H}_3^+$ .Figure 7.48: Coriolis forces acting on the two degenerate components of the  $\nu_2$  mode of  $\text{H}_3^+$ .Figure 7.49: The  $\text{H}_3^+$  molecule has  $\pm|\zeta|\hbar$  units of vibrational angular momentum in the  $\nu_2$  mode.

Linear combinations of  $\nu_{2a}$  and  $\nu_{2b}$  that have vibrational angular momentum (Figure 7.49) can be formed. In this case the magnitude of the vibrational angular momentum<sup>25</sup> is  $\zeta\hbar$  where  $-1 \leq \zeta \leq 1$ . Notice that unlike linear molecules for which the vibrational angular momentum quantum number  $l$  is integral,  $\zeta$  for a symmetric top is not necessarily integral.

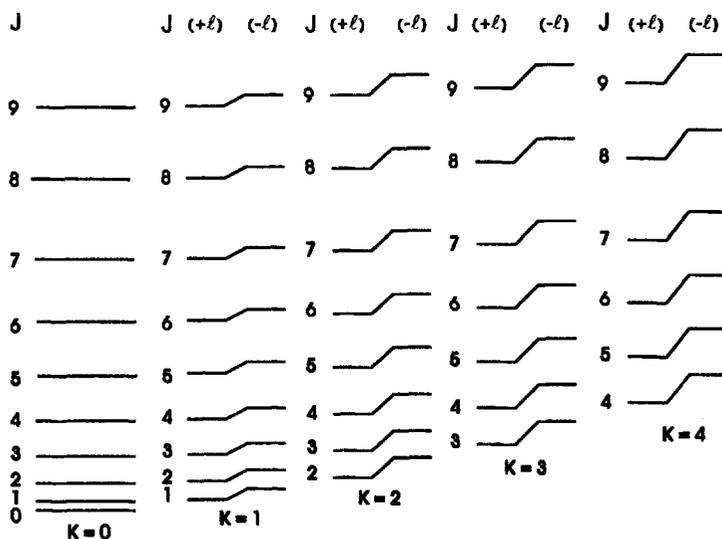


Figure 7.50: Rotational energy levels of a symmetric top molecule in a doubly degenerate vibrational state.

Vibrational angular momentum changes the energy-level formula. When vibrational angular momentum is present in a molecule,  $\hat{\mathbf{J}}$  becomes the vector sum of the rotational angular momentum and the vibrational angular momentum. Note that neither the projection of the vibrational angular momentum ( $\hat{\pi}_z$ ) nor the projection of the rotational angular momentum ( $\hat{J}_z - \hat{\pi}_z$ ) is quantized as an integer about the molecular  $z$ -axis. The projection of the total angular momentum ( $\hat{J}_z$ ), however, is quantized about the molecular  $z$ -axis with a quantum number designated as  $K$ . The rotational Hamiltonian operator becomes<sup>25</sup>

$$\hat{H} = \frac{\hat{j}_x^2}{2I_x} + \frac{\hat{j}_y^2}{2I_y} + \frac{(\hat{J}_z - \hat{\pi}_z)^2}{2I_z}, \quad (7.205)$$

in which  $\hat{\pi}_z$  is the vibrational angular momentum operator about the symmetric top axis. Expanding the last term in equation (7.205), dropping the pure vibrational term containing  $\hat{\pi}_z^2$ , and taking matrix elements of  $\hat{H}$  give the energy levels as

$$E = BJ(J+1) + (A-B)K^2 \mp 2A\zeta K \quad (\text{prolate top}), \quad (7.206a)$$

or

$$E = BJ(J+1) + (C-B)K^2 \mp 2C\zeta K \quad (\text{oblate top}). \quad (7.206b)$$

In equations (7.206), it is assumed that there are  $\zeta\hbar$  units of vibrational angular momentum about the symmetry axis of the symmetric top. The rotational and vibrational angular momenta about the top axis can either be in the same direction ( $-$  sign) or in the opposite direction ( $+$  sign). For historical reasons the  $+|\zeta|$  with  $+|K|$  (and  $-|\zeta|$  with  $-|K|$ ) levels are labeled as  $(+l)$  levels and the  $+|\zeta|$  with  $-|K|$  levels (and  $-|\zeta|$  with  $+|K|$ ) are labeled as  $(-l)$  levels. The energy-level diagram for a doubly degenerate vibrational level of a symmetric top molecule is given in Figure 7.50.

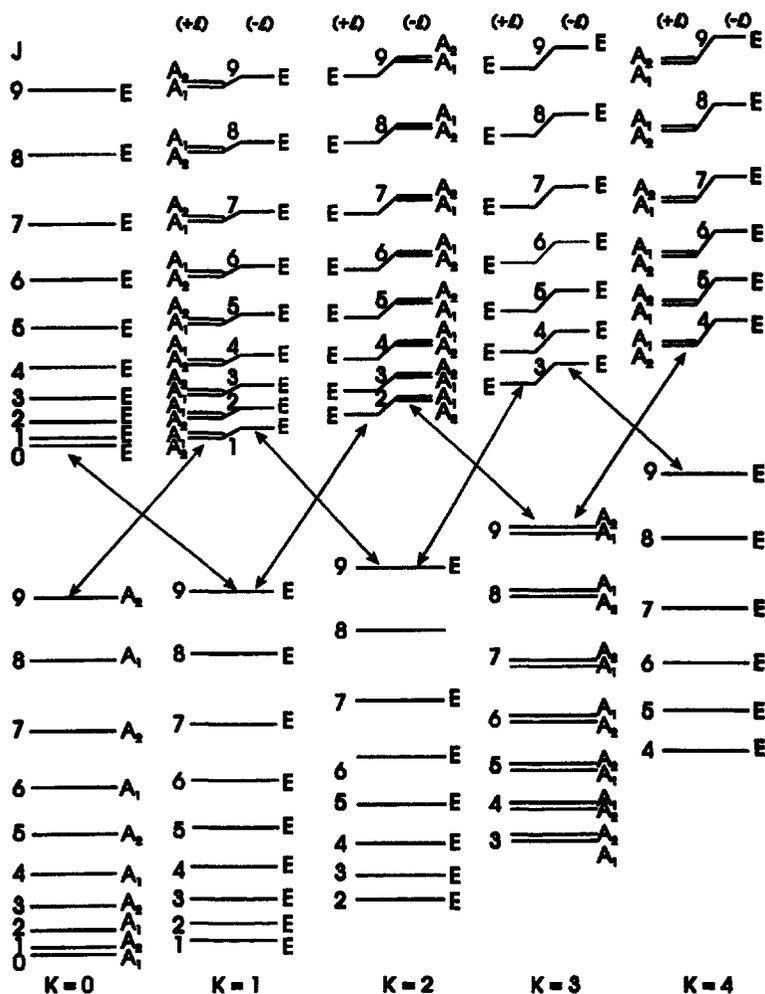


Figure 7.51: The energy-level diagram for the  $E - A_1$  transition of a symmetric top.

The  $E - A_1$  energy-level diagram<sup>25</sup> is given in Figure 7.51. The energy-level structure of an  $E$  vibrational state is complicated by the presence of a first-order Coriolis interaction between the two components. The selection rules are  $\Delta K = \pm 1$  and  $\Delta J = 0, \pm 1$ . Note also that for  $\Delta K = +1$  the transitions connect to the  $(+l)$  stack, while for  $\Delta K = -1$  they connect with the  $(-l)$  stack. The transition can again be represented by a superposition of sub-bands. Notice how the sub-bands do not line up as they do for a parallel transition, but they spread out (Figure 7.52). Each sub-band is separated by approximately  $2(A(1 - \zeta) - B)$ . This gives rise to a characteristic pattern of nearly equally spaced  $Q$  branches (Figure 7.53).

## 7.4 Infrared Transitions of Spherical Tops

Spherical tops such as  $\text{CH}_4$ ,  $\text{NH}_4^+$ ,  $\text{SF}_6$ , and  $\text{C}_{60}$  belong to the point groups  $T_d$ ,  $O_h$ , or  $I_h$ . Let us consider the  $\text{CH}_4$  molecule.<sup>25</sup> There are  $3N - 6 = 9$  modes made up of

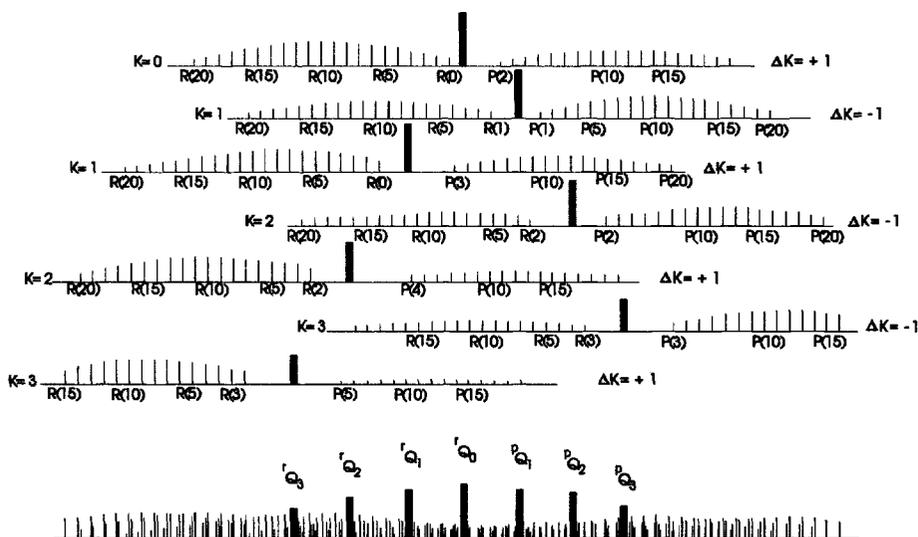


Figure 7.52: Simulated spectrum of an  $E - A_1$  perpendicular transition of a symmetric top. The individual sub-bands are combined to give the total spectrum in the bottom panel. The notation for the  $Q$  branches is  $\Delta^K Q_{K''}$ , with the  $\Delta K = \pm 1$  superscripts denoted as  $r$  and  $p$ .

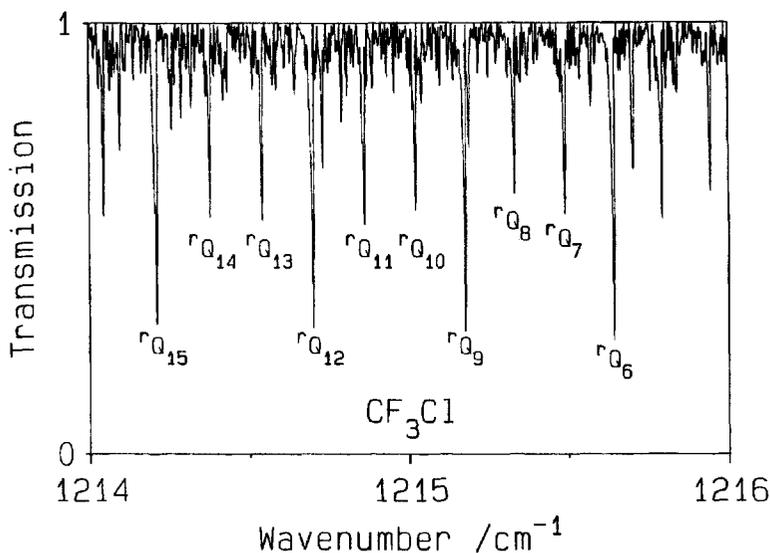


Figure 7.53: The  $\text{CF}_3\text{Cl}$   $\nu_4$  mode exhibiting sub-band  $Q$  structure. The intensity variation in the  $Q$  branches is due to nuclear spin statistics.

four stretching modes and five bending modes (Figure 7.54). The four C—H stretching coordinates can be reduced to a symmetric  $a_1$  stretch ( $\nu_1$ ) and a triply degenerate antisymmetric C—H stretch of  $t_2$  (or  $f_2$ ) symmetry ( $\nu_3$ ). Triply degenerate irreducible representations are labeled as  $t$  (or  $T$ ) by inorganic chemists and electronic spectroscopists, but as  $f$  (or  $F$ ) by many vibrational spectroscopists. The five bends reduce to

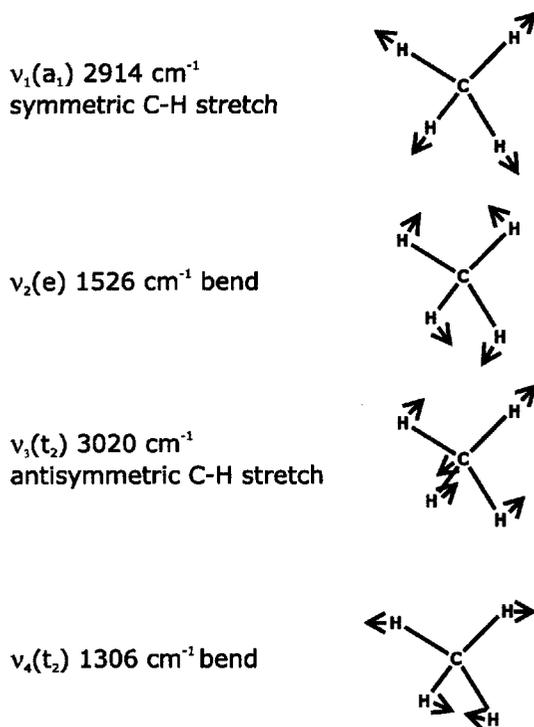


Figure 7.54: The normal modes of vibration of  $\text{CH}_4$  with only one member of each degenerate mode shown.

a pair of  $e$  modes ( $\nu_2$ ) and a triply degenerate  $t_2$  bending mode ( $\nu_4$ ). Only  $\nu_3$  and  $\nu_4$  are infrared active, but all of the modes are Raman active (Chapter 8).

The rotational energy levels of a spherical top are given by  $BJ(J+1)$ ; however, there is both a  $(2J+1)$ -fold  $K$  degeneracy and a  $(2J+1)$ -fold  $M$  degeneracy. The total degeneracy is therefore  $(2J+1)^2$  for each rotational level. A more detailed analysis that takes into account the effects of centrifugal distortion and anharmonicity predicts that the  $K$  degeneracy is partially lifted. The number of levels into which each  $J$  splits can be determined by group theory. These splittings are called cluster splittings and a surprisingly sophisticated theory<sup>31</sup> is required to account for their magnitude. A picture of the effect of some cluster splittings on a transition is presented in Figure 7.55.

It turns out that  $E$  states of a spherical top do not experience first-order Coriolis coupling, so that they have the same energy-level pattern as  $A_1$  states. However,  $T_2$  states experience a first-order Coriolis effect and split into three components,<sup>31</sup>

$$F^+(J) = BJ(J+1) + 2B\zeta J, \quad (7.207)$$

$$F^0(J) = BJ(J+1) - 2B\zeta, \quad (7.208)$$

and

$$F^-(J) = BJ(J+1) - 2B\zeta(J+1). \quad (7.209)$$

The energy-level pattern is given in Figure 7.56 for a  $T_2 - A_1$  transition. Transitions with  $\Delta J = 0, \pm 1$  are allowed, but with the additional restrictions:

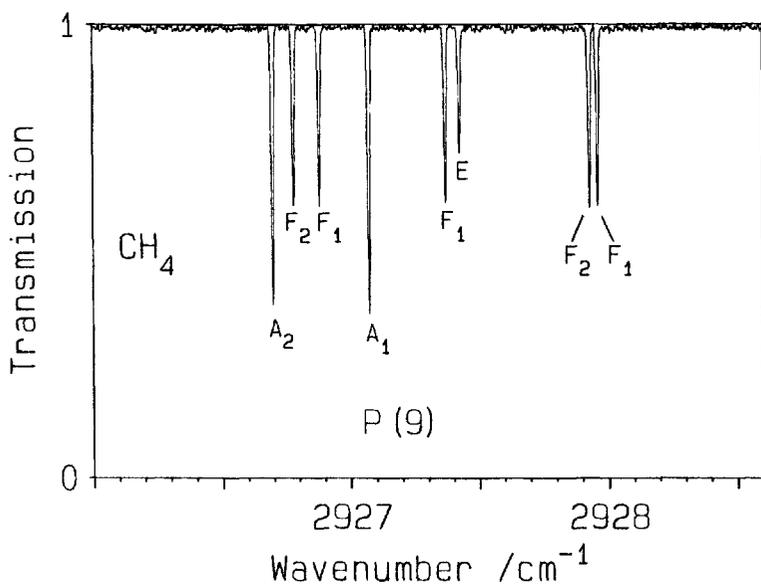


Figure 7.55: The cluster splittings of the  $P(9)$  line of the  $\nu_3$  mode of CH<sub>4</sub>.

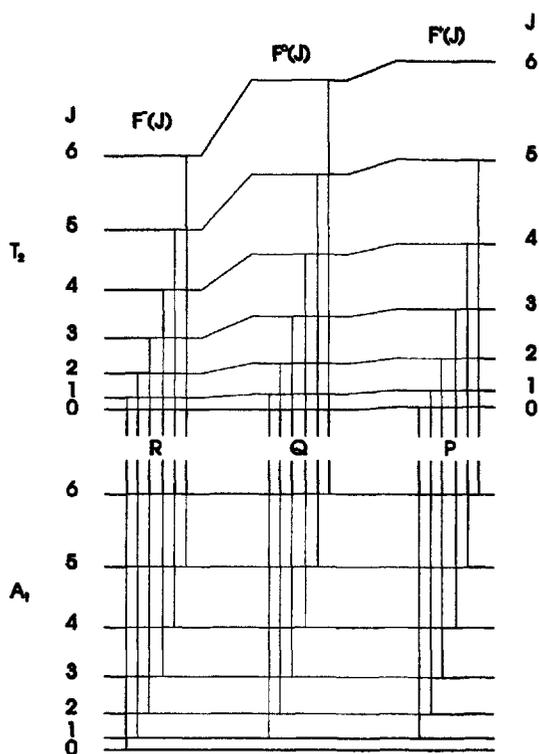


Figure 7.56: The energy-level diagram of a  $T_2 - A_1$  vibration-rotation transition of a spherical top.

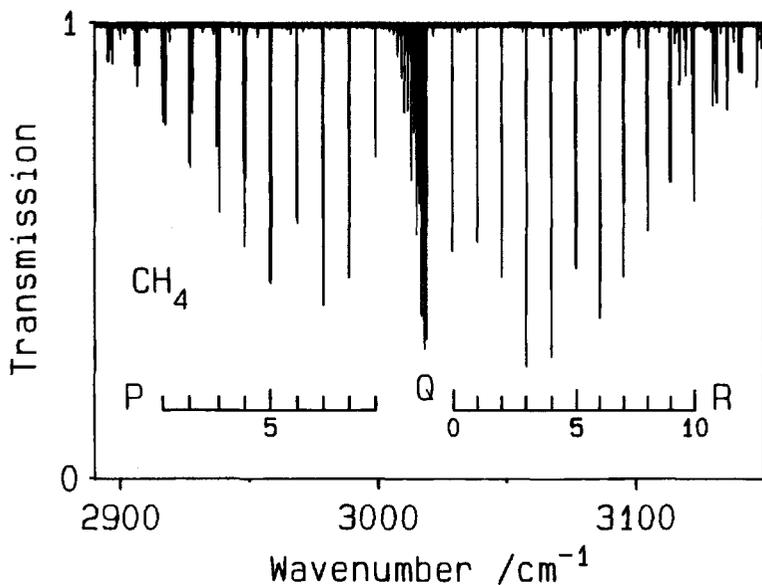


Figure 7.57: The infrared spectrum of the  $\text{CH}_4 \nu_3$  mode.

$$\Delta J = +1 \quad T^- - A_1,$$

$$\Delta J = 0 \quad T^0 - A_1,$$

and

$$\Delta J = -1 \quad T^+ - A_1.$$

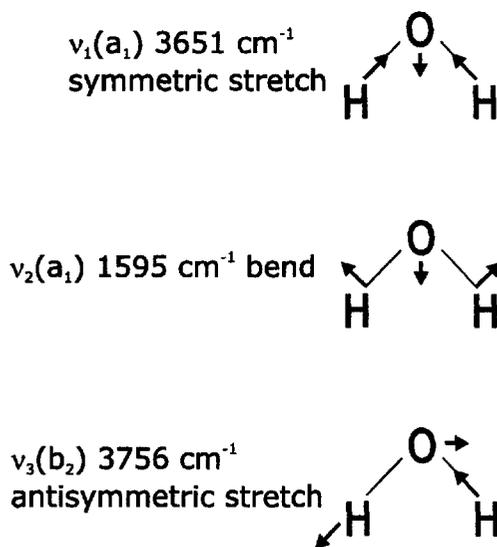
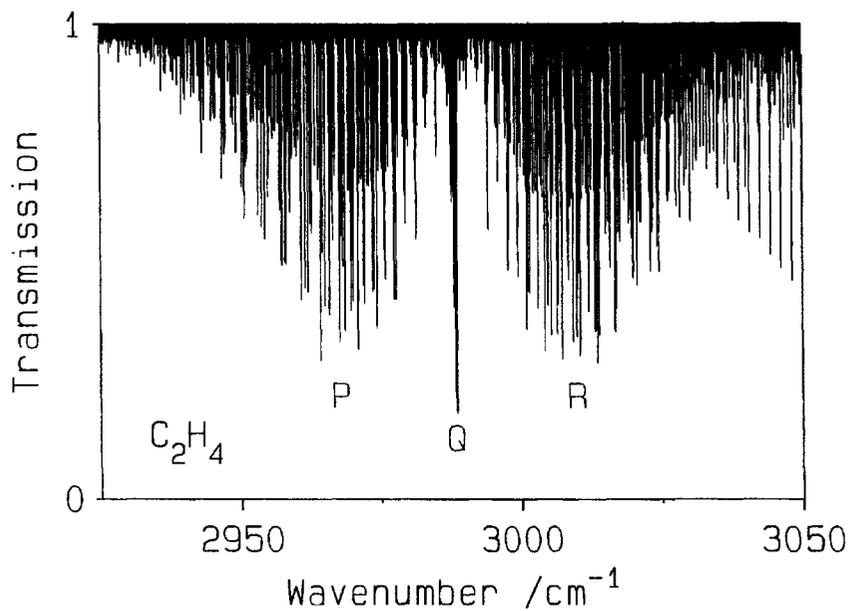
The spectrum of a “typical” spherical top is given in Figure 7.57. At low resolution the spectrum exhibits the characteristic  $PQR$  contour similar to a symmetric top (Figure 7.43).

## 7.5 Vibrational Spectra of Asymmetric Tops

The vast majority of polyatomic molecules are asymmetric tops. The  $\text{H}_2\text{O}$  molecule has three vibrational modes (Figure 7.58), with all modes both infrared and Raman active (Chapter 8).

The vibration-rotation transitions of asymmetric tops are classified as  $a$ -type,  $b$ -type, and  $c$ -type, depending on the orientation of the transition dipole moment relative to the principal axes. For  $\text{H}_2\text{O}$ , the oscillating dipole moments of the  $\nu_1$  and  $\nu_2$  modes are along the  $z(b)$  direction and the transitions are classified as  $b$ -type. The  $\nu_3$  band of  $\text{H}_2\text{O}$  has an oscillating dipole moment along the  $y(a)$  direction giving rise to an  $a$ -type transition. For molecules of sufficiently low symmetry, such as  $\text{HOD}$ , hybrid bands can occur, in this case  $ab$ -hybrid bands.

The selection rules for  $a$ -type,  $b$ -type, and  $c$ -type transitions are the same as for microwave transitions. The selection rules are as follows:

Figure 7.58: The normal modes of vibration of  $\text{H}_2\text{O}$ .Figure 7.59: The  $a$ -type mode  $\nu_{11}$  of ethylene,  $\text{C}_2\text{H}_4$ .

1.  $a$ -type bands, with

$$\Delta K_a = 0, \quad \Delta K_c = \pm 1,$$

and  $\Delta J = 0, \pm 1$ , except for  $K'_a = 0 \leftarrow K''_a = 0$ , for which only  $\Delta J = \pm 1$  is possible;

2. *b*-type bands, with

$$\Delta K_a = \pm 1, \quad \Delta K_c = \pm 1,$$

and  $\Delta J = 0, \pm 1$ ;

3. *c*-type bands, with

$$\Delta K_c = 0, \quad \Delta K_a = \pm 1,$$

and  $\Delta J = 0, \pm 1$ , but for  $K'_c = 0 \leftarrow K''_c = 0$ ,  $\Delta J = \pm 1$  only.

Since many molecules are near-oblate or near-prolate symmetric tops, the general appearance of asymmetric top bands often resembles either parallel or perpendicular bands of a symmetric top. For example, the *a*-type  $\nu_{11}$  band of  $C_2H_4$  is shown in Figure 7.59. This mode is similar in appearance to the parallel transition of a symmetric top.

## 7.6 Vibration-Rotation Line Intensities

All molecules except homonuclear diatomics have at least one allowed vibration-rotation band. Infrared spectroscopy is thus one of the most common techniques used for qualitative and quantitative analysis in chemistry. Infrared spectroscopy is also popular for remote sensing of the earth and in astronomy. To the uninitiated, one of the confusing aspects of quantitative vibration-rotation spectroscopy is the dozens of different units used for line or band intensities,<sup>32-34</sup> including km/mole for vibrational bands by quantum chemists and cm/molecule for lines in the HITRAN<sup>35</sup> database. Another barrier is that most of the relevant equations cited in the literature<sup>32-34</sup> are not in SI units.

As always, the starting point is the set of basic equations that govern the absorption and emission of radiation for a degenerate two-level system (section 5.7). The equation for Beer's law, equation (5.120), for a line at frequency  $\nu$ , including the stimulated emission correction is

$$\begin{aligned} I &= I_0 e^{-\sigma(N_0 - N_1 \frac{2J''+1}{2J'+1})l} \\ &= I_0 e^{-\alpha l}. \end{aligned} \quad (7.210)$$

In this form, Beer's law relates the observed intensity to the concentrations (molecules/m<sup>3</sup>) in the upper state ( $N_1$ ) and lower state ( $N_0$ ) associated with a single transition, rather than the more useful total concentration,  $N = \sum N_i$ . The assumption of thermodynamic equilibrium allows  $N_1$  to be eliminated using the Boltzmann relation (see section 5.7),

$$\frac{N_1}{N_0} = \frac{2J'+1}{2J''+1} e^{-h\nu_{10}/kT}, \quad (7.211)$$

and  $N_0$  can be replaced by  $N$  using a relationship from statistical thermodynamics,<sup>36</sup>

$$N_0 = \frac{N(2J''+1)e^{-E_0/kT}}{q}, \quad (7.212)$$

with  $q$  as the partition function and  $E_0$  the lower state energy (see the discussion near equations (6.41) to (6.43) and near (6.87)). If needed, nuclear spin statistics (or electronic or vibrational degeneracy) appear as an additional degeneracy factor in the numerator and in  $q$ . Taking natural logarithms of equation (7.210) then leads to

$$-\ln\left(\frac{I}{I_0}\right) = \frac{\sigma(2J''+1)}{q} e^{-E_0/kT} (1 - e^{-h\nu_{10}/kT}) Nl \quad (7.213)$$

for Beer's law. The term  $(1 - e^{-h\nu_{10}/kT})$  is the correction due to stimulated emission caused by population in the excited state. For quantitative work in the thermal infrared region ( $\tilde{\nu} \lesssim 2000 \text{ cm}^{-1}$ ) the stimulated emission correction cannot be neglected at room temperature.

For emission from an excited state  $|v'J'\rangle$  down to  $|v''J''\rangle$ , the expression for the rate is given by

$$\frac{dN_1}{dt} = -A_{J' \rightarrow J''} N_1 \quad (7.214)$$

with  $N_1$  in molecules/m<sup>3</sup>. The total power emitted  $P_{J' \rightarrow J''}$  (watts/m<sup>3</sup>) is thus

$$P_{J' \rightarrow J''} = h\nu_{10} A_{J' \rightarrow J''} N_1 \quad (7.215)$$

because each photon carries  $h\nu_{10}$  of energy. Again it is convenient (but not necessary) to assume thermodynamic equilibrium so that the concentration in the excited state,  $N_1$ , can be converted to the total concentration,  $N$ , using equations (7.211) and (7.212). The total power is then

$$P_{J' \rightarrow J''} = \frac{h\nu_{10} A_{J' \rightarrow J''} N (2J'+1) e^{-E_1/kT}}{q} \quad (7.216)$$

and using equation (5.113) for  $A_{J' \rightarrow J''}$  gives

$$P_{J' \rightarrow J''} = \frac{16\pi^3 \nu_{10}^4 N e^{-E_1/kT}}{3\epsilon_0 c^3 q} S_{J'J''} \quad (7.217)$$

or

$$P_{J' \rightarrow J''} = \frac{16\pi^3 \nu_{10}^4 N_1}{3\epsilon_0 c^3 (2J'+1)} S_{J'J''} \quad (7.218)$$

where equation (7.218) does not assume thermodynamic equilibrium. The quantity  $S_{J'J''}$  is defined by equation (5.112) for both atoms and molecules (see below).

In Beer's law for absorption at  $\nu$ , equation (7.213), the product  $Nl$  is defined as the column density,  $x$ , or "optical mass,"

$$x = Nl, \quad (7.219)$$

with dimensions of molecules/m<sup>2</sup>, while the absorption coefficient  $\alpha$  (m<sup>-1</sup>) is given as

$$\alpha = \frac{\sigma(2J''+1)}{q} e^{-E_0/kT} (1 - e^{-h\nu_{10}/kT}) N. \quad (7.220)$$

The lineshape function  $g(\nu - \nu_{10})$  is included in the absorption cross section  $\sigma$  (sometimes written as  $k(\nu)$  in the infrared literature) with the usual equation (section 5.7)

$$\sigma = \frac{2\pi^2\nu_{10}S_{J',J''}}{3\varepsilon_0hc(2J''+1)}g(\nu-\nu_{10}). \quad (7.221)$$

The line strength  $S_{J',J''}$  is defined in exactly the same way for atoms and molecules as

$$S_{J',J''} = \sum_{M'} \sum_{M''} |\langle J' M' | \boldsymbol{\mu} | J'' M'' \rangle|^2. \quad (7.222)$$

The line strength  $S_{J',J''}$  (in  $\text{C}^2 \text{m}^2$  or in the non-SI unit of  $\text{D}^2$ ) is rarely tabulated in favor of the related quantity

$$S' = \frac{2\pi^2\nu_{10}S_{J',J''}}{3\varepsilon_0hcq} e^{-E_0/kT} (1 - e^{-h\nu_{10}/kT}), \quad (7.223)$$

also often confusingly called a line strength or line intensity. The customary notation and terminology uses the symbol  $S$  for three distinct but related quantities: the line strength  $S_{J',J''}$ , equation (7.222), the line intensity  $S'$ , equation (7.223) (often given the symbol  $S$  in the infrared literature), and the Hönl-London rotational factor  $S_{J''}^{J'}$  (Chapter 9). To make matters worse all three quantities are often called "line strengths." Implicit in all these equations is a uniform definition of a line as a transition between two levels,  $J'$  and  $J''$ .

Although the integrated cross section  $\int \sigma d\nu$  obtained by removing the lineshape function  $g(\nu-\nu_{10})$  from equation (7.221) and the line intensity  $S'$  have the same units ( $\text{m}^2 \text{s}^{-1}$ ), they are different quantities. In particular, the cross section  $\sigma$  in equation (7.221) is associated with the form of Beer's law (equation (7.210)) that uses the concentrations  $N_0$  and  $N_1$  of the molecules in the quantum states, while the line strength  $S'$  is associated with Beer's law (equation (7.224)) that uses the total concentration  $N$ .

The definition of  $S'$ , equation (7.223), allows Beer's law to be written as

$$I = I_0 e^{-S'g(\nu-\nu_{10})Nl}. \quad (7.224)$$

Clearly the line intensity,  $S'$ , is an integrated effective cross section that can be determined by solving equation (7.224) as

$$S' = \frac{1}{Nl} \int \ln(I_0/I) d\nu \quad (7.225)$$

with the integration covering a single line to eliminate the lineshape function,  $g(\nu-\nu_{10})$ . The integration could also be over an entire band, in which case  $S'$  is interpreted as a band strength. From equation (7.225), the SI units for  $S'$  are thus  $\text{m}^2 \text{s}^{-1}$  (or  $\text{m}^2 \text{s}^{-1}/\text{molecule}$ ) but these are unfortunately almost never encountered.

The units of the effective integrated cross section  $S'$  are a major headache because of the numerous choices possible for the quantities  $\nu$ ,  $N$ ,  $l$  in equation (7.225). If  $\nu$  is replaced by  $\tilde{\nu}$  in  $\text{cm}^{-1}$ ,  $l$  in  $\text{cm}$  and  $N$  in  $\text{molecules}/\text{cm}^3$ , then  $S'$  has the "HITRAN<sup>35</sup> units" of  $\text{cm}^{-1}/(\text{molecule cm}^{-2})$  or  $\text{cm}/\text{molecule}$ . If the integral (7.225) is taken over an entire infrared band, and the  $\text{cm}$  of the HITRAN unit is converted to  $\text{km}$  and moles are used instead of molecules in concentration, then the units  $\text{km}/\text{mole}$  favored by quantum chemists are obtained for  $S'$ . The conversion from  $\text{cm}/\text{molecule}$  to  $\text{km}/\text{mole}$  requires multiplication by the numerical factor of  $10^{-5}N_A = 6.022\,141\,99 \times 10^{18}$ . Another possible choice is to measure concentration  $N$  in pressure units such as atmospheres, leading to  $S'$  in  $\text{cm}^{-2}/\text{atm}$  (with  $\tilde{\nu}$  in  $\text{cm}^{-1}$ ,  $l$  in  $\text{cm}$ ). The conversion from  $\text{cm}^{-2}/\text{atm}$  to HITRAN units thus requires the use of the ideal gas law. To convert from  $\text{cm}^{-2}/\text{atm}$

to cm/molecule, divide by the factor  $0.101\,325/kT = 7.338\,933 \times 10^{21}/T$ . Finally the conversion from HITRAN units (cm/molecule) to SI units ( $\text{m}^2 \text{s}^{-1}/\text{molecule}$ ) requires multiplication by  $10^{-2}c_0 = 2\,997\,924.58$ . Pugh and Rao<sup>32</sup> provide a convenient table for converting between various possible units for the line intensity  $S'$ .

The HITRAN<sup>35</sup> database includes line positions,  $\tilde{\nu}$  (in  $\text{cm}^{-1}$ ), integrated effective cross sections ("line intensity"),  $S'$  (in  $\text{cm}^{-1}/(\text{molecule cm}^{-2})$ ), the Lorentzian halfwidth for pressure broadening by air (i.e.,  $\Delta\nu_{1/2}/2$  for  $g_L(\nu - \nu_{10})$ ) in  $\text{cm}^{-1}/\text{atm}$  (as well as the self-broadening coefficient), and the lower state energy ( $E_0$  in  $\text{cm}^{-1}$ ) all reported at a standard temperature  $T = 296$  K for some 39 molecules of atmospheric interest. With these quantities the transmission of the atmosphere can be calculated.

The integrated absorption cross section  $\int \sigma d\nu$  can also be used to compute emission spectra. The conversion from the cross section  $\sigma$  to the Einstein  $A_{J' \rightarrow J''}$  value is given by equation (5.121),

$$\sigma = \frac{A_{J' \rightarrow J''} \lambda^2 g(\nu - \nu_{10})}{8\pi} \frac{2J' + 1}{2J'' + 1}. \quad (7.226)$$

Integration over frequency for a line to eliminate the lineshape function  $g(\nu - \nu_{10})$  gives

$$\int \sigma d\nu = \frac{A_{J' \rightarrow J''} \lambda^2}{8\pi} \frac{2J' + 1}{2J'' + 1} = \frac{A_{J' \rightarrow J''} c^2}{8\pi\nu^2} \frac{2J' + 1}{2J'' + 1} \quad (7.227)$$

or

$$A_{J' \rightarrow J''} = \frac{\int \sigma d\nu}{c^2} \frac{8\pi\nu^2}{2J'' + 1} \frac{2J' + 1}{2J'' + 1}. \quad (7.228)$$

As usual for intensities, care with units is required because equation (7.228) assumes SI units. The integrated cross section  $\int \sigma d\nu$  can be converted to  $S'$  using equation (7.223) and the relationship in SI units is

$$S' = \int \sigma d\nu (2J'' + 1) e^{-E_0/kT} (1 - e^{-h\nu_{10}/kT})/q$$

The intensity relationships in Chapter 5 apply, but for  $\int \sigma d\nu$  rather than  $S'$ .

### Line Intensity Calculations

Line intensities  $S'$  can be treated as purely empirical quantities that must be determined by measurement for each individual rotational line in a spectrum. Line intensities, however, can be related back to more basic quantities such as dipole moment functions or transition dipole moments that can be computed by the methods of *ab initio* quantum chemistry or determined from other experiments. A theoretical model for line intensities is also needed for interpolation, extrapolation, and compact representation of line intensity data.

For diatomic molecules the line strength  $S_{J'J''}$ , equation (7.222), can be written as a product of a vibrational part,  $|\mathbf{M}_{v'v''}|^2$ , a rotational part,  $S_{J''}^{\Delta J}$ , called a Hönl–London factor, and a correction term,  $F(m)$ , called a Herman–Wallis<sup>37</sup> factor:

$$\begin{aligned} S_{J'J''} &= \sum_{M'} \sum_{M''} |\langle \psi_{v'J'M'} | \boldsymbol{\mu} | \psi_{v''J''M''} \rangle|^2 \\ &= |\mathbf{M}_{v'v''}|^2 S_{J''}^{\Delta J} F(m). \end{aligned} \quad (7.229)$$

The purely vibrational transition dipole moment

$$\begin{aligned} \mathbf{M}_{v'v''} &= \langle \psi_{v'} | \boldsymbol{\mu} | \psi_{v''} \rangle \\ &= \langle \psi_{\text{vib}}(v', J' = 0) | \boldsymbol{\mu}(r) | \psi_{\text{vib}}(v'', J'' = 0) \rangle \end{aligned} \quad (7.230)$$

has already been discussed earlier in this chapter, equation (7.50). The Hönl–London factors are given in Table 9.4 and  $S_{J''}^R = J'' + 1$  and  $S_{J''}^P = J''$  for the diatomic  ${}^1\Sigma^+$  case. The Herman–Wallis factor  $F(m)$  is usually expressed as a polynomial

$$F(m) = 1 + Cm + Dm^2 + \dots, \quad (7.231)$$

with  $m$  defined as  $J + 1$  for an  $R$  branch and  $-J$  for a  $P$  branch.<sup>38</sup> Equation (7.229) has thus separated the square of the transition dipole moment into a purely vibrational part,  $|\mathbf{M}_{v'v''}|^2$ , a rotational part,  $S_{J''}^{\Delta J}$ , and a correction term,  $F(m)$ , that compensates for errors in separation of vibration from rotation, i.e.,

$$F(m) \equiv \frac{|\langle \psi_{\text{vib}}(v', J') | \boldsymbol{\mu}(r) | \psi_{\text{vib}}(v'', J'') \rangle|^2}{|\langle \psi_{\text{vib}}(v', 0) | \boldsymbol{\mu}(r) | \psi_{\text{vib}}(v'', 0) \rangle|^2}. \quad (7.232)$$

The *Herman–Wallis effect* appears, for example, as a strengthening of an  $R$  branch and a weakening of a  $P$  branch (or vice versa) relative to the expected intensity given by equation (7.229) with  $F(m) = 1$ . The Herman–Wallis factor,  $F(m)$ , quantifies the Herman–Wallis effect and originates from vibration–rotation interaction. Herman–Wallis factors can be computed<sup>38</sup> from the dipole moment function  $\boldsymbol{\mu}(r)$ , equation (7.51), and the Dunham potential,  $V(r)$ , equation (7.35), or simply derived from experimental observations. For linear molecules a similar separation,

$$S_{J'J''} = |\mathbf{M}_{v'v''}|^2 S_{J''}^{\Delta J} F(m) \quad (7.233)$$

can be carried out, but Watson<sup>39</sup> recommends the forms

$$F_{PR}(m) = (1 + A_1 m + A_2^R m^2)^2 \quad (7.234)$$

and

$$F_Q = (1 + A_2^Q J(J + 1))^2, \quad (7.235)$$

with the vibrational angular momentum  $l$  replacing  $\Lambda$  for the Hönl–London factors found in Table 9.4. Similar forms for  $S_{J'J''}$  can be developed for symmetric tops ( $K$  replaces  $\Lambda$  for the Hönl–London factors) and theoretical line intensities for asymmetric rotors such as water are also available.<sup>40</sup>

The simplest possible intensity expression for a diatomic vibration–rotation transition assumes the “double harmonic” approximation, i.e., a harmonic oscillator for the potential and a linear dipole moment function, equations (7.51) to (7.53). Ignoring the Herman–Wallis effect, which is often small, results in the expression<sup>32</sup>

$$\begin{aligned} S_{J'J''} &= |\mathbf{M}_{v'v''}|^2 S_{J''}^{\Delta J} \\ &= \left. \frac{d\boldsymbol{\mu}}{dr} \right|_{r_e}^2 |\langle \psi_{v'} | (r - r_e) | \psi_{v''} \rangle|^2 S_{J''}^{\Delta J} \end{aligned} \quad (7.236)$$

for the line strength,  $S_{J'J''}$ .

The vibrational matrix element in equation (7.236) can be evaluated for the harmonic oscillator using equation (7.56), i.e.,

$$\langle v+1|x|v\rangle = \left(\frac{\hbar}{2\mu_{AB}\omega}\right)^{1/2} \sqrt{v+1} \quad (7.237)$$

with  $x = r - r_e$  for the  $v+1 \leftarrow v$  transition of the A—B molecule with reduced mass  $\mu_{AB}$  and linear dipole moment function  $\mu(r)$ . The equation for the line strength is thus

$$\begin{aligned} S_{J',J''} &= \frac{\hbar(v+1)}{2\mu_{AB}\omega} \left| \frac{d\mu}{dr} \right|_{r_e}^2 S_{J''}^{\Delta J} \\ &= \frac{h(v+1)}{8\pi^2\mu_{AB}\nu_{10}} \left| \frac{d\mu}{dr} \right|_{r_e}^2 S_{J''}^{\Delta J}, \end{aligned} \quad (7.238)$$

and the effective integrated cross section  $S'$  for a line becomes

$$S' = \frac{v+1}{12\varepsilon_0\mu_{ABCq}} \left| \frac{d\mu}{dr} \right|_{r_e}^2 S_{J''}^{\Delta J} e^{-E_0/kT} (1 - e^{-h\nu/kT}), \quad (7.239)$$

using equation (7.223). To obtain a fundamental band intensity,  $S'_{\text{band}}$ , the transition has  $v = 0$  and all lines need to be included by summation over  $J'$  and  $J''$ , i.e.,

$$S'_{\text{band}} = \frac{1}{12\varepsilon_0\mu_{ABCq}} \left| \frac{d\mu}{dr} \right|_{r_e}^2 \sum_{J',J''} S_{J''}^{\Delta J} e^{-E_0/kT} (1 - e^{-h\nu_{10}/kT}) \quad (7.240)$$

The summation over  $J'$  and  $J''$  can be replaced by a sum over  $\Delta J$  and  $J''$ , assuming  $\nu_{10}$  approximately constant, and the Hönl–London factors  $S_{J''}^{\Delta J}$  obey the sum rule (Chapter 9),

$$\sum_{\Delta J} S_{J''}^{\Delta J} = 2J'' + 1 \quad (7.241)$$

plus the definition of  $q$  so that

$$S'_{\text{band}} \approx \frac{1}{12\varepsilon_0 c \mu_{AB}} \left| \frac{d\mu}{dr} \right|_{r_e}^2 (1 - e^{-h\nu_{10}/kT}) \quad (7.242)$$

If the stimulated emission correction,  $1 - e^{-h\nu_{10}/kT}$ , is ignored then the fundamental band intensity is

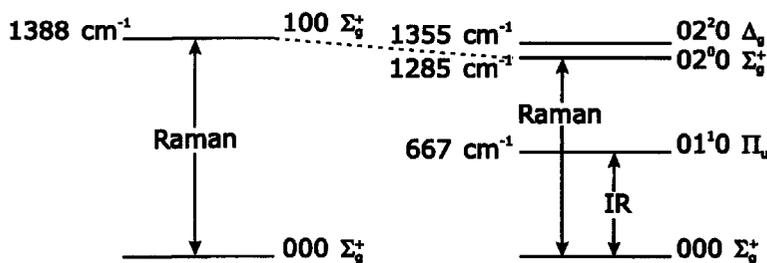
$$S'_{\text{band}} \approx \frac{1}{12\varepsilon_0 c \mu_{AB}} \left| \frac{d\mu}{dr} \right|_{r_e}^2 \quad (7.243)$$

Using numerical values in equation (7.243) gives the equations

$$S'_{\text{band}}/(\text{km/mole}) = 42.256\,078 \frac{1}{\mu_{AB}/(\text{u})} \left| \frac{d\mu}{dr} \right|_{r_e}^2 /(\text{D}/\text{\AA})^2, \quad (7.244)$$

$$S'_{\text{band}}/(\text{cm/molecule}) = 7.016\,785\,6 \times 10^{-18} \frac{1}{\mu_{AB}/(\text{u})} \left| \frac{d\mu}{dr} \right|_{r_e}^2 /(\text{D}/\text{\AA})^2, \quad (7.245)$$

with  $\mu_{AB}$  in atomic mass units (u).

Figure 7.60: Fermi resonance in CO<sub>2</sub>.

For polyatomic molecules an equation similar to (7.243) can be derived, but in this case mass-weighted cartesian coordinates  $q_i$  are used to set up the normal coordinates  $Q_i$  so

$$S'_{\text{band}} \approx \frac{1}{12\varepsilon_0 c} \left| \frac{\partial \mu}{\partial Q_i} \right|_0^2 \quad (7.246)$$

The reduced mass  $\mu_{AB}$  appears explicitly in equation (7.242), but the effective mass is implicitly contained in the  $Q_i$  coordinates in equation (7.246), which have dimensions of  $\text{m kg}^{-1/2}$ . The corresponding numerical equations for (7.246) are the same as (7.244) and (7.245) but with  $\mu_{AB} = 1$  since the masses are included in  $Q_i$ .

The possibility of vibrational, orbital, and electron spin degeneracy is not included in equations (7.243) to (7.246). If necessary, they can be multiplied by an extra degeneracy factor of  $g_i$ . The double harmonic approximation is very popular in *ab initio* quantum chemistry because the required derivatives of the dipole moment can be computed rapidly for the equilibrium geometry, but is of only modest reliability.

## 7.7 Fermi and Coriolis Perturbations

The regular energy-level pattern predicted by  $G(v_i)$  in equation (7.197) rarely exists for real molecules. Deviations from a regular pattern are called *perturbations* by spectroscopists. Consider the Raman spectrum (Chapter 8) for CO<sub>2</sub>. The  $\nu_1$  fundamental mode of CO<sub>2</sub> should be strong, while the  $2\nu_2$  overtone should be weak. In fact they have roughly the same intensity. Moreover, the  $2\nu_2$  ( $\sigma_g^+$ ) mode is not present at  $2 \times 667 \text{ cm}^{-1} = 1334 \text{ cm}^{-1}$ , but at  $1285 \text{ cm}^{-1}$  instead (Figure 7.60). The explanation for these discrepancies was provided by Fermi<sup>25</sup>—the  $\nu_1$  and  $2\nu_2$  vibrational levels have the same symmetry,  $\Sigma_g^+$ , and so there are anharmonic terms in the exact Hamiltonian operator that couple these vibrational modes. The  $02^2_0 \Delta_g$  state is not affected because it has  $\Delta_g$  symmetry. This type of interaction between neighboring vibrational levels of the same symmetry is known as a *Fermi resonance*.

The unperturbed  $\nu_1$  and  $2\nu_2$  levels (dashed lines in Figure 7.61) have almost identical energies, but anharmonic terms that were neglected in the simple harmonic oscillator approximation cause the two levels (Figure 4.1) to be pushed apart. If the original wavefunctions were  $\psi_{100}$  and  $\psi_{02^0_0}$ , then the final mixed wavefunctions are

$$\psi^+ = a\psi_{100} + b\psi_{02^0_0}$$

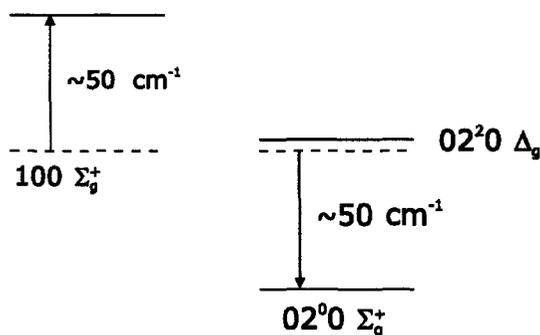


Figure 7.61: Fermi interaction of the  $100 \Sigma_g^+$  and  $02^0 \Sigma_g^+$  vibrational levels of  $\text{CO}_2$ .

$$\psi^- = b\psi_{100} - a\psi_{02^0}$$

with  $a \approx b \approx 1/\sqrt{2}$  in the case of  $\text{CO}_2$ . Using second-order perturbation theory (Chapter 4), the energy shift is given by

$$\Delta E^{(2)} = \frac{V^2}{E_2^{(0)} - E_1^{(0)'}}$$

in which  $V = \langle \psi_2 | \hat{H}^{(1)} | \psi_1 \rangle$  is the interaction matrix element and  $E_2^{(0)}$  and  $E_1^{(0)}$  are the unperturbed energies. Clearly, Fermi resonances are most pronounced when two states of the same symmetry are in close proximity and are coupled by a nonzero anharmonic interaction term.

Other interactions between levels are possible since the simple harmonic oscillator picture has neglected many types of higher-order terms in the vibration-rotation Hamiltonian operator. In addition to the anharmonic terms responsible for Fermi resonances, Coriolis terms can also perturb the expected regular energy-level pattern. "First-order" Coriolis effects have already been discussed for the splittings observed in  $E$  vibrational levels of symmetric tops and  $T$  vibrational levels of spherical tops. These large effects must always be taken into account. In addition "second-order" Coriolis effects are possible between states of different symmetry. Since Coriolis interactions involve rotational motion, they occur only in the gaseous state.

If two vibrational states (with wavefunctions  $\psi_1$  and  $\psi_2$ ) of a molecule are near each other in energy and differ in symmetry, such that

$$\Gamma^{\psi_2} \otimes \Gamma^R \otimes \Gamma^{\psi_1}, \quad R = R_x, R_y, R_z$$

contains a totally symmetric irreducible representation  $A_1$ , then a Coriolis resonance is possible. The explanation<sup>41</sup> of this rule is simple: the lowest-order Coriolis terms neglected in the total Hamiltonian operator have the form  $p\hat{J}_x$ ,  $q\hat{J}_y$ , or  $r\hat{J}_z$  (with  $p$ ,  $q$ , and  $r$  constants) and they behave like the rotations  $R_x$ ,  $R_y$ , and  $R_z$ . Thus if two vibrations differ in symmetry by a rotation about one of the principal axes, then a neglected Coriolis term in the exact Hamiltonian operator can always be found to cause an interaction. Of course, if the states are far apart (hundreds of  $\text{cm}^{-1}$ ), then the effect of the interaction is small since the interaction matrix element is, at most, a few  $\text{cm}^{-1}$  in size.



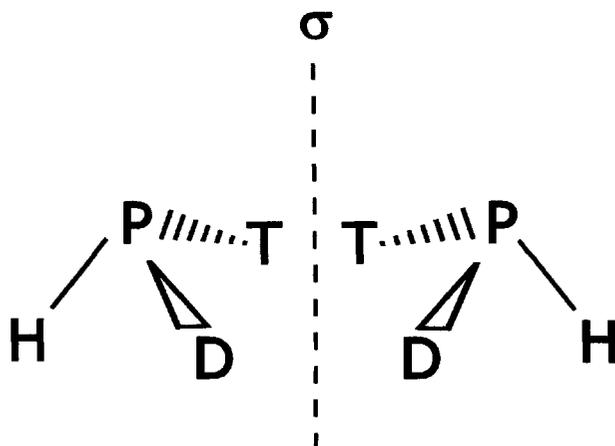


Figure 7.63: The two chiral forms of PHDT.

enantiomeric forms of the molecule. In fact, for a noninverting molecule such as  $\text{PH}_3$  (or  $\text{AsH}_3$ ) the two forms (Figure 7.63) of PHDT could, in principle, be separated.

For chiral molecules the two forms (enantiomers) have identical energy levels, but a large barrier prevents their interconversion. In  $\text{NH}_3$  the barrier for interconversion of these two forms<sup>43</sup> is only  $2009\text{ cm}^{-1}$  (Figure 7.62), allowing facile interconversion by tunneling. The energy-level patterns for the two forms of  $\text{NH}_3$  are no longer identical as a result of their mutual interaction. New approximate wavefunctions need to be constructed

$$\psi^+ = \frac{\psi_L + \psi_R}{\sqrt{2}}$$

$$\psi^- = \frac{\psi_L - \psi_R}{\sqrt{2}}$$

by mixing wavefunctions of the left- and right-handed forms. It turns out that the + levels (sometimes labeled *s* for symmetric) lie below the corresponding – levels (called *a* for antisymmetric). The + or – are added as superscripts (Figure 7.62) to the vibrational quantum number of the inverting normal mode. This +/- or *s/a* notation is not related to the notation for parity discussed previously. Notice that for the energy levels above the barrier, the inversion splitting becomes a vibrational interval and the numbering on the right of the diagram is more appropriate.

Fluxional behavior is an effect commonly observed in weakly bound systems such as van der Waals dimers, for example  $(\text{H}_2\text{O})_2$ . Since the inversion of a molecule changes the handedness of the coordinate system, simple geometric symmetry operations are not adequate to describe the molecular motion. Permutations and inversions of the nuclei need to be considered in order to describe these motions. Permutation-inversion group theory<sup>44</sup> has different group operations and group names, but these groups are often isomorphic with the more familiar point groups. For example the permutation-inversion point group appropriate for the inverting  $\text{NH}_3$  molecule is isomorphic with  $D_{3h}$ . The addition of the inversion operation has increased the order of the group from 6 ( $C_{3v}$ ) to 12 ( $D_{3h}$ ).

## Problems

1. Consider the ethylene molecule of  $D_{2h}$  symmetry.
  - (a) Determine the number and symmetry of the normal modes of vibration. (Choose  $x$  to be out of plane and  $z$  along the C=C axis.)
  - (b) Which modes are infrared active?
  - (c) Use projection operators to find the symmetry of the
    - (i) C=C stretch,
    - (ii) C—H stretches,
    - (iii) In-plane bends,
    - (iv) Out-of-plane bends.
  
2. Determine the symmetry of the normal modes of vibration for the following molecules:
  - (a) H—C≡C—C≡N;
  - (b)  $\text{CO}_3^{2-}$ ;
  - (c)  $\text{PtCl}_4^{2-}$ ;
  - (d) *trans*-glyoxal.
  
3. Fill in the following table:

Molecule	$\tilde{\nu}$ ( $\text{cm}^{-1}$ )	Force Constant ( $\text{N m}^{-1}$ )
NH	3 133	
NF	1 115	
O <sub>2</sub>	1 555	
N <sub>2</sub>	2 331	
NO	1 876	

4. The molecule 1,1-dichloroethylene ( $\text{C}_2\text{H}_2\text{Cl}_2$ ) is planar with  $C_{2v}$  symmetry, the C=C bond coinciding with the  $C_2$ -axis. Take this axis as the  $z$ -axis and the plane containing the molecule as the  $yz$ -plane.
  - (a) Determine the number and symmetry of the normal modes of vibration.
  - (b) How many infrared bands (fundamentals) are there of  $a$ -type,  $b$ -type, and  $c$ -type, respectively? The  $C_2$ -axis is the axis of the intermediate moment of inertia  $I_B$ .
  - (c) Are there any vibrations in this molecule for which the fundamental frequency ( $\Delta v = 1$ ) is forbidden in the infrared, but for which the first overtone ( $\Delta v = 2$ ) is allowed (given sufficient anharmonicity to permit  $\Delta v = 2$ )?

5. The positions of the atoms in the molecule  $N_4S_4$  have been determined by x-ray diffraction. In terms of a set of Cartesian coordinates  $x, y, z$  placed within the molecule, these are

$$\begin{array}{ll} N_1: x = b, y = 0, z = 0; & N_2: 0, b, 0; \\ N_3: -b, 0, 0; & N_4: 0, -b, 0; \\ S_1: a, -a, a; & S_2: -a, a, a; \\ S_3: a, a, -a; & S_4: -a, -a, -a. \end{array}$$

Here the numbers  $a$  and  $b$  are unrelated parameters of the order of a few ångströms in size.

- To what point group does the molecule belong?
- What are the symmetries of the normal modes of vibration?
- How many different vibrational frequencies does the molecule have?
- How many bands should appear in the infrared absorption spectrum as fundamentals?
- Assume the parameter  $a = 3 \text{ \AA} = 1.5b$ . Compute the moments of inertia as well as  $A, B,$  and  $C$ . What kind of rotor is the molecule (linear rotor, spherical top, oblate or prolate symmetric top, near oblate or near prolate asymmetric top)?

The Line Positions of the  $2\nu_3$  Band of  $CH_3F$  (in  $cm^{-1}$ )

$J$	$R(J)$	$P(J)$	$J$	$R(J)$	$P(J)$
0	—	—	18	2 105.29	2 044.05
1	2 084.72	—	19	2 106.09	—
2	2 086.25	—	20	2 106.86	2 039.01
3	2 087.75	2 076.13	21	2 107.56	2 036.42
4	2 089.22	2 074.27	22	2 108.24	2 033.81
5	2 090.65	2 072.41	23	2 108.87	2 031.15
6	2 092.07	2 070.49	24	2 109.45	2 028.44
7	2 093.41	2 068.55	25	2 110.00	2 025.69
8	2 094.75	2 066.55	26	2 110.49	2 022.95
9	2 095.99	—	27	2 110.93	2 020.10
10	2 097.23	2 062.43	28	2 111.40	—
11	2 098.37	2 060.31	29	2 111.70	2 014.33
12	2 099.49	2 058.10	30	2 112.05	2 011.38
13	2 100.58	2 055.88	31	—	2 008.39
14	2 101.59	2 053.60	32	—	2 005.37
15	2 102.60	2 051.27	33	—	2 002.26
16	2 103.54	2 048.92	34	—	—
17	2 104.43	2 046.50	35	—	1 996.00

6. The  $2\nu_3$  band of  $CH_3F$  is observed near  $4.8 \mu m$ . The line positions of this parallel band are provided in the table above.

- Determine  $\nu_0, B', B'', D'_J,$  and  $D''_J$ .

- (b) Assume a tetrahedral geometry and a "standard" C—H bond length to compute  $A'' (= A')$ .
- (c) From the  $B''$  constant and the assumption in (b) compute the C—F bond length.

7. The lines of the fundamental, 1-0, and first overtone, 2-0, vibration-rotation bands of CO are listed in the following table, in  $\text{cm}^{-1}$ .

1-0 Band			2-0 Band		
$J$	$P(J)$	$R(J)$	$J$	$P(J)$	$R(J)$
0	—	2147.0816	0	—	4263.8376
1	—	2150.8565	1	4256.2171	4267.5421
2	2135.5461	2154.5960	2	4252.3023	4271.1774
3	2131.6333	2158.3002	3	4248.3184	4274.7414
4	—	2161.9687	4	4244.2634	4278.2351
5	2123.7001	2165.6015	5	4240.1403	4281.6571
6	2119.6827	2169.1984	6	4235.9477	4285.0096
7	2115.6294	2172.7593	7	4231.6856	4288.2898
8	2111.5434	2176.2840	8	4227.3539	4291.4996
9	2107.4231	2179.7723	9	4222.9549	4294.6379
10	2103.2688	2183.2242	10	4218.4859	4297.7051
11	2099.0815	2186.6395	11	4213.9486	4300.7001
12	2094.8612	2190.0180	12	4209.3431	4303.6240
13	2090.6089	2193.3596	13	4204.6700	4306.4756
14	2086.3215	2196.6642	14	4199.9279	4309.2552
15	2082.0027	2199.9314	15	4195.1186	4311.9619
16	2077.6500	2203.1615	16	4190.2409	4314.5970
17	2073.2647	2206.3539	17	4185.2956	4317.1591
18	2068.8476	2209.5088	18	4180.2830	4319.6487
19	2064.3968	2212.6258	19	4175.2024	4322.0663
20	2059.9148	2215.7040	20	4170.0553	4324.4100
21	2055.4002	2218.7459	21	4164.8411	4326.6807
22	2050.8546	2221.7487	22	4159.5599	4328.8785
23	—	—	23	4154.2115	4331.0029
24	—	2227.6391	24	4148.7969	4333.0537
25	2037.0252	2230.5264			
26	2032.3528	2233.3748			
27	2027.6495	2236.1842			
28	—	2238.9545			

- (a) For each band, determine the five parameters  $\nu_0$ ,  $B''$ ,  $D''$ ,  $B'$ , and  $D'$ .
- (b) From the band origins, determine  $\omega_e$  and  $\omega_e x_e$ .
- (c) Determine  $B_e$  and  $\alpha_e$ .
- (d) Compute  $r_0$  and  $r_e$ . Why are they different?
- (e) Test the Pekeris and Kratzer relationships.
- (f) Predict all of the preceding constants for  $^{13}\text{C}^{16}\text{O}$ .

8. Even when the rotational structure of a vibrational band cannot be resolved, it is sometimes possible to extract the rotational constant  $B$  from the separation between the maxima in the  $P$  and  $R$  branches as

$$\Delta\tilde{\nu}_{PR} = 2.3583(BT)^{1/2},$$

where  $B$  is the rotational constant (in  $\text{cm}^{-1}$ ) and  $T$  is the absolute temperature. Derive this equation.

9. The observed IR bands (in  $\text{cm}^{-1}$ ) of  $^{10}\text{BF}_3$  and  $^{11}\text{BF}_3$  are as follows (vs = very strong; s = strong; m = medium; w = weak):

$^{10}\text{BF}_3$	482	718	1370	1505	1838	1985	2243	2385	3008	3263
$^{11}\text{BF}_3$	480	691	1370	1454	1838	1932	2243	2336	2903	3214
Intensity	s	s	w	vs	w	w	w	w	w	w

The order of increasing vibration frequency of the fundamentals is  $\nu_4 < \nu_2 < \nu_1 < \nu_3$ . Assign the observed bands. (It might be thought that the  $1370\text{ cm}^{-1}$  band is the overtone  $2\nu_2$ , but this can be ruled out. Why?)

10. A spectroscopist is searching for the LiNNN molecule in the gas phase in the infrared region of the spectrum. By analogy with CaNNN (*J. Chem. Phys.* **88**, 2112 (1988)) LiNNN should be linear and quite ionic. The N—N bond distance in crystalline azides ( $\text{M}^+\text{NNN}^-$ ) is  $1.18\text{ \AA}$ .
- Estimate a reasonable Li—N bond length (e.g., from ionic radii) and compute a  $B$  value from the geometry.
  - Determine the number and symmetry of the normal modes of vibration.
  - Estimate frequencies for the normal modes by analogy with other azides.
  - Describe each IR allowed fundamental transition (i.e., parallel or perpendicular, which branches occur, the spacing of the lines, etc.)
11. For the formaldehyde molecule  $\text{H}_2\text{CO}$  of  $C_{2v}$  symmetry:
- Determine the number and symmetries of the normal modes.
  - Determine which modes are infrared active.
  - Number the normal modes and describe each mode (e.g., symmetric/antisymmetric bend/stretch; in plane/out of plane). For each mode provide an estimated vibrational frequency.
12. (a) To what point group does the dichloroacetylene molecule  $\text{Cl—C}\equiv\text{C—Cl}$  belong?
- How many fundamental modes will there be for dichloroacetylene?
  - Sketch the approximate atomic motion of the normal modes.
  - Specify the infrared activity of each.

(e) Why do the first overtones of the infrared active fundamentals not occur in the IR spectrum?

13. The following table gives the fundamentals and combination bands in the infrared spectrum of acetylene. Fundamentals are very strong (vs), combination bands involving only two fundamentals are of medium (m) intensity, and all others are weak (w). The frequency order of the fundamental bands is  $\nu_4 < \nu_5 < \nu_2 < \nu_3 < \nu_1$ . Determine the frequencies of the fundamentals and assign the combination bands.

Position/cm <sup>-1</sup>	730	1328	1961	2703	3295	3308	3898	4091	5260	6556
Intensity	vs	m	w	m	vs	w	m	m	m	m

14. Consider the BF<sub>3</sub> molecule of  $D_{3h}$  symmetry.
- Determine the number and symmetries of the normal modes.
  - Determine the activity of each mode in the infrared.
  - Determine the symmetries of each of the following types of internal modes:
    - B—F stretching modes,
    - Out-of-plane bending motions,
    - F—B—F angle deformations.
  - What are the selection rules for overtone and combination bands in the infrared?
15. Consider the vibrational spectra for the *trans*-difluoroethylene molecule.
- Determine the number and symmetries for the normal modes.
  - Determine the activities of each of the modes in the infrared.
  - Determine the symmetries of each of the following types of internal modes:
    - C—F stretching modes,
    - Out-of-plane bending modes,
    - H—C—F angle deformations,
    - C=C stretching modes.
16. Since the cyanogen molecule C<sub>2</sub>N<sub>2</sub> is linear, it has seven fundamental vibrational modes.
- Determine the IR activity of each of these fundamentals.
  - Sketch the approximate atomic motions for the vibrational modes. Make sure to symmetrize both the parallel and perpendicular modes.
17. For the *cis*-diimide molecule H—N=N—H of  $C_{2v}$  symmetry:
- Determine the number and symmetries of the normal modes of vibration.
  - Which modes are IR active?
  - Number the normal modes and describe each mode (e.g., symmetric N—H stretch, etc.).

(d) For each mode estimate a vibrational frequency.

18. For the IF molecule, the following spectroscopic constants were recently determined:

$$\omega_e = 610.258 \text{ cm}^{-1},$$

$$\omega_e x_e = 3.141 \text{ cm}^{-1},$$

$$B_e = 0.279711 \text{ cm}^{-1},$$

$$\alpha_e = 0.001874 \text{ cm}^{-1}.$$

(a) Determine the IF bond length ( $r_e$ ).

(b) Describe and sketch the fundamental infrared spectrum (at 300 K).

(c) Calculate the frequency of the  $R(2)$  and  $P(2)$  transitions for the fundamental band and the first overtone.

19. The infrared spectrum of  $\text{N}_2\text{O}$  has three fundamental bands. Assuming that the structure of  $\text{N}_2\text{O}$  is linear, explain how this spectrum allows you to distinguish between  $\text{N}-\text{N}-\text{O}$  and  $\text{N}-\text{O}-\text{N}$ . Sketch the approximate atomic motions of the normal modes.

20. Several of the lines in the  $v = 0$  to  $v = 1$  transition for  $\text{H}^{35}\text{Cl}$  have the following wavenumbers:

$P(J)/\text{cm}^{-1}$	$J$	$R(J)/\text{cm}^{-1}$
	0	2906.2464
2865.0977	1	2925.8961
2843.6242	2	2944.9130
2821.5680	3	2963.2849
2798.9423	4	2980.9998
2775.7601	5	2998.0460

(a) Use these data to determine the band origin  $\tilde{\nu}_0$ .

(b) Calculate  $\alpha_e$  and  $B_e$ .

(c) Determine the equilibrium internuclear separation  $r_e$  to as many significant figures as the data justify.

21. The following bands have been measured in the infrared spectrum of a bent  $\text{AB}_2$  molecule:

Wavenumber/ $\text{cm}^{-1}$	Intensity	Wavenumber/ $\text{cm}^{-1}$	Intensity
1200	vs	3600	w
2400	m	3870	m
2670	vs	4700	m
3500	vs	4800	w

Identify the fundamental, overtone, and combination bands.

22. Show that Fermi resonance cannot occur between two levels with different values of  $l$ .
23. Show that the superposition of two vibrations at right angles to one another gives circular motion if the vibrations have equal amplitudes and differ in phase by  $90^\circ$ .
24. (a) Calculate the  $\mathbf{G}$  matrix for  $\text{D}_2\text{O}$ .  
 (b) Using the force constants listed in Table 7.3 calculate the vibrational frequencies of  $\text{D}_2\text{O}$  and compare with the experimental values of  $\nu_1 = 2666 \text{ cm}^{-1}$ ,  $\nu_2 = 1179 \text{ cm}^{-1}$ , and  $\nu_3 = 2789 \text{ cm}^{-1}$ .
25. (a) Using the vibrational constants (7.200) calculate the symmetry and frequency of all possible vibrational levels of  $\text{HCN}$  below  $3500 \text{ cm}^{-1}$ .  
 (b) Calculate all possible allowed vibrational transitions between these levels.
26. A 10-cm-long cell containing 10 mTorr of  $\text{HCl}$  is prepared at 296 K. Consider the  $P(2)$  transition of the fundamental band at  $2843.6247 \text{ cm}^{-1}$ . The HITRAN database<sup>35</sup> lists the pressure-broadening parameter (half width at half maximum) as  $0.2395 \text{ cm}^{-1}/\text{atm}$  for self broadening and the line intensity  $S'$  as  $3.692 \times 10^{-19} \text{ cm}/\text{molecule}$ . Notice that HITRAN uses half width at half maximum for the pressure-broadening parameter rather than the full width used in this book.
  - (a) Is the line predominantly pressure-broadened or Doppler-broadened?
  - (b) What will be the transmission of the cell at line center, i.e., at  $2843.6247 \text{ cm}^{-1}$ ?
  - (c) If the cell is then filled with air to a total pressure of 1 atm, repeat (a) and (b) given that the air-broadening coefficient (half width at half maximum) for  $\text{HCl}$  is  $0.0799 \text{ cm}^{-1}/\text{atm}$ .
27. The first derivative of the dipole moment function for  $\text{HCl}$  was calculated *ab initio* (Meyer and Rosmus, *J. Chem. Phys.* **63**, 2356 (1975)) to be  $|d\mu/dr| = 0.86 \text{ D}/\text{\AA}$  at the calculated equilibrium bond distance,  $r_e = 1.278 \text{ \AA}$ . Other calculated equilibrium constants were  $\omega_e = 2977 \text{ cm}^{-1}$  and  $|\mu_e| = 1.136 \text{ D}$ .
  - (a) What is the *ab initio* band strength  $S'_{\text{band}}$  for the fundamental in HITRAN units ( $\text{cm}/\text{molecule}$ ). How does this compare with the experimental value of  $81.485 \text{ cm}^{-2} \text{ atm}^{-1}$  at 296 K?
  - (b) What is the line intensity,  $S'$ , the integrated cross section  $\int \sigma d\nu$ , and the line strength  $S_{J',J''}$  for the  $R(0)$  line of the fundamental band?
  - (c) What is the Einstein  $A$  value for the  $R(0)$  line? Is  $A^{-1}$  the same as the lifetime?

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## Chapter 8

# Light Scattering and the Raman Effect

### 8.1 Background

The Raman effect is a light-scattering phenomenon. When light of frequency  $\nu_1$  or  $\nu_0$  (usually from a laser or, in the pre-laser era, from a mercury arc lamp) irradiates a sample (Figure 8.1), it can be scattered. The frequency of the scattered light can either be at the original frequency (referred to as *Rayleigh scattering*) or at some shifted frequency  $\nu_S = \nu_1 \pm \nu_{\text{molecular}}$  (referred to as *Raman scattering*). The frequency  $\nu_{\text{molecular}}$  is an internal frequency corresponding to rotational, vibrational, or electronic transitions within a molecule. The vibrational Raman effect is by far the most important, although rotational and electronic Raman effects are also known. For example, the rotational Raman effect provides some of the most accurate bond lengths for homonuclear diatomic molecules.

In discussing the Raman effect some commonly used terms need to be defined (Figure 8.2). Radiation scattering to the lower frequency side (to the “red”) of the exciting line is called *Stokes scattering*, while the light scattered at higher frequencies than that of the exciting line (to the “blue”) is referred to as the *anti-Stokes scattering*. Finally, the magnitude of the shift between the Stokes or the anti-Stokes line and the exciting line is called the *Raman shift*,  $\Delta\nu = |\nu_1 - \nu_S|$ .

#### *Classical Model*

When an electric field is applied to a molecule, the electrons and nuclei respond by moving in opposite directions in accordance with Coulomb’s law. The applied electric field therefore induces a dipole moment in the molecule. As long as the applied electric field is not too strong, the induced dipole moment is linearly proportional to the applied electric field, and is given by

$$\mu_{\text{ind}} = \alpha \mathbf{E}, \quad (8.1)$$

in which the proportionality constant  $\alpha$  is called the *polarizability* and is a characteristic of the molecule.

The intensity of the scattered light is proportional to the square of the magnitude of the induced oscillating dipole moment. If some internal motion of the molecule (vi-

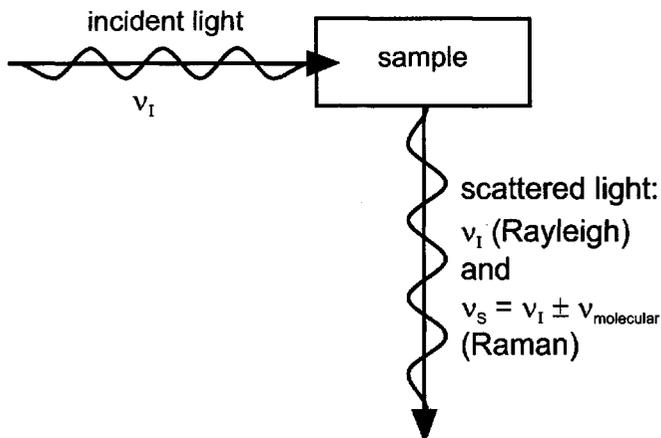


Figure 8.1: Scattering of light by a sample.

brational, rotational, or electronic) modulates this induced oscillating dipole moment, then additional frequencies can appear. Classically, this means that the polarizability has a static term  $\alpha_0$  and a sinusoidal oscillating term with amplitude  $\alpha_1$

$$\alpha = \alpha_0 + \alpha_1 \cos(\omega t) \quad (8.2)$$

with  $\omega (= \omega_{\text{molecular}})$  being some internal angular frequency. As usual, it is convenient to use angular frequency  $\omega (= 2\pi\nu)$  for theoretical work and frequency  $\nu$  (or wavenumber,  $\tilde{\nu}$ ) for experimental work. For example, a vibrational mode  $Q_i$  has

$$\alpha_1 = \left. \frac{\partial \alpha}{\partial Q_i} \right|_{Q_i=0} Q_i, \quad (8.3)$$

so that if the polarizability does not change with vibration, that is, if  $(\partial\alpha/\partial Q_i)|_0 = 0$ , then there is no vibrational Raman effect. Classically, the oscillating polarizability causes the induced dipole moment to oscillate at frequencies other than the incident  $\omega_1$ . To see this, let us represent the applied electric field  $\mathbf{E}$  as  $\mathbf{E}_0 \cos \omega_1 t$ . Upon substituting (8.2) into the magnitude of (8.1), we get

$$\mu_{\text{ind}} = \alpha E = \alpha E_0 \cos \omega_1 t \quad (8.4)$$

$$\begin{aligned} \mu_{\text{ind}} &= (\alpha_0 + \alpha_1 \cos \omega t) E_0 \cos \omega_1 t \\ &= \alpha_0 E_0 \cos \omega_1 t + \alpha_1 E_0 \cos \omega_1 t \cos \omega t \\ &= \alpha_0 E_0 \cos \omega_1 t + \frac{\alpha_1 E_0 \cos(\omega_1 - \omega)t + \alpha_1 E_0 \cos(\omega_1 + \omega)t}{2}. \end{aligned} \quad (8.5)$$

The trigonometric identity

$$\cos \theta \cos \phi = \frac{\cos(\theta - \phi) + \cos(\theta + \phi)}{2} \quad (8.6)$$

has been used in the final step of equation (8.5). The first term is unshifted in frequency and corresponds to Rayleigh scattering (Figure 8.2). The lower frequency term with  $\omega_1 - \omega$  corresponds to Stokes scattering, while the higher frequency term with  $\omega_1 + \omega$

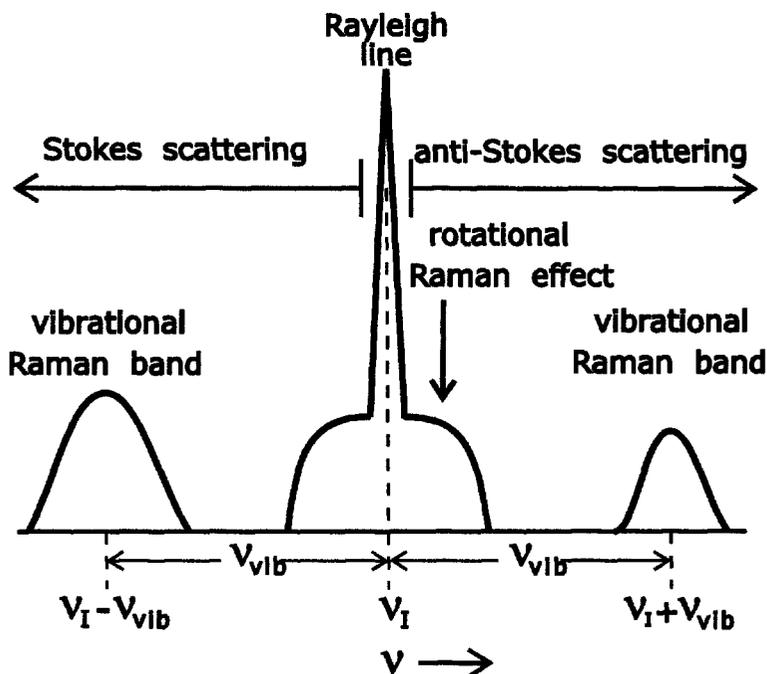


Figure 8.2: Schematic diagram of a Raman spectrum showing vibrational and rotational Raman effects.

corresponds to anti-Stokes scattering (Figure 8.3). This simple classical derivation (8.5) is very deceptive, since it predicts that Stokes and anti-Stokes scattering have the same intensity: this is not usually the case.

The energy-level diagram for Stokes and anti-Stokes scattering shows that anti-Stokes scattering will be weaker because the population in the excited vibrational level is less than that in the ground state (Figure 8.3). For a classical oscillator the scattering (Rayleigh and Raman) is proportional to the fourth power of the frequency (see section 8.4 and problem 7). (The sky is blue because air molecules Rayleigh scatter more blue than red sunlight.) Thus if we introduce the Boltzmann distribution of vibrational populations, the ratio of the intensities of the bands is given by

$$\frac{\text{Anti-Stokes intensity}}{\text{Stokes intensity}} = \frac{(\nu_I + \nu_{\text{vib}})^4 e^{-h\nu_{\text{vib}}/kT}}{(\nu_I - \nu_{\text{vib}})^4} \quad (8.7)$$

for a nondegenerate vibration.

There is one additional complication. For highly symmetric molecules such as  $\text{CH}_4$ , the induced dipole is in the same direction as the applied electric field. For less symmetric molecules, however,  $\mu_{\text{ind}}$  and  $\mathbf{E}$  can point in different directions because the molecular response to the applied electric field can be different along the  $X$ -,  $Y$ - and  $Z$ -axes in the laboratory frame. In matrix notation, equation (8.1) becomes

$$\mu_{\text{ind}} = \alpha \mathbf{E} \quad (8.8)$$

or

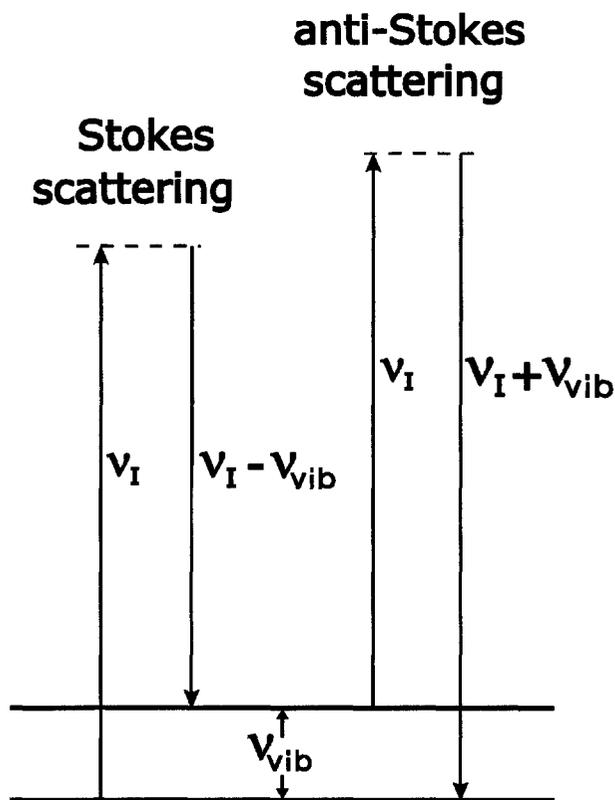


Figure 8.3: Energy-level diagram showing Stokes and anti-Stokes scattering.

$$\begin{pmatrix} \mu_X \\ \mu_Y \\ \mu_Z \end{pmatrix} = \begin{pmatrix} \alpha_{XX} & \alpha_{XY} & \alpha_{XZ} \\ \alpha_{XY} & \alpha_{YY} & \alpha_{YZ} \\ \alpha_{XZ} & \alpha_{YZ} & \alpha_{ZZ} \end{pmatrix} \begin{pmatrix} E_X \\ E_Y \\ E_Z \end{pmatrix}, \quad (8.9)$$

in which  $\alpha$  is a  $3 \times 3$  symmetric matrix. This symmetric matrix is called the *polarizability tensor*.

The polarizability tensor  $\alpha$  can be simplified by working in the appropriate principal axis system of the molecule, analogous to the principal axis system for the moment of inertia tensor (Chapter 6). As the polarizability tensor is a real, symmetric matrix, it is always possible to construct an orthogonal transformation matrix  $\mathbf{X}$  from the normalized eigenvectors of  $\alpha$ . The matrix  $\mathbf{X}$  represents a rotation of the coordinate system,  $\mathbf{r}' = \mathbf{X}^{-1}\mathbf{r}$  or  $\mathbf{r} = \mathbf{X}\mathbf{r}'$ , with  $\mathbf{r}^t = (x, y, z)$ . As discussed in Chapter 3, the diagonalized matrix  $\alpha'$  is related to  $\alpha$  by the similarity transformation

$$\alpha' = \mathbf{X}^{-1}\alpha\mathbf{X}. \quad (8.10)$$

The  $\alpha'$  matrix consists of the eigenvalues of  $\alpha$  and has the form,

$$\alpha' = \begin{pmatrix} \alpha_{x'x'} & 0 & 0 \\ 0 & \alpha_{y'y'} & 0 \\ 0 & 0 & \alpha_{z'z'} \end{pmatrix}, \quad (8.11)$$

or  $\alpha_x = \alpha_{x'x'}$ ,  $\alpha_y = \alpha_{y'y'}$ , and  $\alpha_z = \alpha_{z'z'}$  if the molecular principal axis system is assumed. Note that unless required by symmetry, the principal axes of polarizability of a molecule do not coincide with the principal axes of the moment of inertia.

The polarizability ellipsoid of a molecule is defined by the equation

$$\mathbf{r}^t \boldsymbol{\alpha} \mathbf{r} = 1 \quad (8.12)$$

or

$$\alpha_x x^2 + \alpha_y y^2 + \alpha_z z^2 = 1 \quad (8.13)$$

in the principal axis system. The ellipsoid has maximum total dimensions equal to  $2/\sqrt{\alpha_x}$ ,  $2/\sqrt{\alpha_y}$ , and  $2/\sqrt{\alpha_z}$  along the  $x$ -,  $y$ -, and  $z$ -axes. For a spherical top,  $\alpha_x = \alpha_y = \alpha_z$  so the ellipsoid is a sphere; for a symmetric top or linear molecule,  $\alpha_x = \alpha_y$  and the ellipsoid has a circular cross section in the  $xy$ -plane. For the normal case of light scattering with wavelength  $\lambda$  substantially greater in size than that of the molecule, the molecule behaves as if it were represented in shape by the polarizability ellipsoid.

For a diatomic molecule such as  $\text{H}_2$  or  $\text{HCl}$ , it is convenient to label the polarizability parallel to the molecular  $z$ -axis as  $\alpha_{\parallel}$  ( $= \alpha_z$ ), and the polarizability perpendicular to the  $z$ -axis as  $\alpha_{\perp}$  ( $= \alpha_x = \alpha_y$ ). The mean polarizability is given by

$$\bar{\alpha} = (\alpha_{\parallel} + 2\alpha_{\perp})/3 \quad (8.14)$$

and the polarizability anisotropy  $\gamma$  is defined as

$$\gamma = \alpha_{\parallel} - \alpha_{\perp}. \quad (8.15)$$

The mean polarizability  $\bar{\alpha}$  can be deduced, for example, from a measurement of the optical refractive index (see problem 7) and the anisotropy  $\gamma$  from a measurement of the depolarization ratio  $\rho$  (see below) of Rayleigh scattering.

Polarizability is an important molecular property that plays a role in, for example, intermolecular interactions. The polarizability of a molecule can be calculated by the methods of *ab initio* quantum chemistry. The results of such a calculation<sup>1</sup> of  $\bar{\alpha}$  and  $\gamma$  for  $\text{H}_2$  are displayed in Figure 8.4 as a function of the internuclear distance,  $r$ . At large  $r$ , the value of  $\bar{\alpha}$  approaches that of two H atoms, while at short  $r$  the value of  $\bar{\alpha}$  tends to that of the He atom. At  $r = r_e = 0.742 \text{ \AA}$ ,  $\partial\bar{\alpha}/\partial r \neq 0$  so there will be a vibrational Raman effect for  $\text{H}_2$ . As the bond stretches from equilibrium, the electrons are less tightly held by the nuclei so the polarizability increases. In general for both heteronuclear molecules such as  $\text{HCl}$  and homonuclear molecules such as  $\text{H}_2$ , the polarizability ellipsoid will change as the molecule vibrates, leading to a vibrational Raman effect. The Raman effect is thus less restrictive than normal dipole-allowed infrared vibrational spectroscopy, which has no allowed transitions for a homonuclear molecule because  $\partial\boldsymbol{\mu}/\partial r = 0$ .

Simple arguments based on changes in the polarizability as a function of the normal coordinates  $Q_i$  can be made for a typical polyatomic molecule such as  $\text{CO}_2$ . As shown in Figure 8.5, motion along the symmetric stretching coordinate  $Q_i$ , will change the mean polarizability so that  $\partial\bar{\alpha}/\partial Q_1 \neq 0$  and  $\nu_1$  is Raman active. The situation is different for  $\nu_2$  and  $\nu_3$  because of the high symmetry. The polarizability again changes with  $Q$  but the values at  $+Q$  and  $-Q$  are identical by symmetry, (i.e.,  $\bar{\alpha}(Q)$  is an even function) so at  $Q = 0$ ,  $\partial\bar{\alpha}/\partial Q = 0$  for  $\nu_2$  and  $\nu_3$  (Figure 8.5). For the dipole moments  $\boldsymbol{\mu}$  the opposite situation prevails with  $\partial\boldsymbol{\mu}/\partial Q_1 = 0$ ,  $\partial\boldsymbol{\mu}/\partial Q_2 \neq 0$ , and  $\partial\boldsymbol{\mu}/\partial Q_3 \neq 0$ , so  $\nu_1$  is Raman active but  $\nu_2$  and  $\nu_3$  are infrared active. This is an example of the *rule*

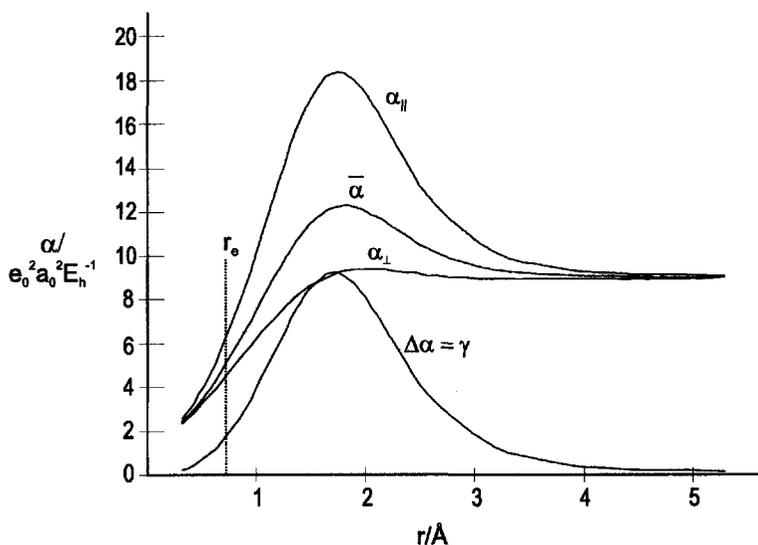


Figure 8.4: The polarizability of  $\text{H}_2$  as a function of internuclear distance  $r$  calculated by *ab initio* methods.<sup>1</sup>

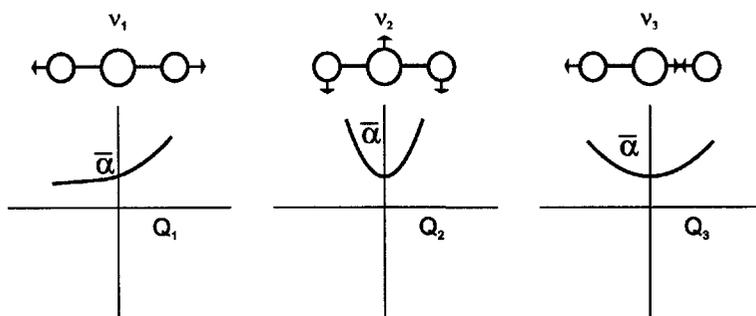


Figure 8.5: The polarizability of  $\text{CO}_2$  as a function of the three normal coordinates  $Q_1$ ,  $Q_2$ , and  $Q_3$  (shown schematically).

of *mutual exclusion* that applies to molecules with a center of symmetry, and will be discussed later.

### Quantum Model

The quantum mechanical theory of the Raman effect was developed in the early 1930s by Placzek.<sup>2</sup> The starting point is the same as in Chapter 1 with a two-level system with energy levels  $E_1$  and  $E_0$  as depicted in Figure 1.8. An oscillating electric field is applied to the system,  $\mathbf{E} = \mathbf{E}_0 \cos \omega t$ , with the wavelength  $\lambda$  assumed to be much bigger than the molecular dimensions. In the case of Rayleigh and Raman scattering, the electric field is not in resonance (i.e.,  $\omega \neq (E_1 - E_0)/\hbar = \omega_{10}$ ), but instead induces an oscillating dipole moment that re-radiates. In quantum mechanics this means that we are looking for (see equation (1.66)) the transition dipole moment,

$$\mathbf{M}_{10}(t) = \langle \Psi_1 | \boldsymbol{\mu} | \Psi_0 \rangle, \quad (8.16)$$

with  $\Psi_1(t)$  and  $\Psi_0(t)$  being the solutions of the time-dependent Schrödinger equation (1.29) for the two-level system. The intensity of the scattered radiation is proportional to  $|\mathbf{M}_{10}|^2$ . The interaction of electromagnetic radiation and the system is taken into account with the electric-dipole interaction term, equation (1.26), namely

$$\hat{H}' = -\boldsymbol{\mu} \cdot \mathbf{E}_0 \cos(\omega t). \quad (8.17)$$

In this case  $\boldsymbol{\mu}$  is an induced moment, and  $\boldsymbol{\mu}$  and  $\mathbf{E}_0$  need not point in the same direction.

Rather than solving the Schrödinger equation as outlined in Chapter 1, perturbation theory (Chapter 4) will be used to obtain an expression for the transition dipole moment,  $\mathbf{M}_{10}(t)$ . In what follows the states in the molecule are labeled as  $|n\rangle = |0\rangle$ ,  $|k\rangle = |1\rangle$ , and  $|r\rangle$ , with  $|r\rangle$  being the additional states in the molecule not depicted in Figure 1.8. The application of the small perturbing electric field, equation (1.6), alters the wavefunction  $\Psi_n$  of the system so that

$$\Psi_n = \Psi_n^{(0)} + \Psi_n^{(1)} + \dots \quad (8.18)$$

The zeroth-order solution to the time-dependent Schrödinger equation,

$$\hat{H}^{(0)}\Psi_n^{(0)} = i\hbar \frac{\partial \Psi_n^{(0)}}{\partial t}, \quad (8.19)$$

is

$$\Psi_n^{(0)} = \psi_n^{(0)} e^{-iE_n t/\hbar} = \psi_n^{(0)} e^{-i\omega_n t}, \quad (8.20)$$

with  $\psi_n^{(0)}$  being the solution of the corresponding time-independent equation,

$$\hat{H}^{(0)}\psi_n^{(0)} = E_n^{(0)}\psi_n^{(0)}. \quad (8.21)$$

The perturbed Schrödinger equation is

$$\left( \hat{H}^{(0)} - \boldsymbol{\mu} \cdot \mathbf{E}_0 \cos(\omega t) \right) \Psi_n = i\hbar \frac{\partial \Psi_n}{\partial t}, \quad (8.22)$$

and using equation (8.18) to first order leads to

$$\left( \hat{H}^{(0)} - \boldsymbol{\mu} \cdot \mathbf{E}_0 \cos(\omega t) \right) \left( \Psi_n^{(0)} + \Psi_n^{(1)} \right) = i\hbar \frac{\partial}{\partial t} \left( \Psi_n^{(0)} + \Psi_n^{(1)} \right), \quad (8.23)$$

or, equivalently, when the zeroth-order equation is subtracted, to

$$\hat{H}^{(0)}\Psi_n^{(1)} - i\hbar \frac{\partial \Psi_n^{(1)}}{\partial t} = \boldsymbol{\mu} \cdot \mathbf{E}_0 \cos(\omega t)\Psi_n^{(0)}. \quad (8.24)$$

The first-order correction can be obtained by assuming (with some foresight) a solution of the form

$$\Psi_n^{(1)} = \psi_n^+ e^{-i(\omega_n + \omega)t} + \psi_n^- e^{-i(\omega_n - \omega)t}. \quad (8.25)$$

Substitution of the assumed solution (8.25) into equation (8.24), using  $\cos(\omega t) = (e^{i\omega t} + e^{-i\omega t})/2$ , and then equating terms with equal time dependence leads to two separate equations,

$$\hat{H}^{(0)}\psi_n^+ - (E_n + h\nu)\psi_n^+ = \boldsymbol{\mu} \cdot \mathbf{E}_0\psi_n^{(0)}/2, \quad (8.26)$$

and

$$\hat{H}^{(0)}\psi_n^- - (E_n - h\nu)\psi_n^- = \boldsymbol{\mu} \cdot \mathbf{E}_0\psi_n^{(0)}/2. \quad (8.27)$$

The right-hand side of equations (8.26) and (8.27) can be manipulated by the trick of inserting unity (Chapter 4), in the form  $1 = \sum |\psi_r^{(0)}\rangle\langle\psi_r^{(0)}|$  so that we obtain

$$\boldsymbol{\mu}|\psi_n^{(0)}\rangle \cdot \mathbf{E}_0 = \sum_r |\psi_r^{(0)}\rangle\langle\psi_r^{(0)}|\boldsymbol{\mu}|\psi_n^{(0)}\rangle \cdot \mathbf{E}_0. \quad (8.28)$$

By defining the matrix element  $\boldsymbol{\mu}_{rn}$  of  $\boldsymbol{\mu}$  by

$$\boldsymbol{\mu}_{rn} = \langle\psi_r^{(0)}|\boldsymbol{\mu}|\psi_n^{(0)}\rangle, \quad (8.29)$$

we may write (8.26) and (8.27) as

$$\hat{H}^{(0)}\psi_n^+ - (E_n + h\nu)\psi_n^+ = \sum_r \boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0\psi_r^{(0)}/2 \quad (8.30)$$

and

$$\hat{H}^{(0)}\psi_n^- - (E_n - h\nu)\psi_n^- = \sum_r \boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0\psi_r^{(0)}/2. \quad (8.31)$$

The  $\psi_n^+$  and  $\psi_n^-$  wavefunctions can also be expanded in terms of the complete, orthonormal set of  $\psi_r^{(0)}$  functions, i.e., as

$$\psi_n^+ = \sum_r c_r^+ \psi_r^{(0)} \quad (8.32)$$

and

$$\psi_n^- = \sum_r c_r^- \psi_r^{(0)}, \quad (8.33)$$

which results in the expansion coefficients  $c_r^+$  and  $c_r^-$  being given by

$$c_r^+ = \frac{\boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0/2}{E_r - E_n - h\nu}, \quad (8.34)$$

and

$$c_r^- = \frac{\boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0/2}{E_r - E_n + h\nu} \quad (8.35)$$

from equations (8.30) and (8.31). The time-dependent first-order correction (equation (8.25)) to the wavefunction is thus

$$\Psi_n^{(1)} = \frac{1}{2\hbar} \sum_r \psi_r^{(0)} \left( \frac{\boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0}{\omega_{rn} - \omega} e^{-i(\omega_n + \omega)t} + \frac{\boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0}{\omega_{rn} + \omega} e^{-i(\omega_n - \omega)t} \right), \quad (8.36)$$

with  $\omega_{rn} \equiv (E_r - E_n)/\hbar$ .

Rayleigh scattering involves no frequency shift of the scattered light (i.e.,  $k = n$  for  $\mathbf{M}_{kn}(t)$ ) and is based on the oscillating part of the quantum mechanical dipole moment, i.e.,

$$\begin{aligned}\mathbf{M}_{nn}(t) &= \langle \Psi_n | \boldsymbol{\mu} | \Psi_n \rangle \approx \langle \Psi_n^{(0)} + \Psi_n^{(1)} | \boldsymbol{\mu} | \Psi_n^{(0)} + \Psi_n^{(1)} \rangle \\ &\approx \langle \Psi_n^{(0)} | \boldsymbol{\mu} | \Psi_n^{(0)} \rangle + \langle \Psi_n^{(0)} | \boldsymbol{\mu} | \Psi_n^{(1)} \rangle + \langle \Psi_n^{(1)} | \boldsymbol{\mu} | \Psi_n^{(0)} \rangle \\ &= \mathbf{M}_{nn}^{(0)} + \mathbf{M}_{nn}^{(1)}(t)\end{aligned}\quad (8.37)$$

to first order. The term  $\mathbf{M}_{nn}^{(0)}$ , given by

$$\langle \Psi_n^{(0)} | \boldsymbol{\mu} | \Psi_n^{(0)} \rangle = \langle \psi_n | \boldsymbol{\mu} | \psi_n \rangle = \boldsymbol{\mu}_{nn} \quad (8.38)$$

has no time dependence and is just the dipole moment of the molecule in state  $|\psi_n\rangle$ . This term does not result in any light scattering and hence can be discarded leaving, to first order,  $\mathbf{M}_{nn}(t) \approx \mathbf{M}_{nn}^{(1)}(t)$ , with  $\mathbf{M}_{nn}^{(1)}(t)$  given as

$$\begin{aligned}\mathbf{M}_{nn}^{(1)}(t) &= \frac{e^{-i\omega t}}{2\hbar} \sum_r \left( \frac{\boldsymbol{\mu}_{nr}(\boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0)}{\omega_{rn} - \omega} + \frac{\boldsymbol{\mu}_{rn}(\boldsymbol{\mu}_{nr} \cdot \mathbf{E}_0)}{\omega_{rn} + \omega} \right) \\ &+ \frac{e^{i\omega t}}{2\hbar} \sum_r \left( \frac{\boldsymbol{\mu}_{nr}(\boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0)}{\omega_{rn} + \omega} + \frac{\boldsymbol{\mu}_{rn}(\boldsymbol{\mu}_{nr} \cdot \mathbf{E}_0)}{\omega_{rn} - \omega} \right).\end{aligned}\quad (8.39)$$

Rayleigh scattering is thus proportional to  $|\mathbf{M}_{nn}^{(1)}|^2$  obtained using equation (8.39).

In exactly the same way, the transition dipole moment  $\mathbf{M}_{kn}^{(1)}(t)$  leads to transitions from state  $|n\rangle$  to state  $|k\rangle$  with

$$\begin{aligned}\mathbf{M}_{kn}^{(1)}(t) &= \frac{e^{i(\omega_{kn}-\omega)t}}{2\hbar} \sum_r \left( \frac{\boldsymbol{\mu}_{kr}(\boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0)}{\omega_{rn} - \omega} + \frac{\boldsymbol{\mu}_{rn}(\boldsymbol{\mu}_{kr} \cdot \mathbf{E}_0)}{\omega_{rk} + \omega} \right) \\ &+ \frac{e^{i(\omega_{kn}+\omega)t}}{2\hbar} \sum_r \left( \frac{\boldsymbol{\mu}_{kr}(\boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0)}{\omega_{rn} + \omega} + \frac{\boldsymbol{\mu}_{rn}(\boldsymbol{\mu}_{kr} \cdot \mathbf{E}_0)}{\omega_{rk} - \omega} \right).\end{aligned}\quad (8.40)$$

Once again, the zeroth order term,

$$\langle \Psi_k^{(0)} | \boldsymbol{\mu} | \Psi_n^{(0)} \rangle = \boldsymbol{\mu}_{kn} e^{i\omega_{kn}t} \quad (8.41)$$

has been discarded because it corresponds to a regular transition dipole moment of the type that has been discussed in Chapter 1, so that it does not contribute to Raman scattering.

For the Raman effect it is assumed that  $\omega_{kn} = (E_k - E_n)/\hbar$  can be positive ( $E_k > E_n$ ) for Stokes scattering or negative ( $E_k < E_n$ ) for anti-Stokes scattering, and that enough energy is available from the incident photon to induce the transition from  $|n\rangle$  to  $|k\rangle$ , i.e.,  $\omega > \omega_{kn}$  or  $\omega - \omega_{kn} > 0$  (sometimes referred to as ‘‘Klein<sup>2-4</sup> conditions’’). In addition, the second term on the right-hand side of equation (8.40) has an oscillating dipole moment at a high angular frequency of  $\omega_{kn} + \omega$ , which is interpreted<sup>3</sup> (somewhat surprisingly!) as a two-photon transition, and will not be considered further. The first

term on the right-hand side of equation (8.40) has the correct angular frequency  $\omega_{kn} - \omega$  associated with the Raman effect.

The expression for the oscillating transition dipole moment for a Raman transition from state  $|n\rangle$  to  $|k\rangle$  is thus given in first-order perturbation theory as

$$\mathbf{M}_{kn}^{(1)}(t) = \frac{e^{i(\omega_{kn} - \omega)t}}{2\hbar} \sum_r \left( \frac{\boldsymbol{\mu}_{kr}(\boldsymbol{\mu}_{rn} \cdot \mathbf{E}_0)}{\omega_{rn} - \omega} + \frac{\boldsymbol{\mu}_{rn}(\boldsymbol{\mu}_{kr} \cdot \mathbf{E}_0)}{\omega_{rk} + \omega} \right). \quad (8.42)$$

Consider the applied electric field to be given in terms of its laboratory Cartesian components as

$$\mathbf{E}_0 = E_{0X}\hat{\mathbf{i}} + E_{0Y}\hat{\mathbf{j}} + E_{0Z}\hat{\mathbf{k}} \quad (8.43)$$

while the oscillating transition dipole moment also has Cartesian components,

$$\mathbf{M}_{kn}^{(1)} = M_{X,kn}\hat{\mathbf{i}} + M_{Y,kn}\hat{\mathbf{j}} + M_{Z,kn}\hat{\mathbf{k}}, \quad (8.44)$$

as do the matrix elements of the dipole moment:

$$\boldsymbol{\mu}_{rn} = \mu_{X,rn}\hat{\mathbf{i}} + \mu_{Y,rn}\hat{\mathbf{j}} + \mu_{Z,rn}\hat{\mathbf{k}}. \quad (8.45)$$

To make the meaning of equation (8.42) clearer, consider for example the  $X$  component of  $\mathbf{M}_{kn}^{(1)}(t)$  on the left-hand side in response to the  $Z$  component of  $\mathbf{E}_0$  on the right-hand side, in which case we may write

$$M_{X,kn}(t) = \frac{e^{i(\omega_{kn} - \omega)t}}{2\hbar} \sum_r \left( \frac{\mu_{X,kr}\mu_{Z,rn}}{\omega_{rn} - \omega} + \frac{\mu_{X,rn}\mu_{Z,kr}}{\omega_{rk} + \omega} \right) E_{0Z}, \quad (8.46)$$

for comparison with the corresponding term from equation (8.9), namely,

$$\mu_X = \alpha_{XZ}E_Z. \quad (8.47)$$

Before the comparison can be made, the substitution of  $E_{0Z} \cos \omega t$  for  $E_Z$  in equation (8.47) needs to be made and then the  $kn$ -matrix element formed using the  $\Psi_k^{(0)}(t)$  and  $\Psi_n^{(0)}(t)$  wavefunctions to give

$$M_{X,kn} = \langle \Psi_k(t) | \hat{\alpha}_{XZ} | \Psi_n(t) \rangle E_{0Z} \cos \omega t. \quad (8.48)$$

Converting the cosine to its exponential form and using  $\Psi_k^{(0)} = \psi_k^{(0)} e^{-i\omega_k t}$  and  $\Psi_n^{(0)} = \psi_n^{(0)} e^{-i\omega_n t}$  leads to

$$M_{X,kn} = \frac{1}{2} \langle \psi_k^{(0)} | \hat{\alpha}_{XZ} | \psi_n^{(0)} \rangle E_{0Z} \left( e^{i(\omega_{kn} - \omega)t} + e^{i(\omega_{kn} + \omega)t} \right), \quad (8.49)$$

and once again the high frequency term with  $\omega_{kn} + \omega$  can be ignored. Comparison of equations (8.46) and (8.49) leads to the conclusion that

$$\alpha_{XZ} = \frac{1}{\hbar} \sum_r \left( \frac{\mu_{X,kr}\mu_{Z,rn}}{\omega_{rn} - \omega} + \frac{\mu_{X,rn}\mu_{Z,kr}}{\omega_{rk} + \omega} \right) \quad (8.50)$$

or in general

$$\alpha_{ij} = \frac{1}{\hbar} \sum_r \left( \frac{\mu_{i,kr}\mu_{j,rn}}{\omega_{rn} - \omega} + \frac{\mu_{i,rn}\mu_{j,kr}}{\omega_{rk} + \omega} \right) \quad (8.51)$$

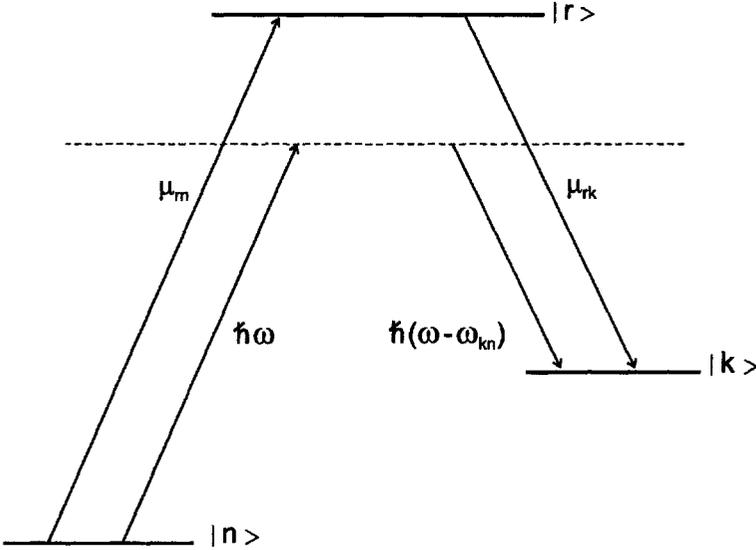


Figure 8.6: Energy-level diagram for the Stokes Raman scattering of  $\hbar\omega$  into  $\hbar(\omega - \omega_{kn})$  and the dipole matrix elements  $\mu_{rn}$  and  $\mu_{rk}$  that contribute to the polarizability tensor elements  $\alpha_{ij}$ .

with  $i, j = X, Y, Z$ . The elements of the polarizability tensor are thus given by a sum of dipole matrix element products divided by energy denominators as depicted in Figure 8.6 for the case of Stokes Raman scattering of an incident photon at  $\hbar\omega$  to  $\hbar(\omega - \omega_{kn})$ .

The polarizability, equation (8.51), contains resonance denominators that cause  $\alpha_{ij}$  to become large if the frequency of the applied electric field approaches that of an atomic or a molecular transition—i.e., if  $\omega$  approaches  $\omega_{rn}$ . In this case, a single term in the sum dominates and results in the *resonance Raman effect*. The resonance Raman effect also leads to an enhancement in the Raman scattering, and with large (but finite!) values of  $\alpha_{ij}$ , when an extra damping term is included in the denominator of equation (8.51).<sup>4</sup>

The selection rules for Raman transitions from state  $|n\rangle$  to state  $|k\rangle$  are, as usual, obtained by inspection of the transition dipole moment integral (8.42)

$$\mathbf{M}_{kn} = \langle \Psi_k | \boldsymbol{\mu} | \Psi_n \rangle, \tag{8.52}$$

which is given in equation (8.46) in terms of the polarizability tensor elements  $\alpha_{ij}$ , equation (8.51). It is convenient to define formally a polarizability operator  $\hat{\alpha}_{ij}$  as

$$\hat{\alpha}_{ij} \equiv \frac{1}{\hbar} \sum_r \left( \frac{\hat{\mu}_i |r\rangle \langle r| \hat{\mu}_j}{\omega_{rn} - \omega} + \frac{\hat{\mu}_j |r\rangle \langle r| \hat{\mu}_i}{\omega_{rk} + \omega} \right) \tag{8.53}$$

so that taking matrix elements

$$\alpha_{ij} = \langle k | \hat{\alpha}_{ij} | n \rangle = \int \psi_k^* \hat{\alpha}_{ij} \psi_n d\tau \tag{8.54}$$

leads to the polarizability tensor values,  $\alpha_{ij}$ , of equation (8.51).

In terms of Raman selection rules, the time dependence of equation (8.52) is of no consequence, and they are determined by the symmetry of  $\psi_k$ ,  $\psi_n$ , and  $\hat{\alpha}_{ij}$  in equation

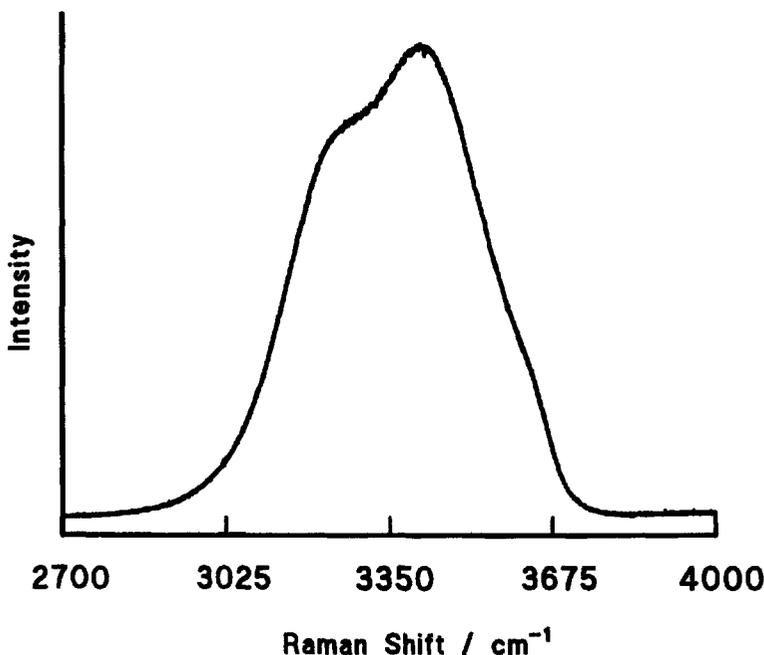


Figure 8.7: Raman spectrum of liquid H<sub>2</sub>O in the O—H stretching region. The peak to the right is the  $\nu_1$  symmetric stretching mode, while the peak to the left is due to the  $2\nu_2$  overtone and the O—H stretching mode of two (or more) hydrogen-bonded H<sub>2</sub>O molecules.

(8.54). The  $\hat{\alpha}_{ij}$  operator is made up of dipole moment operators  $\hat{\mu}_i$  and  $\hat{\mu}_j$ ,  $i, j = x, y, z$ . This means that in the molecular frame the six elements of the polarizability tensor ( $\alpha_{xx}$ ,  $\alpha_{yy}$ ,  $\alpha_{zz}$ ,  $\alpha_{xy}$ ,  $\alpha_{xz}$ , and  $\alpha_{yz}$ ) all transform like the binary products of coordinates  $x^2$ ,  $y^2$ ,  $z^2$ ,  $xy$ ,  $xz$ , and  $yz$  when the symmetry operations of the point group are applied. The symmetry of these binary products (or properly symmetrized combinations) are listed on the right side of character tables. Thus the direct product

$$\Gamma(\psi_1^*) \otimes \Gamma(\alpha_{ij}) \otimes \Gamma(\psi_0) \quad (8.55)$$

must contain the  $A_1$  irreducible representation in order for the corresponding integral to be nonzero and give an allowed Raman transition from  $|0\rangle$  to  $|1\rangle$ .

For example,  $x^2$ ,  $y^2$ , and  $z^2$  for the H<sub>2</sub>O molecule have  $A_1$  symmetry, while  $xy$ ,  $xz$ , and  $yz$  have  $A_2$ ,  $B_1$ , and  $B_2$  symmetry, respectively. Thus the three normal modes of H<sub>2</sub>O,  $\nu_1(a_1)$ ,  $\nu_2(a_1)$ , and  $\nu_3(b_2)$ , are all Raman active (Figure 8.7).

Notice that if a molecule has a center of symmetry, then both  $\psi_0$  (for fundamentals) and  $\alpha_{ij}$  have  $g$  symmetry and consequently  $\psi_1$  must also be of  $g$  symmetry. Thus all Raman active fundamental transitions have  $g$  symmetry, if the molecule has a center of symmetry. Correspondingly, all infrared active fundamentals must have  $u$  symmetry since  $\boldsymbol{\mu}$  has  $u$  symmetry. This leads to the *rule of mutual exclusion*, which states that no fundamental mode of a molecule with a center of symmetry can be both infrared and Raman active. Comparison of infrared and Raman band positions can thus be a simple but powerful tool in deducing molecular geometry.

For the tetrahedral molecule CCl<sub>4</sub> all four vibrational modes ( $\nu_1(a_1)$  459 cm<sup>-1</sup>,  $\nu_2(e)$  218 cm<sup>-1</sup>,  $\nu_3(t_2)$  762 cm<sup>-1</sup>,  $\nu_4(t_2)$  314 cm<sup>-1</sup>) (see Figure 8.8) are Raman active.

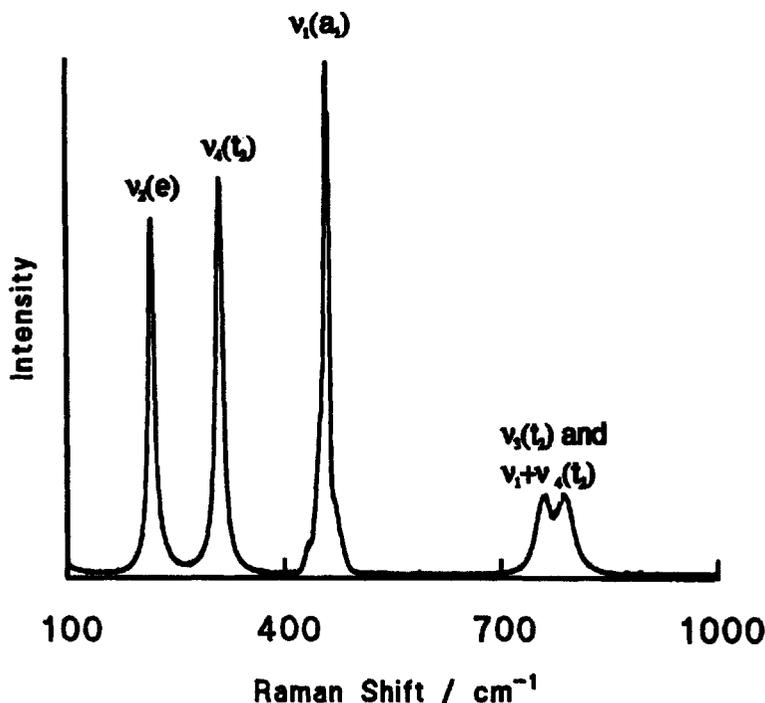


Figure 8.8: Vibrational Raman spectrum of liquid  $\text{CCl}_4$ .

This is in contrast to the infrared spectrum in which only  $\nu_3$  and  $\nu_4$  are observed. The partially resolved doublet near  $775\text{ cm}^{-1}$  in the Raman spectrum is actually two Fermi resonance transitions ( $762\text{ cm}^{-1}$ ,  $790\text{ cm}^{-1}$ ) made up of nearly equal mixtures of  $\nu_3(t_2)$  and  $\nu_1 + \nu_4(t_2)$ .

### Polarization

The typical Raman scattering geometry is shown in Figure 8.9. The intensity of light scattered parallel ( $I_{\parallel}$ ) and perpendicular ( $I_{\perp}$ ) to the incident electric field vector can easily be measured with polarizers. The ratio  $\rho = I_{\perp}/I_{\parallel}$ , called the *depolarization ratio*, is an important clue in the assignment of a vibrational Raman spectrum, because it depends on the symmetry of the vibrational mode.

From the theory of the Raman effect, it is known that a symmetric vibration has  $0 \leq \rho \leq \frac{3}{4}$  for linearly polarized incident light.<sup>5,6</sup> For a non-totally symmetric vibration,  $\rho = \frac{3}{4}$  for linearly polarized incident light, and the band is said to be depolarized. If unpolarized light is used—as was done, for example, using a mercury arc lamp in the prelaser era—then  $\rho = \frac{6}{7}$  for a non-totally symmetric vibration.<sup>5,6</sup> Thus a measurement of the depolarization ratio will often distinguish between totally symmetric and nonsymmetric vibrations. Totally symmetric vibrations, such as the C—Cl stretching mode ( $\nu_1(a_1)$   $459\text{ cm}^{-1}$ ) in  $\text{CCl}_4$ , tend to be strong scatterers with depolarization ratios close to zero (Figure 8.10), whereas this mode is forbidden in the infrared spectrum.

The physical origin of polarized scattering for a symmetric vibration is easy to understand in classical terms. For example, in the case of a symmetric vibration for a spherical top, the induced dipole is always parallel to the incident radiation and the

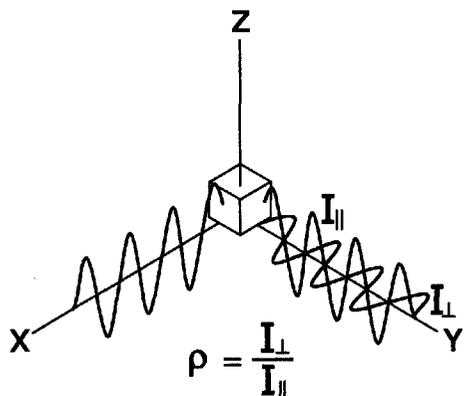


Figure 8.9: Parallel and perpendicular Raman scattering.

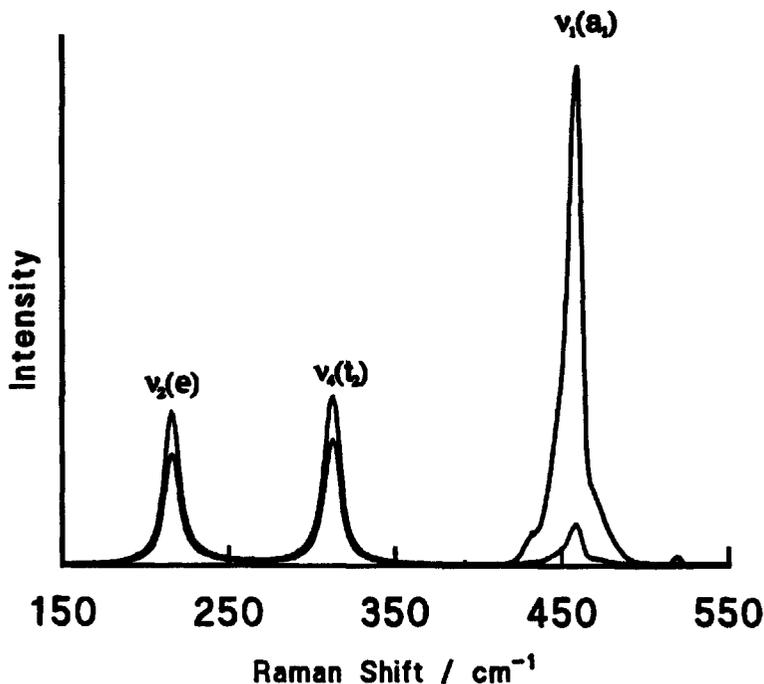


Figure 8.10: Vibrational Raman spectrum of liquid  $\text{CCl}_4$  showing the depolarization of the bands. The upper trace corresponds to  $I_{\parallel}$  and the lower trace to  $I_{\perp}$ .

molecule behaves like a tiny sphere (Figure 8.11): i.e., the polarizability ellipsoid is a sphere. The scattered light is also polarized parallel to the incident light polarization and  $\rho \sim 0$  (Figure 8.11). Molecules with  $O_h$ ,  $T_d$ , or  $I_h$  symmetry behave in this way for totally symmetric ( $a_1$ ) vibrations.

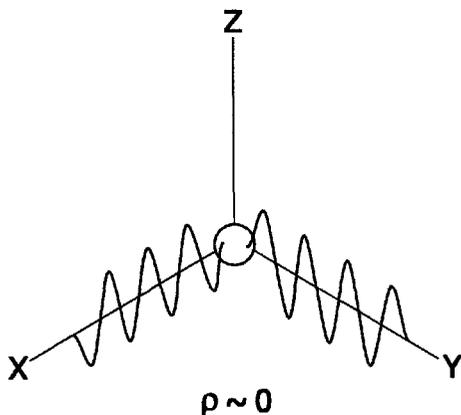


Figure 8.11: Polarized light scattering by a sphere.

## 8.2 Rotational Raman Effect

The dipole moment induced in a nonrotating molecule when an electric field is applied is given in the laboratory frame by

$$\boldsymbol{\mu} = \boldsymbol{\alpha}\mathbf{E}, \quad (8.56)$$

or, written explicitly in matrix format, by

$$\begin{pmatrix} \mu_X \\ \mu_Y \\ \mu_Z \end{pmatrix} = \begin{pmatrix} \alpha_{XX} & \alpha_{XY} & \alpha_{XZ} \\ \alpha_{XY} & \alpha_{YY} & \alpha_{YZ} \\ \alpha_{XZ} & \alpha_{YZ} & \alpha_{ZZ} \end{pmatrix} \begin{pmatrix} E_X \\ E_Y \\ E_Z \end{pmatrix}. \quad (8.57)$$

Since the polarizability tensor, like the moment of inertia tensor, is represented by a real symmetric matrix, it is always possible to find an orthogonal transformation which diagonalizes  $\boldsymbol{\alpha}$ . This new molecular coordinate system is obtained by a rotation of the molecular  $x$ -,  $y$ -, and  $z$ -axes such that the off-diagonal components of  $\boldsymbol{\alpha}$  are eliminated,

$$\boldsymbol{\alpha}' = \begin{pmatrix} \alpha_x & 0 & 0 \\ 0 & \alpha_y & 0 \\ 0 & 0 & \alpha_z \end{pmatrix}. \quad (8.58)$$

As far as light scattering is concerned, the molecule is represented by the polarizability ellipsoid. A spherical top molecule has a spherical polarizability ellipsoid and therefore behaves like a tiny sphere when an electric field is applied. The oscillating electromagnetic field is applied and the scattered light is detected in the laboratory frame of reference. The rotation of the molecule therefore modulates the scattered light for all molecules except spherical top molecules (Figure 8.12).

The rotational Raman effect is less restrictive than is microwave rotational spectroscopy because symmetric linear molecules without dipole moments such as  $\text{Cl}_2$  and  $\text{CO}_2$  have pure rotational Raman spectra. However, spherical tops such as  $\text{CH}_4$ ,  $\text{SF}_6$ , and  $\text{C}_{60}$  will not have observable rotational Raman spectra because an anisotropic polarizability tensor is required. In simple terms, an applied electric field can only exert a torque on a molecule if the molecule is more polarizable along one direction than another.

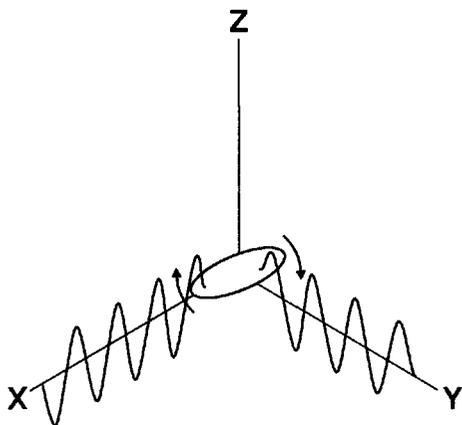


Figure 8.12: Light scattering by a rotating molecule is modulated by the rotational motion.

The rotational selection rules are obtained by evaluating the integrals

$$\begin{aligned}
 \int \psi_1^* \hat{\alpha}_{IJ} \psi_0 d\tau &= \int \psi_1^* \left( \sum_{i,j} \Phi_{Ii} \hat{\alpha}_{ij} \Phi_{jJ} \right) \psi_0 d\tau \\
 &= \sum_{i,j} \alpha_{ij} \int \psi_1^* \Phi_{Ii} \Phi_{jJ} \psi_0 d\tau \quad i, j = x, y, z; \quad I, J = X, Y, Z,
 \end{aligned} \tag{8.59}$$

in which the  $\Phi_{Ii}$  are the direction cosines, the  $\psi_i$  are rotational wavefunctions ( $\psi$  and  $\Phi$  are both functions of the Euler angles  $\theta, \phi, \chi$ , Figure 6.27) and  $\alpha_{ij}$  is the polarizability component in the molecular frame. The direction cosines are required (Chapter 6) in order to transform between the laboratory and molecular coordinate systems. Selection rules for rotational Raman spectroscopy are derived from matrix elements of the *products* of the direction cosine matrix elements. As a result,  $\Delta J = \pm 2$  transitions are possible. In simple terms, since there are two photons involved in a Raman transition, transitions with  $\Delta J = \pm 2$  are possible.

Compare the previous results with pure rotational microwave transitions in which

$$\begin{aligned}
 \int \psi_1^* \hat{\mu}_I \psi_0 d\tau &= \int \psi_1^* \left( \sum_i \Phi_{Ii} \hat{\mu}_i \right) \psi_0 d\tau \\
 &= \sum_i \mu_i \int \psi_1^* \Phi_{Ii} \psi_0 d\tau \quad I = X, Y, Z; \quad i = x, y, z.
 \end{aligned} \tag{8.60}$$

Again the integration is over the Euler angles, and  $\mu_i$  are the dipole moment components averaged over vibrational and electronic variables. In this case the matrix elements of the direction cosines result in the selection rule,  $\Delta J = \pm 1$ .

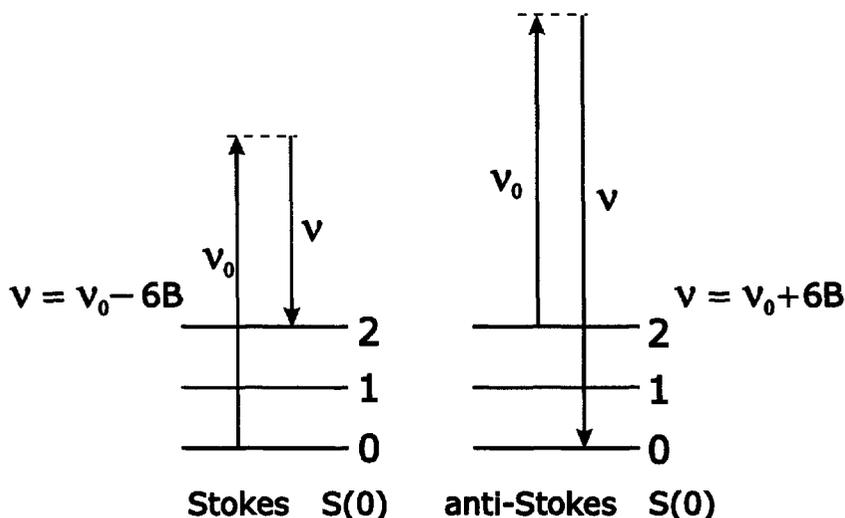


Figure 8.13: Stokes and anti-Stokes  $S(0)$  transitions for the rotational Raman effect.

### *Diatomic Molecules*

The selection rules for the rotational Raman effect in linear  ${}^1\Sigma^+$  molecules are  $\Delta J = 0, \pm 2$ . Only  $S$ -branch transitions ( $\Delta J = +2$ ) are observable since the  $\Delta J = 0$  transitions correspond to the unshifted Rayleigh line. The definition of the  $S$  branch as  $\Delta J = J' - J''$  means that both the Stokes and anti-Stokes transitions are  $S$ -branch lines (Figure 8.13), although this seems confusing at first sight. The definition of  $\Delta J$  is  $J_{\text{upper}} - J_{\text{lower}}$ , not  $J_{\text{final}} - J_{\text{initial}}$ , and as depicted in Figure 8.13,  $J = 2$  is always above  $J = 0$ . As shown in Figure 8.13,  $\Delta J = +2$  for both the Stokes and anti-Stokes  $S(0)$  lines. The situation is analogous to microwave transitions of a linear molecule for which only  $R$  branch ( $\Delta J = +1$ ) transitions occur in both emission  $J + 1 \rightarrow J$  or absorption  $J + 1 \leftarrow J$ , although the initial and final states are different.

The transition frequencies are given by

$$\begin{aligned}\tilde{\nu} &= \tilde{\nu}_0 \pm (B(J+2)(J+3) - BJ(J+1)) \\ &= \tilde{\nu}_0 \pm B(4J+6)\end{aligned}\quad (8.61)$$

where  $\pm$  corresponds to anti-Stokes and Stokes transitions, respectively. The lines are spaced by about  $4B$  from each other. Figure 8.14 shows the rotational Raman spectrum of  $\text{N}_2$ .

## 8.3 Vibration-Rotation Raman Spectroscopy

### *Diatomic Molecules*

The selection rules for vibration-rotation Raman spectroscopy for  ${}^1\Sigma^+$  diatomic molecules are  $\Delta v = \pm 1$  and  $\Delta J = 0, \pm 2$ . The vibrational transitions with  $\Delta v = \pm 2, \pm 3, \dots$  are allowed weakly for the anharmonic oscillator, similar to infrared vibration-rotation spectroscopy.

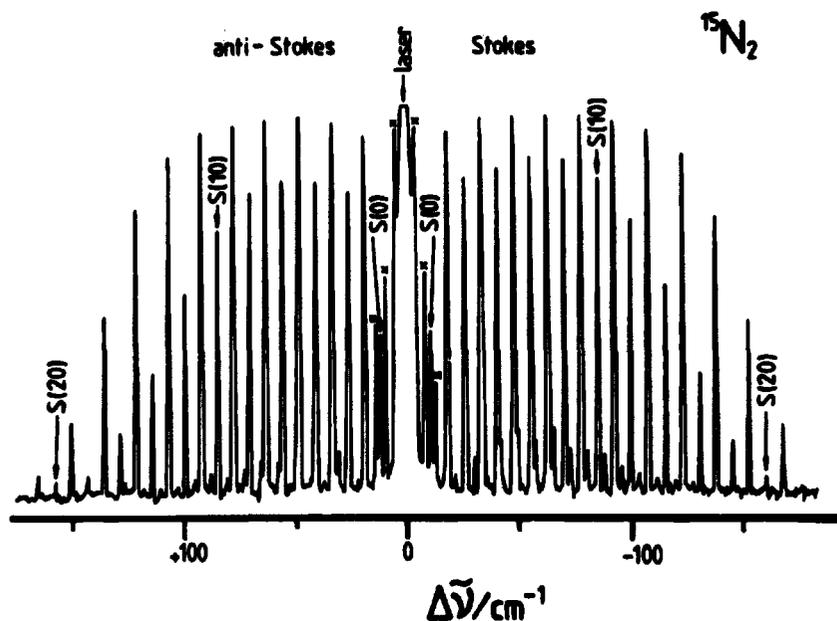


Figure 8.14: Rotational Raman spectrum of  $N_2$ . Note the intensity alternation due to nuclear spin statistics and the x's that mark instrumental artifacts called grating ghosts.

The rotational selection rules  $\Delta J = -2, 0, 2$  result in  $O$ ,  $Q$ , and  $S$  branches, respectively, as shown in Figure 8.15. The vibration-rotation Raman spectrum of  $N_2$  is shown in Figure 8.16.

The equations for the three branches are

$$\tilde{\nu}_S = \tilde{\nu}_0 - (6B' + (5B' - B'')J'' + (B' - B'')(J'')^2) \quad J'' = 0, 1, 2, \dots \quad (8.62)$$

$$\tilde{\nu}_Q = \tilde{\nu}_0 - ((B' - B'')J'' + (B' - B'')(J'')^2) \quad J'' = 0, 1, 2, \dots \quad (8.63)$$

and

$$\tilde{\nu}_O = \tilde{\nu}_0 - (2B' - (3B' + B'')J'' + (B' - B'')(J'')^2) \quad J'' = 2, 3, \dots \quad (8.64)$$

in which  $\tilde{\nu}_0 = \tilde{\nu}_1 - \Delta G_{1/2}$  for the  $1 \leftarrow 0$  Stokes spectrum. Notice that at high resolution (Figure 8.16), the  $Q$ -branch lines can be resolved at high  $J$  because of the  $\Delta B J^2$  term in equation (8.63).

## 8.4 Rayleigh and Raman Intensities

### *Classical Theory*

As discussed in Chapter 1, an oscillating classical dipole moment

$$\boldsymbol{\mu} = \boldsymbol{\mu}_0 \cos \omega t \quad (8.65)$$

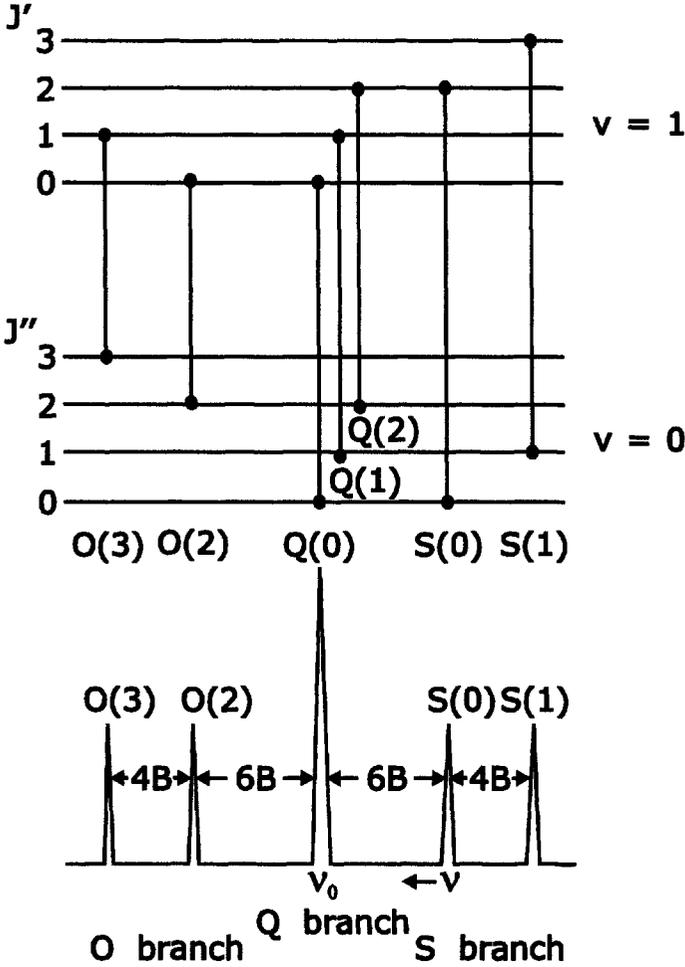


Figure 8.15: Energy-level diagram and spectrum for vibrational Raman scattering of a linear molecule.

radiates with total average power  $P$  (watts) given, from electromagnetic theory,<sup>7</sup> by

$$\begin{aligned}
 P &= \frac{4\pi^3 \nu^4 |\mu_0|^2}{3\epsilon_0 c^3} \\
 &= \frac{4\pi^3 c}{3\epsilon_0 \lambda^4} |\mu_0|^2.
 \end{aligned}
 \tag{8.66}$$

In scattering, the incident electric field polarizes the molecule and induces a dipole moment,  $\mu_{ind} = \alpha E$ , equation (8.1). The electric field oscillation is given by

$$\mathbf{E} = \mathbf{E}_0 \cos \omega t
 \tag{8.67}$$

so  $\mu_{ind}$  oscillates at the same angular frequency  $\omega$  and radiates with a total average power

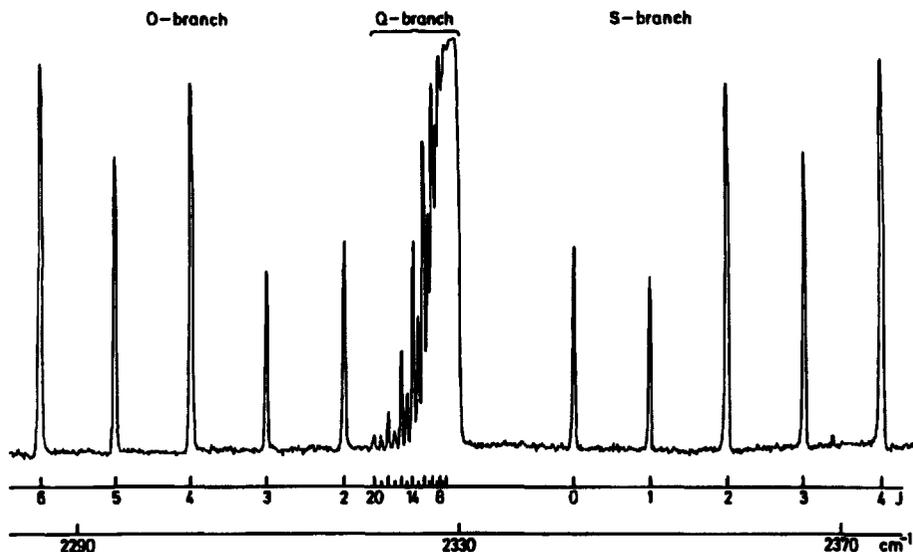


Figure 8.16: Vibration-rotation Raman spectrum of  $N_2$ . Note the intensity alternation due to nuclear spin statistics.

$$P = \frac{4\pi^3 c (\alpha E_0)^2}{3\epsilon_0 \lambda^4}. \quad (8.68)$$

In terms of the incident intensity  $I = \epsilon_0 E_0^2 c / 2$  (equation (1.43)) the electromagnetic wave leads to

$$P = \frac{8\pi^3 \alpha^2 I}{3\epsilon_0^2 \lambda^4} \quad (8.69)$$

for the scattered power per molecule. A scattering cross section  $\sigma_{\text{scat}}$  can be defined as

$$\sigma_{\text{scat}} = \frac{P}{I} = \frac{8\pi^3 \alpha^2}{3\epsilon_0^2 \lambda^4} \quad (8.70)$$

and can be evaluated if a value for the mean polarizability  $\alpha = \bar{\alpha}$  is available.

This scattering causes the extinction of a beam of light of intensity  $I_0$  falling on a sample through an equation similar to Beer's law, as depicted in Figure 1.12,

$$I = I_0 e^{-\sigma_{\text{scat}} N l} = I_0 e^{-\alpha_{\text{scat}} l} \quad (8.71)$$

with the cross section  $\sigma_{\text{scat}}$  due to scattering out of the beam rather than absorption. In general, when a beam of light of intensity  $I_0$  is transmitted through a sample, the light can be absorbed (with  $\sigma = \sigma_{\text{abs}}$ ) as discussed in Chapter 1 or scattered (with  $\sigma_{\text{scat}}$  including both Rayleigh and Raman effects) as discussed here, so that the total extinction of the beam ( $\sigma_{\text{abs}} + \sigma_{\text{scat}}$ ) is given as

$$I = I_0 e^{-(\sigma_{\text{abs}} + \sigma_{\text{scat}}) N l} \quad (8.72)$$

The mean polarizability  $\bar{\alpha}$  can be computed for use in equation (8.70) by *ab initio* methods (e.g., Figure 8.4 for  $H_2$ ) or obtained from refractive index data (Problem 7). In this example, the Rayleigh scattering of light by air leads to the attenuation coefficient

$$\alpha_{\text{scat}} = \frac{32\pi^3(n-1)^2\delta}{3\lambda^4 N}, \quad (8.73)$$

with  $N$  the air density in molecules/m<sup>3</sup>,  $n$  the refractive index of air (Chapter 1, Problem 2) and  $\delta = 1.05$ , a small correction for the anisotropy of the N<sub>2</sub> and O<sub>2</sub> molecules. Rayleigh scattering of sunlight leads to red sunsets and blue skies because of the  $\lambda^{-4}$  dependence of the attenuation coefficient  $\alpha_{\text{scat}}$  of air.

Because Raman and Rayleigh scattering are so weak, a typical laboratory experiment does not involve measurement of the small intensity change in an incident laser beam as given by equation (8.71). Rather, a typical Raman experimental geometry is as shown in Figure 8.9, with the incident electric field  $\mathbf{E}_0$  polarized parallel to the  $Z$ -axis and traveling in the  $X$  direction. The total scattered radiation,  $I'$ , is detected at 90° traveling in the  $Y$  direction, and polarized in the  $X$  and  $Z$  directions. A polarizer could be used to measure  $I'_{\parallel}$  and  $I'_{\perp}$  separately, but generally just the total intensity  $I'$  is detected. The observed intensity of the scattered light,  $I'$ , in this typical 90° scattering geometry is given from electromagnetic theory<sup>7</sup> as

$$I' = \frac{\pi^2\nu^4}{2\epsilon_0 c^3} |\boldsymbol{\mu}_0|^2, \quad (8.74)$$

per molecule. The prime is used on the scattered intensity  $I'$  because the units are watts per steradian (not watts/m<sup>2</sup>). The incident radiation of intensity  $I_0$  is an irradiance (Chapter 1) and has the usual units of watts/m<sup>2</sup>. The oscillating electric field  $\mathbf{E}_0$  is along the  $Z$ -axis so the induced moments needed for scattering are

$$\mu_Z = \alpha_{ZZ} E_{0Z} \quad (8.75)$$

$$\mu_X = \alpha_{XZ} E_{0Z}. \quad (8.76)$$

If there are  $N_0$  molecules per m<sup>3</sup> in the initial state  $|0\rangle$  in the scattering volume  $V_S$ , and the incident intensity is  $I_0 = \epsilon_0 c |E_0|^2 / 2$ , then the scattered intensities become

$$I'_{\parallel} = \frac{\pi^2}{\epsilon_0^2 \lambda^4} \alpha_{ZZ}^2 N_0 V_S I_0, \quad (8.77)$$

$$I'_{\perp} = \frac{\pi^2}{\epsilon_0^2 \lambda^4} \alpha_{XZ}^2 N_0 V_S I_0, \quad (8.78)$$

and

$$I' = I'_{\parallel} + I'_{\perp} = \frac{\pi^2}{\epsilon_0^2 \lambda^4} (\alpha_{XZ}^2 + \alpha_{ZZ}^2) N_0 V_S I_0. \quad (8.79)$$

The molecules in the sample, however, generally have random orientations except in the case of a single crystal, so that the polarizability elements must be averaged over the different molecular orientations.<sup>5,6</sup> The transformation between the laboratory and molecular coordinate systems is given by direction cosines  $\Phi_{J_i}$  (Chapter 6), and as shown in equation (8.59). The resulting average is given in terms of two quantities: the mean polarizability  $\bar{\alpha}$  defined as

$$\bar{\alpha} \equiv (\alpha_x + \alpha_y + \alpha_z) / 3 \quad (8.80)$$

and the anisotropy  $\gamma$ , defined as

$$\gamma^2 \equiv ((\alpha_x - \alpha_y)^2 + (\alpha_y - \alpha_z)^2 + (\alpha_z - \alpha_x)^2) / 2 \quad (8.81)$$

in the principal axis system. For the cases of linear and symmetric top molecules for which  $\alpha_x = \alpha_y$ ,  $\gamma$  reduces to  $\alpha_{\parallel} - \alpha_{\perp}$  (equation (8.15)). In particular, the required orientational averages are

$$\langle \alpha_{ZZ}^2 \rangle = \frac{45\bar{\alpha}^2 + 4\gamma^2}{45}, \quad (8.82)$$

$$\langle \alpha_{XZ}^2 \rangle = \frac{\gamma^2}{15}, \quad (8.83)$$

and

$$\langle \alpha_{ZZ}^2 \rangle + \langle \alpha_{XZ}^2 \rangle = \frac{45\bar{\alpha}^2 + 7\gamma^2}{45}. \quad (8.84)$$

The final intensity expressions for the scattered light are thus given in terms of  $\bar{\alpha}$  and  $\gamma$  as

$$I'_{\parallel} = \frac{\pi^2}{\epsilon_0^2 \lambda^4} \left( \frac{45\bar{\alpha}^2 + 4\gamma^2}{45} \right) N_0 V_S I_0, \quad (8.85)$$

$$I'_{\perp} = \frac{\pi^2}{\epsilon_0^2 \lambda^4} \left( \frac{\gamma^2}{15} \right) N_0 V_S I_0, \quad (8.86)$$

and

$$I' = \frac{\pi^2}{\epsilon_0^2 \lambda^4} \left( \frac{45\bar{\alpha}^2 + 7\gamma^2}{45} \right) N_0 V_S I_0. \quad (8.87)$$

The depolarization ratio  $\rho = I_{\perp}/I_{\parallel}$  has the simple expression

$$\rho = \frac{3\gamma^2}{45\bar{\alpha}^2 + 4\gamma^2}. \quad (8.88)$$

As expected, for spherical tops  $\gamma = 0$  so  $\rho = 0$  and the scattered radiation is linearly polarized as if the molecule was spherical. Expressions (8.81) to (8.88) assume linearly polarized incident radiation as from a laser, but if unpolarized natural light is used, a slightly different set of expressions is obtained.<sup>6</sup>

As always some care with units is needed. The units of  $\bar{\alpha}^2$  and  $\gamma^2$  can be deduced from the basic equation (8.1) and are  $\text{C}^2 \text{m}^4 \text{V}^{-2}$  or  $\text{C}^4 \text{m}^4 \text{J}^{-2}$ . The units to be used for the incident intensity  $I_0$  are  $\text{W m}^{-2}$  and the scattered intensity  $I'$  is in  $\text{W sr}^{-1}$ , with  $N_0$  in molecules per  $\text{m}^3$ .

One of the main problems with scattering (Raman and Rayleigh) intensities,  $I'$ , is that equations such as (8.87) depend upon the particular experimental conditions, such as the  $90^\circ$  viewing geometry, the state of the incident polarization, the size of the scattering volume, and so forth. To remove at least some of the experimental parameters, a quantity called the differential scattering cross section ( $d\sigma/d\Omega$ ) is defined by some authors as

$$\frac{d\sigma}{d\Omega} = \frac{I'}{N_0 V_S I_0} = \frac{\pi^2}{\epsilon_0^2 \lambda^4} \left( \frac{45\bar{\alpha}^2 + 7\gamma^2}{45} \right), \quad (8.89)$$

with units of  $\text{m}^2$  per steradian. The concept of a differential scattering (or Raman) cross section is not particularly useful, and it is just as easy to use the full equations such as (8.87).

### Vibrational Intensity Calculations

The general scattering intensity equations, (8.85) to (8.89), can be applied to the specific case of the vibrational Raman effect. Raman intensities of vibrational bands can be estimated in the "double harmonic" approximation analogous to infrared vibrational band intensities (Chapter 7). In the Raman case, the harmonic oscillator model is assumed for each vibrational mode as in the infrared, but it is the polarizability expansion (rather than the dipole expansion), that is truncated after the linear term: i.e.,

$$\alpha_{ij}(r) = \alpha_{e,ij} + \left. \frac{\partial \alpha_{ij}}{\partial r} \right|_{r_e} (r - r_e) \quad (8.90)$$

for a diatomic molecule. For the Raman transition from  $v$  to  $v'$ , the matrix element  $\langle v' | \alpha_{ij}(r) | v \rangle$  is needed with

$$\langle v' | \alpha_{ij}(r) | v \rangle = \alpha_{e,ij} \langle v' | v \rangle + \left. \frac{\partial \alpha_{ij}}{\partial r} \right|_{r_e} \langle v' | r - r_e | v \rangle. \quad (8.91)$$

The first term on the right-hand side of equation (8.91) is zero because  $\psi_v$  and  $\psi_{v'}$  are orthogonal within a single electronic state, and the second term leads to the usual harmonic oscillator selection rules  $\Delta v = \pm 1$  (Chapter 7). Using equation (7.56) with  $x = r - r_e$  gives

$$\langle v + 1 | r - r_e | v \rangle = \left( \frac{\hbar}{2\mu_{AB}\omega} \right)^{1/2} \sqrt{v + 1} \quad (8.92)$$

for the  $v + 1 \leftarrow v$  transition of the diatomic A—B with reduced mass  $\mu_{AB}$ .

The polarizability tensor elements  $\alpha_{ij}$  for the fundamental vibrational band ( $v = 1 \leftarrow 0$ ) are

$$\alpha_{ij} = \left( \frac{\hbar}{2\mu_{AB}\omega} \right)^{1/2} \left. \frac{\partial \alpha_{ij}}{\partial r} \right|_{r_e} = \left( \frac{h}{8\pi^2 \mu_{AB} \nu_{10}} \right)^{1/2} \left. \frac{\partial \alpha_{ij}}{\partial r} \right|_{r_e}. \quad (8.93)$$

Equation (8.93) can be combined with equation (8.87) to yield the intensity expression

$$I' = \frac{h}{8\varepsilon_0^2 \lambda^4 \mu_{AB} \nu_{10}} \left( \frac{45(\bar{\alpha}')^2 + 7(\gamma')^2}{45} \right) N_0 V_S I_0 \quad (8.94)$$

with

$$\bar{\alpha}' = \left( \frac{\partial \alpha_x}{\partial r} + \frac{\partial \alpha_y}{\partial r} + \frac{\partial \alpha_z}{\partial r} \right) / 3 \quad (8.95)$$

and

$$(\gamma')^2 = \left( \left( \frac{\partial \alpha_x}{\partial r} - \frac{\partial \alpha_y}{\partial r} \right)^2 + \left( \frac{\partial \alpha_y}{\partial r} - \frac{\partial \alpha_z}{\partial r} \right)^2 + \left( \frac{\partial \alpha_z}{\partial r} - \frac{\partial \alpha_x}{\partial r} \right)^2 \right) / 2. \quad (8.96)$$

In equation (8.94), the primes on  $\bar{\alpha}$  and  $\gamma$  denote derivatives, while on  $I$  the prime indicates that the units are watts/sr rather than watts/m<sup>2</sup>.

The lower state population density  $N_0$  can be replaced by the total population density  $N$  using the usual relationship for a harmonic oscillator from statistical thermodynamics,

$$N_0 = \frac{N}{q} = \frac{N}{1 - e^{-h\nu_{10}/kT}}, \quad (8.97)$$

in which  $q$  is the partition function for a simple harmonic oscillator. The final intensity equation for Stokes vibrational Raman scattering for the fundamental band of a diatomic thus becomes

$$I' = \frac{h(\nu - \nu_{10})^4 NV_S I_0}{8\varepsilon_0^2 c^4 \mu_{AB} \nu_{10} (1 - e^{-h\nu_{10}/kT})} \left( \frac{45(\bar{\alpha}')^2 + 7(\gamma')^2}{45} \right). \quad (8.98)$$

For polyatomic molecules the polarizability is expanded in terms of the normal mode  $Q_k$  with

$$\alpha_{ij}(Q_k) = \alpha_{e,ij} + \left. \frac{\partial \alpha_{ij}}{\partial Q_k} \right|_0 Q_k. \quad (8.99)$$

The expression for the scattered intensity for mode  $Q_k$  is then

$$I'_k = \frac{h}{8\varepsilon_0^2 \lambda^4 \nu_k} \left( \frac{45(\bar{\alpha}'_k)^2 + 7(\gamma'_k)^2}{45} \right) N_0 V_S I_0, \quad (8.100)$$

with the definitions

$$\bar{\alpha}'_k \equiv \left( \frac{\partial \alpha_x}{\partial Q_k} + \frac{\partial \alpha_y}{\partial Q_k} + \frac{\partial \alpha_z}{\partial Q_k} \right) / 3 \quad (8.101)$$

and

$$(\gamma'_k)^2 \equiv \left( \left( \frac{\partial \alpha_x}{\partial Q_k} - \frac{\partial \alpha_y}{\partial Q_k} \right)^2 + \left( \frac{\partial \alpha_y}{\partial Q_k} - \frac{\partial \alpha_z}{\partial Q_k} \right)^2 + \left( \frac{\partial \alpha_z}{\partial Q_k} - \frac{\partial \alpha_x}{\partial Q_k} \right)^2 \right) / 2 \quad (8.102)$$

For a polyatomic molecule, the equation corresponding to (8.98) for Stokes Raman scattering by a fundamental mode  $Q_k$  is given similarly by

$$I'_k = \frac{h(\nu - \nu_k)^4 NV_S I_0}{8\varepsilon_0^2 c^4 \nu_k q_v} \left( \frac{45(\bar{\alpha}'_k)^2 + 7(\gamma'_k)^2}{45} \right), \quad (8.103)$$

with  $q_v$  the total vibrational partition function. The formula (8.88) for the depolarization ratio also applies, but with the polarizability derivatives,  $\bar{\alpha}'$  and  $\gamma'$ , replacing the polarizabilities,  $\bar{\alpha}$  and  $\gamma$ .

The units of polarizability,  $\alpha$ , are not always easy to understand because the SI units of C m<sup>2</sup> V<sup>-1</sup> obtained from the basic equation (8.1) are often not encountered. If equation (8.1) is used with cgs units, then surprisingly the dimensions of  $\alpha$  are cm<sup>3</sup> so  $\alpha$  values are traditionally reported in Å<sup>3</sup> (1 Å<sup>3</sup> = 10<sup>-24</sup> cm<sup>3</sup>). One can imagine that the polarizability ellipsoid has this "pseudo volume" in these non-SI units. The conversion from polarizabilities in Å<sup>3</sup> to C m<sup>2</sup> V<sup>-1</sup> involves multiplication by the factor of  $4\pi\varepsilon_0 \times 10^{-6}$ , i.e.,

$$\alpha/(\text{C m}^2 \text{ V}^{-1}) = 1.112\,650\,1 \times 10^{-16} \alpha/(\text{cm}^3) \quad (8.104)$$

$$= 1.112\,650\,1 \times 10^{-40} \alpha/(\text{\AA}^3) \quad (8.105)$$

*Ab initio* computer programs use atomic units internally, and the atomic units for polarizability can be deduced from equation (8.51). The atomic units for  $\alpha$  are  $a_0^2 e^2 / E_h$ , with  $a_0$  the Bohr radius and  $E_h$  the hartree (1 hartree =  $2 R_\infty = 219\,474.631\,2 \text{ cm}^{-1}$ ). The numerical conversion factor from atomic units is given as

$$\alpha/(\text{C m}^2 \text{ V}^{-1}) = 1.648\,777\,2 \times 10^{-41} \alpha/(a_0^2 e^2 E_h^{-1}). \quad (8.106)$$

## 8.5 Conclusions

There has been a renaissance in Raman spectroscopy with the availability of lasers, Fourier transform spectrometers, and sensitive array detectors. Although Rayleigh scattering is weak and Raman scattering even weaker (typically  $10^{-6}$  of the incident radiation), Raman spectroscopy has a number of important attributes.

Raman spectroscopy has different selection rules than do direct electronic, vibrational, and rotational spectroscopies, so it provides complementary information, especially for centrosymmetric molecules. Raman spectroscopy uses visible light to obtain electronic, vibrational, and rotational information about molecules. Since the technology for generating, manipulating, and detecting visible light is often more advanced than the corresponding infrared and millimeter wave technology, this can provide an important experimental advantage. The water molecule is a relatively weak Raman scatterer but a strong infrared absorber. Because of this fact, Raman spectroscopy is often the technique of choice for the vibrational spectroscopy of molecules in aqueous environments. For example, the vibrational spectroscopy of biological samples (which are altered by dehydration) is usually best carried out by Raman scattering.

### Problems

1. Which normal modes of ethylene are Raman active? (See Problem 1 of Chapter 7.)
2. For the molecules in Problem 2 of Chapter 7, which modes are infrared active and which are Raman active?
3. Discuss the Raman activity of the normal modes of the molecules in Problems 4, 5, 12, 14, 15, 16, and 17 of Chapter 7.
4. For the ICl molecule the following spectroscopic constants are listed in Huber and Herzberg's book:

$$\begin{aligned} \omega_e &= 384.293 \text{ cm}^{-1} \\ \omega_e x_e &= 1.501 \text{ cm}^{-1} \\ B_e &= 0.114\,158\,7 \text{ cm}^{-1} \\ \alpha_e &= 0.000\,535\,4 \text{ cm}^{-1}. \end{aligned}$$

- (a) Predict the pure rotational Raman spectrum. What will be the Raman shift of the two lines closest to the exciting laser line?
- (b) Predict the pattern of the Stokes vibration-rotation Raman spectrum for the fundamental band. What will be the Raman shifts of the  $S(0)$  and  $O(2)$  lines from the exciting laser line at  $5145 \text{ \AA}$ ?
5. Fill in the following table with a yes (Y) or a no (N) to indicate allowed spectroscopic transitions. Answer yes if one or more modes or transitions are allowed, and no if all modes or transitions are forbidden.

Molecule	Rotational	Vibrational	Rotational Raman	Vibrational Raman
H <sub>2</sub> O				
SF <sub>6</sub>				
CS <sub>2</sub>				
N <sub>2</sub> O				
Allene				
Benzene				
Cl <sub>2</sub>				

6. The vibrational Raman spectrum of the  $\text{SO}_3^{2-}$  anion of  $C_{3v}$  symmetry exhibits four bands in aqueous solution:  $966 \text{ cm}^{-1}$  (strong, p);  $933 \text{ cm}^{-1}$  (shoulder, dp);  $620 \text{ cm}^{-1}$  (weak, p); and  $473 \text{ cm}^{-1}$  (dp) (p = polarized; dp = depolarized). Assign the symmetries of the bands and describe the motion of the normal modes.
7. The attenuation of sunlight by Rayleigh scattering is described by equation (8.71).
- (a) Derive equation (8.73) using the Lorentz-Lorenz relationship between the mean polarizability  $\bar{\alpha}$  and the refractive index  $n$ :

$$\bar{\alpha}N = 3\epsilon_0 \frac{n^2 - 1}{n^2 + 2}.$$

$N$  is the molecular density in molecules/ $\text{m}^3$ .

- (b) At  $500 \text{ nm}$ , what is the amount of direct sunlight ( $1 - I/I_0$ ) removed by Rayleigh scattering as measured by a person on the earth's surface? Use the refractive index for air (at 1 atm) given in Question 2 of Chapter 1. Assume that the atmospheric pressure  $p$  (and the density  $N$ , the quantity  $n - 1$ , and consequently  $\sigma_{\text{scat}}$ ) obey the barometric law,  $p/p_0 = e^{-z/H}$ , with  $z$  the height above the ground and the atmospheric scale height,  $H$ , taken as  $8 \text{ km}$ . Do the calculation for a solar zenith angle of  $0^\circ$ , i.e., the sun is directly overhead. Take the temperature as  $15^\circ\text{C}$  and ignore its variation with height.
8. The mean polarizability  $\bar{\alpha}$  of  $\text{N}_2$  gas has been found to be  $1.778 \text{ \AA}^3$  by measurement of the refractive index (Problem 7) at  $5145 \text{ \AA}$ . The polarizability anisotropy  $\gamma$  has been determined to be  $0.714 \text{ \AA}^3$  by measurement of the depolarization of scattered light from an argon ion laser operating at  $5145 \text{ \AA}$ .

- (a) What was the measured depolarization ratio?
- (b) The argon ion laser beam has a power of 1 W and is focused to a  $10\ \mu\text{m}$  spot (i.e., approximately a cube with  $10\text{-}\mu\text{m}$  sides) in  $\text{N}_2$  at 1 atm pressure. A 2-cm diameter lens with a focal length of 10 cm collects the photons scattered at  $90^\circ$ . What is the scattered power detected, assuming no optical or detector losses? How many photons/s are detected?
9. Pecul and Coriani (*Chem. Phys. Lett.* **355**, 377 (2002)) have calculated the derivatives of the mean polarizability  $\bar{\alpha}' = \partial\alpha/\partial r$  and  $\gamma' = \partial\gamma/\partial r$  at  $r_e$  for  $\text{N}_2$ . At  $5145\ \text{\AA}$ , they obtained  $\bar{\alpha}' = 6.61$  and  $\gamma' = 7.80$  in atomic units. (*Hint:* Atomic units for  $\bar{\alpha}'$  and  $\gamma'$  are  $a_0 e^2 E_h^{-1}$ , while atomic units for  $\bar{\alpha}$  and  $\gamma$  are  $a_0^2 e^2 E_h^{-1}$ .)
- (a) What is the depolarization ratio for the fundamental Stokes Raman vibrational band?
- (b) For the experimental conditions of Problem 8, compute the scattered power for the Stokes Raman fundamental band at  $2330\ \text{cm}^{-1}$ .

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## Chapter 9

# Electronic Spectroscopy of Diatomics

### 9.1 Orbitals and States

Within the Born–Oppenheimer approximation, the electronic Schrödinger equation for a diatomic molecule A–B is

$$\hat{H}_{\text{el}}\psi_{\text{el}} = E_{\text{el}}\psi_{\text{el}}, \quad (9.1)$$

with the Hamiltonian operator given by

$$\hat{H}_{\text{el}} = \frac{-\hbar^2}{2m_e} \sum_i \nabla_i^2 - \sum_i \frac{Z_A e^2}{4\pi\epsilon_0 r_{Ai}} - \sum_i \frac{Z_B e^2}{4\pi\epsilon_0 r_{Bi}} + \frac{Z_A Z_B e^2}{4\pi\epsilon_0 r_{AB}} + \sum_i \sum_{j>i} \frac{e^2}{4\pi\epsilon_0 r_{ij}}. \quad (9.2)$$

The approximate solution of equation (9.1) is accomplished by assuming that  $\psi_{\text{el}}$  is made up of molecular orbitals (MOs) and that each MO is a linear combination of atomic orbitals (LCAOs). More precisely, an approximate solution is written in the form of a determinant (Chapter 5)

$$\psi_{\text{el}} = |\phi_1(1)\bar{\phi}_1(2)\phi_2(3)\cdots| \quad (9.3)$$

with each  $\phi_i$  being a molecular orbital of the form

$$\phi_i = \sum_j (C_{iA_j} \phi_j^A + C_{iB_j} \phi_j^B), \quad (9.4)$$

in which  $\phi_j^A$  and  $\phi_j^B$  are atomic orbitals localized on atoms A and B, respectively.

Information about the electronic structure of diatomic molecules can therefore be derived from consideration of the shapes of the molecular orbitals constructed as linear combinations of atomic orbitals. The atomic orbitals of the constituent atoms are in an environment with reduced symmetry ( $D_{\infty h}$  or  $C_{\infty v}$ , rather than  $K_h$ ) in the diatomic molecule. As the symmetry is reduced from spherical to axial, each electron with orbital angular momentum  $l$  will begin to precess about the internuclear axis (Figure 9.1). This sort of “intramolecular Stark effect” means that although  $l$  is no longer a good quantum number, the projection of  $\hat{L}$  onto the internuclear axis,  $m_l$ , remains useful. The sign of

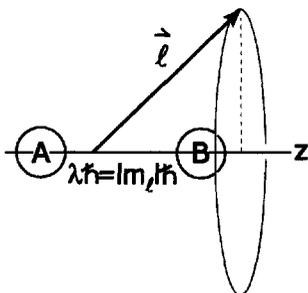
Figure 9.1: The precession of  $\hat{\mathbf{l}}$  about the internuclear axis.

Table 9.1: Correlation of Atomic and Molecular Orbitals

Atomic Orbital $l$	Molecular Orbital $\lambda$
$s$	$\sigma$
$p$	$p\sigma, p\pi$
$d$	$d\sigma, d\pi, d\delta$
$f$	$f\sigma, f\pi, f\delta, f\phi$

$m_l$  is determined by whether the circulation of the electron around the internuclear axis is left- or right-handed, and the electronic wavefunction is an eigenfunction of the  $\hat{l}_z$  operator, that is,

$$\hat{l}_z \psi = -i\hbar \frac{\partial}{\partial \phi} A e^{im_l \phi} = m_l \hbar \psi = \pm |m_l| \hbar \psi. \quad (9.5)$$

The direction of the circulation of an electron around the internuclear axis cannot affect the energy, so there is a double degeneracy for  $\lambda \equiv |m_l| > 0$ . It is useful to label the atomic orbitals in a diatomic molecule by

$$\lambda = |m_l| = 0, 1, 2, \dots, l. \quad (9.6)$$

The irreducible representations of the point group  $K_h$  are  $s, p, d, f, g, h, i, k, l, m, n, o$ , etc.; similarly  $\sigma, \pi, \delta, \phi, \gamma, \eta, \iota, \kappa, \lambda, \mu, \nu, \omega$ , etc., are the irreducible representations of  $C_{\infty v}$  (Table 9.1). The labels  $g$  and  $u$  are appended as subscripts for  $D_{\infty h}$  molecules.

The atomic orbitals are combined to give the molecular orbitals, as shown in Figure 9.2 for a homonuclear diatomic molecule made of atoms of the second row of the periodic table. For  $O_2$  the electrons are added to the molecular orbitals to give the configuration

$$(\sigma_g 1s)^2 (\sigma_u^* 1s)^2 (\sigma_g 2s)^2 (\sigma_u^* 2s)^2 (\sigma_g 2p)^2 (\pi_u 2p)^4 (\pi_g^* 2p)^2$$

or

$$1\sigma_g^2 1\sigma_u^2 2\sigma_g^2 2\sigma_u^2 3\sigma_g^2 1\pi_u^4 1\pi_g^2,$$

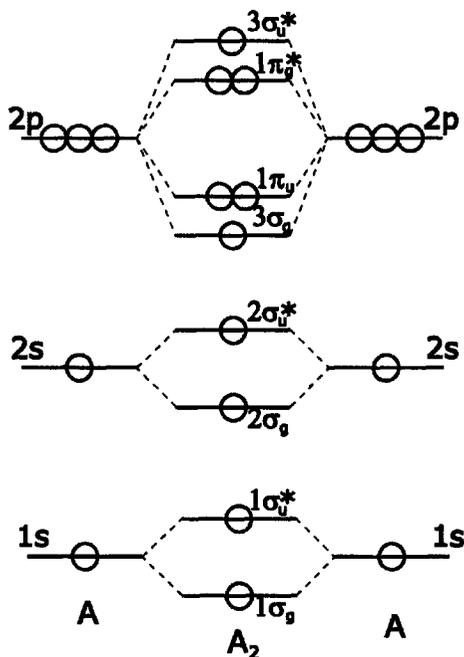


Figure 9.2: Molecular orbital diagram for second-row diatomic molecules. Note that for Li<sub>2</sub> to N<sub>2</sub> the order of the  $1\pi_u$  and  $3\sigma_g$  ( $\pi_u(2p)$  and  $\sigma_g(2p)$ ) orbitals are switched.

depending on the labeling scheme adopted for the molecular orbitals. The net bond order ((number of bonding electrons – number of anti-bonding electrons)/2) is 2 for O<sub>2</sub>. From a  $(\pi_g)^2$  configuration the electronic states that result are given by the direct product

$$\Pi_g \otimes \Pi_g = \Sigma_g^+ \oplus [\Sigma_g^-] \oplus \Delta_g \quad (9.7)$$

with square brackets around the antisymmetric part of the product. Since the two  $\pi_g^*$  electrons are identical in O<sub>2</sub>, care must be taken not to violate the Pauli exclusion principle. On exchange of the two identical electrons (fermions), the total wavefunction  $\psi_{\text{orbital}}\psi_{\text{spin}}$  must be antisymmetric. This means that the symmetric part ( $\Sigma_g^+$  and  $\Delta_g$ ) of the  $\Pi_g \otimes \Pi_g$  product combines with the antisymmetric electron spin part  $((\alpha(1)\beta(2) - \alpha(2)\beta(1))/\sqrt{2})$ , while the antisymmetric orbital part ( $\Sigma_g^-$ ) combines with the symmetric electron spin part  $(\alpha(1)\alpha(2), (\alpha(1)\beta(2) + \alpha(2)\beta(1))/\sqrt{2}, \beta(1)\beta(2))$ . The  $(\pi_g^*)^2$  configuration of oxygen therefore gives rise to the  $^1\Sigma_g^+$ ,  $^3\Sigma_g^-$ , and  $^1\Delta_g$  electronic states. Since Hund's rules apply to molecules as well as to atoms,  $^3\Sigma_g^-$  is expected to lie lowest in energy. If the two electrons are in different orbitals  $(\pi)^1(\pi')^1$ , then twice as many states are possible, namely those associated with the terms  $^1,^3\Sigma^+$ ,  $^1,^3\Sigma^-$ , and  $^1,^3\Delta$ . Note that in the direct product tables in Appendix C, the antisymmetric part of each product is enclosed in square brackets.

The notation for the electronic states of diatomic molecules parallels that for atoms, with the symbol  $^{2S+1}\Lambda_\Omega$  used in place of  $^{2S+1}L_J$ , and with  $\Lambda = \sum \lambda_i$ . Capital Greek letters are used for the multi-electron molecular labels, while lowercase Greek letters are used for one-electron orbital labels. For example, the electronic configuration of the O<sub>2</sub> ground state has two unpaired electrons with  $\lambda_1 = \pm 1$  and  $\lambda_2 = \pm 1$ . The possible

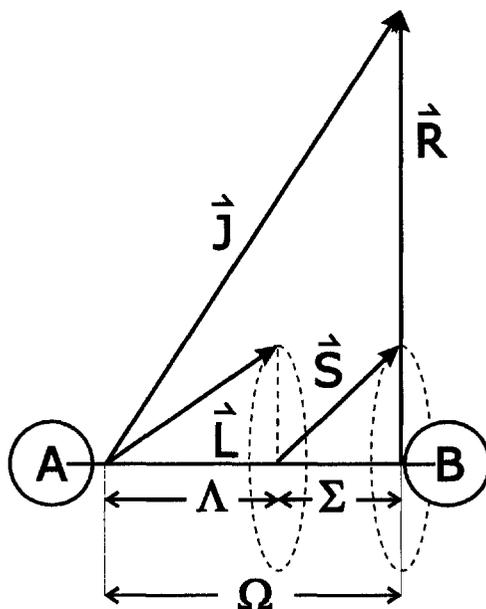


Figure 9.3: Angular momenta in a diatomic molecule.

Table 9.2: Angular Momenta in Diatomic Molecules

Angular Momentum	Projection on Molecular Axis (units of $\hbar$ )
$\hat{\mathbf{J}}$	$\Omega = (\Lambda + \Sigma)$
$\hat{\mathbf{L}}$	$\Lambda$
$\hat{\mathbf{S}}$	$\Sigma$
$\hat{\mathbf{R}}$	—
$\hat{\mathbf{N}} = \hat{\mathbf{R}} + \hat{\mathbf{L}}$	$\Lambda$

values of the projection of total orbital angular momentum  $\Lambda = \lambda_1 + \lambda_2$  are  $\pm 2, 0, 0$ , which translates into  $\Delta, \Sigma^+$ , and  $\Sigma^-$  electronic states. Notice that since the  $\lambda_i$  are projections of the angular momenta of electrons along the internuclear axis, they add as scalars rather than as vectors.

For a diatomic molecule the total angular momentum (exclusive of nuclear spin) is the vector sum of orbital ( $\hat{\mathbf{L}}$ ), spin ( $\hat{\mathbf{S}}$ ), and nuclear rotation ( $\hat{\mathbf{R}}$ ) angular momenta,  $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}} + \hat{\mathbf{R}}$  (Figure 9.3). The total angular momentum  $\hat{\mathbf{J}}$  has a projection of  $\Omega\hbar$  units of angular momentum along the molecular axis and (as always)  $M_J\hbar$  along the space fixed  $Z$ -axis (Figure 9.4). The notation for various angular momenta and their projections on the intermolecular axis are summarized in Table 9.2. Notice that the name given to the projection of  $\hat{\mathbf{S}}$  along the internuclear axis is  $\Sigma$ . (This is completely unrelated to the fact that  $\Lambda = 0$  is also called a  $\Sigma$  state!) The  $\Omega$  quantum number is sometimes appended as a subscript to label a particular spin component.

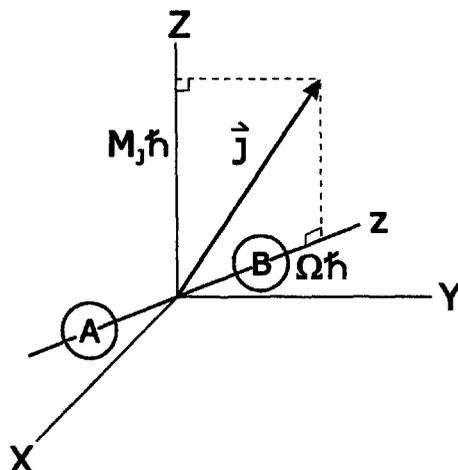


Figure 9.4: Components of  $\hat{\mathbf{J}}$  in the laboratory ( $X, Y, Z$ ) and the molecular ( $x, y, z$ ) frames.

For  $\Lambda > 0$  there is a double orbital degeneracy which we can think of as corresponding to the circulation of the electrons in a clockwise or counterclockwise direction. This degeneracy remains for  $\Omega > 0$ , and it is customary to use  $|\Omega|$  to represent both values. For example, a  ${}^3\Sigma_g^-$  state has  ${}^3\Sigma_{1g}^-$  and  ${}^3\Sigma_{0g}^-$  spin components ( $\Lambda = 0, \Sigma = \pm 1, 0$ ), while a  ${}^2\Pi$  state has  ${}^2\Pi_{3/2}$  and  ${}^2\Pi_{1/2}$  ( $\Lambda = 1, \Sigma = \pm 1/2$ ) spin components. Notice that there are always  $2S + 1$  spin components labeled by their  $|\Omega|$  values except when  $S > |\Lambda| > 0$ . In that case there is a notational problem in labeling the  $2S + 1$  spin components, so  $\Omega = |\Lambda| + \Sigma$  is used instead of  $|\Lambda + \Sigma|$ . For example, for a  ${}^4\Pi$  state ( $S = 3/2, \Lambda = 1$ ) the spin components are labeled as  ${}^4\Pi_{5/2}, {}^4\Pi_{3/2}, {}^4\Pi_{1/2}$ , and  ${}^4\Pi_{-1/2}$ .

The electronic states of diatomic molecules are also labeled with letters:  $X$  is reserved for the ground state, while  $A, B, C$ , and so on, are used for excited states of the same multiplicity ( $2S + 1$ ) as the ground state, in order of increasing energy. States with a multiplicity different from that of the ground state are labeled with lowercase letters  $a, b, c$ , and so on, in order of increasing energy. This convention is illustrated by the energy-level diagram of the low-lying electronic states of  $\text{O}_2$  in Figure 9.5.

The possible electronic transitions among the energy levels are determined by the selection rules:

1.  $\Delta\Lambda = 0, \pm 1$ . The transitions  $\Sigma - \Sigma, \Pi - \Sigma, \Pi - \Pi, \Delta - \Pi$ , and so forth, are allowed.
2.  $\Delta S = 0$ . Transitions that change multiplicity are very weak for molecules formed from light atoms, but as spin-orbit coupling increases in heavy atoms, transitions with  $\Delta S \neq 0$  become more strongly allowed.
3.  $\Delta\Sigma = 0$  (for Hund's case (a), see below).
4.  $\Delta\Omega = 0, \pm 1$ .
5.  $\Sigma^+ - \Sigma^+, \Sigma^- - \Sigma^-$ , but not  $\Sigma^+ - \Sigma^-$ . This selection rule is a consequence of the  $\mu_z$  transition dipole moment having  $\Sigma^+$  symmetry. Notice that  $\Sigma^+ - \Pi$  and  $\Sigma^- - \Pi$  transitions are both allowed.

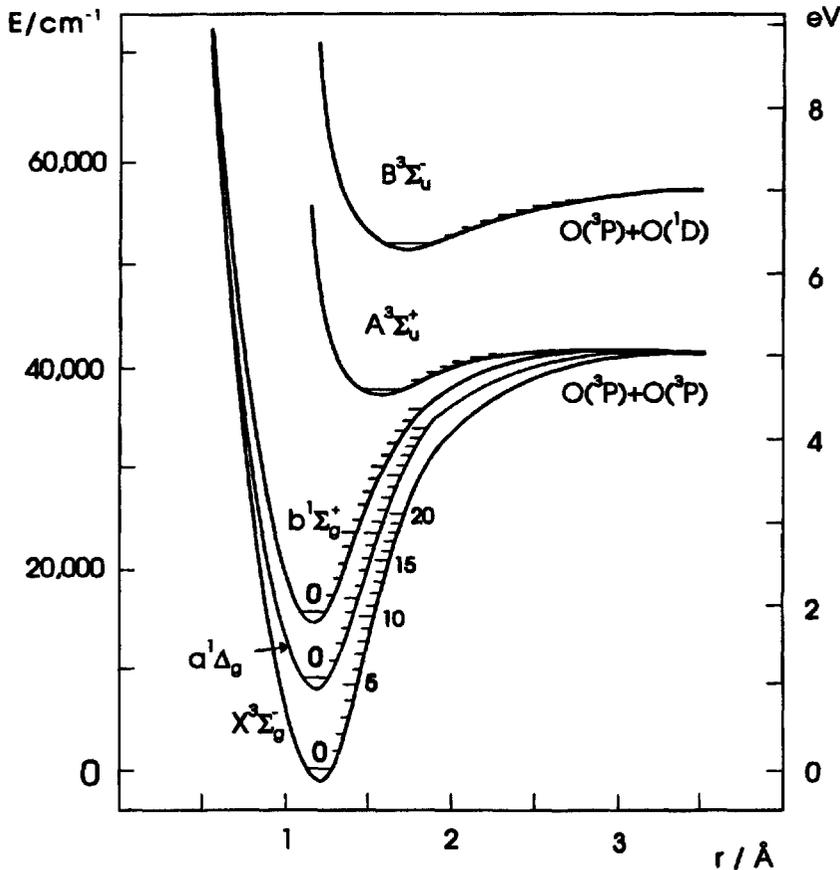


Figure 9.5: The low-lying electronic states of the  $O_2$  molecule.

6.  $g \leftrightarrow u$ . The transitions  ${}^1\Pi_g - {}^1\Pi_u$ ,  ${}^1\Sigma_u^+ - {}^1\Sigma_g^+$ , and so forth, are allowed for centrosymmetric molecules.

For example, transitions among the first three electronic states of  $O_2$  ( $b^1\Sigma_g^+$ ,  $a^1\Delta_g$ , and  $X^3\Sigma_g^-$ , Figure 9.5) are forbidden, but the  $B^3\Sigma_u^- - X^3\Sigma_g^-$  transition is allowed. The  $B - X$  transition of  $O_2$ , which is known as the Schumann-Runge system, is responsible for the absorption of UV light for wavelengths  $\lambda < 200$  nm in the earth's atmosphere. The vacuum UV region is so named because of this absorption of radiation by  $O_2$  in air. Spectroscopic measurements in the vacuum UV region must therefore be carried out under vacuum conditions.

## 9.2 Vibrational Structure

An electronic transition is made up of vibrational bands, each of which in turn is made up of rotational lines. The presence of many vibrational bands, labeled as  $v' - v''$ , explains why electronic transitions are often called band systems. The terms band system, band, and line date from the early days of spectroscopy and refer to the appearance of electronic transitions of gaseous molecules recorded photographically on glass plates



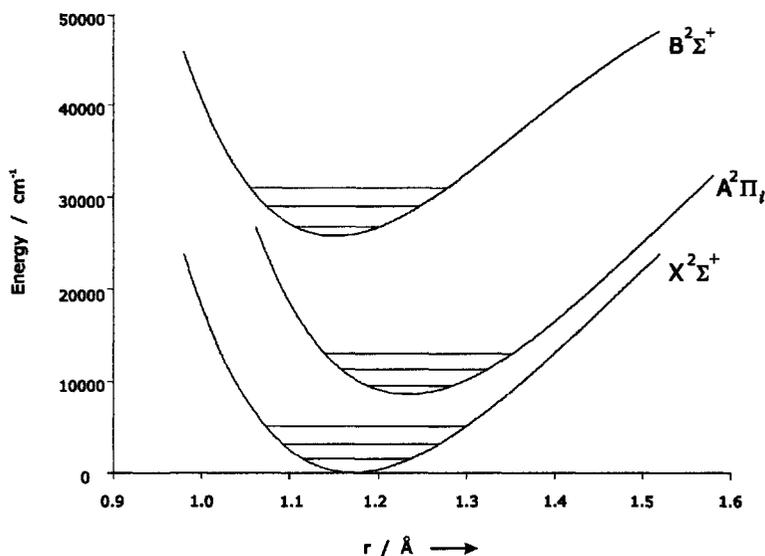


Figure 9.7: Potential energy curves and the lowest few vibrational levels for the  $X^2\Sigma^+$ ,  $A^2\Pi_i$ , and  $B^2\Sigma^+$  states of CN.

$$\tilde{\nu}_{v',v''} = \Delta T_e + \omega'_e(v' + \frac{1}{2}) - \omega'_e x'_e(v' + \frac{1}{2})^2 + \dots - (\omega''_e(v'' + \frac{1}{2}) - \omega''_e x''_e(v'' + \frac{1}{2})^2 + \dots), \quad (9.8)$$

in which  $\Delta T_e = E_{\text{upper}} - E_{\text{lower}}$  is the energy separation (in  $\text{cm}^{-1}$ ) between the potential minima of the two electronic states.

The intensities of the various vibrational bands are determined by three factors: the intrinsic strength of the electronic transition, the populations of the vibrational levels, and the squared overlap integral of the two vibrational wavefunctions, called the *Franck-Condon factor*. Franck-Condon factors result from the application of a more general rule called the *Franck-Condon principle*. This principle has both classical and quantum mechanical versions.

The classical version of the Franck-Condon principle is based on the idea that electronic transitions occur very quickly, in less than  $10^{-15}$  s. In such a short time the nuclei do not have time to move, so vibration, rotation, and translation are “frozen” during an electronic transition. On a potential-energy diagram, therefore, electronic transitions occur vertically at the initial  $r$  value. The kinetic energy (but not the potential energy) is the same immediately before and after an electronic transition.

The presence of vibrational levels can be added to the classical picture (Figure 9.8) by quantizing the energy levels (Figure 9.7). Furthermore, vibrating diatomic molecules (except for  $v = 0$ ) spend more time at the classical inner and outer turning points of the vibrational motion than in the middle, so that transitions can be approximated as occurring near the turning points. For example, Figure 9.9 predicts the 0–18 band of the  $B - X$  transition of  $\text{Br}_2$  to be strong in emission, while Figure 9.7 predicts the 2–0 band of the  $A^2\Pi - X^2\Sigma^+$  transition of CN to be strong in absorption.

The quantum mechanical version of the Franck-Condon principle is based on the fact that the intensity of a given transition is proportional to the square of the transition moment integral

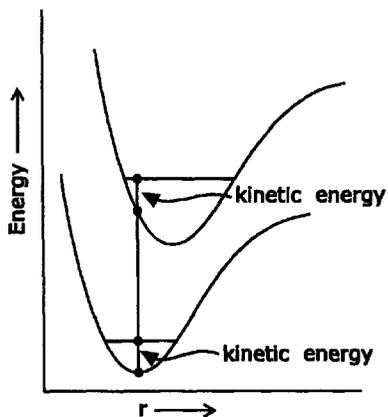


Figure 9.8: Electronic transitions occur vertically on an energy-level diagram and approximately conserve the radial kinetic energy.

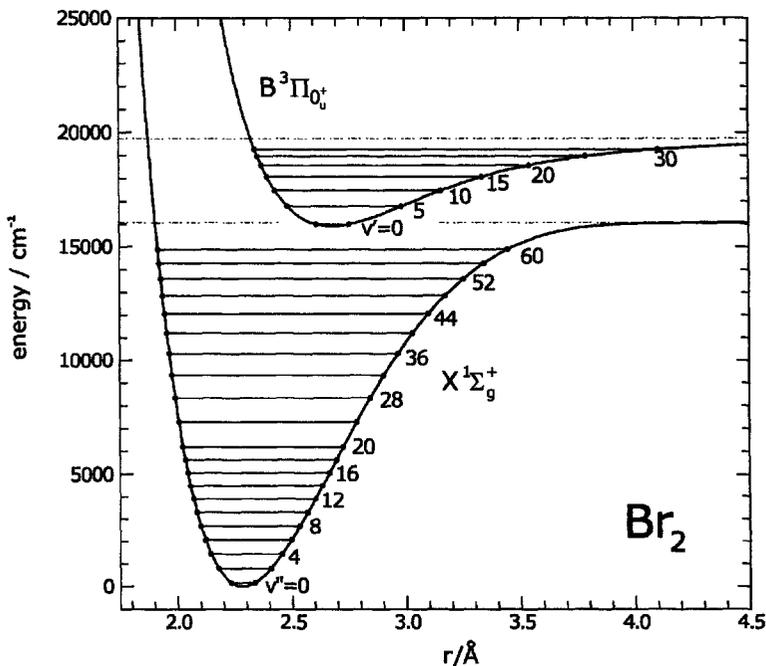


Figure 9.9: The  $\text{Br}_2$   $B^3\Pi_{0+}$  and  $X^1\Sigma_g^+$  potential energy curves.

$$M_{ev} = \int \psi_{e'v'}^* \mu \psi_{e''v''} d\tau, \quad (9.9)$$

and the Born–Oppenheimer approximation that separates electronic and nuclear motion,  $\psi_{ev} = \psi_e \psi_v$ . The rotational motion of the diatomic molecule is ignored here, but its inclusion does not change the derivation. The electronic wavefunctions are a parametric function of the internuclear distance  $r$  so the transition dipole moment  $M_{ev}$

becomes

$$\mathbf{M}_{ev} = \int \psi_{v'}^* \left( \int \psi_{e'}^*(\mathbf{r}_i; r) \boldsymbol{\mu} \psi_{e''}(\mathbf{r}_i; r) d\tau_{el} \right) \psi_{v''} dr \quad (9.10)$$

with  $\mathbf{r}_i$  representing the electronic coordinates. The electronic transition dipole moment  $\mathbf{R}_e(r)$  is defined as

$$\begin{aligned} \mathbf{R}_e(r) &= \int \psi_{e'}^*(\mathbf{r}_i; r) \boldsymbol{\mu} \psi_{e''}(\mathbf{r}_i; r) d\tau_{el} \\ &= \langle \psi_{e'}^* | \boldsymbol{\mu} | \psi_{e''} \rangle. \end{aligned} \quad (9.11)$$

The electronic transition dipole moment integral  $\mathbf{R}_e(r)$  can be expanded in a Taylor series about a convenient value of  $r$ , say  $\bar{r}$ . Using  $\bar{r}$ , the electronic transition dipole moment  $\mathbf{R}_e(r)$  becomes

$$\mathbf{R}_e(r) = \mathbf{R}_e(\bar{r}) + \left. \frac{\partial \mathbf{R}_e(r)}{\partial r} \right|_{\bar{r}} (r - \bar{r}) + \frac{1}{2} \left. \frac{\partial^2 \mathbf{R}_e}{\partial r^2} \right|_{\bar{r}} (r - \bar{r})^2 + \dots \quad (9.12)$$

Substitution of equation (9.12) into equation (9.10), and retaining only the first term on the right-hand side of equation (9.12) leads to the transition dipole moment  $\mathbf{M}_{ev}$  expression

$$\mathbf{M}_{ev} = \mathbf{R}_e(\bar{r}) \langle v' | v'' \rangle. \quad (9.13)$$

In other words,  $\mathbf{M}_{ev}$  has been factored (9.13) into the product of an electronic and a vibrational part, with the electronic transition dipole moment  $\mathbf{R}_e$  given by

$$\mathbf{R}_e = \mathbf{R}_e(\bar{r}) = \int \psi_{e'}^*(\mathbf{r}_i; \bar{r}) \boldsymbol{\mu} \psi_{e''}(\mathbf{r}_i; \bar{r}) d\tau_{el}, \quad (9.14)$$

and the vibrational overlap integral  $\langle v' | v'' \rangle$  given as

$$\langle v' | v'' \rangle = \int \psi_{v'}^* \psi_{v''} dr. \quad (9.15)$$

The expansion center  $\bar{r}$  could be chosen as  $r_e''$  or as  $(r_e' + r_e'')/2$ , but the best approximation is to let  $\bar{r}$  be the  $r$ -centroid value for a particular  $v' - v''$  vibrational band. The  $r$ -centroid,  $\bar{r}_{v'v''}$ , is defined as

$$\bar{r}_{v'v''} \equiv \frac{\langle \psi_{v'} | r | \psi_{v''} \rangle}{\langle \psi_{v'} | \psi_{v''} \rangle}. \quad (9.16)$$

In this case, the Franck-Condon separation of electronic and vibrational motion, equation (9.13), becomes

$$\mathbf{M}_{ev} = \mathbf{R}_e(\bar{r}_{v'v''}) \langle v' | v'' \rangle. \quad (9.17)$$

The  $r$ -centroid approximation, equation (9.17), is mainly of pedagogic interest and in practice either equation (9.13) or the full  $r$ -dependent transition moment  $\mathbf{R}_e(r)$  is used. In fact, if the transition dipole moment function  $\mathbf{R}_e(r)$  is available from either *ab initio* calculation or experiment, then it should be used explicitly with numerical vibrational wavefunctions as discussed at the end of this chapter.

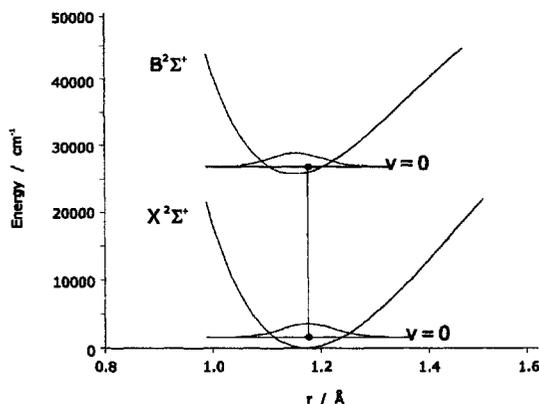


Figure 9.10: For the  $B^2\Sigma^+ - X^2\Sigma^+$  transition of CN, the 0-0 band will be strong because of favorable vibrational overlap.

The intensity of a vibronic transition is proportional to the square of the transition moment integral, namely

$$I_{e'v'e''v''} \propto |\mathbf{R}_e|^2 q_{v'-v''}, \quad (9.18)$$

in which  $q_{v'-v''} = |\langle v' | v'' \rangle|^2$  is called the *Franck-Condon factor*. Note that although the vibrational wavefunctions are all orthogonal within one electronic state, they are not orthogonal between two different electronic states. The electronic transition dipole moment  $|\mathbf{R}_e|$  has a magnitude of about 1 debye for an allowed transition, while  $q$  ranges between 0 and 1 depending on the extent of overlap. The value of  $|\mathbf{R}_e|^2$  measures the intrinsic strength of an electronic transition, while the Franck-Condon factor determines how the intensity is distributed among the vibrational bands.

As discussed in more detail in Chapter 7, potential curves  $V(r)$  can be obtained for each electronic state using the RKR procedure and the  $B_v$  and  $G_v$  constants.<sup>3</sup> The one-dimensional vibrational Schrödinger equation (7.13), can be solved numerically to obtain the vibrational wavefunctions, which are then used to integrate equation (9.15).<sup>4</sup> Franck-Condon factors can thus easily be calculated with freely available computer programs.<sup>3,4</sup>

The intensity of the vibrational bands of an electronic transition is determined by the population of the vibrational levels, the intrinsic strength of a transition ( $\mathbf{R}_e$ ), and the Franck-Condon factors. In the case of a common initial vibrational level, the relative intensity of two bands is given by a ratio of Franck-Condon factors. The magnitudes of the vibrational overlap integrals can be estimated from a picture of vibrational wavefunctions (e.g., Figure 7.5). As shown in Figure 9.10, when the two electronic states have similar  $r_e$  and  $\omega_e$  values, then the  $\Delta v = 0$  sequence of diagonal bands 0-0, 1-1, 2-2 are strong because of the optimal overlap of the  $\Delta v = 0$  vibrational wavefunctions of the upper and lower electronic states. In contrast, when the equilibrium internuclear separation  $r_e$  of the two electronic states is significantly different, the off-diagonal bands ( $\Delta v \neq 0$ ) have particularly good vibrational overlaps (Figure 9.11).

The vibrational band heads or origins of an electronic band system can be conveniently organized into a *Deslandres table*. A Deslandres table is a two-dimensional array of vibrational band energies, constructed as shown in Table 9.3. The terms diagonal and off-diagonal bands refer implicitly to a Deslandres table. The differences between

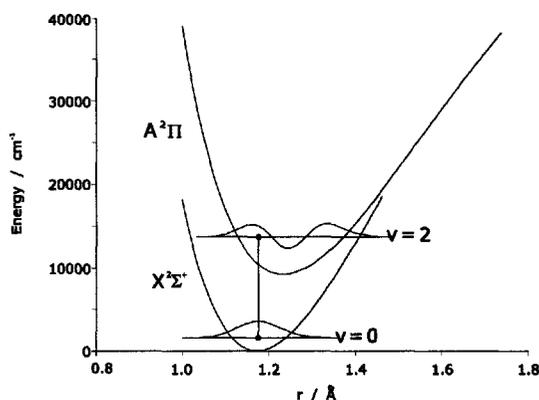


Figure 9.11: The 2-0 band of the  $A^2\Pi - X^2\Sigma^+$  transition of CN has a large Franck-Condon factor.

Table 9.3: Deslandres Table of the Band Heads of the  $N_2 C^3\Pi_u - B^3\Pi_g$  Second Positive System<sup>2</sup>

$v' / v''$	0	1	2	3	4				
0	29 652.8 1 990.7	1 703.6 1 937.1	27 949.2 1 992.2	1 674.7 1 945.0	26 274.5 1 992.4	1 647.2 1 939.1	24 627.3 1 992.8	1 611.4 1 939.2	23 015.9 1 987.1
1	31 643.5 1 939.8	1 702.1 1 869.8	29 941.4 1 872.6	1 674.5 1 864.0	28 266.9 1 864.0	1 646.8 1 870.7	26 620.1 1 870.7	1 617.1 1 870.7	25 003.0 1 876.3
2	33 583.4 1 869.8	1 704.8 1 752.1	31 878.5 1 771.3	1 666.6 1 771.3	30 211.9 1 775.8	1 652.7 1 775.8	28 559.2 1 777.3	1 617.0 1 777.3	26 942.2 1 771.5
3	35 453.1 1 752.1	1 702.0 1 682.8	33 751.1 1 682.8	1 675.2 1 670.7	32 075.9 1 670.7	1 646.0 1 644.5	30 429.9 1 644.5	1 611.4 1 617.2	28 818.5 30 590.0
4	37 205.2	1 682.8	35 522.4	1 670.7	33 815.7	1 644.5	32 207.2	1 617.2	30 590.0

elements in adjacent rows and columns give the apparent vibrational intervals of the ground and excited electronic states.

## 9.3 Rotational Structure of Diatomic Molecules

### *Singlet-Singlet Transitions*

The rotational structure of singlet-singlet electronic transitions is identical to that of the vibrational transitions of a linear molecule, as discussed in Chapter 7. In Chapter 7 it was shown for a linear polyatomic molecule that the projection of the total angular momentum along the internuclear axis originates from the vibrational angular momentum  $\hat{I}$ . For singlet electronic states the angular momentum projection along the  $z$ -axis originates from the orbital motion of the electrons ( $\hat{L}$ ). As shown in Figure 9.3, the projection of  $\hat{L}$  along the internuclear axis is denoted  $\Lambda$  and is analogous to the vibrational angular momentum quantum number  $l$ . As discussed in Chapter 7, three types of transitions are possible:

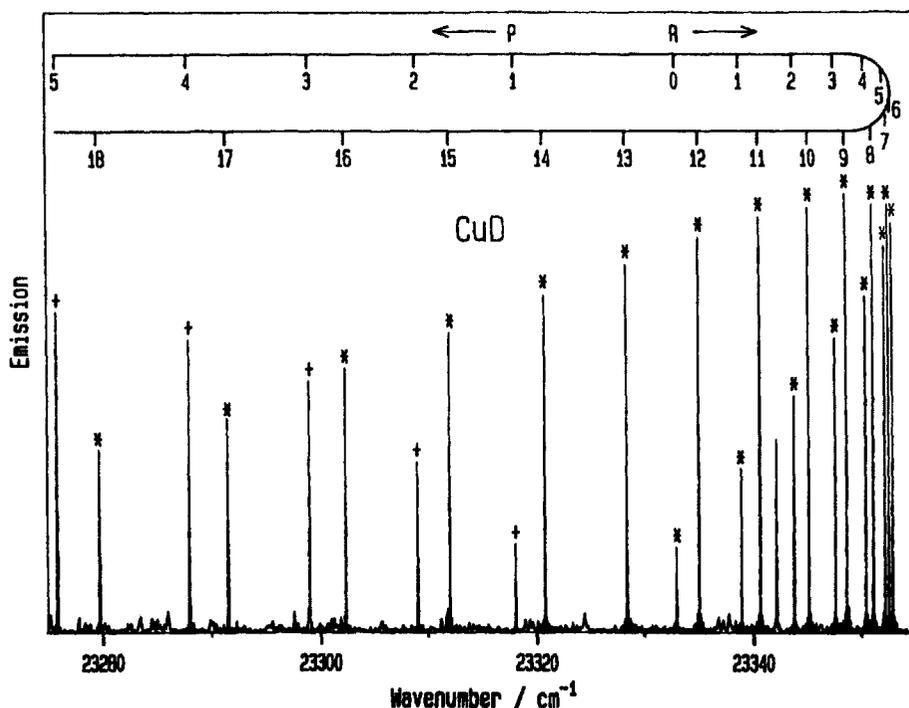


Figure 9.12: The 0-0 band of the CuD  $A^1\Sigma^+ - X^1\Sigma^+$  system.<sup>5</sup> The peaks marked with + are P branch transitions, while those marked with \* belong to the R branch.

1.  $\Delta\Lambda = 0$ ,  $\Lambda'' = \Lambda' = 0$ . As shown in Figure 9.12,  $^1\Sigma^+ - ^1\Sigma^+$  (or  $^1\Sigma^- - ^1\Sigma^-$ ) transitions have only P and R branches ( $\Delta J = \pm 1$ ).  $^1\Sigma^- - ^1\Sigma$  transitions are parallel transitions, defined as those for which the transition dipole moment lies along the z-axis.
2.  $\Delta\Lambda = \pm 1$ . As illustrated by Figure 9.13,  $^1\Pi - ^1\Sigma^+$ ,  $^1\Pi - ^1\Sigma^-$ ,  $^1\Delta - ^1\Pi$ , and so on, transitions have strong Q branches ( $\Delta J = 0$ ) as well as P and R branches, with  $\Delta J = \pm 1$ . These transitions have a transition dipole moment perpendicular to the molecular axis, and hence are designated as perpendicular transitions.
3.  $\Delta\Lambda = 0$ ,  $\Lambda' = \Lambda'' \neq 0$ . Transitions such as  $^1\Pi - ^1\Pi$ ,  $^1\Delta - ^1\Delta$ , and so on, are characterized by weak Q branches and strong P and R branches ( $\Delta J = 0, \pm 1$ ).

The total power  $P_{J',J''}$  (in watts/m<sup>3</sup>) emitted by an excited rovibronic level  $|nv'J'\rangle$  making a transition to the lower level  $|n''v''J''\rangle$  is given by the expression<sup>7</sup>

$$P_{J',J''} = \frac{16\pi^3}{3\epsilon_0 c^3} \frac{N_{J'}}{(2J'+1)} \nu_{J',J''}^4 q_{v'-v''} |\mathbf{R}_e|^2 S_{J',J''}^{\Delta J}, \quad (9.19)$$

in which  $N_{J'}$  is the excited-state population in molecules/m<sup>3</sup>,  $\nu_{J',J''}$  is the transition frequency in Hz,  $q_{v'-v''}$  is the Franck-Condon factor,  $\mathbf{R}_e$  is the electronic transition dipole moment in coulomb meters, and  $S_{J',J''}^{\Delta J}$  is a rotational line strength term called a Hönl-London factor (Table 9.4). This equation (9.19) is obtained from the expression for the Einstein A factor (equation (1.53)) by multiplication by  $h\nu$  to convert from

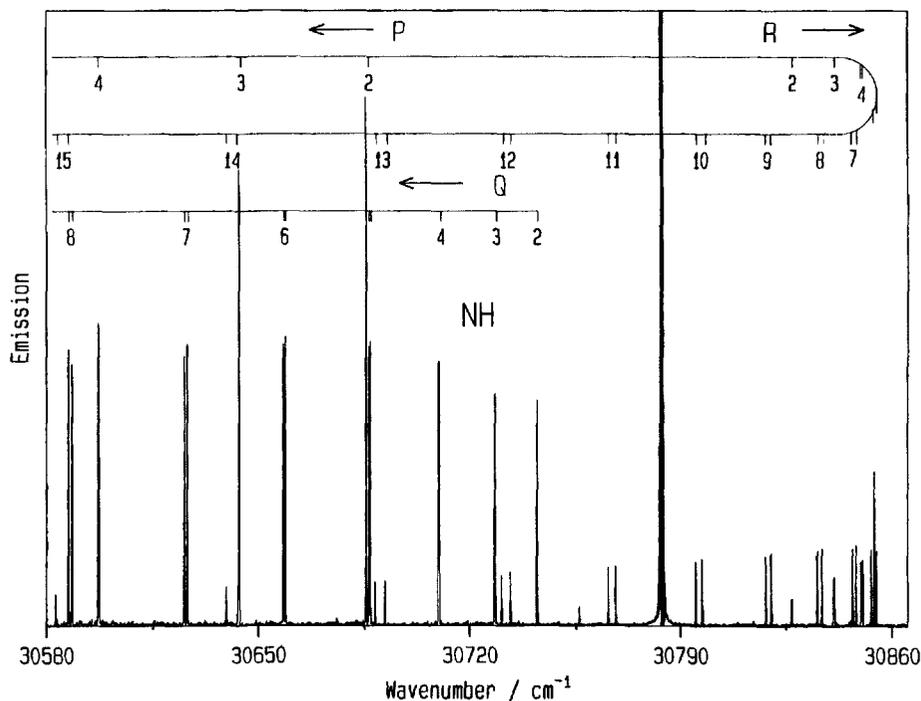


Figure 9.13: The 0-0 band of the NH  $c^1\Pi - a^1\Delta$  system.<sup>6</sup> The doubling of the lines at high  $J$  in the  $R$  and  $Q$  branches is due to  $\Lambda$ -doubling in the  $c^1\Pi$  state. The very intense line near  $30790\text{ cm}^{-1}$  is due to emission from the He atom.

photons/s to watts, then multiplication by  $N_{J'}$  to account for the number density of excited states, and finally by the substitution of  $q_{v'-v''} |\mathbf{R}_e|^2 S_{J',J}^{\Delta,J} / (2J' + 1)$  for  $\mu_{10}^2$  (see Chapter 5 and the end of this chapter for details). The Hönl-London factors are derived from the properties of symmetric top wavefunctions.<sup>8</sup> The relative intensities of the rotational lines in a band of an electronic transition are given by the Hönl-London factors  $S_{J',J}^{\Delta,J}$  of Table 9.4.

For electronic transitions the rotational constants of the two states can differ significantly. Consider the expressions for  $P$ ,  $Q$ , and  $R$  branches,

$$\tilde{\nu}_P = \tilde{\nu}_0 - (B' + B'')J'' + (B' - B'')(J'')^2, \quad (9.20a)$$

$$\tilde{\nu}_R = \tilde{\nu}_0 + 2B' + (3B' - B'')J'' + (B' - B'')(J'')^2, \quad (9.20b)$$

or

$$\tilde{\nu}_R = \tilde{\nu}_0 + (B' + B'')(J'' + 1) + (B' - B'')(J'' + 1)^2. \quad (9.20c)$$

and

$$\tilde{\nu}_Q = \tilde{\nu}_0 + (B' - B'')J''(J'' + 1). \quad (9.21)$$

If  $B' < B''$ , then the spacing between the lines in the  $P$  branch will increase as  $J''$  increases (9.20a), while the spacings between the lines of the  $R$  branch (9.20c) will

Table 9.4: Hönl–London Factors

$$\Delta\Lambda = 0$$

$$S_{J''}^R = \frac{(J'' + 1 + \Lambda)(J'' + 1 - \Lambda)}{J'' + 1}$$

$$S_{J''}^Q = \frac{(2J'' + 1)\Lambda^2}{J''(J'' + 1)}$$

$$S_{J''}^P = \frac{(J'' + \Lambda)(J'' - \Lambda)}{J''}$$

$$\Delta\Lambda = +1^a$$

$$S_{J''}^R = \frac{(J'' + 2 + \Lambda'')(J'' + 1 + \Lambda'')}{2(J'' + 1)}$$

$$S_{J''}^Q = \frac{(J'' + 1 + \Lambda'')(J'' - \Lambda'')(2J'' + 1)}{2J''(J'' + 1)}$$

$$S_{J''}^P = \frac{(J'' - 1 - \Lambda'')(J'' - \Lambda'')}{2J''}$$

$$\Delta\Lambda = -1^a$$

$$S_{J''}^R = \frac{(J'' + 2 - \Lambda'')(J'' + 1 - \Lambda'')}{2(J'' + 1)}$$

$$S_{J''}^Q = \frac{(J'' + 1 - \Lambda'')(J'' + \Lambda'')(2J'' + 1)}{2J''(J'' + 1)}$$

$$S_{J''}^P = \frac{(J'' - 1 + \Lambda'')(J'' + \Lambda'')}{2J''}$$

<sup>a</sup>These values are twice those listed in Herzberg's *Spectra of Diatomic Molecules* because of a different definition of the perpendicular transition dipole moment (see section 9.5).

decrease (Figure 9.12). At some point the lines in the  $R$  branch will pile up and then turn around. This pile up of lines is called a *band head* and is a characteristic feature of many electronic transitions. At low resolution (Figure 9.14) a band head has a characteristic edge structure due to the overlap of many rotational lines. Conversely, if  $B' > B''$ , then the band head will be in the  $P$  branch and the band is said to be blue (or violet) degraded (or degraded to shorter wavelengths). If the band head is in the  $R$  branch, then the band is described as red degraded (or degraded to longer wavelengths), as illustrated in Figure 9.14. Band heads also occur in vibration-rotation spectra, but because  $B' - B''$  is relatively small the head often occurs at sufficiently high  $J$  that it escapes observation.

The expressions for  $P$  and  $R$  branches (9.20) can be combined into a single expression by defining an index  $m = J'' + 1$  for the  $R$  branch and  $m = -J''$  for the  $P$  branch,

$$\tilde{\nu}_{P,R} = \tilde{\nu}_0 + (B' + B'')m + (B' - B'')m^2. \quad (9.22)$$

For the  $Q$  branch  $m = J''$  and the expression for the line positions is

$$\tilde{\nu}_Q = \tilde{\nu}_0 + (B' - B'')m(m + 1). \quad (9.23)$$

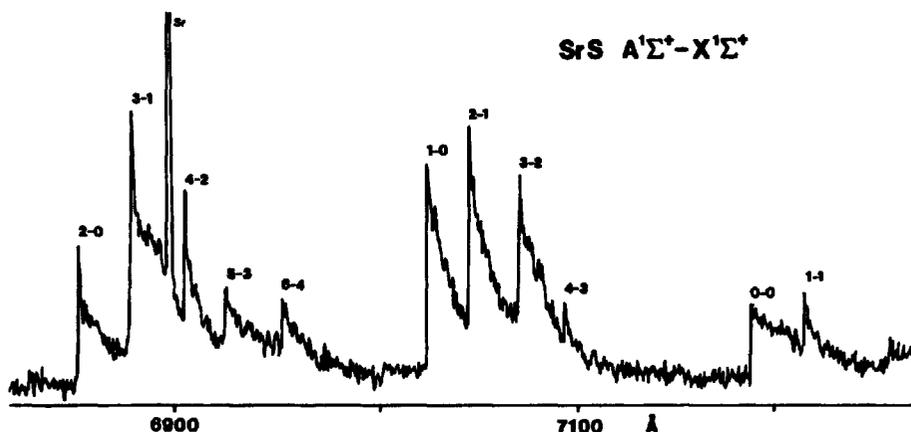


Figure 9.14: The low-resolution laser excitation spectrum of the  $A^1\Sigma^+ - X^1\Sigma^+$  transition of SrS.<sup>9</sup>

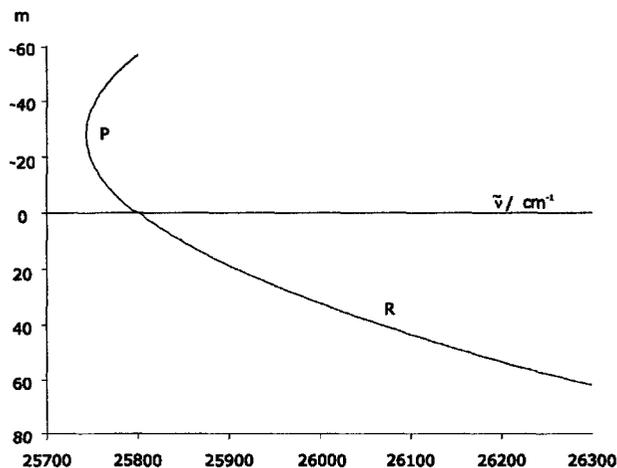


Figure 9.15: Fortrat parabola for the 0-0 band of the  $B^2\Sigma^+ - X^2\Sigma^+$  transition of CN.<sup>1</sup>

The  $\tilde{\nu}_{P,R}$  expression (9.22) can be plotted as a function of  $m$ , or more commonly  $m$  is plotted as a function of  $\tilde{\nu}$ , to give what is called a *Fortrat parabola* (Figure 9.15). A Fortrat parabola is helpful in visualizing the rotational structure of a vibrational band. The head will occur in the Fortrat parabola when

$$\frac{d\tilde{\nu}}{dm} = 0 = (B' + B'') + 2m(B' - B''), \quad (9.24)$$

or when

$$m_H = -\frac{(B' + B'')}{2(B' - B'')}, \quad (9.25)$$

with the head-origin separation being given by

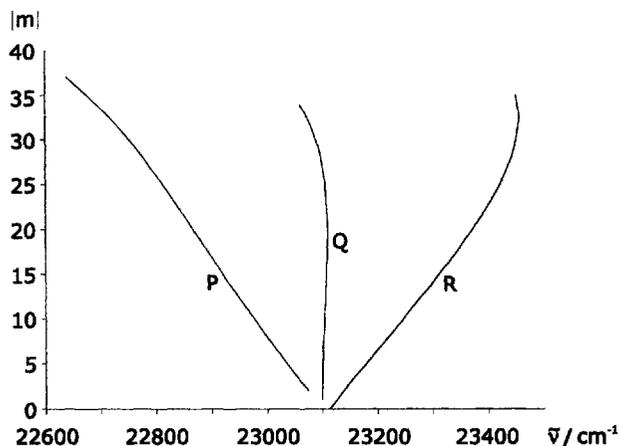


Figure 9.16: The Fortrat diagram for the  $A^1\Pi - X^1\Sigma^+$  transition of  $\text{BD}$ .<sup>10</sup>

$$\tilde{\nu}_H - \tilde{\nu}_0 = -\frac{(B' + B'')^2}{4(B' - B'')} \quad (9.26)$$

Sometimes the Fortrat diagram is plotted as a function of  $|m|$  so that the  $P$  and  $R$  branches no longer fall on the same parabola (Figure 9.16). In addition the effect of centrifugal distortion terms, which become significant for high- $J$  lines, distorts the curve from a pure parabolic shape (Figure 9.16).

For  $^1\Pi$  states (and all states with  $\Lambda > 0$ ) there is a degeneracy associated with the two possible values  $\pm|\Lambda|$ . The effect of this double degeneracy on a  $\Pi - \Sigma$  transition has already been discussed in Chapter 7. In the case of an electronic transition the small splitting in the levels of a  $^1\Pi$  state is called  $\Lambda$ -doubling<sup>11</sup> rather than  $l$ -type doubling, but the effects are completely analogous. Only the names and the physical origins of the splittings are different. Thus the energy-level diagram for a  $\Pi - \Sigma$  transition (Figure 7.33) applies to both vibrational transitions of linear polyatomic molecules and electronic transitions of diatomic (and linear polyatomic) molecules. The energy level splitting due to  $\Lambda$ -doubling increases with  $J$  and is given by  $qJ(J+1)$  for  $^1\Pi$  states.<sup>11</sup>

### *Nonsinglet Transitions*

The rotational structure of nonsinglet states is more complex than that of singlet states because of the effect on the rotational structure of both spin and orbital angular momentum. In general, the Hamiltonian operator is separated into electronic, vibrational, and rotational parts, with the effect of spin-orbit coupling within a state being accounted for by the addition of the spin-orbit operator,

$$\hat{H}_{\text{so}} = A\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}, \quad (9.27)$$

so that

$$\hat{H} = \hat{H}_{\text{el}} + \hat{H}_{\text{vib}} + \hat{H}_{\text{rot}} + \hat{H}_{\text{so}}. \quad (9.28)$$

The methodology for determining the rotational energies for nonsinglet states is to formulate this Hamiltonian operator in terms of the various angular momentum

operators  $\hat{\mathbf{J}}$ ,  $\hat{\mathbf{N}}$ ,  $\hat{\mathbf{S}}$ ,  $\hat{\mathbf{R}}$ , and  $\hat{\mathbf{L}}$  (Table 9.2) and to select suitable basis functions. In the next step a matrix representation of the Hamiltonian operator can be constructed and diagonalized to obtain the energy eigenvalues and wavefunctions (eigenvectors).

Since the focus of this section is on the rotational structure, it is convenient simply to treat the vibronic expectation value,

$$\langle \hat{H}_{\text{el}} + \hat{H}_{\text{vib}} \rangle = E_{ev}, \quad (9.29)$$

as a constant. Explicit forms for  $\hat{H}_{\text{rot}}$  and  $\hat{H}_{\text{so}}$  in the molecular coordinate system are<sup>11-13</sup>

$$\begin{aligned} \hat{H}_{\text{rot}} &= B(\hat{\mathbf{R}})^2 = B(\hat{\mathbf{J}} - \hat{\mathbf{L}} - \hat{\mathbf{S}})^2 \\ &= B(\hat{J}^2 - \hat{J}_z^2) + B(\hat{S}^2 - \hat{S}_z^2) + B(\hat{L}^2 - \hat{L}_z^2) \\ &\quad - B(\hat{J}^+ \hat{L}^- + \hat{J}^- \hat{L}^+) - B(\hat{J}^+ \hat{S}^- + \hat{J}^- \hat{S}^+) + B(\hat{L}^+ \hat{S}^- + \hat{L}^- \hat{S}^+) \end{aligned} \quad (9.30)$$

and

$$\hat{H}_{\text{so}} = A\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = A\hat{L}_z \hat{S}_z + \frac{1}{2}A(\hat{L}^+ \hat{S}^- + \hat{L}^- \hat{S}^+), \quad (9.31)$$

in which

$$\hat{J}^\pm = \hat{J}_x \pm i\hat{J}_y, \quad \hat{L}^\pm = \hat{L}_x \pm i\hat{L}_y, \quad \hat{S}^\pm = \hat{S}_x \pm i\hat{S}_y$$

are raising and lowering operators. In equations (9.30) and (9.31) *all* operators are in the molecular frame so that  $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$ ,  $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$ , but  $[\hat{J}_x, \hat{J}_y] = -i\hbar\hat{J}_z$  (anomalous). Notice that the commutators associated with the components of  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  are the same in both the laboratory and molecular frames, while those associated with  $\hat{\mathbf{J}}$  are anomalous in the molecular frame. This means that  $\hat{J}^+$  is a lowering operator and  $\hat{J}^-$  is a raising operator—that is,

$$\hat{J}^- |\Omega JM\rangle = \hbar\sqrt{J(J+1) - \Omega(\Omega+1)} |\Omega+1, JM\rangle \quad (9.32)$$

$$\hat{J}^+ |\Omega JM\rangle = \hbar\sqrt{J(J+1) - \Omega(\Omega-1)} |\Omega-1, JM\rangle. \quad (9.33)$$

Next a suitable basis set needs to be chosen. Since the Hamiltonian operator is comprised of electronic, vibrational, and rotational terms, it is convenient to use a simple product basis set,

$$|\text{el}\rangle|\text{vib}\rangle|\text{rot}\rangle = |n\Lambda\Sigma\rangle|v\rangle|\Omega JM\rangle, \quad (9.34)$$

in which  $n$  and  $v$  label the electronic state and vibrational level. The basis functions are simultaneous eigenfunctions of the operators  $\hat{H}_{\text{el}}$ ,  $\hat{H}_{\text{vib}}$ ,  $\hat{L}_z$ ,  $\hat{S}^2$ ,  $\hat{S}_z$ ,  $\hat{J}_z$ ,  $\hat{J}^2$ , and  $\hat{J}_z$ : that is,

$$\hat{H}_{\text{el}}|n\Lambda S\Sigma\rangle = E_{\text{el},n}|n\Lambda S\Sigma\rangle \quad (9.35)$$

$$\hat{H}_{\text{vib}}|v\rangle = E_{\text{vib}}|v\rangle \quad (9.36)$$

$$\hat{L}_z|n\Lambda S\Sigma\rangle = \Lambda\hbar|n\Lambda S\Sigma\rangle \quad (9.37)$$

$$\hat{S}^2|n\Lambda S\Sigma\rangle = S(S+1)\hbar^2|n\Lambda S\Sigma\rangle \quad (9.38)$$

$$\hat{S}_z|n\Lambda S\Sigma\rangle = \Sigma\hbar|n\Lambda S\Sigma\rangle \quad (9.39)$$

$$\hat{J}_z|\Omega JM\rangle = \Omega\hbar|\Omega JM\rangle \quad (9.40)$$

$$\hat{J}_z|\Omega JM\rangle = M\hbar|\Omega JM\rangle \quad (9.41)$$

$$\hat{J}^2|\Omega JM\rangle = J(J+1)\hbar^2|\Omega JM\rangle. \quad (9.42)$$

The only non-zero off-diagonal matrix elements associated with  $\hat{J}^\pm$  and  $\hat{S}^\pm$  are

$$\langle \Omega \mp 1 JM | \hat{J}^\pm | \Omega JM \rangle = \hbar \sqrt{J(J+1) - \Omega(\Omega \mp 1)} \quad (9.43)$$

$$\langle \Lambda S \Sigma \pm 1 | \hat{S}^\pm | \Lambda S \Sigma \rangle = \hbar \sqrt{S(S+1) - \Sigma(\Sigma \pm 1)}. \quad (9.44)$$

Notice that a label for  $\hat{\mathbf{L}}$  is not present in the basis set because the electronic wavefunction is generally no longer an eigenfunction of  $\hat{L}^2$ , that is,

$$\hat{L}^2\psi \neq L(L+1)\hbar^2\psi, \quad (9.45)$$

even though the projection of  $\hat{\mathbf{L}}$  along the internuclear axis is well defined, with

$$\hat{L}_z\psi = \Lambda\hbar\psi. \quad (9.46)$$

The rapid precessional motion of  $\hat{\mathbf{L}}$  around the internuclear axis (Figure 9.1) prevents the experimental determination of the magnitude of  $\hat{\mathbf{L}}$ . This means that matrix elements of  $\hat{L}^2$  or  $\hat{L}^+$  are simply constants whose values could be determined by *ab initio* calculation. Usually the expectation values of these operators are just absorbed into the electronic band origins. An alternate approach<sup>11</sup> eliminates  $\hat{\mathbf{L}}$  from the rotational Hamiltonian operator to give an effective Hamiltonian operator in terms of  $\hat{\mathbf{N}}^2$  (see Table 9.2) rather than  $\hat{\mathbf{R}}^2$ .

### ${}^2\Sigma^+$ States

For  ${}^2\Sigma^+$  states there are two basis functions,

$$|n, \Lambda = 0, S = \frac{1}{2}, \Sigma = \frac{1}{2}\rangle |v\rangle | \Omega = \frac{1}{2} JM \rangle = |{}^2\Sigma_{1/2}\rangle \quad (9.47)$$

and

$$|n, \Lambda = 0, S = \frac{1}{2}, \Sigma = -\frac{1}{2}\rangle |v\rangle | \Omega = -\frac{1}{2} JM \rangle = |{}^2\Sigma_{-1/2}\rangle, \quad (9.48)$$

in which  $\Omega$ ,  $\Lambda$ , and  $\Sigma$  are signed quantum numbers. The diagonal matrix elements of  $\hat{H}_{\text{rot}}$ , equation (9.30), are (in units of  $\text{cm}^{-1}$ ):

$$\begin{aligned} \langle {}^2\Sigma_{1/2} | \hat{H}_{\text{rot}} | {}^2\Sigma_{1/2} \rangle &= \langle {}^2\Sigma_{-1/2} | \hat{H}_{\text{rot}} | {}^2\Sigma_{-1/2} \rangle \\ &= B(J(J+1) - \frac{1}{4}) + B(\frac{3}{4} - \frac{1}{4}) \\ &= B(J + \frac{1}{2})^2. \end{aligned} \quad (9.49)$$

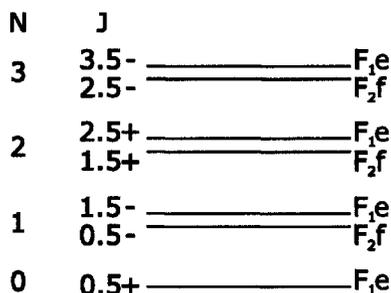


Figure 9.17: Energy-level diagram of a  ${}^2\Sigma^+$  state with a positive spin-rotation constant,  $\gamma$ , that puts the  $e$  level above  $f$  for each  $N$ .

The off-diagonal matrix elements of  $-B(\hat{J}^+\hat{S}^- + \hat{J}^-\hat{S}^+)$  couple the  ${}^2\Sigma_{1/2}$  state to the  ${}^2\Sigma_{-1/2}$  state—namely

$$\begin{aligned} \langle {}^2\Sigma_{-1/2} | -B(\hat{J}^+\hat{S}^-) | {}^2\Sigma_{1/2} \rangle &= -B(J(J+1) - \Omega(\Omega-1))^{1/2} \times \\ & (S(S+1) - \Sigma(\Sigma-1))^{1/2} = -B(J + \frac{1}{2}). \end{aligned} \quad (9.50)$$

Collecting the diagonal and off-diagonal elements of  $\hat{H}$  (in  $\text{cm}^{-1}$ ) gives

$$\hat{H} = \begin{pmatrix} B(J + \frac{1}{2})^2 & -B(J + \frac{1}{2}) \\ -B(J + \frac{1}{2}) & B(J + \frac{1}{2})^2 \end{pmatrix}. \quad (9.51)$$

Although this matrix can be diagonalized as it stands, the diagonalization can be avoided by transforming the basis functions as

$$|{}^2\Sigma^+(e)\rangle = \frac{|{}^2\Sigma_{1/2}^+\rangle + |{}^2\Sigma_{-1/2}^+\rangle}{\sqrt{2}} \quad (9.52)$$

$$|{}^2\Sigma^+(f)\rangle = \frac{|{}^2\Sigma_{1/2}^+\rangle - |{}^2\Sigma_{-1/2}^+\rangle}{\sqrt{2}}, \quad (9.53)$$

in which  $e$  and  $f$  are parity basis functions (see section 9.4). The Hamiltonian matrix in the transformed basis set becomes

$$\hat{H}' = \begin{pmatrix} B(J + \frac{1}{2})^2 & 0 \\ 0 & B(J + \frac{1}{2})^2 + B(J + \frac{1}{2}) \end{pmatrix}. \quad (9.54)$$

It turns out that if the diagonal elements of  $\hat{H}'$  are expressed in terms of the integral quantum number  $N$  (i.e., the total angular momentum exclusive of electronic spin, Table 9.2) rather than  $J$ , then they simplify to

$$F_1(e) = B(N+1)^2 - B(N+1) = BN(N+1) \quad (9.55a)$$

$$F_2(f) = BN^2 + BN = BN(N+1). \quad (9.55b)$$

In equation (9.55a)  $N$  is  $J - \frac{1}{2}$  for the  $e$  levels, while in equation (9.55b)  $N$  is  $J + \frac{1}{2}$  for  $f$  levels. The energy-level pattern for  ${}^2\Sigma^+$  states is thus identical to that for  ${}^1\Sigma^+$

states if  $N$  is used rather than  $J$  (Figure 9.17). The subscripts 1 and 2 in equations (9.55a–b) are chosen in accordance with the spectroscopic custom that  $F_1$  has  $J$  equal to the maximum value of  $(N + S)$ . Although the energy-levels corresponding to the two  $J$  values for each  $N$ —( $F_1(N)$  and  $F_2(N)$ )—are exactly degenerate at this level of theory, inclusion of the spin-rotation interaction term<sup>11,13</sup>

$$\hat{H}_{sr} = \gamma \hat{N} \cdot \hat{S}$$

splits them by the amount  $\gamma(N + \frac{1}{2})$ .

### <sup>2</sup>Π States

There are four possible basis functions for a <sup>2</sup>Π state,

$$|^2\Pi_{3/2}\rangle = |n, \Lambda = 1, S = \frac{1}{2}, \Sigma = \frac{1}{2}\rangle |v\rangle |\Omega = \frac{3}{2}JM\rangle, \quad (9.56a)$$

$$|^2\Pi_{1/2}\rangle = |n, \Lambda = 1, S = \frac{1}{2}, \Sigma = -\frac{1}{2}\rangle |v\rangle |\Omega = \frac{1}{2}JM\rangle, \quad (9.56b)$$

$$|^2\Pi_{-1/2}\rangle = |n, \Lambda = -1, S = \frac{1}{2}, \Sigma = \frac{1}{2}\rangle |v\rangle |\Omega = -\frac{1}{2}JM\rangle, \quad (9.56c)$$

$$|^2\Pi_{-3/2}\rangle = |n, \Lambda = -1, S = \frac{1}{2}, \Sigma = -\frac{1}{2}\rangle |v\rangle |\Omega = -\frac{3}{2}JM\rangle. \quad (9.56d)$$

The derivation of the sixteen possible matrix elements of  $\hat{H}_{rot} + \hat{H}_{so}$ , equations (9.30) and (9.31), is slightly more involved than for <sup>2</sup>Σ<sup>+</sup> states but results in the Hamiltonian matrix (in units of cm<sup>-1</sup>):

$$\begin{pmatrix} \frac{A}{2} + B((J + \frac{1}{2})^2 - 2) & -B\sqrt{(J + \frac{1}{2})^2 - 1} & 0 & 0 \\ -B\sqrt{(J + \frac{1}{2})^2 - 1} & -\frac{A}{2} + B(J + \frac{1}{2})^2 & 0 & 0 \\ 0 & 0 & -\frac{A}{2} + B(J + \frac{1}{2})^2 & -B\sqrt{(J + \frac{1}{2})^2 - 1} \\ 0 & 0 & -B\sqrt{(J + \frac{1}{2})^2 - 1} & \frac{A}{2} + B((J + \frac{1}{2})^2 - 2) \end{pmatrix}. \quad (9.57)$$

To obtain the full matrix  $\hat{H}$ , the vibronic term energy  $E_{ev}$  needs to be added to each term along the diagonal. Note that the constant matrix element of  $\langle B(\hat{L}^2 - \hat{L}_z^2) \rangle$  has been included in  $E_{ev}$ . In this case the  $e/f$  parity basis functions are written as

$$|^2\Pi_{3/2}(e/f)\rangle = \frac{|^2\Pi_{3/2}\rangle \pm |^2\Pi_{-3/2}\rangle}{\sqrt{2}}, \quad (9.58)$$

$$|^2\Pi_{1/2}(e/f)\rangle = \frac{|^2\Pi_{1/2}\rangle \pm |^2\Pi_{-1/2}\rangle}{\sqrt{2}}, \quad (9.59)$$

in which the upper (lower) sign refers to  $e(f)$  parity. The resultant transformed Hamiltonian matrix for the  $e$ -parity block is

$$\hat{H}' = \begin{pmatrix} |^2\Pi_{3/2}(e/f)\rangle & |^2\Pi_{1/2}(e/f)\rangle \\ \frac{A}{2} + B((J + \frac{1}{2})^2 - 2) & -B\sqrt{(J + \frac{1}{2})^2 - 1} \\ -B\sqrt{(J + \frac{1}{2})^2 - 1} & -\frac{A}{2} + B(J + \frac{1}{2})^2 \end{pmatrix} \quad (9.60)$$

with identical matrix elements for the  $f$ -parity block.

There are two limiting cases for the energy levels of a  ${}^2\Pi$  state depending upon the extent of spin-orbit coupling, as measured by the relative size of the diagonal  $A\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$  term in the Hamiltonian operator. When  $A$  is large ( $A \gg BJ$ ), it is referred to as *Hund's case (a)* coupling, while when  $A$  is small ( $A \ll BJ$ ), it is referred to as *Hund's case (b)* coupling.

The Hund's case (a)  ${}^2\Pi$  Hamiltonian matrix is characterized by diagonal matrix elements with the large separation  $\Delta E = H'_{11} - H'_{22} \approx A$ , and by a relatively small off-diagonal matrix element  $H_{12} = -B\sqrt{(J + \frac{1}{2})^2 - 1} \approx -BJ$ . In this case, second-order perturbation theory (Chapter 4) predicts that the higher upper energy level is shifted upward by  $V^2/\Delta E = (H_{12})^2/\Delta E$  and the lower energy level downward by the same amount. There are thus two widely separated spin components,  ${}^2\Pi_{3/2}$  and  ${}^2\Pi_{1/2}$ , with energies given by

$$E_{3/2}(J) = \frac{A}{2} + B((J + \frac{1}{2})^2 - 2) + \frac{B^2((J + \frac{1}{2})^2 - 1)}{A}, \quad (9.61)$$

and

$$E_{1/2}(J) = -\frac{A}{2} + B(J + \frac{1}{2})^2 - \frac{B^2((J + \frac{1}{2})^2 - 1)}{A}. \quad (9.62)$$

The energy-level expressions (9.61) and (9.62) can be simplified by defining

$$A_{\text{eff}} = A - 2B, \quad (9.63)$$

$$B_{3/2(\text{eff})} = B + \frac{B^2}{A} = B \left( 1 + \frac{B}{A} \right), \quad (9.64)$$

and

$$B_{1/2(\text{eff})} = B - \frac{B^2}{A} = B \left( 1 - \frac{B}{A} \right), \quad (9.65)$$

so that the energy-level expressions become

$$E_{3/2}(J) = \frac{A_{\text{eff}}}{2} + B_{3/2(\text{eff})}((J + \frac{1}{2})^2 - 1) \quad (9.66)$$

and

$$E_{1/2}(J) = -\frac{A_{\text{eff}}}{2} + B_{1/2(\text{eff})}((J + \frac{1}{2})^2 - 1). \quad (9.67)$$

Thus, the rotational levels of the  ${}^2\Pi_{1/2}$  and  ${}^2\Pi_{3/2}$  spin components are well separated for each  $J$  value, and each component has its own effective  $B$  value, while the actual mechanical rotational constant is the average

$$B = (B_{1/2(\text{eff})} + B_{3/2(\text{eff})})/2. \quad (9.68)$$

At this stage of development the  $e$  and  $f$  components for each rotational energy level are exactly degenerate. If  $\Lambda$ -doubling<sup>14</sup> is taken into account, the  ${}^2\Pi_{1/2}$  rotational levels are split by  $-p(J + \frac{1}{2})$  and the  ${}^2\Pi_{3/2}$  rotational levels are split by  $f(p, q)(J + \frac{1}{2})^3$ , in which  $f(p, q)$  is a parameter which depends on  $p$  and  $q$ . The two  $\Lambda$ -doubling constants,  $p$  and  $q$ , account for interactions with distant  ${}^2\Sigma$  states. As illustrated in Figure 9.18

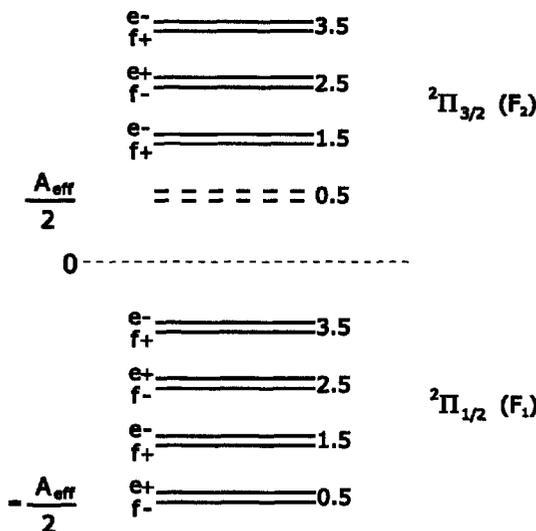


Figure 9.18: The energy-level pattern for a Hund’s case (a)  ${}^2\Pi$  state with negative  $\Lambda$ -doubling parameters,  $p$  and  $q$ , that put  $e$  levels above  $f$  for each  $\Lambda$ -doublet.

each rotational energy-level  $J$  is twofold degenerate and this degeneracy is lifted by  $\Lambda$ -doubling interactions in a rotating molecule.

If the splitting between the  ${}^2\Pi_{1/2}$  and  ${}^2\Pi_{3/2}$  spin components is small, then the Hund’s case (b) energy-level pattern applies. In this case,  $A - 2B \ll BJ$  (i.e.,  $(H_{11} - H_{22}) \ll H_{12}$ ) so that the two spin components are strongly coupled. In this case, the  $2 \times 2$  Hamiltonian matrix (9.60) must be diagonalized (Chapter 4) in order to obtain the two energy levels.

For Hund’s case (b), the spin-orbit coupling constant is zero ( $A = 0$ ). The two energy levels are given by the expression

$$\begin{aligned}
 E_{\pm} &= \frac{H_{11} + H_{22}}{2} \pm \frac{((\Delta E)^2 + 4H_{12}^2)^{1/2}}{2} \\
 &= B((J + \frac{1}{2})^2 - 1) \pm \frac{(4B^2 + 4B^2((J + \frac{1}{2})^2 - 1))^{1/2}}{2} \\
 &= B((J + \frac{1}{2})^2 - 1) \pm B(J + \frac{1}{2}).
 \end{aligned}
 \tag{9.69}$$

For a Hund’s case (b)  ${}^2\Pi$  state it is again useful, as in the  ${}^2\Sigma^+$  case, to introduce the integral quantum number  $N = J + \frac{1}{2}$  for the  $E_+$  level and  $N = J - \frac{1}{2}$  for the  $E_-$  level. Thus from equation (9.69) the  $F_1$  and  $F_2$  energy levels are given by

$$F_2(N) = E_+ = B(N^2 - 1) + BN = B(N(N + 1) - 1)
 \tag{9.70a}$$

and

$$F_1(N) = E_- = B((N + 1)^2 - 1) - B(N + 1) = B(N(N + 1) - 1).
 \tag{9.70b}$$

The energy levels for a Hund’s case (b)  ${}^2\Pi$  state are therefore given by  $B(N(N + 1) - 1)$ , so that at this level of treatment, for each  $N$  there is a fourfold degeneracy,  $F_{1e}$ ,  $F_{1f}$ ,

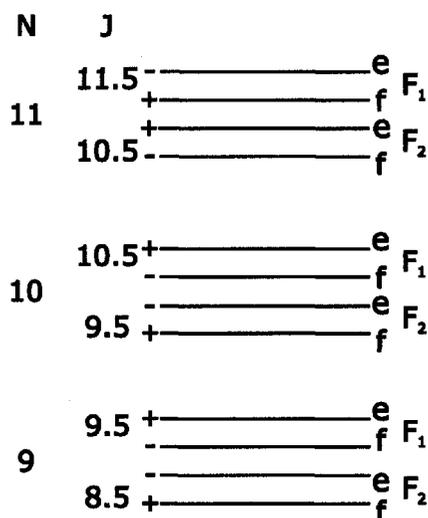


Figure 9.19: Energy-level diagram for a Hund's case (b)  ${}^2\Pi$  state.

$F_{2e}$ , and  $F_{2f}$  (Figure 9.19). The  $F_1$  and  $F_2$  labels are defined so that  $J = N + \frac{1}{2}$  for  $F_1$  and  $J = N - \frac{1}{2}$  for  $F_2$ . If  $\Lambda$ -doubling is considered, then the  $e$  and  $f$  levels are split by  $qJ(J+1)$  as for a  ${}^1\Pi$  state.<sup>14</sup>

In general the rotational energy levels for Hund's case (b)  ${}^{2S+1}\Lambda$  states are given by  $BN(N+1)$  and there are  $2(2S+1)$  degenerate levels for  $\Lambda \neq 0$  and  $2S+1$  degenerate levels for  $\Lambda = 0$  ( $\Sigma^+$  or  $\Sigma^-$  states). In contrast (Figure 9.20) for a Hund's case (a)  ${}^{2S+1}\Lambda$  state there are  $2S+1$  well-separated spin components in which the rotational energy levels for each component are given by  $B_{\Omega(\text{eff})}J(J+1)$ . For Hund's case (a) each  $J$  is doubly degenerate ( $\Lambda > 0$ ), corresponding to  $\pm\Lambda$  (or  $e/f$  parity states). For  $\Sigma$  states, Hund's case (a) coupling is rare since there is no contribution by the diagonal spin-orbit term to the energy levels. Second-order spin-orbit coupling can, however, split a  ${}^{2S+1}\Sigma$  state into  $S+1$  spin components, labeled by their  $\Omega$  values. A molecule such as BiH with nominally a  $X^3\Sigma^-$  ground state is split into  $X_10^+$  and  $X_21$  substates by a large second-order spin-orbit coupling (cf. Hund's case (c) below).

The energy-level patterns described by Hund's case (a) and (b) are extreme limiting cases. The accurate description of molecular energy levels also requires the addition of centrifugal distortion, spin-rotation coupling, spin-spin coupling, and  $\Lambda$ -doubling terms, as well as the numerical diagonalization of the Hamiltonian matrix.<sup>11,13</sup> There are also two additional Hund's coupling cases that are not uncommon: Hund's cases (c) and (d).

In *Hund's case (c)*, the spin-orbit coupling becomes larger than  $BJ$  as well as larger than the separation between neighboring electronic states—that is,  $A \gg BJ$  and  $A \gg \Delta E_{\text{states}}$ . In this case the various spin components from several  ${}^{2S+1}\Lambda$  terms occur in the same energy region and their wavefunctions become mixed through off-diagonal (i.e., between  ${}^{2S+1}\Lambda$  terms) spin-orbit coupling. The complete spin-orbit Hamiltonian operator,

$$\hat{H}_{\text{so}} = \sum a_i \hat{\mathbf{l}}_i \cdot \hat{\mathbf{s}}_i, \quad (9.71)$$

causes both large spin-orbit splittings and extensive mixing of the electronic wavefunctions.

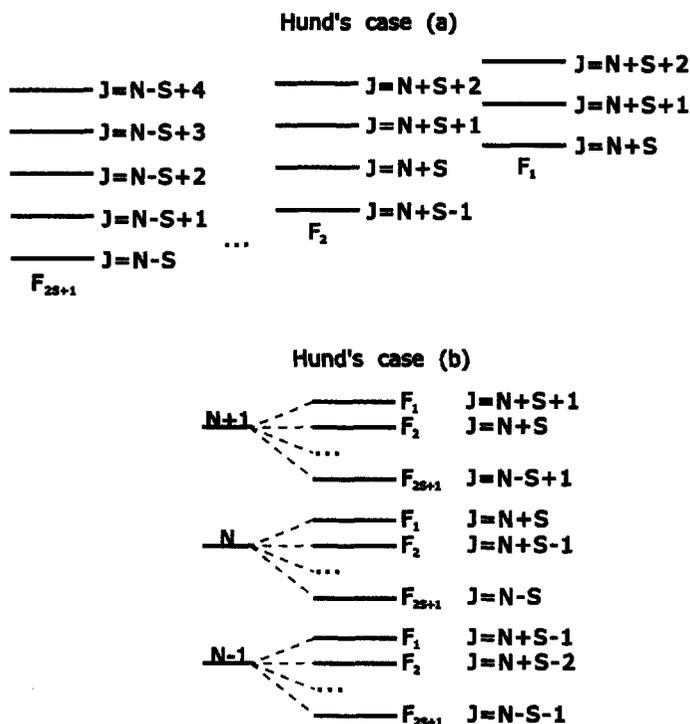


Figure 9.20: Hund's case (a) and Hund's case (b) energy-level patterns for a  $^{2S+1}\Lambda$  state.

The strong coupling of  $\hat{L}$  and  $\hat{S}$  means that only  $J$  and  $\Omega$  are good quantum numbers for pure case (c) coupling. Since  $L$ ,  $S$ ,  $\Lambda$ , and  $\Sigma$  no longer have any meaning, the conventional  $^{2S+1}\Lambda_{\Omega}$  notation is misleading; the Hund's case (c) notation is just  $\Omega$ . For example, the  $B^3\Pi_{0+}$  state of  $I_2$  should be labeled as  $B0^+$  using Hund's case (c) notation. Since spin-orbit coupling increases rapidly with increasing atomic number  $Z$ , electronic states of diatomic molecules containing heavy elements tend toward Hund's case (c). For Hund's case (c) coupling, the  $2S+1$  spin components of a given  $^{2S+1}\Lambda$  term are like independent electronic states and are labeled by the good quantum number  $\Omega$ .

The rotational energy-level pattern for a Hund's case (c) state is similar to the Hund's case (a) pattern (Figure 9.21). The rotational energy-level expression is  $B_{\text{eff}}J(J+1)$  for each  $\Omega$  state. For  $\Omega \neq 0$  each level is doubled due to the  $\pm\Omega$  degeneracy, which can be lifted by  $\Omega$ -type doubling interactions. The  $\Omega$ -doubling splitting increases as  $J$  increases with a  $J$ -dependence that scales as  $J^{2\Omega}$ . For both Hund's cases (a) and (c) the spin components are labeled by  $\Omega$ , but only for Hund's case (a) does a spin component originate from a specific  $^{2S+1}\Lambda$  term. In Hund's case (c) the  $\Omega$  state is a mixture of many  $^{2S+1}\Lambda_{\Omega}$  basis functions.

*Hund's case (d)* applies to Rydberg electronic states in which an electron is excited to an orbital with a large principal quantum number. In this case, the Rydberg electron is so distant from the nuclei that  $\hat{L}$  and  $\hat{s}$  for the Rydberg electron couple only weakly to the internuclear axis. For a pure Hund's case (d) state the rotational energy-level expression is  $BR(R+1)$  and each level has a  $(2S+1)(2L+1)$  degeneracy. The  $2L+1$  and  $2S+1$  degeneracy is associated with the  $M_L$  and  $M_S$  degeneracy of a very atomic-like Rydberg electron. The atomic-like character of the Rydberg molecular orbital allows  $L$

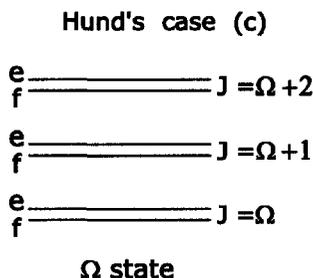


Figure 9.21: Hund's case (c) rotational energy-level pattern.

Table 9.5: Hund's Coupling Cases

Coupling Case	Good Quantum Numbers	Rotational Energy Expression	Orbital and Spin Degeneracy of Each Rotational State
Hund's case (a)	$\Lambda, S, \Sigma, \Omega$	$BJ(J+1)$	2 (1 for $\Sigma$ states)
Hund's case (b)	$\Lambda, S, \Sigma$	$BN(N+1)$	$2(2S+1)$ ( $2S+1$ for $\Sigma$ states)
Hund's case (c)	$\Omega$	$BJ(J+1)$	2 (1 for $\Sigma$ states)
Hund's case (d)	$L, \Omega, S, \Sigma$	$BR(R+1)$	$(2L+1)(2S+1)$

to be a good quantum number in this case. The various Hund's coupling cases<sup>12</sup> are summarized in Table 9.5, keeping only the main rotational term.

## 9.4 The Symmetry of Diatomic Energy Levels: Parity

Nothing causes as much confusion in the study of the spectra of diatomic molecules as does the concept of parity. The difficulty lies in there being several different types of parity such as  $g/u$ ,  $e/f$ ,  $+/-$ , and  $s/a$ . The basic idea is quite simple, however: if the Hamiltonian operator  $\hat{H}$  and a symmetry operator  $\hat{O}_S$  commute, then a set of simultaneous eigenfunctions of the two operators can be found; that is,

$$[\hat{H}, \hat{O}_S] = 0 \quad (9.72)$$

implies that

$$\hat{H}\psi_{\pm} = E_{\pm}\psi_{\pm} \quad (9.73)$$

and that

$$\hat{O}_S\psi_{\pm} = \pm\psi_{\pm}. \quad (9.74)$$

If a molecule has a certain symmetry, then the effect of the associated symmetry operator can be used to label wavefunctions and energy levels. Care must be taken, however,

to specify *both* the symmetry operator and the part of the total Hamiltonian operator under consideration. Failure to do this is the real source of confusion about parity.

### Total (+/-) Parity

If the symmetry operation is  $\hat{E}^*$  (sometimes called  $\hat{I}$ ) and the total Hamiltonian operator including electronic, vibrational, and rotational parts (but not nuclear spin) are used, then one obtains total parity. The  $\hat{E}^*$  operator inverts all of the coordinates of the particles (nuclei and electrons) in the laboratory frame with the origin at the center of mass (Figure 9.4)—that is,

$$\begin{aligned}\hat{E}^* \psi(X_i, Y_i, Z_i) &= \psi(-X_i, -Y_i, -Z_i) \\ &= \pm \psi(X_i, Y_i, Z_i).\end{aligned}\quad (9.75)$$

The  $\hat{E}^*$  operator is a symmetry operator because all of the relative positions of the particles are the same before and after inversion, and consequently the energy levels are unchanged by application of  $\hat{E}^*$ . Note, however, that the sign of the wavefunction can change under  $\hat{E}^*$  since

$$\hat{E}^*(\hat{E}^*)\psi = \psi. \quad (9.76)$$

Total parity is the parity often used in nuclear and atomic physics. Physicists have long noted that the properties of a right-handed system are identical to those of a left-handed system (except for the process of  $\beta$ -decay) since the  $\hat{E}^*$  symmetry operator converts one into the other.

The  $\hat{E}^*$  symmetry operator is used to divide all rovibronic energy states into two groups by means of the equation

$$\hat{E}^* \psi = \hat{E}^*(\psi_{\text{el}}\psi_{\text{vib}}\psi_{\text{rot}}) = \pm \psi. \quad (9.77)$$

All rovibronic energy levels for which the upper sign applies have positive (+) total parity, while all rovibronic levels for which the lower sign applies have negative (-) total parity. The effects of  $\hat{E}^*$  on the electronic, vibrational, and rotational parts of the total wavefunction need to be determined individually.

The effect of  $\hat{E}^*$  on the vibrational part of the diatomic wavefunction is easy to ascertain. The inversion operation leaves the vibrational part of the wavefunction unchanged because  $\psi_{\text{vib}}$  is a function only of the magnitude of the internuclear separation  $r$ . Inversion of the coordinates of all particles leaves  $r$  unchanged, so that

$$\hat{E}^* \psi_{\text{vib}}(r) = \psi_{\text{vib}}(r). \quad (9.78)$$

The effect of  $E^*$  on the rotational wavefunction is more complicated. For instance, Figure 9.22 shows that the  $\hat{E}^*$  operation replaces  $\theta$  by  $\pi - \theta$  and  $\phi$  by  $\phi + \pi$  in  $\psi_{\text{rot}}(\theta, \phi)$ , and for  ${}^1\Sigma^+$  states  $\psi_{\text{rot}} = Y_{JM}(\theta, \phi)$ . We have already discussed the effect of inversion in the laboratory frame on spherical harmonics, the result being

$$\hat{E}^* Y_{JM} = (-1)^J Y_{JM}. \quad (9.79)$$

In general, however,  $\psi_{\text{rot}} = |\Omega JM\rangle$ , and the properties of these rotational wavefunctions need to be considered. The rotational wavefunctions  $|\Omega JM\rangle$  are identical to the symmetric top wavefunctions  $|K JM\rangle$  discussed in Chapter 6, and it is found that<sup>12</sup>

$$\hat{E}^* |\Omega JM\rangle = (-1)^{J-\Omega} |-\Omega JM\rangle. \quad (9.80)$$

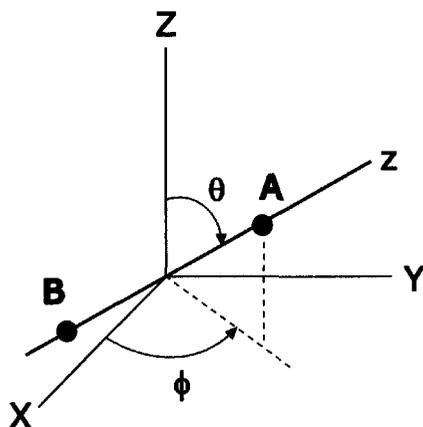


Figure 9.22: Diatomic molecule coordinate system.

The final part of the problem, the effect of inversion on  $\psi_{el}$ , is the most difficult because  $\psi_{el}$  is known in the molecular frame. The calculation of  $\psi_{el}$  is performed under the Born–Oppenheimer approximation in which the nuclei are fixed in space; thus  $\psi_{el}(x_i, y_i, z_i)$  is known but  $\psi_{el}(X_i, Y_i, Z_i)$  is not. Since the inversion operation changes the sign of the laboratory coordinates  $(X_i, Y_i, Z_i)$  of the particles, the effect of  $\hat{E}^*$  on  $\psi_{el}(x_i, y_i, z_i)$  is not obvious. Hougén has considered this problem<sup>12,15</sup> and has shown that  $\hat{E}^*$  in the laboratory frame is equivalent to the symmetry operation of reflection  $\hat{\sigma}_v$  in the molecular frame. The equivalence of the permutation-inversion operation  $\hat{E}^*$  in the laboratory frame and the ordinary point group operation of reflection  $\hat{\sigma}_v$  in the symmetry plane of the diatomic molecule is an important (and not obvious!) result. The effect of  $\hat{\sigma}_v$  on the coordinates of an electron is established by replacing  $(x_i, y_i, z_i)$  by  $(x_i, -y_i, z_i)$ , if the reflection is arbitrarily chosen to be in the  $xz$ -plane of the molecule. Without considering the detailed form of the spin and orbital parts of  $\psi_{el}$ , it is not possible to extend the treatment further. It turns out that the effect of  $\hat{\sigma}_v$  on the spin and orbital parts is<sup>12</sup>

$$\hat{\sigma}_v |S\Sigma\rangle = (-1)^{S-\Sigma} |S, -\Sigma\rangle, \quad (9.81)$$

and

$$\hat{\sigma}_v |\Lambda\rangle = \pm (-1)^\Lambda |-\Lambda\rangle. \quad (9.82)$$

The effect of  $\hat{\sigma}_v$  on a  $|\Lambda = 0\rangle$  orbital part of the electronic wavefunction is particularly interesting since one obtains

$$\hat{\sigma}_v |\Lambda = 0\rangle = \pm |\Lambda = 0\rangle.$$

These  $\pm$  signs are written as superscripts on the term symbols,  $\Sigma^+$  and  $\Sigma^-$ ; they correspond to

$$\hat{\sigma}_v |\Sigma^\pm\rangle = \pm |\Sigma^\pm\rangle. \quad (9.83)$$

The superscript  $\pm$  sign in the term symbol for  $\Sigma$  states indicates the effect of the  $\hat{\sigma}_v$  symmetry operator on only the orbital part of the electronic wavefunction. The addition

of superscript  $\pm$  to the term symbol for  $\Lambda > 0$  ( $\Pi$ ,  $\Delta$ ,  $\Phi$ , etc., states) is not necessary since the levels always occur as a  $\pm$  pair because of the twofold orbital degeneracy (Figure 9.18).

The effect of the  $\hat{\sigma}_v$  operator (equivalent to  $\hat{E}^*$ ) on the total wavefunction is determined by combining equations (9.80), (9.81), and (9.82):

$$\begin{aligned}\hat{\sigma}_v(\psi_{\text{el}}\psi_{\text{vib}}\psi_{\text{rot}}) &= \hat{\sigma}_v(|n\Lambda S\Sigma\rangle|v\rangle|\Omega JM\rangle) \\ &= (-1)^{J-2\Sigma+S+\sigma}|n, -\Lambda, S, -\Sigma\rangle|v\rangle|-\Omega JM\rangle,\end{aligned}\quad (9.84)$$

in which  $\sigma = 0$  for all states, except  $\Sigma^-$  states for which  $\sigma = 1$ . Since the  $\hat{\sigma}_v$  operation changes the signs of  $\Lambda$ ,  $\Sigma$ , and  $\Omega$ , the parity eigenfunctions are linear combinations of the basis functions, namely

$$|^{2S+1}\Lambda_{\Omega\pm}\rangle = \frac{|^{2S+1}\Lambda_{\Omega}\rangle \pm (-1)^{J-2\Sigma+S+\sigma}|^{2S+1}\Lambda_{-\Omega}\rangle}{\sqrt{2}}\quad (9.85)$$

with

$$\hat{\sigma}_v|^{2S+1}\Lambda_{\Omega\pm}\rangle = \pm|^{2S+1}\Lambda_{\Omega\pm}\rangle.\quad (9.86)$$

The selection rules on total parity are derived, as usual, by requiring a totally symmetric integrand for the transition moment integral,

$$\int \psi_f^* \boldsymbol{\mu} \psi_i d\tau.\quad (9.87)$$

The transition moment operator has  $(-)$  parity since

$$\hat{E}^* \boldsymbol{\mu} = -\boldsymbol{\mu},\quad (9.88)$$

so that only  $+\leftrightarrow -$  transitions are allowed for one-photon electric dipole transitions.

### **Rotationless (e/f) Parity**

Notice that the total parity changes sign with  $J$  (Figure 9.23) because of the phase factor  $(-1)^J$  in equation (9.80). This alternation of the total parity with  $J$  is always present, and hence it is useful to factor it out by defining  $e$  and  $f$  parity as<sup>16</sup>

$$\hat{E}^* \psi = +(-1)^J \psi \quad \text{for } e,\quad (9.89a)$$

and

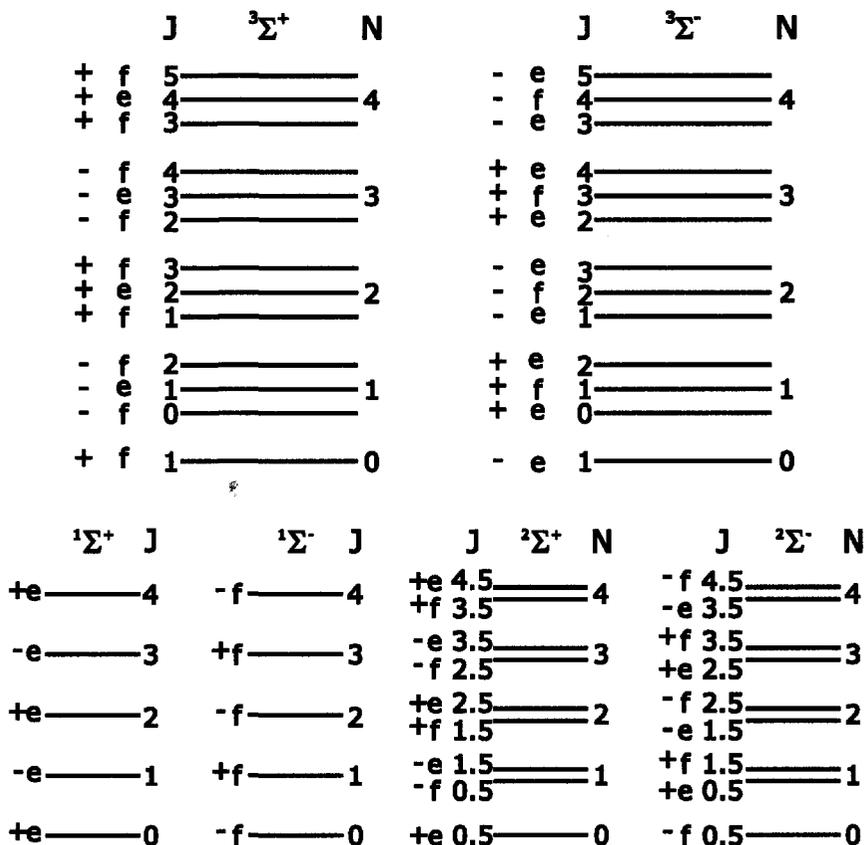
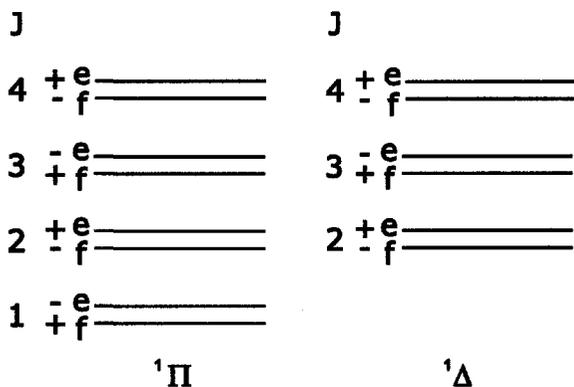
$$\hat{E}^* \psi = -(-1)^J \psi \quad \text{for } f\quad (9.89b)$$

for integer  $J$ . Similarly, for half-integer  $J$ ,

$$\hat{E}^* \psi = +(-1)^{J-1/2} \psi \quad \text{for } e,\quad (9.90a)$$

and

$$\hat{E}^* \psi = -(-1)^{J-1/2} \psi \quad \text{for } f,\quad (9.90b)$$

Figure 9.23: Parity of  $^1\Sigma$ ,  $^2\Sigma$ , and  $^3\Sigma$  rotational energy levels.Figure 9.24: Parity of  $^1\Pi$  and  $^1\Delta$  rotational energy levels with positive  $\Lambda$ -doubling constants,<sup>14,17</sup>  $q$  and  $q_\Delta$  (respectively), that put  $e$  levels above  $f$  for each  $\Lambda$ -doublet.

in which  $\psi$  is the total rovibronic wavefunction. In the molecular frame,  $\hat{E}^*$  is replaced by  $\hat{\sigma}_v$ . Notice that  $e$  and  $f$  parity is a “residual parity” or a rotationless parity describing the total parity with the rotational part removed.

Because the alternation of sign with  $J$  has been removed,  $e/f$  parity is more convenient to use than total parity. All rotational energy levels of  ${}^1\Sigma^+$  states have  $e$  parity, while all  ${}^1\Sigma^-$  rotational energy levels have  $f$  parity (Figure 9.23). For  ${}^1\Pi$  states all of the rotational energy levels occur as  $e/f$  pairs (Figure 9.24). The  $+ \leftrightarrow -$  selection rule for total parity becomes  $e \leftrightarrow e$  and  $f \leftrightarrow f$  for  $P$  and  $R$  branches, while  $e \leftrightarrow f$  for  $Q$  branches.

The use of  $e/f$  parity also suppresses an annoying  $(-1)^{J-2\Sigma+S+\sigma}$  factor in the definition of the parity eigenfunctions (9.85) since

$$|{}^2\Pi_{3/2}(e/f)\rangle = \frac{|{}^2\Pi_{3/2}\rangle \pm |{}^2\Pi_{-3/2}\rangle}{\sqrt{2}}, \quad (9.91)$$

$$|{}^2\Pi_{1/2}(e/f)\rangle = \frac{|{}^2\Pi_{1/2}\rangle \pm |{}^2\Pi_{-1/2}\rangle}{\sqrt{2}}, \quad (9.92)$$

$$|{}^2\Sigma_{1/2}^+(e/f)\rangle = \frac{|{}^2\Sigma_{1/2}^+\rangle \pm |{}^2\Sigma_{-1/2}^+\rangle}{\sqrt{2}}, \quad (9.93)$$

$$|{}^2\Sigma_{1/2}^-(e/f)\rangle = \frac{|{}^2\Sigma_{1/2}^-\rangle \mp |{}^2\Sigma_{-1/2}^-\rangle}{\sqrt{2}}, \quad (9.94)$$

in which  $e$  corresponds to the upper sign and  $f$  to the lower.

### Gerade/Ungerade (g/u) Parity

For homonuclear diatomic molecules, the point group  $D_{\infty h}$  contains an inversion operation  $\hat{i}$ . This inversion operation is applied in the molecular frame, unlike the  $\hat{E}^*$  symmetry operation that is applied in the laboratory frame. Moreover,  $\hat{E}^*$  (or  $\hat{\sigma}_v$ ) is a symmetry operation for all diatomic molecules, while  $\hat{i}$  is a symmetry operation only for homonuclear diatomic molecules.

It is useful to classify only the electronic orbital part of the wavefunction with the aid of  $\hat{i}$ . (The operation  $\hat{i}$  acts only on the spatial coordinates of electrons and leaves the vibrational, rotational, and electron spin parts of the wavefunction unchanged in any case.) Now the location of the nuclei define the molecular  $z$ -axis, so the inversion operation  $\hat{i}$  means that only the electrons (but not the nuclei or the coordinate system) are inverted through the center of the molecule,

$$\hat{i}\psi_{\text{el}}(x_i, y_i, z_i) = \psi_{\text{el}}(-x_i, -y_i, -z_i) = \pm\psi_{\text{el}}(x_i, y_i, z_i) \quad (9.95)$$

or

$$\hat{i}|\Lambda\rangle = \pm|\Lambda\rangle, \quad (9.96)$$

where the  $+$  sign corresponds to  $g$  (*gerade*) parity and  $-$  to  $u$  (*ungerade*) parity. The  $g$  or  $u$  parity is appended as a subscript to the term symbol of a diatomic molecule, for example,  ${}^2\Sigma_g^+$ ,  ${}^3\Sigma_u^-$ ,  ${}^1\Delta_u$ . Thus  $g$  and  $u$  are used to classify just the electron orbital part of the total wavefunction. The selection rule  $g \leftrightarrow u$  applies for electric dipole-allowed transitions since the transition dipole moment  $\mu$  is of  $u$  parity.

### Symmetric/Antisymmetric (s/a) Parity

For homonuclear diatomic molecules an additional symmetry called *s* (for symmetric) and *a* (for antisymmetric) can be used to classify the rotational energy levels. The Pauli exclusion principle demands that the total wavefunction *including nuclear spin* be symmetric or antisymmetric with respect to interchange of the two identical nuclei. This interchange of two identical nuclei is described by the operator  $\hat{P}_{12}$  in the laboratory frame. Experimentally it is found that if the identical nuclei are bosons ( $I = 0, 1, 2, \dots$ ), then the total wavefunction (including nuclear spin) is symmetric with respect to the  $\hat{P}_{12}$  operator, while if the identical nuclei are fermions ( $I = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ ), then the total wavefunction is antisymmetric with respect to  $\hat{P}_{12}$ . The total wavefunction is written as the product of the "normal" wavefunction  $\psi$  which includes electron spin, orbital, vibrational, and rotational parts (as discussed earlier) and a nuclear spin wavefunction  $\psi_{\text{nuc}}$ , so that we obtain

$$\hat{P}_{12}(\psi\psi_{\text{nuc}}) = +(\psi\psi_{\text{nuc}}) \quad \text{for bosons,} \quad (9.97)$$

and

$$\hat{P}_{12}(\psi\psi_{\text{nuc}}) = -(\psi\psi_{\text{nuc}}) \quad \text{for fermions.} \quad (9.98)$$

Consider the molecules  $\text{H}_2$  or  $\text{F}_2$ , in which the nuclei are fermions with  $I = \frac{1}{2}$ . In this case there are four nuclear spin wavefunctions, three of them symmetric,

$$\psi_{\text{nuc(sym.)}} = \begin{cases} \frac{\alpha(1)\alpha(2)}{\sqrt{2}} \\ \frac{\alpha(1)\beta(2) + \beta(1)\alpha(2)}{\sqrt{2}} \\ \beta(1)\beta(2) \end{cases} \quad (9.99)$$

and one of them antisymmetric,

$$\psi_{\text{nuc(antisym.)}} = \frac{\alpha(1)\beta(2) - \beta(1)\alpha(2)}{\sqrt{2}} \quad (9.100)$$

The symmetric  $\psi_{\text{nuc}}$  wavefunctions must be combined with antisymmetric normal wavefunctions (excluding nuclear spin)  $\psi$  to give an overall antisymmetric product  $\psi\psi_{\text{nuc}}$ . These rovibronic wavefunctions are therefore antisymmetric with respect to  $\hat{P}_{12}$  and are labeled *a*. Similarly, the single antisymmetric nuclear spin function must be combined with a symmetric normal wavefunction (excluding nuclear spin) to give an overall antisymmetric product  $\psi\psi_{\text{nuc}}$ . These rovibronic energy levels are therefore symmetric (with respect to  $\hat{P}_{12}$ ) and are labeled *s*. In the case of  $\text{H}_2$  the *a* rotational energy levels (called *ortho* levels) have three times the statistical weight of the *s* rotational levels (called *para* levels). By convention *ortho* levels always have the larger statistical weight and *para* levels the smaller (see Chapter 7).

Which rotational energy levels of a homonuclear diatomic molecule are *s* and which are *a* with respect to the  $\hat{P}_{12}$  operation? The problem is that  $\hat{P}_{12}$  is a permutation-inversion operator that switches (permutes) the two nuclei of a homonuclear diatomic molecule so that  $\hat{P}_{12}$  is not an ordinary group symmetry operator.

The  $\hat{P}_{12}$  operator can be expressed in terms of group symmetry operations and applied to the wavefunction in two steps. First, all of the electrons and all of the nuclei are inverted through the origin by applying the  $\hat{E}^*$  operator in the laboratory frame; then only the electrons are inverted back by applying  $\hat{i}$  in the molecular frame. Thus all + rotational energy levels have *s* symmetry for *g* electronic states or *a* symmetry

${}^1\Sigma_g^+$ J	${}^1\Sigma_g^-$ J	${}^1\Sigma_u^+$ J	${}^1\Sigma_u^-$ J
+ s e ——— 4	- a f ——— 4	+ a e ——— 4	- s f ——— 4
- a e ——— 3	+ s f ——— 3	- s e ——— 3	+ a f ——— 3
+ s e ——— 2	- a f ——— 2	+ a e ——— 2	- s f ——— 2
- a e ——— 1	+ s f ——— 1	- s e ——— 1	+ a f ——— 1
+ s e ——— 0	- a f ——— 0	+ a e ——— 0	- s f ——— 0

Figure 9.25: Parity for  ${}^1\Sigma_g$  and  ${}^1\Sigma_u$  rotational energy levels.

for  $u$  electronic states. Similarly all  $-$  rotational energy levels have  $a$  symmetry for  $g$  electronic states and  $s$  symmetry for  $u$  electronic states (Figure 9.25).

Now since the  $\hat{E}^*$  operator is represented by  $\hat{\sigma}_v$  in the molecular frame so that

$$\hat{P}_{12} = \hat{\sigma}_v^{xz} \hat{i} = \hat{\sigma}_v^{xz} \hat{\sigma}_v^{xz} \hat{C}_2(y) = \hat{C}_2(y), \quad (9.101)$$

this means that the application of  $\hat{P}_{12}$  in the laboratory frame is equivalent to  $\hat{C}_2(y)$  in the molecular frame.<sup>15</sup> Similarly the  $\hat{i}$  operator in the molecular frame ( $\hat{i} = \hat{\sigma}_v^{xz} \hat{C}_2(y)$ ) is equivalent to  $\hat{E}^* \hat{P}_{12}$  in the laboratory frame.<sup>15</sup> This is a very plausible result since the  $\hat{E}^*$  operator inverts the electrons and nuclei, while  $\hat{P}_{12}$  switches the nuclei back leaving only the electrons inverted.

Consider the ground state of the  $O_2$  molecule of  ${}^3\Sigma_g^-$  symmetry. The even  $N$  values have  $-$  parity ( $a$ ) and the odd  $N$  values  $+$  parity ( $s$ ) (Figure 9.26). The nuclear spin of  ${}^{16}O$  is zero, so only symmetric nuclear spin wavefunctions  $\psi_{\text{nuc}}$  are possible. The symmetric  $\psi_{\text{nuc}}$  wavefunction must be combined with the  $s$  symmetry  $\psi$  function because oxygen nuclei are bosons. This means that the  $a$  levels of  $O_2$  (even  $N$ ) cannot exist and transitions involving them are therefore missing in the spectrum. The  $s$  and  $a$  symmetry labels are thus very useful in establishing the relative intensities of rotational lines. In general the relative nuclear spin weight of *para* levels relative to *ortho* levels is given by  $(2I+1)I / ((2I+1)(I+1)) = I / (2I+1)$ . The electric dipole selection rule for  $s$  and  $a$  symmetry is  $s \leftrightarrow s$  and  $a \leftrightarrow a$  since electronic transitions cannot simultaneously flip nuclear spins.

## 9.5 Rotational Line Intensities

Quantitative spectroscopy is based on interpreting the intensities as well as the positions of lines. The starting point for diatomic intensities is the set of equations for a  $J' \leftrightarrow J''$  transition of an atom in Chapter 5. The line intensity equations for atoms (Section 5.7) all apply provided that the atomic line strength  $S_{J',J''}$  is replaced by the  $q_{v'-v''} |\mathbf{R}_e|^2 S_{J',J''}^{\Delta J}$  product. The molecular line strength  $S_{J',J''}$  has the same definition as the atomic line strength, i.e.,

$$S_{J',J''} \equiv \sum_{M',M''} |\langle J' M' | \boldsymbol{\mu} | J'' M'' \rangle|^2, \quad (9.102)$$

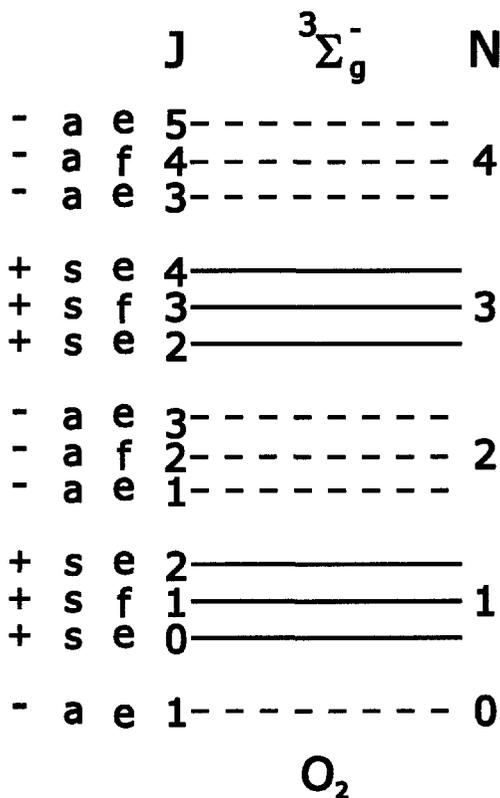


Figure 9.26: The parity of the  $O_2 X^3\Sigma_g^-$  rotational energy levels.

although in the molecular case,  $S_{J',J''}$  applies to a single rotational line in a rovibronic transition. Thus, each  $\Lambda$ -doublet is composed of two lines and the spin multiplicity adds  $2S + 1$  additional lines, even if the  $\Lambda$ -doubling or spin splitting is not resolved.

As in atoms, a line is defined as a transition between two energy levels,  $J'$  and  $J''$ .<sup>7</sup> Thus in both atoms and molecules each energy level has a degeneracy of  $2J + 1$  (in the absence of electric and magnetic fields) and for a  ${}^{2S+1}\Lambda$  molecular electronic state, there are  $(2S + 1)(2 - \delta_{0,\Lambda})$  energy levels for each  $J$ .

The rotational line strength is approximately factored into vibrational, electronic, and rotational components with

$$S_{J',J''} = q_{v'-v''} |\mathbf{R}_e|^2 S_{J',J''}^{\Delta J}. \quad (9.103)$$

Implicit in equation (9.103) is an arbitrary recipe<sup>7,18</sup> for separating the electronic and rotational components or, in other words, a simultaneous definition of the Hönl–London factors  $S_{J',J''}^{\Delta J}$  and the electronic transition dipole moment  $\mathbf{R}_e$ . For transitions with  $\Delta\Lambda = 0$  or  $\Delta\Omega = 0$  it is customary to use the  $\hat{\mu}_z \equiv \hat{\mu}^0$  operator so that for  ${}^1\Sigma^+ - {}^1\Sigma^+$ ,  ${}^1\Sigma^- - {}^1\Sigma^-$ ,  ${}^1\Pi - {}^1\Pi$ , etc., transitions we have

$$|\mathbf{R}_e|^2 = |\langle n' \Lambda S \Sigma | \hat{\mu}_z | n'' \Lambda S \Sigma \rangle|^2 = |\langle n' \Lambda | \hat{\mu}_z | n'' \Lambda \rangle|^2, \quad (9.104)$$

with  $S = \Sigma = 0$ . For these parallel singlet transitions, the Hönl–London factors of the upper third of Table 9.4 apply.

The situation is different for perpendicular transitions with  $\Delta\Lambda = \pm 1$  (or  $\Delta\Omega = \pm 1$ ) such as  ${}^1\Pi-{}^1\Sigma^+$ ,  ${}^1\Pi-{}^1\Delta$ , and so forth. If the square of the transition moment is taken as

$$|\mathbf{R}_e|^2 = |\langle n'\Lambda \pm 1S\Sigma | \hat{\mu}_x | n''\Lambda S\Sigma \rangle|^2 + |\langle n'\Lambda \pm 1S\Sigma | \hat{\mu}_y | n''\Lambda S\Sigma \rangle|^2, \quad (9.105)$$

then the Hönl-London factors of the lower two thirds of Table 9.2 must be divided by an additional factor of 2. However, the recommended definitions<sup>7</sup> of perpendicular transition moments are

$$\hat{\mu}^+ = \frac{1}{\sqrt{2}}(\hat{\mu}_x + i\hat{\mu}_y) \quad (9.106)$$

for  $\Delta\Lambda = +1$ , and

$$\hat{\mu}^- = \frac{1}{\sqrt{2}}(\hat{\mu}_x - i\hat{\mu}_y) \quad (9.107)$$

for  $\Delta\Lambda = -1$ . The electronic transition dipole matrix elements are then

$$\begin{aligned} R_e^+ &= \langle n\Lambda + 1S\Sigma | \hat{\mu}^+ | n\Lambda S\Sigma \rangle, \\ R_e^- &= \langle n\Lambda - 1S\Sigma | \hat{\mu}^- | n\Lambda S\Sigma \rangle, \\ R_e^0 &= \langle n\Lambda S\Sigma | \hat{\mu}^0 | n\Lambda S\Sigma \rangle. \end{aligned} \quad (9.108)$$

The transition moment components  $R_e^+$ ,  $R_e^-$ , and  $R_e^0$  can always be chosen to be real numbers by a suitable phase convention.<sup>18</sup>

Electromagnetic radiation is applied to the molecule in the laboratory frame and, for convenience, assumed to be polarized along the  $Z$ -axis. Derivation of rotational line strength factors thus requires that the transformation from laboratory to molecular coordinates be considered. The  $Z$  component of the transition dipole moment in the laboratory frame,  $\hat{\mu}_Z$ , is related to the components in the molecular frame by the  $\Phi_{Ij}$  direction cosines of the  $\mathbf{S}^t$  matrix (Chapter 6):

$$\begin{aligned} \hat{\mu}_Z &= \Phi_{Zx}\hat{\mu}_x + \Phi_{Zy}\hat{\mu}_y + \Phi_{Zz}\hat{\mu}_z \\ &= \frac{1}{2}(\Phi_{Zx} - i\Phi_{Zy})(\hat{\mu}_x + i\hat{\mu}_y) + \frac{1}{2}(\Phi_{Zx} + i\Phi_{Zy})(\hat{\mu}_x - i\hat{\mu}_y) \\ &\quad + \Phi_{Zz}\hat{\mu}_z, \end{aligned} \quad (9.109)$$

and using the definitions of  $\hat{\mu}^0$  and  $\hat{\mu}^\pm$  leads to the equation,

$$\begin{aligned} \hat{\mu}_Z &= \frac{1}{\sqrt{2}}(\Phi_{Zx} - i\Phi_{Zy})\hat{\mu}^+ + \frac{1}{\sqrt{2}}(\Phi_{Zx} + i\Phi_{Zy})\hat{\mu}^- \\ &\quad + \Phi_{Zz}\hat{\mu}^0. \end{aligned} \quad (9.110)$$

The basis functions for a singlet electronic state  ${}^1\Lambda$  are taken as the usual  $\psi = \psi_{el}\psi_{vib}\psi_{rot}$  product, i.e.,

$$\begin{aligned} |\psi\rangle &= |n\Lambda S = 0, \Sigma = 0\rangle |v\rangle |\Omega = \Lambda JM\rangle \\ &= |n\Lambda\rangle |v\rangle |\Lambda JM\rangle, \end{aligned} \quad (9.111)$$

so the transition dipole moment component  $M_Z$  for an allowed  ${}^1\Lambda' - {}^1\Lambda''$  transition is

$$\begin{aligned} M_Z &= \langle \Lambda' | \langle v' | \langle \Lambda' J' M' | \hat{\mu}_Z | \Lambda'' J'' M'' \rangle | v'' \rangle | \Lambda'' \rangle \\ &= \langle v' | v'' \rangle \langle \Lambda' | \langle \Lambda' J' M' | \hat{\mu}_Z | \Lambda'' J'' M'' \rangle | \Lambda'' \rangle, \end{aligned} \quad (9.112)$$

assuming that the transition dipole moment has no  $r$  dependence. Substitution of equation (9.110) for  $\hat{\mu}_Z$  into equation (9.112) gives

$$\begin{aligned} M_Z &= \mathbf{R}_e^+ \langle v' | v'' \rangle \langle \Lambda + 1 J' M | \frac{\Phi_{Zx} - i\Phi_{Zy}}{\sqrt{2}} | \Lambda J'' M \rangle \\ &\quad + \mathbf{R}_e^- \langle v' | v'' \rangle \langle \Lambda - 1 J' M | \frac{\Phi_{Zx} + i\Phi_{Zy}}{\sqrt{2}} | \Lambda J'' M \rangle \\ &\quad + \mathbf{R}_e^0 \langle v' | v'' \rangle \langle \Lambda J' M | \Phi_{Zz} | \Lambda J'' M \rangle, \end{aligned} \quad (9.113)$$

using the selection rules  $\Delta\Lambda = 0, \pm 1$  and  $\Delta M = 0$ .

Only one term on the right-hand side of equation (9.113) applies at any one time because  $R_e^+$  only has nonzero matrix elements for  $\Delta\Lambda = +1$  (e.g., a  ${}^1\Pi - {}^1\Sigma^+$  transition),  $R_e^-$  only has nonzero matrix elements for  $\Delta\Lambda = -1$  (e.g., a  ${}^1\Sigma^+ - {}^1\Pi$  transition), and  $R_e^0$  for  $\Delta\Lambda = 0$  (e.g., a  ${}^1\Delta - {}^1\Delta$  transition). Values for the transition dipole moments  $R_e^+$ ,  $R_e^-$ , and  $R_e^0$  can be derived from experiment—for example, by measuring the lifetime  $\tau$  of a transition and then converting to an Einstein  $A_{J' \rightarrow J''}$  value (and hence a transition dipole moment). Care must be taken in the interpretation of experimental lifetimes because all possible decay channels contribute to a measured lifetime, not just the desired  $J' \rightarrow J''$  transition. Alternately, the methods of modern *ab initio* quantum chemistry can be used to compute transition dipole moments.

The Hönl–London factors  $S_{J', J''}^{\Delta J}$  of Table 9.4 can be derived using equations (9.113) and (9.103). For example, the  $S_{J', J''}^{\Delta J}$  values for  $\Delta\Lambda = 1$  correspond to using the first term on the right of equation (9.113) and are

$$S_{J', J''}^{\Delta J} = 3/2 \sum_M |\langle \Lambda + 1 J' M | \Phi_{Zx} + i\Phi_{Zy} | \Lambda J'' M \rangle|^2. \quad (9.114)$$

The factor of 3 has appeared in equation (9.114) because for unpolarized light  $\hat{\mu}_X$ ,  $\hat{\mu}_Y$ , and  $\hat{\mu}_Z$  in the laboratory frame need to be considered, not just  $\hat{\mu}_Z$ . The results of the calculation must be independent of the choice of  $X$ ,  $Y$ , or  $Z$  so each transition dipole component makes an identical contribution—hence the factor of 3. The sum over  $M$  is needed in (9.114) because of the definition of the line strength  $S_{J', J''}$  given by equation (9.102). The direction cosine matrix elements are given in many sources<sup>12,19</sup> and the sum over  $M$  then results in the  $S_{J', J''}^{\Delta J}$  values of Table 9.4.<sup>18</sup>

There is a possibility of a factor of two error because the Hönl–London factors of Table 9.4 must be used with the definitions of  $R_e^+$  and  $R_e^-$  given by equation (9.108). If the “traditional” equation (9.105) is used instead for the square of the transition dipole moment, then the Hönl–London factors for  $\Delta\Lambda = +1$  and  $\Delta\Lambda = -1$  of Table 9.4 need to be divided by two.

The set of equations in SI units for line intensities of singlet electronic transitions (see section 5.7) are:

$$A_{J' \rightarrow J''} = \frac{16\pi^3 \nu^3 q_{v' - v''} |\mathbf{R}_e|^2 S_{J''}^{\Delta J}}{3\epsilon_0 \hbar c^3 (2J' + 1)}, \quad (9.115)$$

$$B_{J' \leftarrow J''} = \frac{2\pi^2 q_{v' - v''} |\mathbf{R}_e|^2 S_{J''}^{\Delta J}}{3\epsilon_0 \hbar^2 (2J'' + 1)}, \quad (9.116)$$

$$f_{J' \leftarrow J''} = \frac{8\pi^2 m_e \nu q_{v' - v''} |\mathbf{R}_e|^2 S_{J''}^{\Delta J}}{3\hbar e^2 (2J'' + 1)}, \quad (9.117)$$

$$\sigma = \frac{2\pi^2 \nu q_{v' - v''} |\mathbf{R}_e|^2 S_{J''}^{\Delta J}}{3\epsilon_0 \hbar c (2J'' + 1)} g(\nu - \nu_0). \quad (9.118)$$

The normalized lineshape function  $g(\nu - \nu_0)$  (Chapter 1) has not been included in equations (9.115) to (9.117), but can be appended to the equations. If a lineshape function  $g(\nu - \nu_0)$  is included, then a subscript  $\nu$  should be added, i.e.,

$$(A_{J' \rightarrow J''})_\nu = A_{J' \rightarrow J''} g(\nu - \nu_0), \quad (9.119)$$

but this practice is not usually followed.

The cross section  $\sigma$  (9.118) in Beer's law (Chapter 5),

$$I = I_0 e^{-\sigma(N_0 - N_1 \frac{2J''+1}{2J'+1})l}, \quad (9.120)$$

includes a lineshape function. For electronic transitions, the stimulated emission correction,  $N_1(2J'' + 1)/(2J' + 1)$  in equation (9.120) is rarely needed because  $N_1$  is usually small and can be neglected. The various equations in section 5.7 that interconvert Einstein  $A$  and  $B$  coefficients, absorption cross sections and oscillator strengths all apply because of the common definition of the line strength factor  $S_{J', J''}$  (equation (9.102)) for atoms and molecules. The line strength factor  $S_{J', J''}$ , which approximately equals  $q_{v' - v''} |\mathbf{R}_e|^2 S_{J''}^{\Delta J}$  when applied to electronic transitions of diatomic molecules, should not be confused with the Hönl-London factor (rotational line strength factor),  $S_{J''}^{\Delta J}$ , which is represented by a similar symbol. The line intensity  $S'$  often encountered in infrared work (obtained by assuming thermodynamic equilibrium and absorbing the stimulated emission correction, see Chapter 7) should not be confused with integrated cross section  $\int \sigma d\nu$  or with Hönl-London factors  $S_{J''}^{\Delta J}$  or with the line strength  $S_{J', J''}$ .

The separation of electronic and vibrational coordinates is not necessary if a bond-length dependent transition dipole moment function  $\mathbf{R}_e(r)$  is available. *Ab initio* calculations, for example, can provide electronic wavefunctions  $\psi_e(\mathbf{r}_i; r)$  at various internuclear distances,  $r$ . These wavefunctions can be used by many *ab initio* program suites to compute the electronic transition dipole moment as a function of  $r$ ,

$$\mathbf{R}_e(r) = \int \psi_{el}^*(\mathbf{r}_i; r) \boldsymbol{\mu}(\mathbf{r}_i; r) \psi_{el}(\mathbf{r}_i; r) d\tau_{el}, \quad (9.121)$$

with  $\mathbf{r}_i$  representing the electronic coordinates. Rather than separating the electronic and vibrational parts as

$$\mathbf{M}_{ev} = \mathbf{R}_e(\bar{r}) \langle v' | v'' \rangle \quad (9.122)$$

using the Franck-Condon approximation, the  $\mathbf{R}_e(r)$  transition dipole moment can be used directly. Computer programs such as Le Roy's LEVEL<sup>4</sup> can accept  $\mathbf{R}_e(r)$  functions and compute relative vibrational band intensities

$$I_{e'v'e''v''} \propto \left| \int_0^\infty \psi_{v'}^*(r) \mathbf{R}_e(r) \psi_{v''}(r) dr \right|^2 \quad (9.123)$$

from the vibrational wavefunctions  $\psi_v(r)$  for each electronic state or better yet, if  $V_{\text{eff}}(r)$  (equation (7.15)) is used, from the vibrational wavefunction for each rovibronic level with  $\psi_{vJ}(r)$ .

Integrated (i.e., no  $g(\nu - \nu_0)$ ) Einstein  $A_{J' \rightarrow J''}$  values can also be computed<sup>4</sup> from the equation,

$$A_{J' \rightarrow J''} = \frac{16\pi^3 \nu^3 S_{J''}^{\Delta J}}{3\varepsilon_0 hc^3 (2J' + 1)} |\langle \psi_{v'J'} | \mathbf{R}_e(r) | \psi_{v''J''} \rangle|^2$$

or

$$A_{J' \rightarrow J''} = 3.136 186 1 \times 10^{-7} \tilde{\nu}^3 \frac{S_{J''}^{\Delta J}}{2J' + 1} |\langle \psi_{v'J'} | \mathbf{R}_e(r) | \psi_{v''J''} \rangle|^2, \quad (9.124)$$

with  $A_{J' \rightarrow J''}$  in  $\text{s}^{-1}$ ,  $\mathbf{R}_e$  in debye,  $\tilde{\nu}$  in  $\text{cm}^{-1}$ , and  $S_{J''}^{\Delta J}$  from Table 9.4. The corresponding equation for the integrated absorption cross section  $\sigma$  (section 5.7) for a line is

$$\int \sigma d\nu = \frac{2\pi^2 \nu S_{J''}^{\Delta J}}{3\varepsilon_0 hc (2J'' + 1)} |\langle \psi_{v'J'} | \mathbf{R}_e(r) | \psi_{v''J''} \rangle|^2 \quad (9.125)$$

or

$$\int \sigma d\tilde{\nu} = 4.162 379 9 \times 10^{-19} \frac{\tilde{\nu} S_{J''}^{\Delta J}}{2J'' + 1} |\langle \psi_{v'J'} | \mathbf{R}_e(r) | \psi_{v''J''} \rangle|^2 \quad (9.126)$$

with  $\int \sigma d\tilde{\nu}$  in HITRAN units (Chapter 7) of  $\text{cm}/\text{molecule}$ ,  $\tilde{\nu}$  in  $\text{cm}^{-1}$ , and the transition moment  $\mathbf{R}_e$  in debye. Equations for conversions to other quantities and units are available in Chapter 5 and Chapter 7.

The intensities in diatomic electronic spectroscopy obey a number of useful “sum rules.” These sum rules can be used for a variety of purposes including being a sanity check to find potential errors. The vibrational sum rule for Franck–Condon factors is

$$1 = \sum_{v'} q_{v'-v''} = \sum_{v''} q_{v'-v''}. \quad (9.127)$$

For a real molecule, implicit in equation (9.127) is a sum over both discrete states and continuum states, as can be seen from the derivation of the sum rule using  $\sum |v\rangle\langle v| = 1$ , the identity for completeness of the basis set (Chapter 4), equation (4.6):

$$\begin{aligned} \sum_{v''} q_{v'-v''} &= \sum_{v''} |\langle v' | v'' \rangle|^2 \\ &= \sum_{v''} \langle v' | v'' \rangle \langle v'' | v' \rangle \\ &= \langle v' | \left( \sum_{v''} |v''\rangle \langle v''| \right) | v' \rangle \\ &= \langle v' | v' \rangle \\ &= 1 \end{aligned} \quad (9.128)$$

The Hönl–London factors of Table 9.4 also obey a rotational sum rule,

$$\sum_{\Delta J} S_{J''}^{\Delta J} = S_{J''}^P + S_{J''}^Q + S_{J''}^R = 2J'' + 1 \quad (9.129)$$

for  $\Delta\Lambda = +1$ , or  $-1$ , or  $0$ . As an example, for  $\Delta\Lambda = 0$  the sum is

$$\begin{aligned} S_{J''}^P + S_{J''}^Q + S_{J''}^R &= \frac{(J'' + 1 + \Lambda)(J + 1 - \Lambda)}{J'' + 1} \\ &\quad + \frac{(2J'' + 1)\Lambda^2}{J''(J'' + 1)} + \frac{(J + \Lambda)(J'' - \Lambda)}{J''} \\ &= \frac{J''(J'' + 1)(2J'' + 1)}{J''(J'' + 1)} = 2J'' + 1. \end{aligned} \quad (9.130)$$

The rotational sum rule applies to all rotational transitions connected to a given  $J'$  or  $J''$ , and the sum is over all possible connected  $J''$  or  $J'$  (respectively) to yield either  $2J' + 1$  or  $2J'' + 1$ .

In the general case of a nonsinglet  $2S'+1\Lambda' - 2S''+1\Lambda''$  transition it is possible to derive<sup>18</sup> a set of Hönl–London factors in the Hund's case (a) or Hund's case (b) limit. The computation of line intensities for nonsinglet molecules<sup>7,12,18</sup> will not be discussed in detail. The basic idea, though, is simple. First a set of transition dipole moments and associated Hönl–London factors are derived for the transitions between the basis set functions (Hund's case (a) or (b)) of the two states. After diagonalization of the Hamiltonian matrices for the upper and lower states, the upper and lower state wavefunctions are known as linear combinations of their basis functions. Using these linear combinations and the known line strength factors for transitions between the basis functions, the line strength for any particular rovibronic line can be computed. The calculation is somewhat similar to computing the rotational line intensities for asymmetric rotors using symmetric top basis functions with the three components of the dipole moment.

In the general case the sum rule  $\sum S_{J''}^{\Delta J}$  for an allowed  $2S'+1\Lambda' - 2S''+1\Lambda''$  transition becomes

$$\sum S_{J''}^{\Delta J} = (2 - \delta_{0\Lambda'}\delta_{0\Lambda''})(2S + 1)(2J + 1), \quad (9.131)$$

with the sum over all  $\Delta J$  for a given state and over all spin-components and any  $\Lambda$ -doublets. In essence for each  $J''$  (or  $J'$ ) of an electronic state, the spin-components contribute  $2S + 1$  levels and  $\Lambda$ -doubling contributes either 1 ( $\Lambda = 0$ ) or 2 ( $\Lambda > 0$ ) levels to the sum, in addition to the usual  $2J'' + 1$  or  $2J' + 1$  from equation (9.130).

## 9.6 Dissociation, Photodissociation, and Predissociation

Under certain conditions the dissociation energies of diatomic molecules can be determined by electronic spectroscopy. If the two potential curves involved in an electronic transition have significantly different  $r_e$  values (Figure 9.9), then the Franck–Condon factors favor long progressions in the upper state  $v'$  for absorption or in the lower state  $v''$  in emission (Figure 9.27). The resulting progressions can be extrapolated using the Le Roy–Bernstein method (Chapter 7) to determine the dissociation energies. These

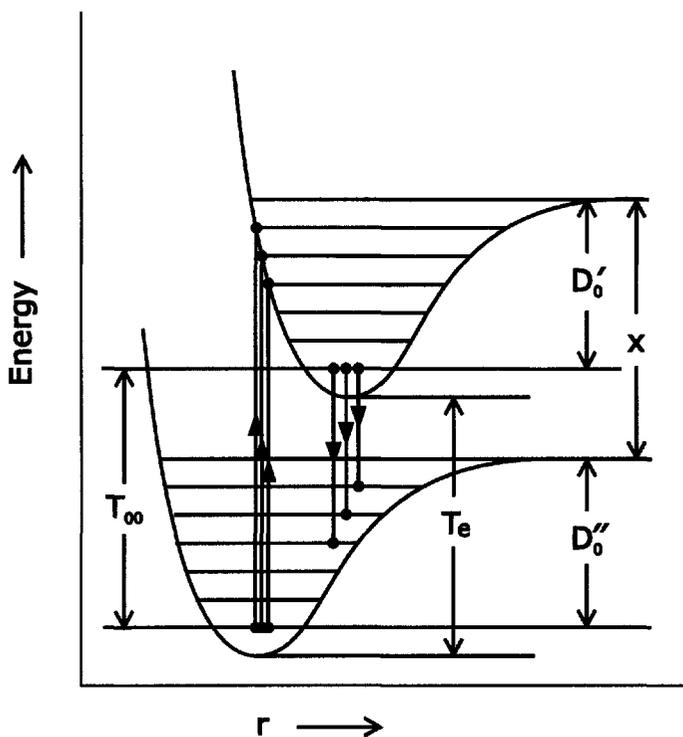


Figure 9.27: Absorption and emission progressions.

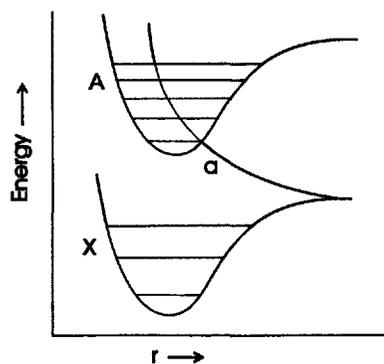


Figure 9.28: Predissociation of a state labeled A by a state labeled a.

studies are carried out by electronic spectroscopy. Note that if  $D'_0$  is known,  $D''_0$  can be calculated by using the equation

$$T_{00} = x - D'_0 + D''_0, \quad (9.132)$$

provided the atomic transition energy  $x$  is known (Figure 9.27).

If the initial state of a molecule is excited by light of energy  $h\nu > D''_0 + x$  (Figure 9.27), it can cause transitions into the vibrational continuum above the dissociation

asymptote of the excited state potential energy function. The spectrum in this region will be a continuum with very broad peaks with widths of hundreds or even thousands of  $\text{cm}^{-1}$ . For absorption from a single vibrational level of the lower state, the structure will roughly mimic the shape of the square of the initial state radial wavefunction. Observation of the low-energy onset of this continuum can be used as a means of determining the absolute energy of the upper state asymptote, and hence also the ground state dissociation energy,  $D_0''$ .

The transition intensity for photodissociation<sup>20,21</sup> from a particular initial state ( $v'', J''$ ) may be defined in terms of the frequency-dependent absorption cross section  $\sigma$  in SI units of  $\text{m}^2$ ,

$$\sigma(v'', J'') = \frac{2\pi^2\nu}{3\epsilon_0\hbar c} \sum_{J'} \frac{S_{J''}^{\Delta J}}{2J'' + 1} |\langle \psi_{E, J'}(\mathbf{r}) | \mathbf{R}_e(\mathbf{r}) | \psi_{v'', J''}(\mathbf{r}) \rangle|^2. \quad (9.133)$$

For practical calculations equation (9.133) for  $\sigma$  (in  $\text{\AA}^2$ ) becomes

$$\sigma(v'', J'') = 3.22696 \times 10^{-4} \bar{\nu} \sqrt{\mu/K(E)} \sum_{J'} \frac{S_{J''}^{\Delta J}}{2J'' + 1} \left( \int_0^\infty \psi_{E, J'}^*(r) \mathbf{R}_e(r) \psi_{v'', J''}(r) dr \right)^2, \quad (9.134)$$

in which the transition dipole moment has units of debye,  $\bar{\nu}$  is the transition wavenumber in  $\text{cm}^{-1}$ ,  $K(E) = E - V(r = \infty)$  is the asymptotic radial kinetic energy of the molecular fragments dissociating along the final state potential energy curve, in  $\text{cm}^{-1}$ ,  $\mu$  is the reduced mass in atomic mass units (u), and the final state continuum wavefunction  $\psi_{E, J'}(r)$  is normalized to unit asymptotic amplitude. Note, however, that the quantity normally reported experimentally is the decadic molar extinction coefficient  $\epsilon$  (equation (1.63)) in units of  $\text{liter mole}^{-1} \text{cm}^{-1}$ ,  $\epsilon(v, J; \bar{\nu}) = (10^{-19} N_A / \ln(10)) \sigma(v, J)$ .

It is quite common for two potential curves to cross (Figure 9.28). Of course, if the two crossing curves have the same symmetry, then an avoided crossing occurs because of the noncrossing rule. (The noncrossing rule states that when two potential energy curves of the same symmetry try to cross, there will always be a mixing of the two wavefunctions to give an avoided crossing.) In any case, in the vicinity of the crossing point an interaction, no matter how small, exists between the diabatic (nonmixed) curves. The interaction acts as a perturbation to mix the wavefunctions of the two states near the crossing point. This means that the “bound” wavefunctions of the  $A$  state (Figure 9.28) have some “free”  $a$ -state character near the crossing point (and vice versa). The  $A - X$  transition, therefore, will display predissociation, in which some levels of the  $A$  state are able to dissociate. This type of predissociation is present in  $\text{NH}$ ,<sup>22</sup> for which the states are  $X^3\Sigma^-$ ,  $A^3\Pi$ , and the repulsive state is  $^5\Sigma^-$ . In general, predissociation occurs whenever there is a substantial radial matrix element between a bound vibrational wavefunction and a continuum radial wavefunction (approximately a sine wave) of another electronic state.<sup>20</sup> These overlap matrix elements tend to be largest if the crossing point occurs near the outer turning point of the bound vibrational level (Figure 9.28), but a curve crossing is not necessary for predissociation.

The *Golden Rule* expression for such predissociation rates  $k(v, J)$  is, in units of  $\text{s}^{-1}$ ,

$$k(v, J) = (4\pi^2/h) |\langle \psi_{E, J'}(\mathbf{r}) | H_{A, a}(\mathbf{r}) | \psi_{v, J}(\mathbf{r}) \rangle|^2, \quad (9.135)$$

or

$$k(v, J) = 9.175\,553\,9 \times 10^{10} \sqrt{\mu/K(E)} \left( \int_0^\infty \psi_{E,J}(r) H_{A,a}(r) \psi_{v,J}(r) dr \right)^2, \quad (9.136)$$

in which  $H_{A,a}(r)$  is the coupling strength function in units of  $\text{cm}^{-1}$ , and the other symbols are defined above for equation (9.134).

Predissociation is surprisingly common in the spectroscopy of diatomic molecules. The presence of predissociation is evident when rotational or vibrational structure abruptly terminates, or if the lines become broad (Chapter 1), in accordance with the Heisenberg energy-time uncertainty principle ( $\Delta E \Delta t \geq \hbar$ ). The energy corresponding to the wavelength at which predissociation occurs provides an upper limit to the ground-state dissociation energy.

### Problems

- For the diatomic molecules  $\text{Na}_2$ ,  $\text{CO}$ ,  $\text{CO}^+$ ,  $\text{SO}$ , and  $\text{NO}$ :
  - What are the lowest energy configurations and the bond order?
  - What terms arise from each configuration and which term lies lowest in energy?
  - For each term, what levels arise ( $\Omega$  components) and which level lies lowest in energy?
  - Predict the lowest energy excited electronic level for each molecule.
- The  $\text{CrO}$  molecule has a  $\sigma\delta^2\pi$  configuration.
  - What electronic states arise from this configuration?
  - What is the ground state of  $\text{CrO}$ , and is it regular or inverted?
  - If the spin-orbit coupling constant  $A$  is  $63 \text{ cm}^{-1}$  for the ground state, calculate the energy-level pattern of the spin components.
- For the  $^{63}\text{Cu}_2$  molecule the following spectroscopic constants were determined for the  $B^1\Sigma^+ - X^1\Sigma^+$  electronic transition:

$$\begin{array}{ll} T_e = 21\,757.619 \text{ cm}^{-1} & \\ \omega'_e = 246.317 \text{ cm}^{-1} & \omega''_e = 266.459 \text{ cm}^{-1} \\ \omega_e x'_e = 2.231 \text{ cm}^{-1} & \omega_e x''_e = 1.035 \text{ cm}^{-1} \\ B'_e = 0.098\,847 \text{ cm}^{-1} & B''_e = 0.108\,781 \text{ cm}^{-1} \\ \alpha'_e = 0.000\,488 \text{ cm}^{-1} & \alpha''_e = 0.000\,620 \text{ cm}^{-1} \end{array}$$

- Construct the Deslandres table for  $0 \leq v' \leq 3$  and  $0 \leq v'' \leq 3$ .
  - Is the  $1 - 0$  band degraded to longer or shorter wavelengths?
  - At what  $J$  and wavenumber will the  $1 - 0$  band head occur?
- The following spectroscopic constants were determined for the  $\text{SrS } A^1\Sigma^+ - X^1\Sigma^+$  transition:

$$\begin{aligned}
 T_e &= 13\,932.707 \text{ cm}^{-1} \\
 \omega'_e &= 339.145 \text{ cm}^{-1} & \omega''_e &= 388.264 \text{ cm}^{-1} \\
 \omega_e x'_e &= 0.5524 \text{ cm}^{-1} & \omega_e x''_e &= 1.280 \text{ cm}^{-1} \\
 r'_e &= 2.511\,60 \text{ \AA} & r''_e &= 2.439\,68 \text{ \AA} \\
 D'_0 &= 3.48 \text{ eV} & D''_0 &\approx 3 \text{ eV}
 \end{aligned}$$

- (a) Calculate the Morse potential parameters for the  $X^1\Sigma^+$  and  $A^1\Sigma^+$  states of SrS, and graph the potential curves.
- (b) Predict the strongest band in the emission spectrum from  $v' = 0$  and in the absorption spectrum from  $v'' = 0$ .
5. The following vibrational bands were observed for the first negative system of  $\text{N}_2$  (in  $\text{cm}^{-1}$ ):

17 046.5	23 992.5	28 299.7
17 373.2	24 144.9	30 221.1
19 121.4	25 542.1	30 306.3
19 416.6	25 739.4	30 355.0
19 692.7	25 913.5	30 371.6
21 229.1	26 065.5	32 489.8
21 491.1	26 183.6	
21 734.5	26 261.4	
21 952.1	27 908.6	
23 368.3	28 051.2	
23 600.6	28 169.7	
23 807.9	28 254.1	

- (a) Assign the vibrational quantum numbers to the bands and construct a Deslandres table. (*Hint*: Draw a stick spectrum of the data.)
- (b) Derive a set of spectroscopic constants that reproduce this data set.
6. Derive the Hamiltonian matrices (9.57) and (9.60).
7. Obtain programs to calculate RKR potentials<sup>3</sup> and Franck–Condon factors<sup>4</sup> (see <http://leroy.uwaterloo.ca/>). Using the data for  $\text{Cu}_2$  in problem 3, calculate the Franck–Condon factors for the Deslandres table of part 3(a).
8. Using the data for  $\text{Cu}_2$  in problem 3, calculate approximate Franck–Condon factors for the Deslandres table of part 3(a). Assume that the vibrational wavefunctions are represented by harmonic oscillator wavefunctions and then compute the overlaps numerically using a computer program such as Maple, Mathcad, or Mathematica.
9. In general, Franck–Condon factors are best calculated numerically, but some analytical approximations are available (Nicholls, *J. Chem. Phys.* **74**, 6980 (1981)). For example, for a diatomic molecule A–B with  $\omega'_e$ ,  $r'_e$ ,  $\omega''_e$ ,  $r''_e$ , and  $\mu_{AB}$ , the Franck–Condon factor for the  $0-v$  bands are given approximately by the Poisson probability equation

$$q_{0-v} = u^v e^{-u} / v!$$

with  $u = (\Delta r_e)^2 \mu_{AB} \bar{\omega}_e / 67.44$ ,  $\Delta r_e$  in Å,  $\mu_{AB}$  in atomic mass units u and  $\bar{\omega}_e$  in  $\text{cm}^{-1}$ . The average vibrational frequency  $\bar{\omega}_e$  is defined by

$$\bar{\omega}_e^{1/2} = 2(\omega'_e \omega''_e)^{1/2} / ((\omega'_e)^{1/2} + (\omega''_e)^{1/2}).$$

For the  $B-X$  transition of  $^{79}\text{Br}_2$  (Figure 9.9), the required constants are  $\omega'_e = 168 \text{ cm}^{-1}$ ,  $\omega''_e = 325 \text{ cm}^{-1}$ ,  $r'_e = 2.678 \text{ Å}$ ,  $r''_e = 2.281 \text{ Å}$ . Compute the Franck–Condon factors for the  $0-15$  to  $0-25$  bands and check the graphical prediction from Figure 9.9 that the  $0-18$  band is the strongest band in emission.

10. The lifetime of the  $v' = 0$  level of the  $A^1\Sigma_u^+$  state of  $\text{Li}_2$  has been measured to be 18 ns. The  $0-0$  band is at  $14020 \text{ cm}^{-1}$ .
  - (a) Calculate the oscillator strength,  $f_{\text{abs}}$ , and the electronic transition dipole moment  $|\mathbf{R}_e|$  (in debye) for the  $A^1\Sigma_u^+ - X^1\Sigma_g^+$  electronic transition. How do the vibrational and rotational sum rules help in the calculation?
  - (b) Given that the Franck–Condon factor for the  $0-0$  band of the  $A^1\Sigma_u^+ - X^1\Sigma_g^+$  transition at  $14020 \text{ cm}^{-1}$  is 0.052, calculate the Einstein  $A$  value for the band. What would the Einstein  $A$  value be for the  $R(2)$  transition of this band?
  - (c) Assuming Doppler broadening, what would the peak transmission ( $I/I_0$ ) be for the  $R(2)$  line of the  $0-0$  band of  $A^1\Sigma_u^+ - X^1\Sigma_g^+$  transition? The  $^7\text{Li}_2$  dimer is in a heated furnace 10 cm long at  $750^\circ\text{C}$  and has a partial pressure of 17 mTorr. ( $B''_e = 0.672 \text{ cm}^{-1}$  and  $\omega''_e = 351 \text{ cm}^{-1}$  are needed to compute the partition function and the population in  $v'' = 0$ ,  $J'' = 2$ .)

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## Chapter 10

# Electronic Spectroscopy of Polyatomics

### 10.1 Orbitals and States

The spectroscopy of polyatomic molecules has much more variety than the spectroscopy of diatomic molecules. Rather than try to survey this vast field, only a few topics will be considered in order to provide the reader with a glimpse of it.

Molecular orbital theory is the key to understanding the electronic structure of polyatomic molecules. The electronic Schrödinger equation

$$\hat{H}_{\text{el}}\psi = E_{\text{el}}\psi \quad (10.1)$$

can be solved approximately by constructing a set of molecular orbitals in which each molecular orbital is a linear combination of atomic orbitals

$$\phi_{\text{MO}} = \sum c_i \phi_i, \quad (10.2)$$

in the same manner as has been discussed for diatomic molecules. The total wavefunction is a Slater determinant of MOs,

$$\psi = |\phi_{\text{MO}}(1)\bar{\phi}_{\text{MO}}(2)\cdots|. \quad (10.3)$$

These electronic wavefunctions are constructed in such a way that they are eigenfunctions of the symmetry operators for a specific molecular point group. Moreover, since the symmetry operators commute with the electronic Hamiltonian operator, that is,

$$[\hat{H}_{\text{el}}, \hat{O}_R] = 0, \quad (10.4)$$

these wavefunctions are simultaneously eigenfunctions of the electronic Hamiltonian operator. Thus, the electronic wavefunctions are classified by the irreducible representations of the appropriate molecular point group, and the electronic wavefunction belongs to a particular irreducible representation. For example, because the point group symmetry of water is  $C_{2v}$ , the electronic wavefunctions of water must have  $A_1$ ,  $A_2$ ,  $B_1$ , or  $B_2$  symmetry.

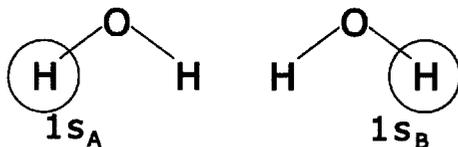


Figure 10.1: The  $1s$  orbitals on the H atoms in the  $\text{H}_2\text{O}$  molecule.

### *Walsh's Rules: Qualitative Molecular Orbital Theory*

A great deal of useful insight into the electronic structure of molecules can be gained through the use of qualitative molecular orbital theory. (Of course, even more information is provided by quantitative molecular orbital theory, but this is itself a subfield of chemistry as extensive as molecular spectroscopy.) The application of rudimentary molecular orbital theory in predicting the geometry and electronic structure of molecules was pioneered by A. D. Walsh in a series of influential papers published in the early 1950s. In honor of Walsh's work, particularly in deducing whether the geometry of a triatomic molecule is linear or bent, qualitative molecular orbital predictions can be obtained from a set of simple rules known as "Walsh's rules."

The dihydride triatomics with the structural formula  $\text{AH}_2$  can be either linear ( $D_{\infty h}$ ) or bent ( $C_{2v}$ ). Consider the possible molecular orbitals of the  $\text{H}_2\text{O}$  molecule formed by simple linear combination of the valence  $1s$  orbitals of H (Figure 10.1) and the  $2s$  and  $2p$  orbitals of O. The two  $1s$  atomic hydrogen orbitals are not suitable wavefunctions since they do not have the correct  $C_{2v}$  symmetry since, for example, the  $\hat{C}_2$  symmetry operator converts the  $1s_A$  orbital into the  $1s_B$  orbital, i.e.,

$$\hat{C}_2 1s_A = 1s_B. \quad (10.5)$$

The first step is to form symmetry-adapted linear combinations (SALCs) of atomic orbitals either by inspection or by the use of projection operators.

The four irreducible representations of the  $C_{2v}$  point group give rise to four projection operators:

$$\hat{P}^{A_1} = \sum \chi_R^{A_1} \hat{O}_R = \hat{O}_E + \hat{O}_{C_2} + \hat{O}_{\sigma_v} + \hat{O}_{\sigma'_v} \quad (10.6)$$

$$\hat{P}^{A_2} = \hat{O}_E + \hat{O}_{C_2} - \hat{O}_{\sigma_v} - \hat{O}_{\sigma'_v} \quad (10.7)$$

$$\hat{P}^{B_1} = \hat{O}_E - \hat{O}_{C_2} + \hat{O}_{\sigma_v} - \hat{O}_{\sigma'_v} \quad (10.8)$$

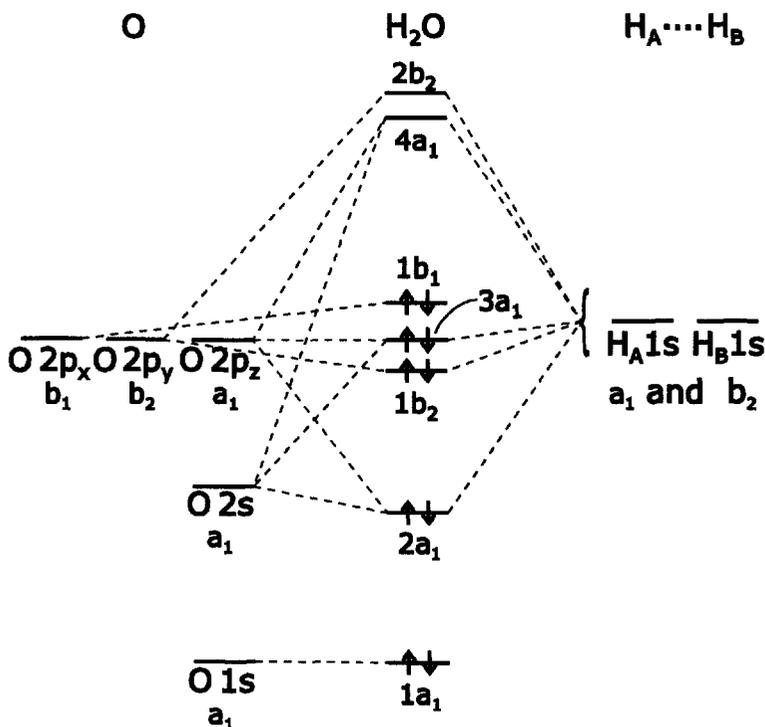
$$\hat{P}^{B_2} = \hat{O}_E - \hat{O}_{C_2} - \hat{O}_{\sigma_v} + \hat{O}_{\sigma'_v}. \quad (10.9)$$

The application of these four operators to the  $1s_A$  function projects out two functions of the appropriate symmetry, namely

$$\phi_{a_1} = \frac{1s_A + 1s_B}{\sqrt{2}}, \quad (10.10)$$

$$\phi_{b_2} = \frac{1s_A - 1s_B}{\sqrt{2}}, \quad (10.11)$$

upon normalization (assuming orthonormal atomic basis functions). With the oxygen atom situated at the origin of the coordinate system, the symmetries of the oxygen

Figure 10.2: Molecular orbital diagram for the  $\text{H}_2\text{O}$  molecule.

valence atomic orbitals  $s$ ,  $p_x$ ,  $p_y$ , and  $p_z$  are  $a_1$ ,  $b_1$ ,  $b_2$ , and  $a_1$ , respectively. The hydrogen and oxygen symmetry orbitals are then combined by forming linear combinations to produce the molecular orbitals shown in Figure 10.2. The O  $1s$  core orbital is a nonbonding core orbital with  $a_1$  symmetry, while the O  $2s$  orbital is a valence orbital and is the second orbital of  $a_1$  symmetry. Since the O  $1s$   $a_1$  orbital is lower in energy than O  $2s$   $a_1$  it is labeled  $1a_1$ , while the O  $2s$  orbital is labeled  $2a_1$ . The remaining orbitals are labeled in a similar fashion, in which the numbering starts with the lowest energy orbital for each symmetry type.

The electronic configuration of the ground state of  $\text{H}_2\text{O}$  is thus

$$(1a_1)^2(2a_1)^2(1b_2)^2(3a_1)^2(1b_1)^2,$$

and is denoted  $\tilde{X}^1A_1$ . The labeling scheme for states,  $\tilde{X}$ ,  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{a}$ ,  $\tilde{b}$ , and so on, parallels that of diatomic molecules, but a tilde is added in order to distinguish the  $\tilde{A}$  or  $\tilde{B}$  states from the  $A$  or  $B$  labels of the irreducible representations. For consistency, the tilde is used in the state labels of all polyatomic molecules, even linear ones. The orbitals are labeled according to the appropriate irreducible representations using lowercase letters, but the overall symmetry of an electronic state is capitalized. The multiplicity  $2S + 1$  appears as a left superscript.

The highest occupied molecular orbital (HOMO) in  $\text{H}_2\text{O}$  is the nonbonding, out-of-plane O  $2p_x$  orbital. The lowest unoccupied molecular orbital (LUMO) is the strongly antibonding  $4a_1$  orbital (Figure 10.2). As is often the case, the first excited configuration of  $\text{H}_2\text{O}$  is obtained by transferring an electron from the HOMO to the LUMO to give

the excited electronic configuration

$$(1a_1)^2(2a_1)^2(1b_2)^2(3a_1)^2(1b_1)^1(4a_1)^1.$$

This configuration gives rise to states with  $b_1 \otimes a_1 = b_1$  orbital symmetry and a total electronic spin of  $S = 0$  or 1. Qualitative molecular orbital theory thus predicts that there are two states associated with this excited LUMO configuration,  $\tilde{A}^1B_1$  and  $\tilde{a}^3B_1$ . Furthermore, Hund's first rule, which states that among all states arising from the same configuration, the one with the highest multiplicity is lowest in energy, predicts that  $\tilde{a}^3B_1$  is lower in energy than  $\tilde{A}^1B_1$ .

The selection rules for electronic transitions are derived through the use of the transition moment integral

$$\int \psi_{el,f}^* \boldsymbol{\mu} \psi_{el,i} d\tau. \quad (10.12)$$

In the case of  $H_2O$ , the components of  $\boldsymbol{\mu}$  have  $B_1$ ,  $B_2$ , and  $A_1$  symmetry, while the initial and final electronic states have  $A_1$  and  $B_1$  symmetry. Thus the  $\tilde{A}^1B_1 - \tilde{X}^1A_1$  transition should be a fully allowed, electric-dipole transition in which the electron is transferred out of a nonbonding  $b_1$  orbital into an antibonding  $a_1$  orbital. The  $H_2O \tilde{A}^1B_1 - \tilde{X}^1A_1$  transition is present in the 1860 to 1450 Å region of the vacuum ultraviolet.<sup>1</sup> The diffuse nature of this particular electronic transition, typical of electronic transitions observed in the vacuum ultraviolet (VUV) region of the spectrum, is due to molecular photodissociation.

An example of a linear  $AH_2$  molecule is the molecule  $BeH_2$ .  $BeH_2$  is known to exist as a polymeric solid,<sup>2</sup> and it has recently been prepared in the gas phase.<sup>3</sup> The symmetry-adapted linear combinations of the two hydrogen  $1s$  orbitals for linear  $AH_2$  are by inspection

$$\phi_{\sigma_g} = \frac{1s_A + 1s_B}{\sqrt{2}}, \quad (10.13)$$

and

$$\phi_{\sigma_u} = \frac{1s_A - 1s_B}{\sqrt{2}}. \quad (10.14)$$

The molecular orbital diagram for  $BeH_2$  is given in Figure 10.3. Both the  $Be 1s$  (not shown) and  $2s$  orbitals have  $\sigma_g$  symmetry so that the valence  $\sigma_g$  orbital is labeled as  $2\sigma_g$ . The ground-state configuration for  $BeH_2$  is  $(1\sigma_g)^2(2\sigma_g)^2(1\sigma_u)^2$ , which gives an  $\tilde{X}^1\Sigma_g^+$  ground state.

Why is  $H_2O$  bent while  $BeH_2$  is linear? Geometric predictions can be obtained either from detailed *ab initio* calculations or by constructing a Walsh molecular orbital diagram. A Walsh MO diagram is a correlation diagram based on the change in orbital overlap caused by a change in geometry. The Walsh MO diagram for the  $AH_2$  case is given in Figure 10.4. The  $2\sigma_g$  orbital decreases slightly in energy as the molecule bends due to the increased overlap between the hydrogen  $1s_A$  and  $1s_B$  orbitals, while the bending of the  $A-H$  bonds lifts the degeneracy of the nonbonding  $1\pi_u$  orbital. The out-of-plane  $1b_1$  component remains nonbonding while the in-plane  $3a_1$  component becomes strongly bonding as the molecule bends.

The Walsh MO diagram predicts that all  $AH_2$  molecules with four or less valence electrons will be linear in their ground states while all  $AH_2$  molecules with five or more

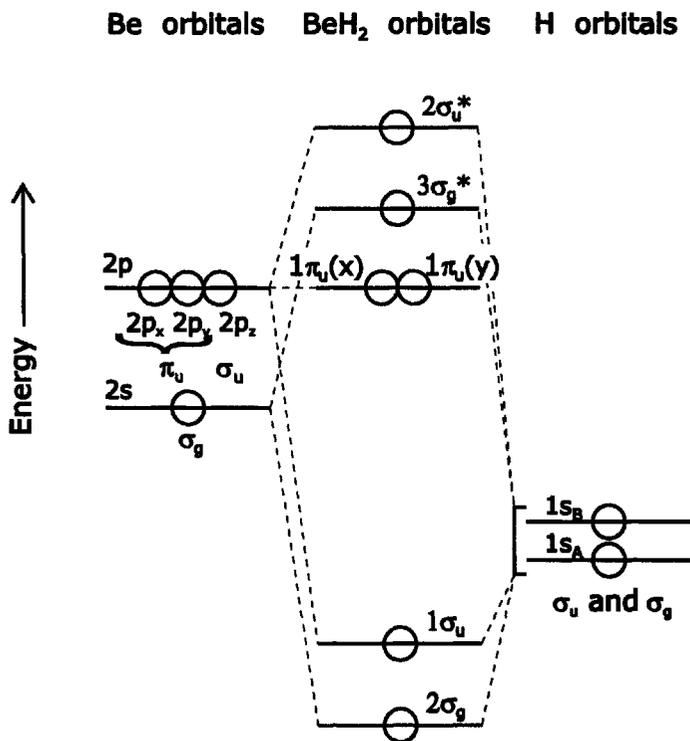


Figure 10.3: Molecular orbital diagram for BeH<sub>2</sub>.

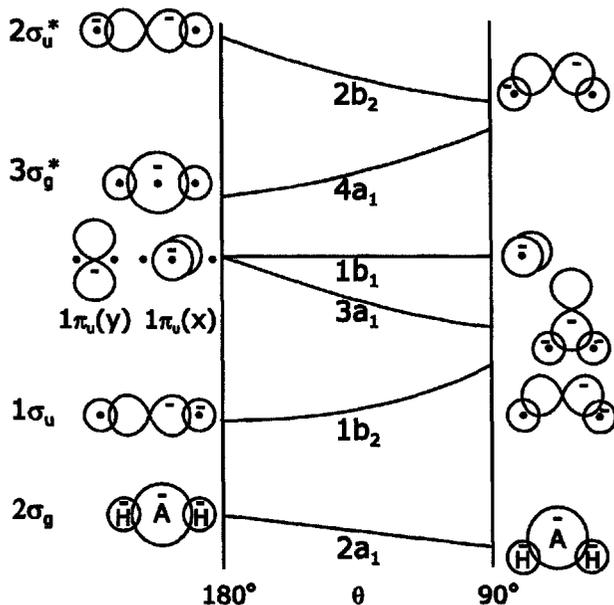


Figure 10.4: Bent-linear correlation diagram for AH<sub>2</sub>-type molecules.

valence electrons will be bent in their ground states. This prediction is confirmed by experimental evidence.

The  $\text{NH}_2$  molecule with seven valence electrons is another  $\text{AH}_2$  example. The ground-state configuration is

$$(1a_1)^2(2a_1)^2(1b_2)^2(3a_1)^2(1b_1)^1,$$

appropriate for a bent molecule. The ground state of  $\text{NH}_2$  is predicted and observed to be  $\tilde{X}^2B_1$  with the unpaired electron in a nonbonding, out-of-plane  $p$  orbital. The first excited configuration is obtained by promoting a  $3a_1$  electron to the  $1b_1$  orbital, namely

$$(1a_1)^2(2a_1)^2(1b_2)^2(3a_1)^1(1b_1)^2,$$

which gives the  $\tilde{A}^2A_1$  state. The  $\tilde{A}^2A_1 - \tilde{X}^2B_1$  transition is electric dipole-allowed and occurs in the visible region of the spectrum. Notice that the  $\text{NH}_2$  molecule in the  $\tilde{X}^2B_1$  state is predicted to be strongly bent (the observed angle is  $103.4^\circ$ , similar to the bond angle for  $\text{H}_2\text{O}$ ), while the first excited state has only one electron in the  $3a_1$  orbital so that the geometry in the  $\tilde{A}^2A_1$  state is predicted to be considerably less bent. Experimentally the bond angle for the  $\tilde{A}^2A_1$  state has been determined to be  $144^\circ$ ; this result indicates that  $\text{NH}_2$  in  $\tilde{A}^2A_1$  state is closer to being linear than bent at a right angle.<sup>1</sup>

The symmetric nonhydride triatomic BAB is another common type of molecular species. The Walsh MO diagram (bent-linear correlation diagram) is constructed for  $\text{AB}_2$  by using the same principles that were used for  $\text{AH}_2$ . The presence of valence  $s$  and  $p$  orbitals on all three centers, however, complicates the picture somewhat (see Figure 10.5).

As electrons are added to the  $\text{AB}_2$  molecular orbitals, the molecule should be linear as long as the number of valence electrons does not exceed 16. The  $\text{C}_3$  molecule with 12 valence electrons and the  $\text{CO}_2$  molecule with 16 valence electrons are linear, in agreement with this prediction.  $\text{AB}_2$  molecules with 17 or more electrons are predicted to be bent. Accordingly,  $\text{NO}_2$  with 17 valence electrons and  $\text{O}_3$  with 18 valence electrons are both bent.<sup>1</sup>

While predictions based on qualitative molecular orbital theory are in most cases reliable, there are some important exceptions. For example, the molecule  $\text{SiC}_2$ , which is isovalent with linear  $\text{C}_3$ , is found to be T-shaped.<sup>4</sup> Walsh's rules are only applicable to covalently bonded molecules, not to ionically bonded molecules. The fact that the bonding in  $\text{SiC}_2$  is best represented by the ionic species,  $\text{Si}^+\text{C}_2^-$ , is an excellent example of why Walsh's rules can fail to predict the correct geometry.

### *Hückel Molecular Orbital Theory*

Aromatic molecules such as benzene and naphthalene can be thought of as containing two types of valence electrons,  $\sigma$  and  $\pi$ . There are localized C—C and C—H  $\sigma$ -bonds, which hold the molecule together ( $\sigma$ -framework), and delocalized  $\pi$ -bonds formed from the out-of-plane carbon  $p_z$  orbitals. Hückel MO theory is based on simple principles involving the electronic properties of the  $\pi$ -molecular orbitals. Although Hückel theory is very simple, it manages to capture the essence of  $\pi$ -electronic structure in aromatic molecules.

Hückel MO theory is based on several approximations. The first is that the  $\pi$ -electrons can be treated separately from the  $\sigma$ -electrons in an aromatic molecule. The

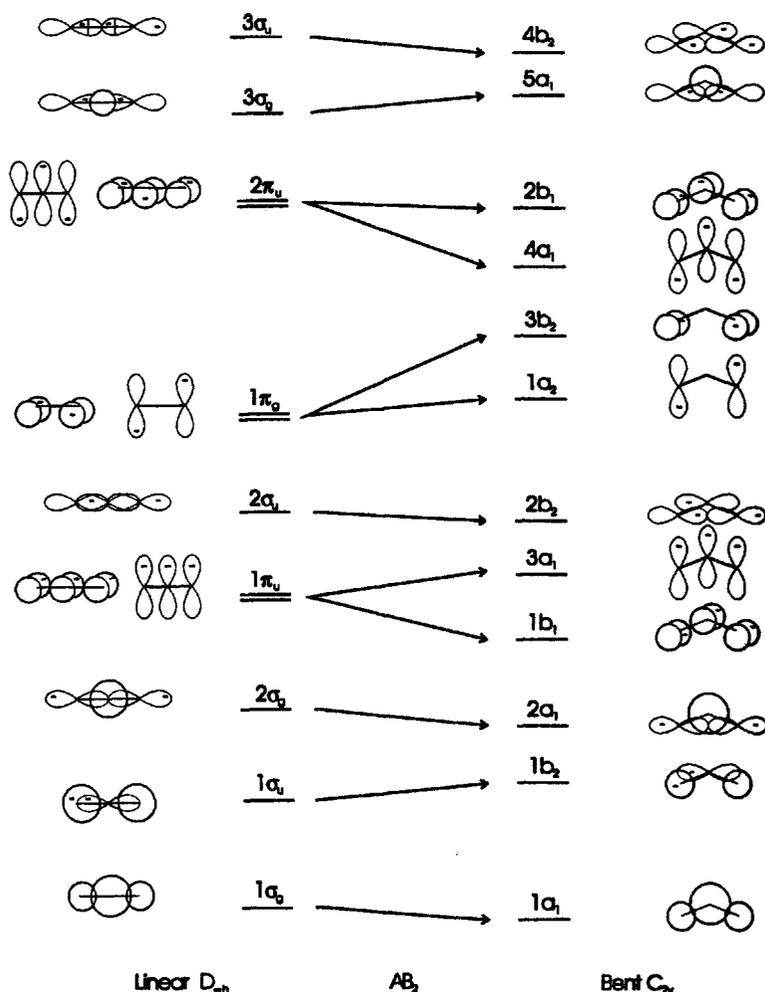


Figure 10.5: Bent-linear correlation diagram for the BAB molecule. In this case the numbering of the molecular orbitals does not include the core orbitals.

assumption of  $\sigma - \pi$  separability is equivalent to assuming that the electronic Hamiltonian operator can be separated into two parts as

$$\hat{H}_{el} = \hat{H}_{\pi} + \hat{H}_{\sigma}, \quad (10.15)$$

with wavefunctions correspondingly factored as  $\psi = \psi_{\pi}\psi_{\sigma}$ . Similarly, the  $\pi$ -electron Hamiltonian operator can be broken into a sum of separate one-electron effective Hamiltonian operators (one for each  $\pi$ -electron), namely

$$\hat{H}_{\pi} = \sum_{k=1}^{n_{\pi}} \hat{H}^{eff}(k). \quad (10.16)$$

The molecular orbitals are expressed as linear combinations of the atomic  $p_z$ -orbitals  $f_i$ , viz.,

$$\phi = \sum_{j=1}^{n_\pi} c_j f_j. \quad (10.17)$$

Substitution of equation (10.17) into the electronic Schrödinger equation then yields the set of homogeneous linear equations

$$\sum_{j=1}^{n_\pi} (H_{ij}^{\text{eff}} - S_{ij} E) c_j = 0. \quad (10.18)$$

The variational principle can then be applied to determine optimal values for the set of coefficients  $c_j$ . The variational principle states that the optimum coefficients for an approximate ground-state electronic wavefunction can be obtained by minimizing the electronic energy of the system. In matrix notation, equation (10.18) becomes

$$(\hat{\mathbf{H}}^{\text{eff}} - \mathbf{S}E)\mathbf{c} = 0 \quad (10.19)$$

in which the elements of  $\hat{\mathbf{H}}^{\text{eff}}$  and the overlap matrix  $\mathbf{S}$  are given by

$$H_{ij}^{\text{eff}} = \langle f_i | \hat{H} | f_j \rangle, \quad S_{ij} = \langle f_i | f_j \rangle.$$

A nontrivial solution to the secular equation,

$$|\hat{\mathbf{H}}^{\text{eff}} - \mathbf{S}E| = 0, \quad (10.20)$$

consists of  $n_\pi$  eigenvalues  $E_n$  with corresponding eigenfunctions

$$\phi_n = \sum_{j=1}^{n_\pi} c_{nj} f_j, \quad n = 1, \dots, n_\pi. \quad (10.21)$$

In Hückel theory the integrals  $H_{ij}^{\text{eff}}$  and  $S_{ij}$  are not determined by *ab initio* methods, but instead are determined empirically. Additional approximations are invoked in order to reduce the number of unknown parameters to just two, the integrals  $\alpha$  and  $\beta$ , defined as

$$H_{ii}^{\text{eff}} = \int f_i \hat{H}^{\text{eff}} f_i d\tau \equiv \alpha, \quad (10.22)$$

and

$$H_{ij}^{\text{eff}} = \int f_i \hat{H}^{\text{eff}} f_j d\tau \equiv \beta \quad \text{or} \quad 0. \quad (10.23)$$

If the two carbon atoms are adjacent, then  $H_{ij}^{\text{eff}} = \beta$ , while if the two carbon atoms are not adjacent, then  $H_{ij}^{\text{eff}} = 0$ . Furthermore, the overlap matrix is assumed to be the unit matrix,  $S_{ij} = \delta_{ij}$ . With these assumptions the secular equation takes a simple form in terms of the two parameters  $\alpha$  (the Coulomb integral) and  $\beta$  (the resonance or bond integral).

To illustrate Hückel molecular orbital theory, consider the  $\pi$ -electronic structure of the *s-trans*-butadiene molecule of  $C_{2h}$  symmetry (Figure 10.6). The Hückel approximations give rise to the secular equation

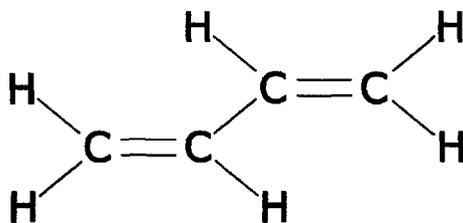


Figure 10.6: The butadiene molecule.

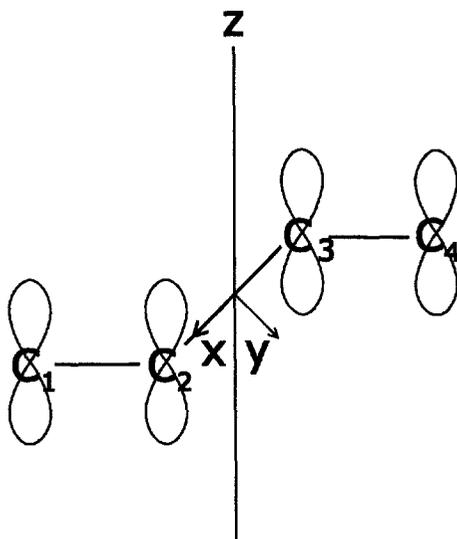


Figure 10.7: Coordinate system for butadiene.

$$\begin{vmatrix} \alpha - E & \beta & 0 & 0 \\ \beta & \alpha - E & \beta & 0 \\ 0 & \beta & \alpha - E & \beta \\ 0 & 0 & \beta & \alpha - E \end{vmatrix} = 0 \quad (10.24)$$

for the four molecular orbitals constructed from the four  $p_z$  orbitals, one on each carbon atom (Figure 10.7).

As usual, the application of molecular symmetry simplifies the problem. The four  $p_z$  orbitals form a four-dimensional reducible representation of the  $C_{2h}$  point group with characters

$$\begin{array}{c|cccc} & \hat{E} & \hat{C}_2 & \hat{i} & \hat{\sigma}_h \\ \hline \chi^{\text{red}} & 4 & 0 & 0 & -4 \end{array}$$

which can be reduced as  $\Gamma^{\text{red}} = 2\Gamma^{a_u} \oplus 2\Gamma^{b_g}$ . The symmetry-adapted linear combinations of atomic orbitals can be found by inspection, or by the use of projection operators, to be

$$g_1 = \frac{p_{z1} + p_{z4}}{\sqrt{2}}, \quad (a_u) \quad (10.25a)$$

$$g_2 = \frac{p_{z2} + p_{z3}}{\sqrt{2}}, \quad (a_u) \quad (10.25b)$$

$$g_3 = \frac{p_{z1} - p_{z4}}{\sqrt{2}}, \quad (b_g) \quad (10.25c)$$

$$g_4 = \frac{p_{z2} - p_{z3}}{\sqrt{2}}, \quad (b_g). \quad (10.25d)$$

Using these symmetry-adapted basis functions  $\{g_i\}$  rather than  $\{p_{zi}\}$  gives matrix elements

$$\begin{aligned} H'_{11} &= \left\langle \frac{p_{z1} + p_{z4}}{\sqrt{2}} \left| \hat{H} \right| \frac{p_{z1} + p_{z4}}{\sqrt{2}} \right\rangle \\ &= \alpha \end{aligned} \quad (10.26)$$

$$H'_{22} = \left\langle \frac{p_{z2} + p_{z3}}{\sqrt{2}} \left| \hat{H} \right| \frac{p_{z2} + p_{z3}}{\sqrt{2}} \right\rangle = \alpha + \beta, \quad (10.27)$$

and so on. There is also a new secular determinant,

$$\begin{vmatrix} a_u & \alpha - E & \beta & 0 & 0 \\ a_u & \beta & \alpha + \beta - E & 0 & 0 \\ b_g & 0 & 0 & \alpha - E & \beta \\ b_g & 0 & 0 & \beta & \alpha - \beta - E \end{vmatrix} = 0. \quad (10.28)$$

Notice that the secular determinant in equation (10.28) has been partitioned into two smaller symmetry blocks because all matrix elements between functions of different symmetry are zero. The four solutions of the secular equation are thus

$$E_1(1a_u) = \alpha + \frac{(1 + \sqrt{5})\beta}{2} = \alpha + 1.618\beta, \quad (10.29a)$$

$$E_2(1b_g) = \alpha + \frac{(-1 + \sqrt{5})\beta}{2} = \alpha + 0.618\beta, \quad (10.29b)$$

$$E_3(2a_u) = \alpha + \frac{(1 - \sqrt{5})\beta}{2} = \alpha - 0.618\beta, \quad (10.29c)$$

$$E_4(2b_g) = \alpha + \frac{(-1 - \sqrt{5})\beta}{2} = \alpha - 1.618\beta, \quad (10.29d)$$

while the associated normalized wavefunctions are

$$\phi_1(1a_u) = 0.3718(p_{z1} + p_{z4}) + 0.6015(p_{z2} + p_{z3}), \quad (10.30a)$$

$$\phi_2(1b_g) = 0.6015(p_{z1} - p_{z4}) + 0.3718(p_{z2} - p_{z3}), \quad (10.30b)$$

$$\phi_3(2a_u) = 0.6015(p_{z1} + p_{z4}) - 0.3718(p_{z2} + p_{z3}), \quad (10.30c)$$

$$\phi_4(2b_g) = 0.3718(p_{z1} - p_{z4}) - 0.6015(p_{z2} - p_{z3}). \quad (10.30d)$$

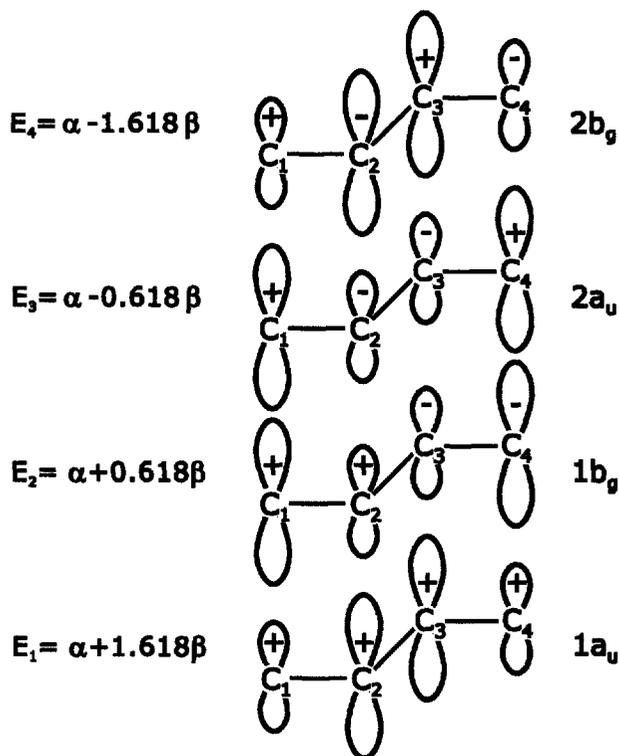


Figure 10.8: Hückel molecular orbitals of butadiene.

The atomic orbital composition of the wavefunctions is shown in Figure 10.8.

The lowest energy  $\pi$ -electron configuration of butadiene is  $(1a_u)^2(1b_g)^2$ , giving an  $\tilde{X}^1A_g$  ground state. The promotion of an electron from the  $1b_g$  HOMO to the  $2a_u$  LUMO gives the  $(1a_u)^2(1b_g)^1(1a_u)^1$  configuration and the  $\tilde{A}^1B_u$  and  $\tilde{A}^3B_u$  states. The very diffuse  $\tilde{A}^1B_u - \tilde{X}^1A_g$  electronic transition has been observed at 217 nm.<sup>1</sup> Simple Hückel theory predicts the HOMO to LUMO transition to be at  $2(0.618)\beta$ , giving a value of  $-37300 \text{ cm}^{-1}$  for  $\beta$ . Note that both Coulomb integrals ( $\alpha$ ) and resonance integrals ( $\beta$ ) are negative numbers due to the choice of the zero of energy (cf. the hydrogen atom, Chapter 5).

In general the electronic Hamiltonian operator for a linear polyene with  $n$  atoms is a symmetric tridiagonal matrix with elements along the diagonal equal to  $\alpha$  and subdiagonal and superdiagonal elements equal to  $\beta$ . For this special form of the tridiagonal matrix, the solution to the secular equation is well known and given by energies<sup>5</sup>

$$E_i = \alpha - 2\beta \cos\left(\frac{\pi i}{n+1}\right), \quad i = 1, 2, \dots, n, \quad (10.31)$$

and wavefunctions

$$\phi_i = \left(\frac{2}{n+1}\right)^{1/2} \sum_{j=1}^n p_{2j} \sin\left(\frac{i\pi j}{n+1}\right). \quad (10.32)$$

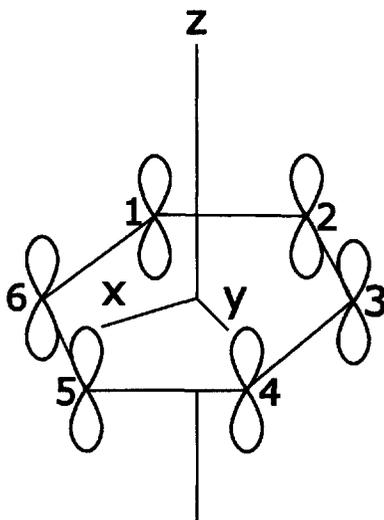


Figure 10.9: Coordinate system for the benzene molecule.

Cyclic  $\pi$ -electron molecules have a different secular equation in Hückel theory. For example, the secular equation for benzene is

$$\begin{vmatrix} \alpha - E & \beta & 0 & 0 & 0 & \beta \\ \beta & \alpha - E & \beta & 0 & 0 & 0 \\ 0 & \beta & \alpha - E & \beta & 0 & 0 \\ 0 & 0 & \beta & \alpha - E & \beta & 0 \\ 0 & 0 & 0 & \beta & \alpha - E & \beta \\ \beta & 0 & 0 & 0 & \beta & \alpha - E \end{vmatrix} = 0. \quad (10.33)$$

The nonzero elements  $H_{16} = H_{61} = \beta$  must be added in order to satisfy the cyclic boundary condition that the last carbon atom in the loop around the ring must join to the first carbon atom in the loop.

The application of molecular symmetry is again helpful in simplifying the secular equation in (10.33). The benzene molecule has  $D_{6h}$  symmetry (Figure 10.9), but it is simpler to use the rotational subgroup  $D_6$  ( $D_{6h} = D_6 \otimes C_i$ ) for the problem. The appropriate  $g$  and  $u$  labels for the wavefunctions can be determined either by inspection or by using the symmetry operators of the  $D_{6h}$  point group. The six  $p_z$  orbitals form a reducible representation with characters

$$\chi^{\text{red}} \begin{array}{c|cccccc} & \hat{E} & \hat{C}_6 & \hat{C}_3 & \hat{C}_2 & \hat{C}'_2 & \hat{C}''_2 \\ \hline & 6 & 0 & 0 & 0 & -2 & 0 \end{array}$$

in which the  $C_6$ -,  $C_3$ -, and  $C_2$ -axes lie along the  $z$ -axis, while the  $C'_2$ - and  $C''_2$ -axes lie in the molecular plane. The three  $C'_2$ -axes bisect carbon atoms, while the three  $C''_2$ -axes bisect the carbon-carbon bonds. This representation can be reduced to give

$$\Gamma^{\text{red}} = \Gamma^{a_2} \oplus \Gamma^{b_2} \oplus \Gamma^{e_1} \oplus \Gamma^{e_2}. \quad (10.34)$$

Since the  $p_z$  orbitals change sign on application of the  $\hat{\sigma}_h$  operator, inspection of the  $D_{6h}$  character table (Appendix B) for the presence of a negative sign for this operation gives

$$\Gamma^{\text{red}} = \Gamma^{a_{2u}} \oplus \Gamma^{b_{2g}} \oplus \Gamma^{e_{1g}} \oplus \Gamma^{e_{2u}}. \quad (10.35)$$

Symmetry-adapted linear combinations of the six  $p_z$  orbitals are determined by the use of projection operators, together with the orthogonality condition in the case of the degenerate  $e_1$  and  $e_2$  functions. The six symmetrized functions are

$$f_1 = \frac{p_{z1} + p_{z2} + p_{z3} + p_{z4} + p_{z5} + p_{z6}}{\sqrt{6}} \quad (a_2) \quad (10.36a)$$

$$f_2 = \frac{p_{z1} - p_{z2} + p_{z3} - p_{z4} + p_{z5} - p_{z6}}{\sqrt{6}} \quad (b_2) \quad (10.36b)$$

$$f_3 = \frac{p_{z1} - p_{z2} - 2p_{z3} - p_{z4} + p_{z5} + 2p_{z6}}{2\sqrt{3}} \quad (e_1) \quad (10.36c)$$

$$f_4 = \frac{p_{z1} + p_{z2} - p_{z4} - p_{z5}}{2} \quad (e_1) \quad (10.36d)$$

$$f_5 = \frac{p_{z1} + p_{z2} - 2p_{z3} + p_{z4} + p_{z5} - 2p_{z6}}{2\sqrt{3}} \quad (e_2) \quad (10.36e)$$

$$f_6 = \frac{p_{z1} - p_{z2} + p_{z4} - p_{z5}}{2} \quad (e_2). \quad (10.36f)$$

Constructing the secular determinant in this basis set gives

$$\begin{vmatrix} \alpha + 2\beta - E & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha - 2\beta - E & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha + \beta - E & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha + \beta - E & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha - \beta - E & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha - \beta - E \end{vmatrix} = 0, \quad (10.37)$$

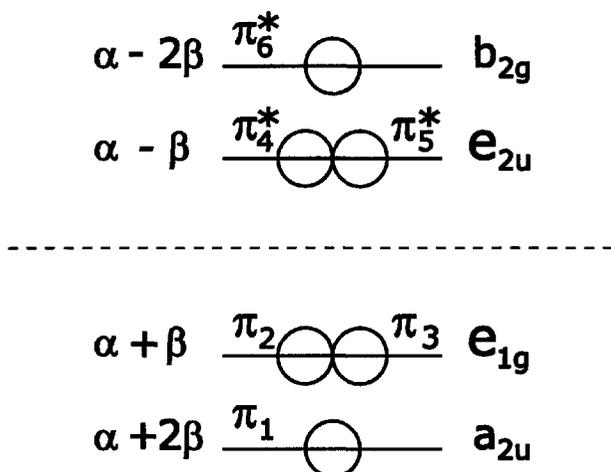
which is already diagonal. The energy levels are at  $\alpha + 2\beta$ ,  $\alpha - 2\beta$ ,  $\alpha + \beta$  (doubly degenerate) and  $\alpha - \beta$  (doubly degenerate) as shown in Figure 10.10.

The  $\pi$ -electron ground-state configuration of benzene is  $(a_{2u})^2(e_{1g})^4$  with  $\tilde{X}^1A_{1g}$  being the ground state. The lowest energy excitation promotes one electron out of the  $e_{1g}$  orbital into the  $e_{2u}$  orbital, which results in the configuration  $(a_{2u})^2(e_{1g})^3(e_{2u})^1$ . For this configuration the direct product  $e_{1g} \otimes e_{2u}$ , which reduces to  $b_{1u} \oplus b_{2u} \oplus e_{1u}$ , gives rise to the states,  $^1B_{1u}$ ,  $^3B_{1u}$ ,  $^1B_{2u}$ ,  $^3B_{2u}$ ,  $^1E_{1u}$ , and  $^3E_{1u}$ . Since  $\mu_z$  has  $A_u$  symmetry and  $(\mu_x, \mu_y)$  have  $E_{1u}$  symmetry, however, only the  $^1E_{1u} - ^1A_{1g}$  transition of benzene is allowed for the  $D_{6h}$  point group.

The famous ultraviolet transition of benzene at 260 nm turns out to be the forbidden  $\tilde{A}^1B_{1u} - \tilde{X}^1A_{1g}$  transition,<sup>1</sup> which becomes allowed as a result of vibronic coupling (see section 10.3 on vibronic coupling).

## 10.2 Vibrational Structure of Electronic Transitions

Within the Born–Oppenheimer approximation, the separation of vibrational and electronic motion leads to the concept of associating electronic states with potential energy

Figure 10.10: Energy-level diagram for the  $\pi$ -electrons of benzene.

surfaces. For a diatomic molecule, the potential energy function  $V(r)$  is a function of a single variable, the internuclear separation  $r$ . For a polyatomic molecule the potential energy function  $V(Q_i)$  is a function of  $3N - 6$  (or 5) internal coordinates, usually expressed in terms of normal modes. The simple one-dimensional diatomic potential energy curve is replaced by a multidimensional potential energy surface for each polyatomic electronic state. Shown in Figure 10.11 is a simple example of a polyatomic potential energy surface for the He(CO) van der Waals molecule.

The solution of the Schrödinger equation for nuclear motion on each potential energy surface of a polyatomic molecule provides the corresponding vibrational frequencies and anharmonicities for each electronic state, namely<sup>1</sup>

$$G(v_i) = \sum_r \omega_r \left( v_r + \frac{d_r}{2} \right) + \sum_{r,s>r} x_{rs} \left( v_r + \frac{d_r}{2} \right) \left( v_s + \frac{d_s}{2} \right) + \sum_{t,t'>t} g_{tt'} l_t l_{t'}. \quad (10.38)$$

In this equation, the  $\omega_r$  are the harmonic vibrational frequencies, each with a corresponding degeneracy  $d_r$ , the  $x_{rs}$  and  $g_{tt'}$  are anharmonic corrections, and the index  $t$  refers to degenerate modes with vibrational angular momentum  $l_t$ . The polyatomic vibrational equation (10.38) may be compared with the much simpler vibrational energy-level expression for a diatomic molecule, namely

$$G(v) = \omega_e \left( v + \frac{1}{2} \right) - \omega_e x_e \left( v + \frac{1}{2} \right)^2. \quad (10.39)$$

A vibronic transition frequency is given by the difference between two vibronic term values, that is

$$\tilde{\nu} = \Delta T_e + G'(v'_1, v'_2, \dots) - G''(v''_1, v''_2, \dots), \quad (10.40)$$

in which  $\Delta T_e$  is the minimum potential energy difference between the two states. A vibronic transition can be specified by noting the electronic transition and the vibrational quantum numbers for each of the states, for example,  $\tilde{A} - \tilde{X} (v'_1 v'_2 \dots) - (v''_1 v''_2 \dots)$ . A convenient shorthand notation for a vibronic transition of the  $\tilde{A} - \tilde{X}$  electronic transition is denoted as

## He-CO Potential Surface Recommended Fit

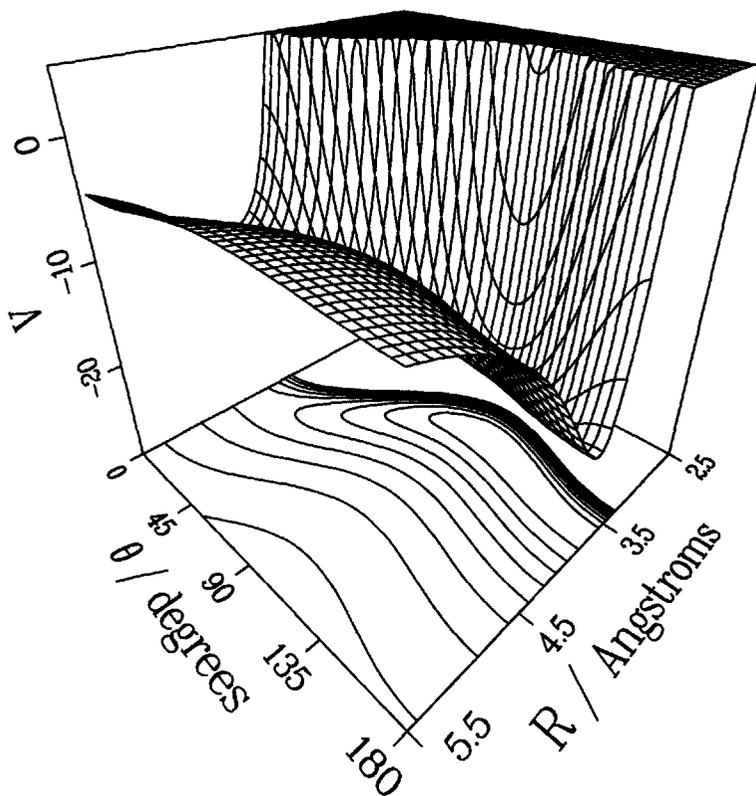


Figure 10.11: The He(CO) potential energy surface as a function of the radial distance  $R$  of the He atom from the center of mass of the CO molecule, and the angular position  $\theta$  of the He atom relative to the CO internuclear axis. The origin at  $0^\circ$  represents a colinear HeCO geometry.

$$\tilde{A} - \tilde{X} \quad 1_{v_1'}^{\nu_1'} \quad 2_{v_2'}^{\nu_2'} \cdots ,$$

with the numbers 1, 2, etc. referring to the  $\nu_1, \nu_2$ , etc., vibrational modes. For example, the formaldehyde transition from  $\tilde{X}^1A_1$  ( $v_i = 0$ ) to  $\tilde{A}^1A_2$  ( $v_2 = 2, v_4 = 1$ ) can be written either as  $\tilde{A}^1A_2 - \tilde{X}^1A_1(020100) - (000000)$  or as  $\tilde{A}^1A_2 - \tilde{X}^1A_1, 2_0^2 4_0^1$ . Except for triatomic molecules, the second notation is preferable and is now often used for infrared as well as vibronic transitions of polyatomic molecules.

The vibrational selection rules for an allowed electronic transition are determined from the Franck-Condon principle (Chapter 9). The intensity of a vibronic transition is proportional to the square of the transition moment integral,

$$\begin{aligned}
\mathbf{M}_{e'v'e''v''} &= \int \psi_{e'v'}^* \boldsymbol{\mu} \psi_{e''v''} d\tau_{ev} \\
&= \int \psi_{e'}^* \psi_{v'}^* \boldsymbol{\mu} \psi_{e''} \psi_{v''} d\tau_{ev} \\
&= \int \psi_{e'}^* \boldsymbol{\mu} \psi_{e''} d\tau_{el} \int \psi_{v'}^* \psi_{v''} d\tau_v \\
&= \mathbf{M}_{e'e''} \int \psi_{v'_1}^* \psi_{v''_1} dQ_1 \int \psi_{v'_2}^* \psi_{v''_2} dQ_2 \cdots
\end{aligned} \tag{10.41}$$

Within the realm of the Born–Oppenheimer and normal mode approximations the transition moment integral is comprised of an electronic transition dipole moment

$$\mathbf{R}_{e'e''} = \mathbf{R}_e = \langle \psi_{e'} | \boldsymbol{\mu} | \psi_{e''} \rangle$$

and a product of  $3N - 6$  (or 5) vibrational overlap integrals. For a totally symmetric vibration the selection rule on  $v$  is therefore

$$\Delta v_i = 0, \pm 1, \pm 2, \dots,$$

and the intensity of the transition determined by the Franck–Condon factor

$$q_{v'_i-v''_i} = \left| \int \psi_{v'_i}^* \psi_{v''_i} dQ_i \right|^2 \tag{10.42}$$

for that mode. For non-totally symmetric vibrations, the Franck–Condon factor vanishes for  $v_i$  values such that

$$\Delta v_i = \pm 1, \pm 3, \pm 5, \dots,$$

because the product  $\Gamma^{\psi_{v'}} \otimes \Gamma^{\psi_{v''}}$  does not contain the totally symmetric irreducible representation. For an allowed electronic transition, the nonsymmetric vibrational modes obey the selection rule

$$\Delta v_i = \pm 2, \pm 4, \pm 6, \dots$$

### 10.3 Vibronic Coupling: The Herzberg–Teller Effect

Often nonsymmetric vibrational transitions occur in an electronic transition with the selection rule

$$\Delta v = \pm 1, \pm 3, \pm 5 \dots,$$

although, as discussed above, this is forbidden for electric dipole-allowed electronic transitions. The fact that these transitions tend to be relatively weak is indicative of an electronically forbidden character. The  $\tilde{A}^1B_{2u} - \tilde{X}^1A_{1g}$  transition of benzene at 260 nm is a classic example. The  $\tilde{A} - \tilde{X}$  transition is forbidden (because  $x$ ,  $y$ , and  $z$  belong to the  $E_{1u}$  and  $A_{2u}$  irreducible representations), but has been observed with moderate intensity in the near UV region (Figure 10.12). The 0–0 origin band is not observed, but  $\Delta v_6 = 1$  transitions ( $\nu_6$  has  $e_{2g}$  symmetry) are prominent. In addition a long progression in  $\nu_1$ , the ring breathing mode, is present. (Note that, by convention,

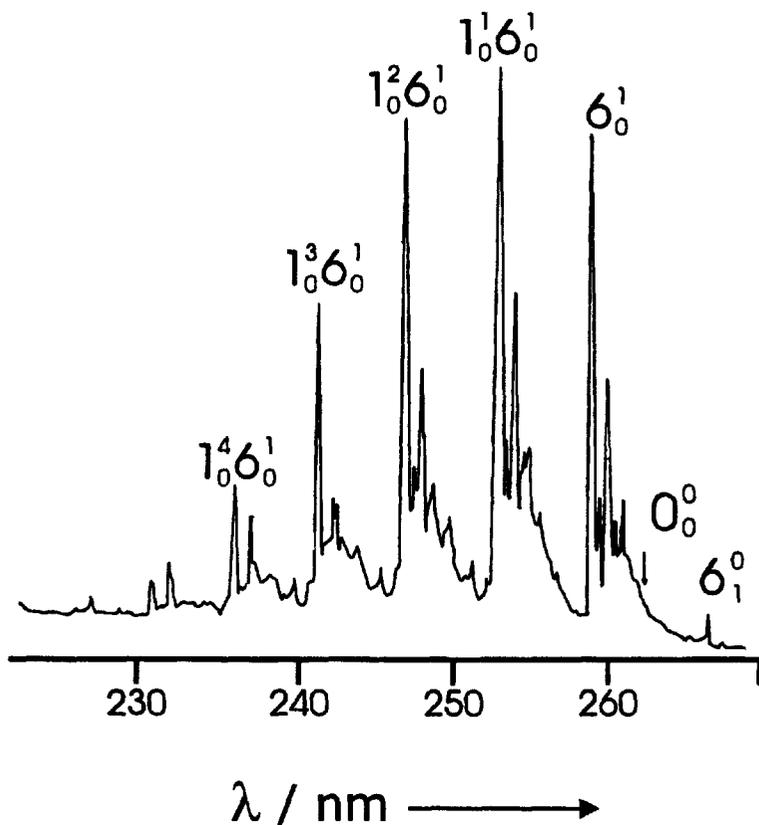


Figure 10.12: The  $\tilde{A}^1B_{2u} - \tilde{X}^1A_{1g}$  transition of benzene near 260 nm.

the Wilson numbering scheme<sup>6,7</sup> for the vibrational modes of benzene is commonly used rather than the Herzberg numbering scheme recommended by Mulliken.<sup>8</sup>) The  $\nu_1$  mode is totally symmetric ( $a_{1g}$ ), so there are no symmetry restrictions on  $\Delta v_1$ . The progression in  $\nu_1$  (Figure 10.12) means that the benzene ring has  $D_{6h}$  symmetry in both ground and  $\tilde{A}$  states but the size of the ring is different in the two states. A detailed rotational analysis confirms that the C—C bond length increases by  $+0.0353 \text{ \AA}$  from  $1.397 \text{ \AA}$  in the  $\tilde{X}^1A_{1g}$  ground state.<sup>9</sup> The  $\nu_1$  mode, however, always occurs in combination with an odd number of non-totally symmetric vibrations, often  $\nu_6$  (Figures 10.12 and 10.13).

The  $\tilde{A} - \tilde{X}$  transition is a forbidden electronic transition that becomes allowed by vibronic coupling, as first explained by Herzberg and Teller.<sup>1,10</sup> In this case, the total vibronic symmetry ( $\Gamma^{\text{el}} \otimes \Gamma^{\text{vib}} = \Gamma^{\text{vibronic}}$ ) must be examined. In the case of the benzene  $\tilde{A}^1B_{2u} - \tilde{X}^1A_{1g} 6_0^1$  transition, this requires that the transition moment integral

$$\int \psi_{\text{vibronic}}^* \boldsymbol{\mu} \psi_{\text{vibronic}}' d\tau \quad (10.43)$$

be considered.

The vibronic symmetry of the ground state is  $A_{1g}$ , while the vibronic symmetry of the excited state is  $B_{2u} \otimes e_{2g} = E_{1u}$ . Since  $\mu_x$  and  $\mu_y$  have  $E_{1u}$  symmetry, the transition moment integral is now nonzero. Provided that the vibrational and electronic degrees



$$\psi_{e'} = \psi_f^0 + \sum_{k \neq f} c_k \psi_k^0, \quad (10.47)$$

with the expansion coefficients  $c_k$  given by

$$c_k = \frac{\langle \psi_k^0 | \hat{H}' | \psi_f^0 \rangle}{E_f^0 - E_k^0}. \quad (10.48)$$

The degree of mixing is determined by the ratio of the magnitude of the vibronic coupling matrix element  $\langle \psi_k^0 | \hat{H}' | \psi_f^0 \rangle$  and the separation  $(E_f^0 - E_k^0)$  between the interacting states. The electronic transition moment integral then becomes

$$\begin{aligned} \mathbf{M}_{e'e''} &= \int \psi_{e'}^* \boldsymbol{\mu} \psi_{e''} d\tau \\ &= \int (\psi_f^*)^0 \boldsymbol{\mu} \psi_{e''}^0 d\tau_e + \sum c_k \int (\psi_k^*)^0 \boldsymbol{\mu} \psi_{e''}^0 d\tau_e. \end{aligned} \quad (10.49)$$

For the benzene  $\tilde{A}^1B_{2u} - \tilde{X}^1A_{1g}$  transition the first term in (10.49) vanishes and the sum is dominated by interaction with the nearby  $\tilde{C}^1E_{1u}$  state. The  $\tilde{C}^1E_{1u} - \tilde{X}^1A_{1g}$  electronic transition is fully allowed. The mode  $\nu_6$  of  $e_{2g}$  symmetry mixes with the  $\tilde{A}^1B_{2u}$  electronic state to give a vibronic state of  $E_{1u}$  symmetry. The  $\tilde{A}^1B_{2u} \nu_6 = 1$  vibronic state mixes with the  $\tilde{C}^1E_{1u}$  electronic state of the same symmetry. The  $\tilde{A}^1B_{2u} - \tilde{X}^1A_{1g}$  electronic transition<sup>1,9</sup> is observed because of intensity “borrowing” or “stealing” from the nearby strong  $\tilde{C}^1E_{1u} - \tilde{X}^1A_{1g}$  transition through vibronic coupling involving mainly  $\nu_6$ .

The interaction matrix element between the two excited states determines the magnitude of the Herzberg–Teller effect. This electronic matrix element can be expressed as

$$\langle \psi_k^0 | \hat{H}' | \psi_f^0 \rangle = \langle \psi_k^0 | \left( \frac{\partial \hat{H}}{\partial Q_i} \right)_{Q_i=0} Q_i | \psi_f^0 \rangle, \quad (10.50)$$

and is nonzero only if the vibronic symmetry  $\Gamma^{\psi_f^0} \otimes \Gamma^{Q_i}$  of the  $\tilde{A}$  state matches the electronic symmetry of the  $\tilde{C}$  state,  $\Gamma^{\psi_k^0}$ . In this electronic integral  $(\partial \hat{H} / \partial Q_i)_{Q_i=0}$  is a function of the electron coordinates and has the same symmetry as the normal mode  $Q_i$  since  $\hat{H}$  itself is totally symmetric. This leads to the selection rule

$$\Delta v_i = \pm 1 (\pm 3, \pm 5, \dots)$$

for the  $\tilde{A} - \tilde{X}$  vibronically active mode(s).

## 10.4 Jahn–Teller Effect

The Jahn–Teller<sup>1,10,11</sup> effect also violates the selection rule  $\Delta v_i = \pm 2, \pm 4, \pm 6, \dots$  for nonsymmetric vibrations in an electronic transition. This effect in molecules is a consequence of the much celebrated Jahn–Teller theorem. Jahn and Teller<sup>11</sup> proved that any nonlinear molecule in an orbitally degenerate electronic state will always distort in such a way as to lower the symmetry and remove the degeneracy. This is a perfectly general

result, but it conveys no information as to the size of the distortion. Both infinitesimal and massive distortions are possible, and both are consistent with the Jahn–Teller theorem.

For nonsinglet molecules, spin-orbit coupling competes with the Jahn–Teller effect, since spin-orbit splitting also lifts orbital degeneracies independently of the Jahn–Teller effect. The Jahn–Teller theorem is also inapplicable to linear molecules for which undistorted orbitally degenerate states ( $\Pi$ ,  $\Delta$ ,  $\Phi$ , ...) are possible.

The Jahn–Teller effect is a consequence of Born–Oppenheimer breakdown, and it is convenient to use the same approach that was used to describe vibronic coupling. There is a zeroth-order Born–Oppenheimer electronic Hamiltonian operator and a perturbation operator,

$$\hat{H}' = \sum_i \left( \frac{\partial \hat{H}_e}{\partial Q_i} \right)_{Q_i=0} Q_i, \quad (10.51)$$

responsible for the mixing of vibrational and electronic wavefunctions. If  $\psi_a$  and  $\psi_b$  are linearly independent orbitally degenerate electronic wavefunctions, then  $\hat{H}'$  will lift their degeneracy if there is a nonzero matrix element between them, that is,

$$\int \psi_a^* \hat{H}' \psi_b d\tau_e = \sum_i \int \psi_a^* \left( \frac{\partial \hat{H}_e}{\partial Q_i} \right)_{Q_i=0} Q_i \psi_b d\tau_e \neq 0. \quad (10.52)$$

This electronic integral will be nonzero only if  $\Gamma^{Q_i} \otimes (\Gamma^{\psi_a} \otimes \Gamma^{\psi_b})_{\text{sym}}$  contains the totally symmetric irreducible representation because  $(\partial \hat{H}_e / \partial Q_i)_{Q_i=0}$  is a function of the electron coordinates and has the same symmetry as  $Q_i$ . Since  $\psi_a$  and  $\psi_b$  belong to the same irreducible representation, the symmetric product is used to ensure that the Pauli exclusion principle is not violated. Jahn and Teller<sup>11</sup> exhaustively considered all degenerate states of all point groups with respect to all non-totally symmetric vibrational distortions. In all cases, except for linear molecules, a vibrational distortion ( $Q_i$ ) could be found that broke the symmetry of the molecule. As shown schematically in Figure 10.14, a distortion from the symmetric configuration can be found that lowers the total energy of the molecule. For example, the hexafluorobenzene cation  $C_6F_6^+$  has an  $\tilde{X}^2E_{1g}$  ground state with a small spin-orbit splitting. The  $\tilde{X}^2E_{1g}$  state of  $C_6F_6^+$  distorts, lowering the symmetry from  $D_{6h}$  and lifting the degeneracy in the  $E_{1g}$  electronic state. The size of the Jahn–Teller distortion can be estimated from the vibronic activity associated with the non-totally symmetric  $e_{2g}$  modes in the  $\tilde{B}^2A_{2u} - \tilde{X}^2E_{1g}$  electronic transition.<sup>12</sup>

## 10.5 Renner–Teller Effect

Although linear molecules are not subject to the Jahn–Teller effect, they do experience another “name” effect—the Renner–Teller effect.<sup>1,10,13</sup> (Linear molecules are also subject to the Herzberg–Teller effect.) The Renner–Teller effect is the interaction of vibrational and electronic angular momenta in a linear molecule. The levels associated with bending modes are shifted in energy by an interaction that couples vibrational motion to electronic motion for states in which  $\Lambda \neq 0$  (i.e.,  $\Pi$ ,  $\Delta$ ,  $\Phi$ , ...).

The Renner–Teller effect occurs because the double orbital degeneracy is lifted as a linear molecule bends during vibrational motion (Figure 10.15). As the linear molecule bends, the two potential curves  $V^+$  and  $V^-$  (corresponding, for example, to the  $p$

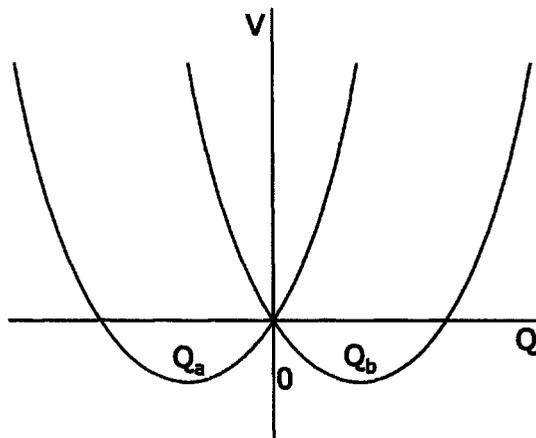


Figure 10.14: A Jahn–Teller distortion along a doubly degenerate vibrational coordinate lifts the degeneracy and stabilizes the molecule.

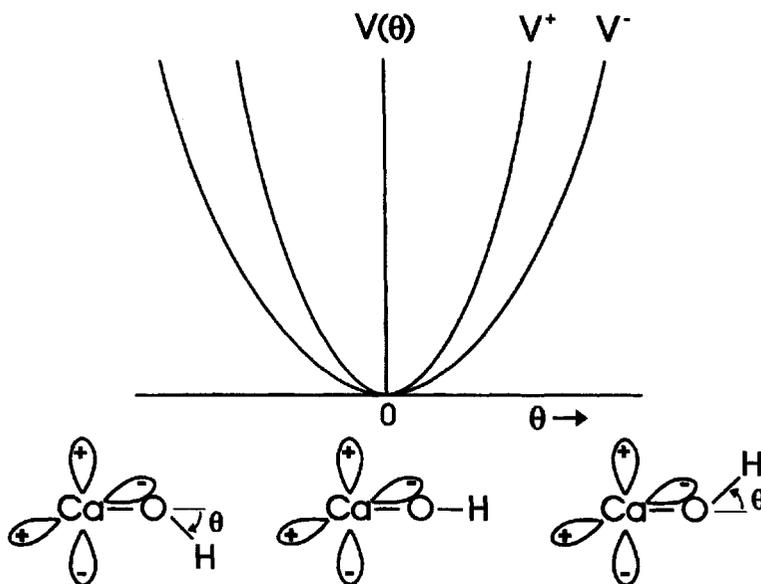


Figure 10.15: The bending motion of a linear molecule lifts the orbital degeneracy of the  $\tilde{A}^2\Pi$  state of CaOH.

orbital in the plane of the molecule and the  $p$  orbital out of the plane) become distinct. The combined vibrational and electronic motion on these two potential surfaces mixes the zeroth-order vibrational and electronic wavefunctions associated with the linear configuration. The Renner–Teller effect is again a consequence of the breakdown of the Born–Oppenheimer approximation.

The bending motion of a linear molecule has vibrational angular momentum  $l\hbar$  along the  $z$ -axis (for a state with  $v_2$  bending quanta  $l = v_2, v_2 - 2, \dots, 0$  or  $1$ ). A linear molecule can also have electronic orbital angular momentum with a projection of  $\Lambda\hbar$  along the  $z$ -

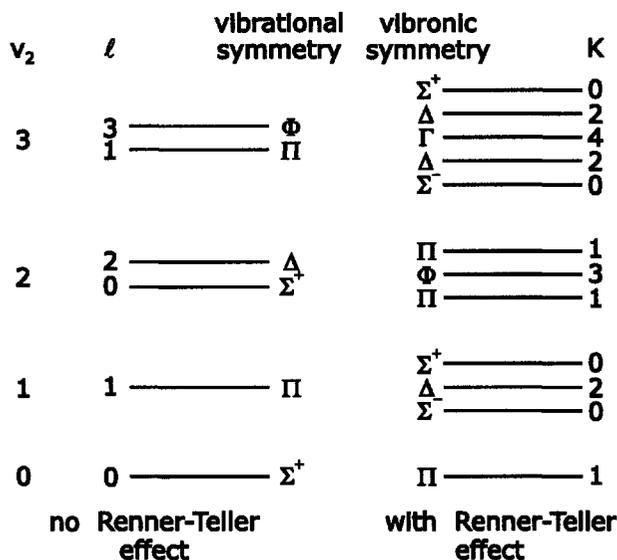


Figure 10.16: Vibronic energy levels of a  $^1\Pi$  electronic state of a linear triatomic molecule. The vibrational symmetry is on the left while the vibronic symmetry (vibrational  $\otimes$  electronic) is on the right.

axis. For the Renner–Teller effect, the coupling of these two angular momenta requires a new quantum number,  $K$ , given by

$$K = \Lambda + l. \quad (10.53)$$

Notice that  $K$ ,  $\Lambda$ , and  $l$  are signed numbers, but by convention only the magnitudes are usually quoted. The vibrational energy-level pattern appropriate for a  $^1\Pi$  electronic state is shown in Figure 10.16. In this figure the vibronic symmetries are obtained from the direct product of the vibrational symmetry with the electronic orbital symmetry ( $\Pi$ )

$$\Gamma^{\text{vib}} \otimes \Gamma^{\text{el}} = \Gamma^{\text{vibronic}}. \quad (10.54)$$

Without the Renner–Teller effect the excited bending levels in a triatomic molecule are given by the expression  $gl^2$ , where  $g$  ( $=g_2$ ) has a typical value of a few  $\text{cm}^{-1}$  because of anharmonicity. With the Renner–Teller effect the vibrational pattern is more complex, with splittings that are typically on the order of tens or hundreds of  $\text{cm}^{-1}$ , depending upon the magnitude of the electronic-vibrational interaction.

## 10.6 Nonradiative Transitions: Jablonski Diagram

Molecules in liquids, solids, and gases can exchange energy through collisions. Energy deposited in a specific molecule soon dissipates throughout the system because of these *intermolecular* interactions. Interestingly, if energy is deposited in a large, isolated molecule it can also be dissipated through *intramolecular* interactions. For example, electronic excitation can be converted into vibrational excitation in a large molecule without collisions. The various possible processes that can occur for a large organic

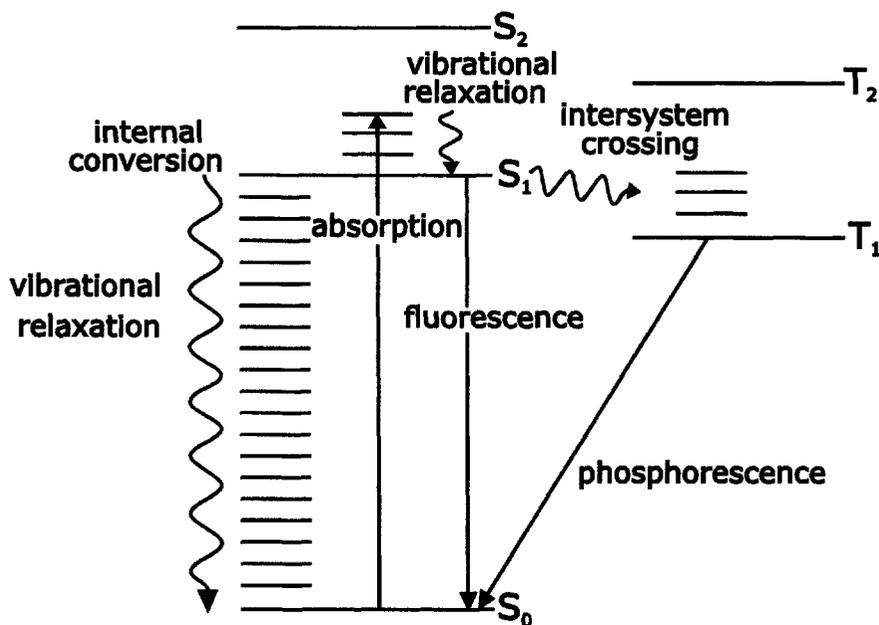


Figure 10.17: Photoprocesses in a large molecule: a Jablonski diagram.

molecule, such as benzene or naphthalene, are summarized in a *Jablonski diagram* (see Figure 10.17).

In the Jablonski diagram the ground state is a singlet and is labeled  $S_0$  with  $S_1, S_2, S_3, \dots$  used for excited singlet states with increasing energy. For an aromatic molecule such as benzene the  $S_1 \leftarrow S_0$  transition is typically a  $\pi^* \leftarrow \pi$  excitation. For example, the first excited electronic configuration of benzene  $(a_{2u})^2(e_{1g})^3(e_{2u})^1$  gives rise to singlet and triplet states; according to Hund's rules, the lowest energy triplet lies below the lowest energy singlet. The triplet states are labeled as  $T_1, T_2, \dots$  in order of increasing energy.

The fate of an absorbed photon in a large molecule can be described with the aid of the Jablonski diagram. The  $S_1 \leftarrow S_0$  absorption is followed by rapid (picosecond) vibrational relaxation to the bottom of the  $S_1$  state in any condensed phase. In the absence of collisions this particular vibrational relaxation process cannot occur without violating the principle of conservation of energy.

There are four possible fates of a large molecule in the electronic state  $S_1$ : reaction, fluorescence, internal conversion, or intersystem crossing. From a chemical point of view perhaps the most important possibility is the reaction of the excited molecule with other molecules. This possibility is studied in the vast field of photochemistry and is beyond the scope of this book.

If the molecule in  $S_1$  re-emits a photon,  $S_1 \rightarrow S_0$ , this process is known as *fluorescence*. If the emitted photon has the same energy as the absorbed photon, then the process is known as *resonance fluorescence*; otherwise, if the emitted photon has less energy than the absorbed photon, then the process is known as *relaxed fluorescence*. Small, gas-phase molecules emit resonance fluorescence but large, condensed phase molecules emit relaxed fluorescence. Fluorescence lifetimes are typically on the order of nanoseconds.

Singlet to triplet conversion for excited states is also possible in which the  $S_1 \rightarrow T_1$  process is known as *intersystem crossing*. Emission from the triplet state back down to the ground state is weakly allowed by spin-orbit mixing. The  $T_1 \rightarrow S_0$  emission is called *phosphorescence* and typically has a lifetime on the order of milliseconds to seconds for large organic molecules.

A molecule in  $S_1$  can bypass the triplet manifold and transfer directly to high vibrational levels of the ground state. This nonradiative  $S_1 - S_0$  process is called *internal conversion*. The processes of internal conversion and intersystem crossing in an isolated gas-phase molecule can only occur because a high density of vibrational levels exists, allowing energy conservation to be satisfied exactly (within the limits of the uncertainty principle based on the lifetime of the excited state).

## 10.7 Photoelectron Spectroscopy

Photoelectron spectroscopy allows the orbital energies of a molecule to be measured directly. In photoelectron spectroscopy the molecule is bombarded with electromagnetic radiation of sufficient energy to ionize the molecule. Typically, vacuum ultraviolet radiation is used to liberate valence electrons, while x-ray radiation is used to dislodge core electrons. Valence photoelectron spectroscopy is of particular interest to chemists since this process provides a means whereby the energies of the bonding orbitals can be measured directly.

The He atom provides a convenient source of vacuum ultraviolet radiation. For instance, one source is the He  $^1P - ^1S$  resonance transition at  $171\,129.148\text{ cm}^{-1}$  ( $21.217\,307\text{ eV}$  or  $584.353\,99\text{ \AA}$ ), which involves the promotion of the electron from  $1s$  to  $2p$ . The energy of this transition is more than sufficient to ionize any outer valence electron according to the process



for any molecule,  $M$ . The excess energy that results from the energy difference of the photon minus the binding energy of the electron is distributed as internal energy in  $M^+$  or as kinetic energies of the molecular ion  $M^+$  and the electron. Since an electron is much lighter than a molecule, conservation of momentum requires that the ionized electron move at a high speed relative to the molecule. In other words, nearly all of the kinetic energy is carried by the electron, so a measurement of the electron energy gives the internal energy of the molecule from the Einstein equation,

$$h\nu_0 = E_{\text{molecule}} + KE_e. \quad (10.56)$$

Ignoring vibrational and rotational energy for the moment, this means that if the photon energy ( $h\nu_0$ ) is known and the electron kinetic energy is measured ( $KE_e$ ), then the electron binding energy ( $E_{\text{molecule}}$ ) in a particular electronic orbital of the molecule is determined. This means that the photoelectric effect directly measures the binding energies of electrons in a molecule.

For example, the valence photoelectron spectrum of  $\text{H}_2\text{O}$  recorded with He  $I$  excitation is shown in Figure 10.18. As is customary, the spectrum is not plotted as a function of the measured electron kinetic energy ( $KE_e$ ) but as a function of  $h\nu_0 - KE_e$ . In this way the scale directly reads the binding energy of the electron orbital (or the ionization energy of the orbital). The lowest energy ionization at  $12.61\text{ eV}$  corresponds to the ionization from the nonbonding out-of-plane orbital of  $b_1$  symmetry (see Figure 10.2).

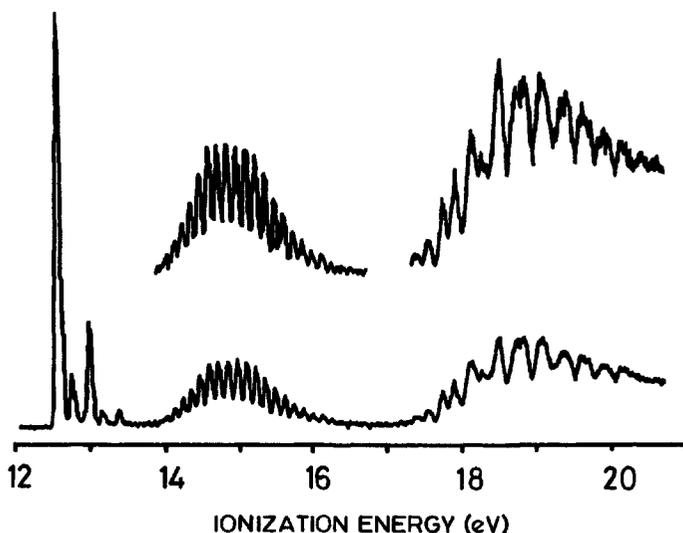


Figure 10.18: Valence photoelectron spectrum of H<sub>2</sub>O recorded with He I excitation.

The next ionization (14–16 eV) corresponds to the in-plane H—O bonding orbital of  $a_1$  symmetry. The removal of this bonding electron changes considerably the geometry of the final H<sub>2</sub>O<sup>+</sup> ion relative to the geometry of the ground-state molecule. Since the ionization process is very fast, the Franck–Condon principle applies, and substantial vibrational structure is observed. The origin band for the ionization of the  $a_1$  electron is at 13.7 eV. Finally, removal of the in-plane  $b_2$  (Figure 10.2) bonding electron with 18–20 eV ionization energy also results in extensive vibrational structure near the origin at 17.2 eV. The final valence orbital at 32 eV, corresponding to the ionization of a  $2s$  ( $a_1$ ) O electron, is not shown in Figure 10.18 because the binding energy is greater than the 21.2 eV of energy that is available from the He I source.

The photoelectron spectrum of H<sub>2</sub>O gives the binding energies of the four occupied valence molecular orbitals. The  $y$ -axis of the qualitative molecular orbital diagram (Figure 10.2) is thus made quantitative. The photoelectron spectrum of H<sub>2</sub>O also suggests that molecular orbitals are more than figments of a quantum chemist's imagination. Orbitals exist and their properties can be measured.

## 10.8 Rotational Structure: H<sub>2</sub>CO and HCN

The rotational energy levels of linear, symmetric top, spherical top, and asymmetric top molecules have been discussed in previous chapters. The general selection rules have also been considered in the sections on infrared spectroscopy (Chapter 7) and electronic spectroscopy of diatomic molecules (Chapter 9).

Electronic spectra of polyatomic molecules display much more variety than is found in the infrared transitions of polyatomic molecules or electronic transitions of diatomic molecules. The main reason for this diversity is the possibility that an electronic transition induces large changes in the geometry. It is not uncommon for a molecule, such as HCN, to change point groups or to dissociate into fragments upon electronic excitation. HCN has a linear ground state, while the first excited electronic state has a

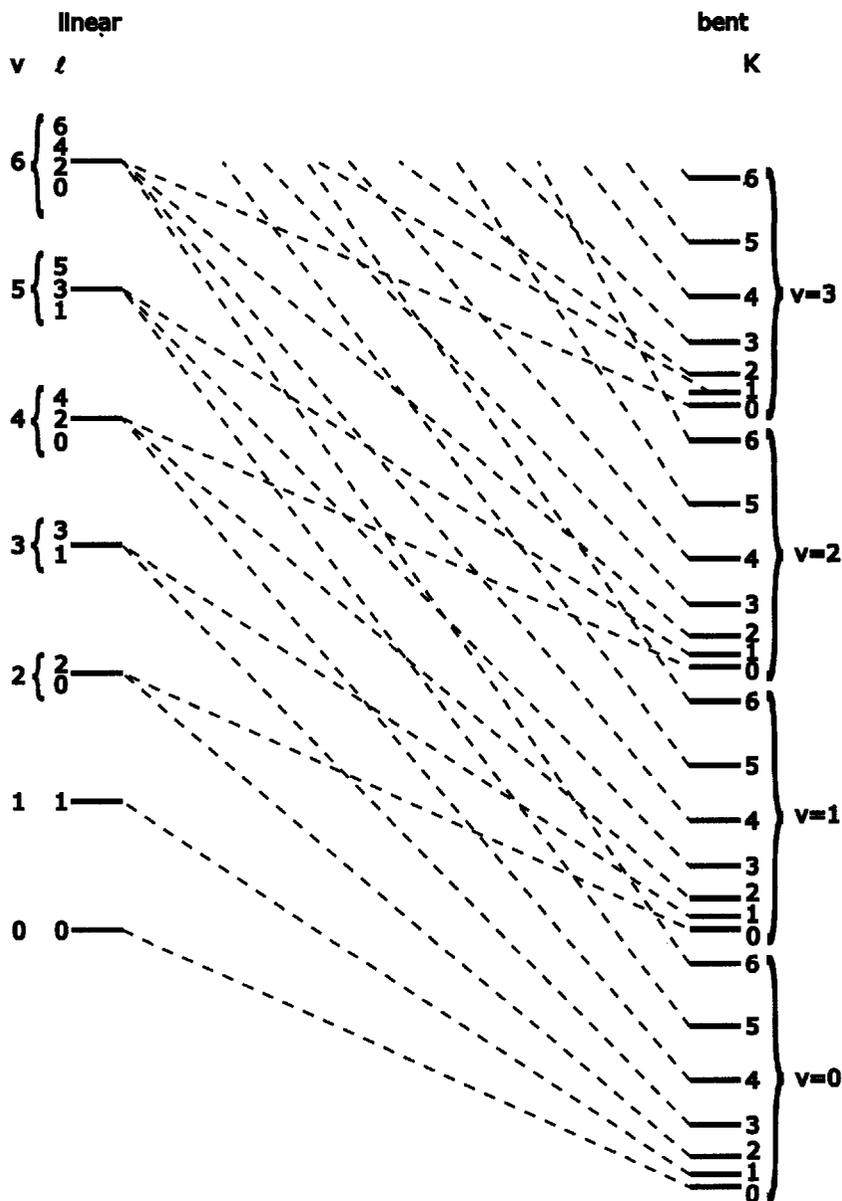


Figure 10.19: Bent-linear correlation diagram for the rotational energy levels of a triatomic molecule.

bent structure so it is necessary to consider a bent-linear rotational level correlation diagram.

The ground state of HCN is  $\tilde{X}^1\Sigma^+$  ( $C_{\infty v}$ ) and has four degrees of vibrational freedom plus two degrees of rotational freedom. The  $\tilde{A}^1A''$  ( $C_s$ ) excited electronic state has three degrees of vibrational freedom and three degrees of rotational freedom. It is necessary, therefore, to convert one bending vibration in linear HCN into rotation about the  $a$ -axis for bent HCN. A correlation diagram is very helpful in obtaining a

qualitative understanding of how energy levels and quantum numbers must change as a function of bending angle. For bent HCN the energy levels are given approximately by (ignoring stretching vibrations)

$$BJ(J+1) + (A-B)K_a^2 + h\nu_b(v_b + \frac{1}{2}), \quad (10.57)$$

since the molecule is a near prolate top. In equation (10.57) the  $K_a$  quantum number gives the projection of rotational angular momentum about the top axis and  $h\nu_b$  is the vibrational quantum of bending. For the linear molecule, the corresponding energy level expression is

$$BJ(J+1) + gl^2 + h\nu(v+1). \quad (10.58)$$

In equation (10.58) the  $l$  quantum number gives the projection of vibrational angular momentum along the  $z$ -axis and  $h\nu$  is the quantum of bending for the linear molecule. Since  $J$  is always a good quantum number, it must remain invariant to any changes in geometry. However,  $l$  is transformed into  $K$  as the molecule bends (Figure 10.19). The vibrational angular momentum  $l\hbar$  becomes  $K_a\hbar$  units of rotational angular momentum as the HCN molecule bends.

The  $\tilde{A}^1A'' - \tilde{X}^1\Sigma^+$  transition<sup>1,14</sup> of HCN occurs near 1800 Å. This transition shows a long progression in the  $\nu_2$  bending mode (and in  $\nu_3$ , the CN stretch) in both absorption<sup>14</sup> and laser-excited emission. From the Franck–Condon principle, linear–linear transitions will not exhibit a progression in a bending mode. The bands are found to obey the selection rule  $K' - l'' = \pm 1$  (in general,  $K' - l''$  could be 0,  $\pm 1$ ), consistent with the transition dipole moment pointing out of the plane of the bent molecule. The excited  $\tilde{A}$  state must therefore have  $\tilde{A}^1A''$  symmetry.

The rotational structure of the bands is relatively simple because the excited state is approximately a prolate top with rotational spacings given by  $BJ(J+1)$ . The selection rule  $\Delta J = 0, \pm 1$  appropriate for a perpendicular electronic transition gives a simple  $P$ ,  $Q$ ,  $R$  structure for each band (Figure 10.20).

Perhaps the most famous electronic spectrum is the  $\tilde{A}^1A_2 - \tilde{X}^1A_1$  transition of formaldehyde, observed in the region 3530–2300 Å.<sup>1,15</sup> The  $\tilde{A} - \tilde{X}$  transition was the first electronic transition of an asymmetric top molecule in which the rotational structure was understood in great detail.<sup>16</sup> In addition, formaldehyde is the simplest molecule with the carbonyl chromophore, and therefore serves as a prototype for more complex aldehydes and ketones.

The electronic structure of formaldehyde can be rationalized with a simple localized molecular orbital picture. The carbonyl chromophore has a set of localized molecular orbitals<sup>16</sup> given in Figure 10.21.

The six valence outer electrons (two of the four carbon electrons form the C–H bonds and the two O(2s) electrons are not considered) give a  $(5a_1)^2(1b_1)^2(2b_2)^2$  ground-state configuration. The HOMO–LUMO transition corresponds to the transfer of a nonbonding, in-plane  $O_{2p_y}(2b_2)$  electron to an antibonding C–O  $\pi^*(2b_1)$  orbital. This type of transition is associated with =C=O, =C=S, –N=O, –NO<sub>2</sub>, and –O–N=O chromophores and is called a  $\pi^* \leftarrow n$  transition. For H<sub>2</sub>CO, the first excited-state configuration is  $(5a_1)^2(1b_1)^2(2b_2)^1(2b_1)^1$ , which gives rise to the  $\tilde{A}^1A_2$  and  $\tilde{a}^3A_2$  electronic states.

The  $\tilde{A}^1A_2 - \tilde{X}^1A_1$  electronic transition is electric-dipole forbidden because the  $A_2$  irreducible representation is the only one in the  $C_{2v}$  point group not associated with one of the components of the dipole moment vector. The  $\tilde{A} - \tilde{X}$  transition appears weakly through vibronic coupling with  $\nu_4$ , the out-of-plane bending mode of  $b_1$  symmetry

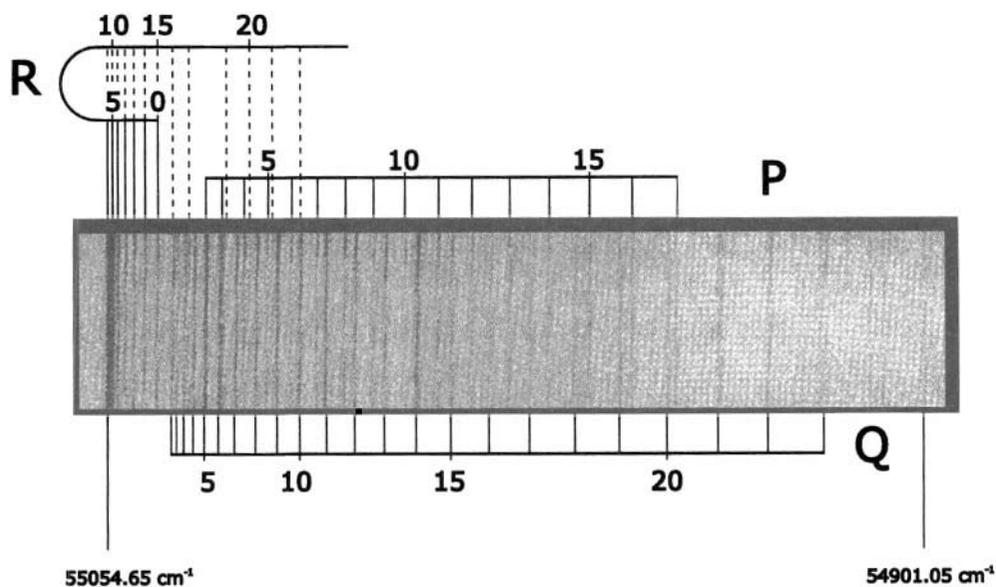


Figure 10.20: The  $030(K' = 1) - 000$  absorption band of the  $\tilde{A}'^1A'' - \tilde{X}'^1\Sigma^+$  transition of HCN.

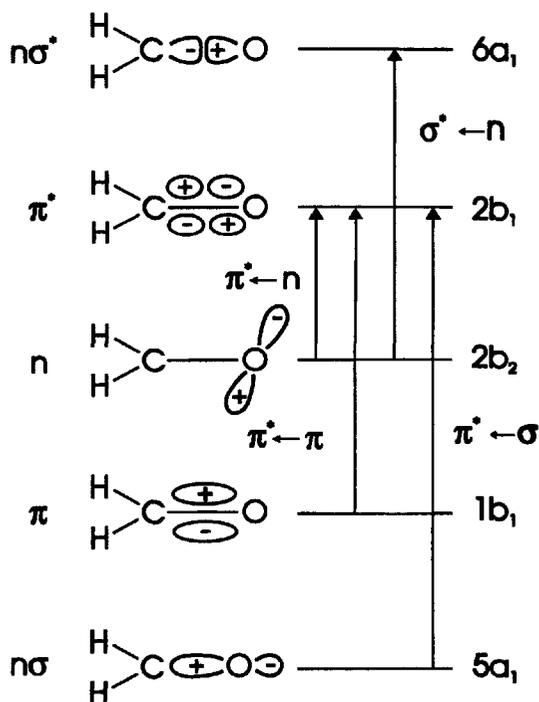


Figure 10.21: Localized molecular orbitals associated with the carbonyl chromophore of formaldehyde.

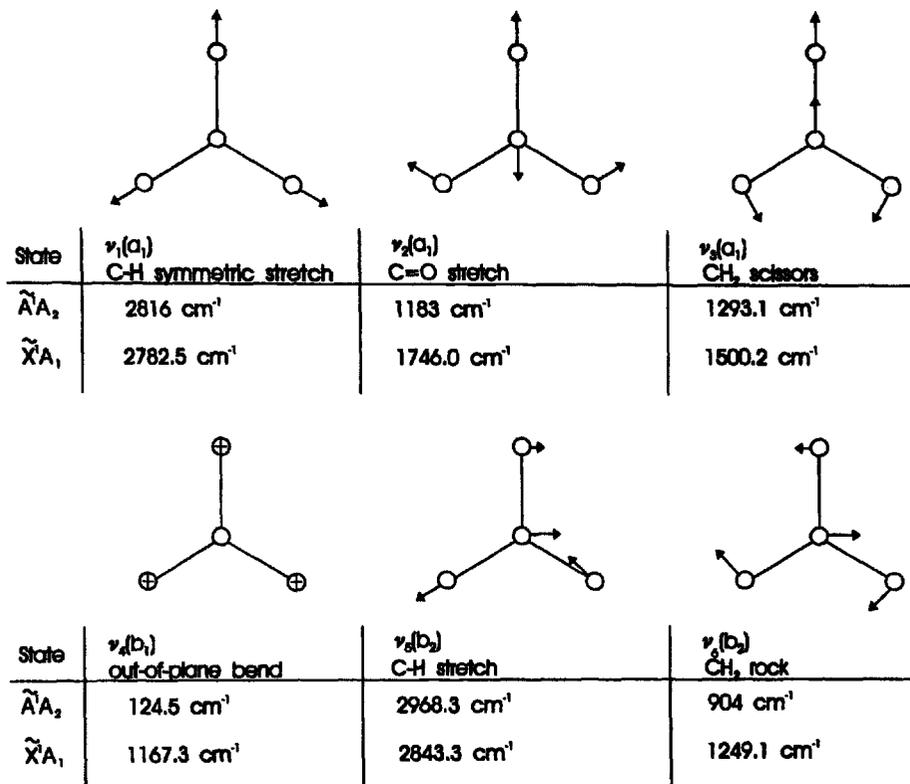


Figure 10.22: The vibrational modes<sup>15,16</sup> of formaldehyde in the  $\tilde{A}^1A_2$  and  $\tilde{X}^1A_1$  states.

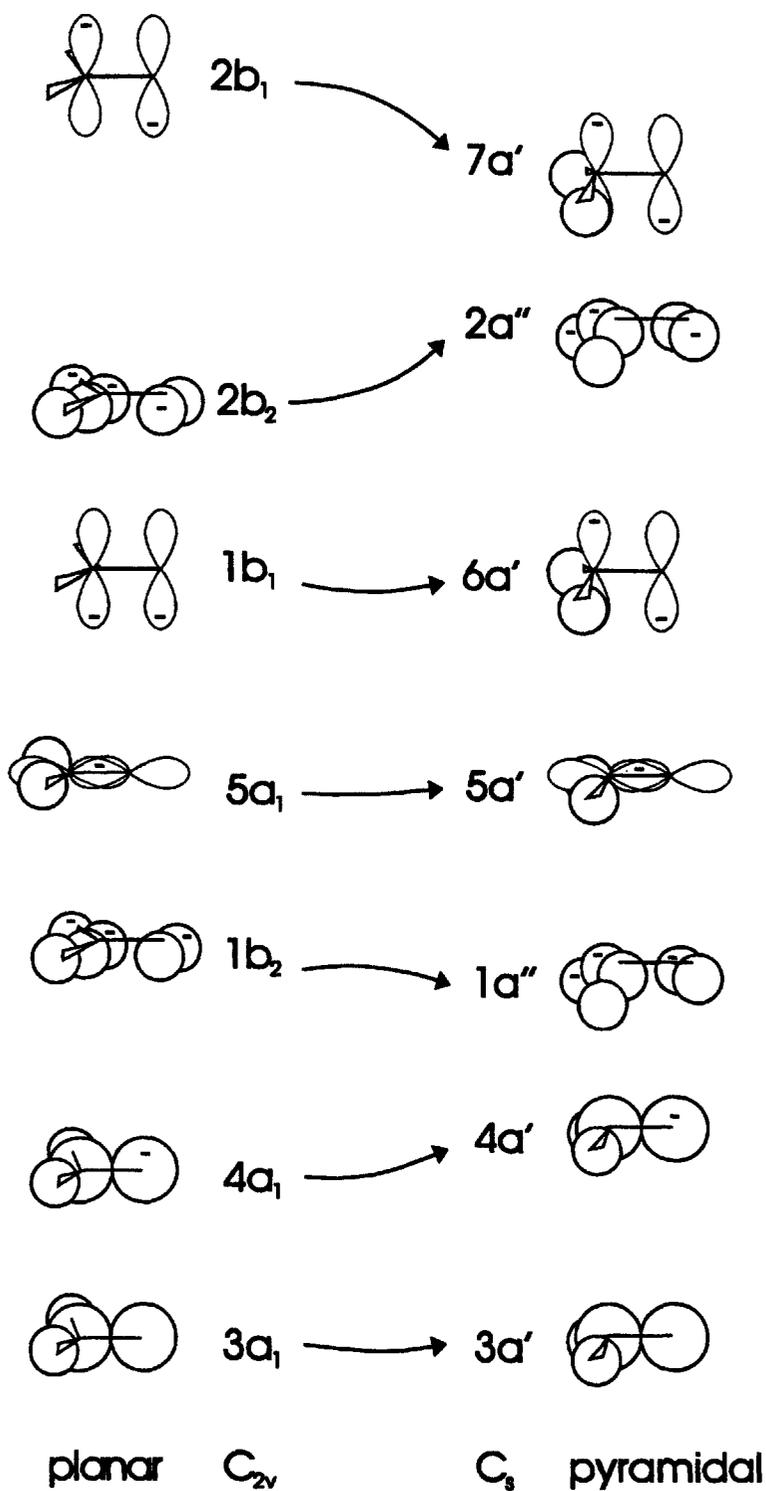
(Figure 10.22). Since the direct product  $A_2 \otimes B_1$  gives  $B_2$ , the  $\tilde{A}^1A_2 - \tilde{X}^1A_1$  transition borrows intensity from the allowed  $\tilde{B}^1B_2 - \tilde{X}^1A_1$  transition<sup>16</sup> near 1750 Å.

The observed  $\tilde{A}^1A_2 - \tilde{X}^1A_1$  laser excitation spectrum of formaldehyde<sup>17</sup> is shown in Figure 10.23. A tunable laser was used to excite the vibronic levels of the  $\tilde{A}$  state of formaldehyde, and the resulting undispersed total fluorescence was monitored as a function of laser wavelength.<sup>17</sup> The simple picture of the electronic structure of H<sub>2</sub>CO described above is unable to account for some features of the spectra. The long progression of 1182 cm<sup>-1</sup> is assigned as the  $\nu_2'$  carbonyl stretching mode. This long progression appears because of a lengthening of the C—O bond. The vibrational intervals associated with the out-of-plane bending mode in the excited electronic state ( $\nu_4'$ ) are very peculiar, however. In the  $\tilde{A}$  state the frequency of the  $\nu_4'$  mode is 125 cm<sup>-1</sup>, as compared to the 1167 cm<sup>-1</sup> value for the  $\nu_4''$  mode of the  $\tilde{X}$  state.

Surprisingly, the rotational analysis of H<sub>2</sub>CO preceded the vibronic analysis. From the moments of inertia, the inertial defect (Chapter 6),  $\Delta (= I_C - I_A - I_B)$  was found to be -0.265 u Å<sup>2</sup> in the  $\tilde{A}$  state, but 0.057 u Å<sup>2</sup> in the  $\tilde{X}$  state.<sup>15</sup> For a perfectly rigid planar molecule,  $\Delta$  would be exactly zero, but for a nonrigid planar molecule (such as the ground state of H<sub>2</sub>CO),  $\Delta$  is generally a small positive number due to vibrational and electronic motion.<sup>18</sup>

The vibronic and rotational structure of the spectra associated with the  $\tilde{A}$  state of formaldehyde can be understood if the  $\tilde{A}$  state is allowed to be slightly nonplanar. This possibility is in accord with the prediction of the Walsh diagram for H<sub>2</sub>CO (Figure



Figure 10.24: Walsh diagram for H<sub>2</sub>CO.

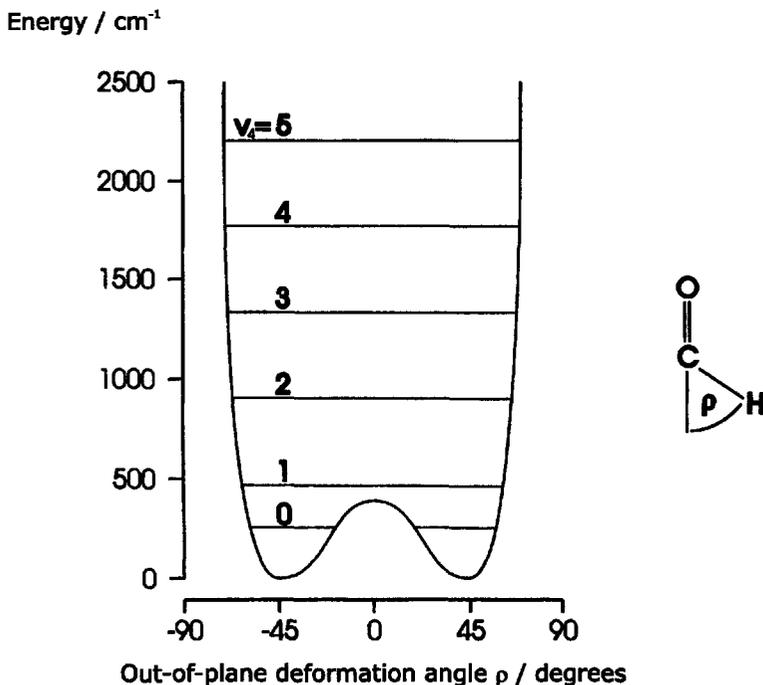


Figure 10.25: The potential energy curve as a function of the out-of-plane bending angle in the  $\tilde{A}^1A_2$  state of  $H_2CO$ .

Some perpendicular  $c$ -type bands can also be found in the absorption spectra<sup>15,16</sup> with the selection rules  $\Delta K_a = \pm 1$ ,  $\Delta K_c = 0$ ,  $\Delta J = 0, \pm 1$ . They must be associated with  $B_1 - A_1$  vibronic bands and contain odd quanta of  $\nu'_5(b_2)$  and  $\nu'_6(b_2)$ , since the direct product  $A_2 \otimes b_2$  gives the  $B_1$  irreducible representation. These bands, in fact, account for as much as one-quarter of the strength of the formaldehyde  $\tilde{A} - \tilde{X}$  electronic transition.

Finally a few very weak parallel bands ( $a$ -type) are found in the  $\tilde{A} - \tilde{X}$  transition. For these bands the selection rules are  $\Delta K_a = 0$ ,  $\Delta K_c = \pm 1$ ,  $\Delta J = 0, \pm 1$ . Surprisingly, the origin band ( $0_0^0$ ) is found in the spectrum with  $a$ -type selection rules. The origin band has  $A_2 - A_1$  vibronic character and is forbidden by electric dipole selection rules in a vibronically-induced electronic transition. Magnetic dipole transitions are possible, however, and the magnetic dipole moment behaves like the rotations  $R_x$ ,  $R_y$ , and  $R_z$  in the character table. Callomon and Innes<sup>21</sup> proved that the origin band of the  $\tilde{A}^1A_2 - \tilde{X}^1A_1$  transition has magnetic-dipole character.

## 10.9 Intensity of Transitions

Once again the formulas presented in Chapter 5 for atoms can be used because of the common definition (equation (5.112)) of the line strength

$$S_{J',J''} = \sum_{J',J''} |\langle J' M' | \boldsymbol{\mu} | J'' M'' \rangle|^2. \quad (10.59)$$

For a polyatomic molecule  $J$  remains a good quantum number and the  $M_J$  degeneracy is

still  $2J+1$ . As for diatomic molecules,  $S_{J',J''}$  can be factored into electronic, vibrational, and rotational parts as long as the wavefunction can be written as  $\psi = \psi_{\text{el}}\psi_{\text{vib}}\psi_{\text{rot}}$ . The separation of the electronic and vibrational motion has already been discussed, equation (10.41), and leads to polyatomic Franck–Condon factors for each normal mode,  $Q_i$ . The line strength  $S_{J',J''}$  is thus

$$\begin{aligned} S_{J',J''} &= |\mathbf{R}_e|^2 S_{J''}^{\Delta J} \left| \int \psi_{v'_1}^* \psi_{v''_1} dQ_1 \right|^2 \left| \int \psi_{v'_2}^* \psi_{v''_2} dQ_2 \right|^2 \dots \\ &= |\mathbf{R}_e|^2 S_{J''}^{\Delta J} q_{v'_1-v''_1} q_{v'_2-v''_2} \dots, \end{aligned} \quad (10.60)$$

with the definition for Franck–Condon factors being

$$q_{v'_i-v''_i} = \left| \int \psi_{v'_i}^* \psi_{v''_i} dQ_i \right|^2 \quad (10.61)$$

The rotational line strength part,  $S_{J''}^{\Delta J}$ , is a Hönl–London factor (Table 9.4) for linear singlet molecules or for singlet symmetric tops (with  $K' = \Lambda'$  and  $K'' = \Lambda''$  in Table 9.4), but is in general more complicated. The polyatomic line strength expression, equation (10.60), is the same as that for a diatomic molecule, equation (9.103), with the exception that the  $3N-6$  (or 5) Franck–Condon products have replaced the single  $q_{v'-v''}$  diatomic Franck–Condon factor.

The simple line strength expression (10.60) fails frequently, for example, for transitions affected by vibronic coupling, or between states that experience the Jahn–Teller or Renner–Teller effects. There are even more subtle problems associated with the separation of  $S_{J',J''}$  into electronic, vibrational, and rotational parts. For example, in a polyatomic molecule the principal axis system of the upper electronic state is not always oriented in the same direction as in the lower state because of changes in geometry. This “axis switching” allows nominally forbidden rotational branches to appear in the electronic spectrum.<sup>1</sup> The rotation of the principal axes will also affect the Franck–Condon factors because this corresponds to a relative rotation of the inertial coordinate systems. The normal modes  $Q_i$  have different equilibrium positions in the upper and lower states, and can also be “rotated” relative to one another. In this case the *Dushinsky effect*<sup>22</sup> of coordinate rotation needs to be considered when the Franck–Condon factors, equation (10.60), are computed.

The diatomic equations (9.115) to (9.118) apply as long as  $q_{v'-v''}$  is interpreted as a product of Franck–Condon factors and as long as the rotational part is not necessarily a Hönl–London factor from Table 9.4.

A particularly useful relationship can be derived connecting the integrated absorption cross section and the oscillator strength of a transition. Many electronic transitions of polyatomics, for example, suffer from predissociation and the upper state lifetimes then reflect the rates of various nonradiative processes rather than the radiative lifetime. This means that the equations involving the radiative lifetimes and the Einstein  $A$  values are sometimes not very useful. The integrated absorption coefficient of an entire electronic system, however, is easily measured in any phase and can be converted to an oscillator strength  $f_{\text{abs}}$  or to an Einstein  $A$  value. If the excited state has unit *quantum efficiency* (i.e., the probability of emission of a photon is one) and the excited state only emits to the ground state, then the radiative lifetime,  $\tau$ , is  $A^{-1}$ .

The individual rotational lines of an electronic transition of a large molecule may not be resolved or the molecule may be in solution or in the form of a solid. In this

case, some of the intensity equations in Chapter 5 may not be very useful because they implicitly apply to individual lines that are not observed for a large molecule. For example, the integrated absorption coefficient for an electronic band system cannot be treated in the same way as that for a line because the band system usually covers a substantial frequency range.

For a broad electronic transition the integrated Einstein  $B$  value is given by

$$B = \int B_\nu d\nu, \quad (10.62)$$

with

$$B_\nu = \frac{c\sigma}{h\nu} \quad (10.63)$$

from equation (5.135) for the cross section  $\sigma$  or

$$B = \frac{c}{h} \int \frac{\sigma}{\nu} d\nu = \frac{c}{h} \int \sigma d \ln \nu. \quad (10.64)$$

The Einstein  $A_\nu$  value is related to the Einstein  $B_\nu$  value (equation (1.22), ignoring degeneracy) by

$$B_\nu = \frac{c^3}{8\pi h\nu^3} A_\nu, \quad (10.65)$$

so that an integration over frequency gives the rather approximate formula

$$B \approx \frac{c^3}{8\pi h} \left\langle \frac{1}{\nu^3} \right\rangle A, \quad (10.66)$$

from which the Einstein  $A$  value can be expressed as

$$A \approx \frac{8\pi}{c^2} \left\langle \frac{1}{\nu^3} \right\rangle^{-1} \int \frac{\sigma}{\nu} d\nu. \quad (10.67)$$

The approximation occurs because the Einstein  $A$  and  $B$  coefficients have different frequency dependences and when the transition covers a substantial frequency range, it is no longer possible to convert one to the other exactly. Indeed the usual intensity formulas in Chapter 5 are all defined for individual lines and their successful application to unresolved bands is not assured.

The measurement of the absorption cross section  $\sigma$  need not be in the gas phase, in which case  $c = c_0/n$  (Chapter 1) and thus

$$A \approx \frac{8\pi n^2}{c_0^2} \left\langle \frac{1}{\nu^3} \right\rangle^{-1} \int \frac{\sigma}{\nu} d\nu \quad (10.68)$$

or

$$\frac{1}{\tau} \approx 2.880 \times 10^{-9} n^2 \langle \tilde{\nu}^{-3} \rangle^{-1} \int \frac{\varepsilon}{\tilde{\nu}} d\tilde{\nu}, \quad (10.69)$$

if the molar absorption coefficient  $\varepsilon$  (see equation (1.63)) is used. Equations (10.67) and (10.69) were derived by Strickler and Berg,<sup>23</sup> and tested successfully with the solution spectra of a number of strongly absorbing large organic molecules. Equation (10.68) applies to strong, electric dipole-allowed transitions and assumes that there

is no orbital or spin degeneracy to consider. If orbital or electron spin degeneracy is present, then the ratio  $g_{\text{lower}}/g_{\text{upper}}$  appears in equations (10.67) to (10.69).

Similar equations for other integrated quantities can be derived using equation (10.63) and the careful application of some of the relationships given in equations (5.134) to (5.138). For example, the transition strength  $S$  in  $\text{C}^2 \text{ m}^2$  is given by

$$S = \frac{3\epsilon_0 hc}{2\pi^2} \int \frac{\sigma}{\nu} d\nu, \quad (10.70)$$

and the oscillator strength  $f_{\text{abs}}$  by

$$f = \frac{4\epsilon_0 m_e c}{e^2} \int \sigma d\nu. \quad (10.71)$$

Again for equation (10.70), orbital or spin degeneracy is not considered, and if present a factor of  $g_{\text{lower}}$  appears in the numerator. Although often discussed, the oscillator strength and other related quantities are not well defined for an electronic band system.

### Problems

1. Construct a planar-pyramidal correlation diagram (Walsh diagram) for an  $\text{AH}_3$ -type of molecule. Predict the geometries of the  $\text{BH}_3$ ,  $\text{CH}_3$ ,  $\text{NH}_3$ , and  $\text{CH}_3^-$  molecules.
2. With the aid of the Walsh diagram (Figure 10.4) predict the geometry and electronic symmetry of the  $\text{BH}_2$ ,  $\text{BH}_2^+$ , and  $\text{BH}_2^-$  molecules. For  $\text{BH}_2$  predict the lowest energy electronic transition.
3. Predict the ground electronic states for the following BAB-type molecules,  $\text{CNC}$ ,  $\text{NCN}$ ,  $\text{BO}_2$ ,  $\text{N}_3$ ,  $\text{CO}_2^+$ , and  $\text{NO}_2^+$ .
4. The ethylene molecule has a strong diffuse absorption beginning at  $2100 \text{ \AA}$  and extending into the vacuum ultraviolet. A very long progression in the excited-state  $\text{CH}_2$  torsional mode is observed. On the basis of simple  $\pi$ -molecular orbital theory and the Franck-Condon principle, account for these observations and suggest an assignment for the electronic transition.
5. Construct a molecular orbital diagram for the  $\pi$ -electrons of the linear molecule  $\text{C}_4$ . For the  $\text{C}_4$ ,  $\text{C}_4^+$ , and  $\text{C}_4^-$  molecules predict the ground electronic states. (*Hint:* To obtain the correct  $\pi$ -electron count, construct a filled  $\sigma$ -framework from the 8 valence  $sp$  hybrid orbitals.)
6. The  $\pi$ -electrons of naphthalene ( $\text{C}_{10}\text{H}_8$ ) can be considered to be confined to a rectangular box of dimension  $4 \text{ \AA}$  by  $7 \text{ \AA}$  ("particle-in-a-box").
  - (a) Set up and solve the Schrödinger equation to find the energy levels.
  - (b) Add the electrons to the energy-level diagram.
  - (c) Which levels correspond to HOMO and LUMO? At what wavelength will the lowest energy transition occur?
7. Consider the  $\pi$ -electron system of the cyclopentadienyl radical,  $\text{C}_5\text{H}_5$ , taken to be of  $D_{5h}$  symmetry.

- (a) Determine to which irreducible representation the MOs belong.
  - (b) Assuming orthogonality between the  $2p_z$  atomic orbitals, determine the normalized MOs.
  - (c) Use the Hückel approximation to determine the MO energies.
  - (d) What are the ground and first excited state configurations for the  $\pi$ -electron system?
  - (e) What transitions will be symmetry allowed?
8. Consider the hydrogen peroxide molecule H—O—O—H, which for the purposes of this problem is assumed to have  $C_{2h}$  symmetry (rather than  $C_2$  symmetry).
- (a) For the basis set of 10 atomic orbitals including the  $1s$  on the two hydrogen atoms, and  $2s$ ,  $2p_x$ ,  $2p_y$ , and  $2p_z$  on the oxygens, determine the symmetries of the 10 molecular orbitals.
  - (b) Use the projection operator technique to obtain functions that form bases for these irreducible representations.
  - (c) Specify the irreducible representations corresponding to each of the molecular orbitals in the following table of Intermediate Neglect of Differential Overlap (INDO) eigenvalues and eigenvectors:
  - (d) Determine the symmetries of the excited configurations formed by promoting an electron from the highest occupied MO to the lowest empty (number 8) and to the next highest (number 9) MO. Will the electronic transitions from the ground ( $A_g$ ) state to excited states of these symmetries be allowed? Explain your reasoning.

Energy/a.u.	-1.610	-1.278	-0.772	-0.752	-0.706	-0.577	-0.543	0.165	0.322	0.342
	1	2	3	4	5	6	7	8	9	10
1 O <sub>1</sub> 2s	-0.642	-0.581	0.300	0.040	0.000	-0.218	0.000	-0.260	0.196	-0.069
2 O <sub>1</sub> 2p <sub>x</sub>	0.138	-0.171	0.051	-0.555	0.000	-0.396	0.000	0.550	0.125	-0.406
3 O <sub>1</sub> 2p <sub>y</sub>	-0.064	-0.160	-0.574	-0.282	0.000	0.502	0.000	-0.224	0.406	-0.308
4 O <sub>1</sub> 2p <sub>z</sub>	0.000	0.000	0.000	0.000	-0.707	0.000	-0.707	0.000	0.000	0.000
5 O <sub>2</sub> 2s	-0.642	0.581	-0.300	0.040	0.000	-0.218	0.000	0.260	0.196	0.069
6 O <sub>2</sub> 2p <sub>x</sub>	-0.138	-0.171	0.051	0.555	0.000	0.396	0.000	0.550	-0.125	-0.406
7 O <sub>2</sub> 2p <sub>y</sub>	0.064	-0.160	-0.574	0.282	0.000	-0.502	0.000	-0.224	-0.406	-0.308
8 O <sub>2</sub> 2p <sub>z</sub>	0.000	0.000	0.000	0.000	-0.707	0.000	0.707	0.000	0.000	0.000
9 H <sub>1</sub> 1s	-0.254	-0.328	-0.278	-0.332	0.000	0.210	0.000	0.282	-0.530	0.485
10 H <sub>2</sub> 1s	-0.254	0.328	0.279	-0.332	0.000	0.210	0.000	-0.282	-0.530	-0.485

9. For the *s-cis*-butadiene molecule  $\text{CH}_2=\text{CH}-\text{CH}=\text{CH}_2$  of  $C_{2v}$  symmetry:
  - (a) What are the symmetries of the molecular orbitals for the  $\pi$ -electrons?
  - (b) Using Hückel theory, derive the energies and wavefunctions of the  $\pi$ -molecular orbitals.
  - (c) What are the ground-state electronic symmetries of the cation, neutral, and anion of *s-cis*-butadiene?
10. For the allyl free radical  $\text{CH}_2\text{CHCH}_2$  of  $C_{2v}$  symmetry, what are the symmetries of the  $\pi$ -orbitals? For the  $\pi$ -electrons, derive the Hückel molecular orbitals and energies. What is the ground-state  $\pi$ -electron configuration and the electronic symmetry of the allyl free radical? (Pick the  $x$ -axis to be out of the plane of the molecule.)
11. Derive equation (10.69) from (10.68).
12. The absorption spectrum of Rhodamine B dye in ethanol was measured and the value of  $\int \epsilon/\tilde{\nu}d\tilde{\nu}$  was found to be  $5937 \text{ L mole}^{-1} \text{ cm}^{-1}$ , while  $\langle \tilde{\nu}^{-3} \rangle^{-1}$  was found to be  $0.51 \times 10^{13} \text{ cm}^{-3}$ . The effective refractive index of ethanol is 1.360. Calculate the radiative lifetime  $\tau$  and the transition dipole moment in debye for this dye (assuming a quantum efficiency of 1). From the transition dipole moment calculate an approximate absorption oscillator strength ( $\tilde{\nu}_{\text{max}}$  is about  $17800 \text{ cm}^{-1}$ ).

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# Appendix A

## Units, Conversions, and Physical Constants

The Mohr and Taylor (1998) recommended values of the fundamental physical constants. The digits in parentheses are the one standard deviation uncertainty in the last digits of the given value. Mohr, P. J. and Taylor, B. N., CODATA Recommended Values for the Fundamental Physical Constants: 1998, *Rev. Mod. Phys.* **72**, 35 (2000). See <http://www.codata.org/>.

Quantity	Symbol	Value	Unit
speed of light in vacuum	$c, c_0$	299 792 458	$\text{m s}^{-1}$
permeability of vacuum	$\mu_0$	$4\pi \times 10^{-7}$ $=12.566\,370\,614\dots \times 10^{-7}$	$\text{N A}^{-2}$ $\text{N A}^{-2}$
permittivity of vacuum	$\epsilon_0$	$1/\mu_0 c^2$ $=8.854\,187\,817\dots \times 10^{-12}$	$\text{F m}^{-1}$
Newtonian constant of gravitation	$G$	$6.673(10) \times 10^{-11}$	$\text{m}^3 \text{kg}^{-1} \text{s}^{-2}$
Planck constant	$h$	$6.626\,068\,76(52) \times 10^{-34}$	$\text{J s}$
$h/2\pi$	$\hbar$	$1.054\,571\,596(82) \times 10^{-34}$	$\text{J s}$
elementary charge	$e$	$1.602\,176\,462(63) \times 10^{-19}$	$\text{C}$
Bohr magneton, $e\hbar/2m_e$	$\mu_B$	$9.274\,008\,99(37) \times 10^{-24}$	$\text{J T}^{-1}$
nuclear magneton, $e\hbar/2m_p$	$\mu_N$	$5.050\,783\,17(20) \times 10^{-27}$	$\text{J T}^{-1}$
proton mass	$m_p$	$1.672\,621\,58(13) \times 10^{-27}$ 1.007 276 466 88(13)	$\text{kg}$ $\text{u}$
proton-electron mass ratio	$m_p/m_e$	1 836.152 667 5(39)	
proton magnetic moment	$\mu_p$	$1.410\,606\,633(58) \times 10^{-26}$	$\text{J T}^{-1}$
proton gyromagnetic ratio, $2m_p/\hbar$	$\gamma_p$	$2.675\,222\,12(11) \times 10^8$	$\text{s}^{-1} \text{T}^{-1}$
fine-structure constant, $e^2/4\pi\epsilon_0\hbar c$	$\alpha$	$7.297\,352\,533(27) \times 10^{-3}$	
Rydberg constant, $m_e c \alpha^2 / 2h$	$R_\infty$	10 973 731.568 549(83)	$\text{m}^{-1}$
in hertz, $R_\infty c$		$3.289\,841\,960\,368(25) \times 10^{15}$	$\text{Hz}$
in joules, $R_\infty \hbar c$		$2.179\,871\,90(17) \times 10^{-18}$	$\text{J}$
in electron volts, $R_\infty \hbar c / \{e\}$		13.605 691 72(53)	$\text{eV}$
Bohr radius, $\alpha/4\pi R_\infty$	$a_0$	$0.529\,177\,208\,3(19) \times 10^{-10}$	$\text{m}$
Hartree energy, $2R_\infty \hbar c$	$E_h$	$4.359\,743\,81(34) \times 10^{-18}$	$\text{J}$
in eV, $E_h / \{e\}$		27.211 383 4(11)	$\text{eV}$

Quantity	Symbol	Value	Unit
electron mass	$m_e$	$9.109\,381\,88(72) \times 10^{-31}$	kg
		$5.485\,799\,110(12) \times 10^{-4}$	u
electron magnetic moment <sup>a</sup>	$\mu_e$	$-928.476\,362(37) \times 10^{-26}$	J T <sup>-1</sup>
in Bohr magnetons <sup>a</sup>	$\mu_e/\mu_B$	-1.001 159 652 186 9(41)	
in nuclear magnetons <sup>a</sup>	$\mu_e/\mu_N$	-1 838.281 966 0(39)	
electron $g$ -factor <sup>a</sup>	$g_e$	-2.002 319 304 373 7(82)	
Avogadro constant	$N_A$	$6.022\,141\,99(47) \times 10^{23}$	mol <sup>-1</sup>
atomic mass constant			
$1 m_u = m(^{12}\text{C})/12 = 1 \text{ u}$	u	$1.660\,538\,73(13) \times 10^{-27}$	kg
Faraday constant, $N_A e$	$F$	96 485.341 5(39)	C mol <sup>-1</sup>
molar gas constant	$R$	8.314 472(15)	J mol <sup>-1</sup> K <sup>-1</sup>
Boltzmann constant, $R/N_A$	$k$	$1.380\,650\,3(24) \times 10^{-23}$	J K <sup>-1</sup>
in electron volts, $k/\{e\}$		$8.617\,342(15) \times 10^{-5}$	eV K <sup>-1</sup>
in hertz, $k/h$		$2.083\,664\,4(36) \times 10^{10}$	Hz K <sup>-1</sup>
in wavenumbers, $k/hc$		69.503 56(12)	m <sup>-1</sup> K <sup>-1</sup>
molar volume (ideal gas), $RT/p$			
$T = 273.15 \text{ K}$ , $p = 101\,325 \text{ Pa}$	$V_m$	$22.413\,996(39) \times 10^{-3}$	m <sup>3</sup> mol <sup>-1</sup>
$T = 273.15 \text{ K}$ , $p = 100 \text{ kPa}$	$V_m$	$22.710\,981(40) \times 10^{-3}$	m <sup>3</sup> mol <sup>-1</sup>
Stefan–Boltzmann			
constant, $(\pi^2/60)k^4/\hbar^3c^2$	$\sigma$	$5.670\,400(40) \times 10^{-8}$	W m <sup>-2</sup> K <sup>-4</sup>
first radiation constant, $2\pi\hbar c^2$	$c_1$	$3.741\,771\,07(29) \times 10^{-16}$	W m <sup>2</sup>
second radiation constant, $hc/k$	$c_2$	0.014 387 752(25)	m K
Wien displacement law constant,			
$b = \lambda_{\max}T = c_2/4.965\,114\,231\dots$	$b$	$2.897\,768\,6(51) \times 10^{-3}$	m K
electron volt, $(e/C) \text{ J} = \{e\} \text{ J}$	eV	$1.602\,176\,462(63) \times 10^{-19}$	J
standard atmosphere	atm	101 325	Pa
standard acceleration of gravity	$g_n$	9.806 65	m s <sup>-2</sup>

<sup>a</sup>Mohr and Taylor have used negative values for these quantities as recommended by Brown *et al.*, *Mol. Phys.* **98**, 1597 (2000). The equations in this book follow the traditional definitions and require positive values.

# Appendix B

## Character Tables

$C_s$	$\hat{E}$	$\hat{\sigma}_h$		
$A'$	1	1	$x; y; R_z$	$x^2; y^2; z^2; xy$
$A''$	1	-1	$z; R_x; R_y$	$xz; yz$

$C_i$	$\hat{E}$	$\hat{i}$		
$A_g$	1	1	$R_x; R_y; R_z$	$x^2; y^2; z^2; xy; xz; yz$
$A_u$	1	-1	$x; y; z$	

$C_2$	$\hat{E}$	$\hat{C}_2$		
$A$	1	1	$z; R_z$	$x^2; y^2; z^2; xy$
$B$	1	-1	$x; y; R_x; R_y$	$xz; yz$

$C_3$	$\hat{E}$	$\hat{C}_3$	$\hat{C}_3^2$		$\epsilon = \exp(2\pi i/3)$
$A$	1	1	1	$z; R_z$	$x^2 + y^2; z^2$
$E$	$\begin{Bmatrix} 1 & \epsilon \\ 1 & \epsilon^* \end{Bmatrix}$	$\begin{Bmatrix} \epsilon & \epsilon^* \\ \epsilon^* & \epsilon \end{Bmatrix}$	$\begin{Bmatrix} \epsilon^* & \epsilon \\ \epsilon & \epsilon^* \end{Bmatrix}$	$(x, y); (R_x, R_y)$	$(x^2 - y^2, xy); (xz, yz)$

After D. Bishop, *Group Theory and Chemistry*, Dover, New York, 1993.

For groups which can be written as direct products  $G = G_1 \otimes C_i$  or  $G = G_1 \otimes C_s$ , i.e.,  $C_{nh}$ ,  $D_{nh}$ ,  $D_{3d}$ ,  $D_{5d}$ ,  $S_6$ ,  $O_h$ ,  $D_{\infty h}$ , and  $I_h$ , the order in which the irreducible representations are listed in the Herzberg character tables (Vol. II, *Infrared and Raman Spectra*; Vol. III, *Electronic Spectra of Polyatomic Molecules*) differs from that given here. By convention it is the order in the Herzberg character tables that is to be used in numbering normal modes of vibration.

$C_4$	$\hat{E}$	$\hat{C}_4$	$\hat{C}_2$	$\hat{C}_4^3$		
A	1	1	1	1	$z; R_z$	$x^2 + y^2; z^2$
B	1	-1	1	-1		$x^2 - y^2; xy$
E	$\begin{Bmatrix} 1 & i & -1 & -i \\ 1 & -i & -1 & i \end{Bmatrix}$				$(x, y); (R_x, R_y)$	$(xz, yz)$

$C_5$	$\hat{E}$	$\hat{C}_5$	$\hat{C}_5^2$	$\hat{C}_5^3$	$\hat{C}_5^4$		$\varepsilon = \exp(2\pi i/5)$
A	1	1	1	1	1	$z; R_z$	$x^2 + y^2; z^2$
$E_1$	$\begin{Bmatrix} 1 & \varepsilon & \varepsilon^2 & \varepsilon^{2*} & \varepsilon^* \\ 1 & \varepsilon^* & \varepsilon^{2*} & \varepsilon^2 & \varepsilon \end{Bmatrix}$					$(x, y); (R_x, R_y)$	$(xz, xy)$
$E_2$	$\begin{Bmatrix} 1 & \varepsilon^2 & \varepsilon^* & \varepsilon & \varepsilon^{2*} \\ 1 & \varepsilon^{2*} & \varepsilon & \varepsilon^* & \varepsilon^2 \end{Bmatrix}$						$(x^2 - y^2, xy)$

$C_6$	$\hat{E}$	$\hat{C}_6$	$\hat{C}_3$	$\hat{C}_2$	$\hat{C}_3^2$	$\hat{C}_6^5$		$\varepsilon = \exp(2\pi i/6)$
A	1	1	1	1	1	1	$z; R_z$	$x^2 + y^2; z^2$
B	1	-1	1	-1	1	-1		
$E_1$	$\begin{Bmatrix} 1 & \varepsilon & -\varepsilon^* & -1 & -\varepsilon & \varepsilon^* \\ 1 & \varepsilon^* & -\varepsilon & -1 & -\varepsilon^* & \varepsilon \end{Bmatrix}$						$(x, y); (R_x, R_y)$	$(xz, yz)$
$E_2$	$\begin{Bmatrix} 1 & -\varepsilon^* & -\varepsilon & 1 & -\varepsilon^* & -\varepsilon \\ 1 & -\varepsilon & -\varepsilon^* & 1 & -\varepsilon & -\varepsilon^* \end{Bmatrix}$							$(x^2 - y^2, xy)$

$D_2 = V$	$\hat{E}$	$\hat{C}_2(z)$	$\hat{C}_2(y)$	$\hat{C}_2(x)$		
A	1	1	1	1		$x^2; y^2; z^2$
$B_1$	1	1	-1	-1	$z; R_z$	$xy$
$B_2$	1	-1	1	-1	$y; R_y$	$xz$
$B_3$	1	-1	-1	1	$x; R_x$	$yz$

$D_3$	$\hat{E}$	$2\hat{C}_3$	$3\hat{C}_2'$		
$A_1$	1	1	1		$x^2 + y^2; z^2$
$A_2$	1	1	-1	$z; R_z$	
E	2	-1	0	$(x, y); (R_x, R_y)$	$(x^2 - y^2, xy); (xz, yz)$

$D_4$	$\hat{E}$	$2\hat{C}_4$	$\hat{C}_2$	$2\hat{C}_2'$	$2\hat{C}_2''$	
$A_1$	1	1	1	1	1	$x^2 + y^2; z^2$
$A_2$	1	1	1	-1	-1	$z; R_z$
$B_1$	1	-1	1	1	-1	
$B_2$	1	-1	1	-1	1	$xy$
E	2	0	-2	0	0	$(x, y); (R_x, R_y)$ $(xz, yz)$

$D_5$	$\hat{E}$	$2\hat{C}_5$	$2\hat{C}_5^2$	$5\hat{C}_2'$		$\alpha = 72^\circ$
$A_1$	1	1	1	1		$x^2 + y^2; z^2$
$A_2$	1	1	1	-1	$z; R_z$	
$E_1$	2	$2\cos \alpha$	$2\cos 2\alpha$	0	$(x, y); (R_x, R_y)$	$(xz, yz)$
$E_2$	2	$2\cos 2\alpha$	$2\cos \alpha$	0		$(x^2 - y^2, xy)$

$D_6$	$\hat{E}$	$2\hat{C}_6$	$2\hat{C}_3$	$\hat{C}_2$	$3\hat{C}_2'$	$3\hat{C}_2''$	
$A_1$	1	1	1	1	1	1	$x^2 + y^2; z^2$
$A_2$	1	1	1	1	-1	-1	$z; R_z$
$B_1$	1	-1	1	-1	1	-1	
$B_2$	1	-1	1	-1	-1	1	
$E_1$	2	1	-1	-2	0	0	$(x, y); (R_x, R_y)$ $(xz, yz)$
$E_2$	2	-1	-1	2	0	0	$(x^2 - y^2, xy)$

$C_{2v}$	$\hat{E}$	$\hat{C}_2$	$\hat{\sigma}_v(xz)$	$\hat{\sigma}_v(yz)$		
$A_1$	1	1	1	1	$z$	$x^2; y^2; z^2$
$A_2$	1	1	-1	-1	$R_z$	$xy$
$B_1$	1	-1	1	-1	$x; R_y$	$xz$
$B_2$	1	-1	-1	1	$y; R_x$	$yz$

$C_{3v}$	$\hat{E}$	$2\hat{C}_3$	$3\hat{\sigma}_v$		
$A_1$	1	1	1	$z$	$x^2 + y^2; z^2$
$A_2$	1	1	-1	$R_z$	
$E$	2	-1	0	$(x, y); (R_x, R_y)$	$(x^2 - y^2, xy); (xz, yz)$

$C_{4v}$	$\hat{E}$	$2\hat{C}_4$	$\hat{C}_2$	$2\hat{\sigma}_v$	$2\hat{\sigma}_d$	
$A_1$	1	1	1	1	1	$z$ $x^2 + y^2; z^2$
$A_2$	1	1	1	-1	-1	$R_z$
$B_1$	1	-1	1	1	-1	$x^2 - y^2$
$B_2$	1	-1	1	-1	1	$xy$
$E$	2	0	-2	0	0	$(x, y); (R_x, R_y)$ $(xz, yz)$

$C_{5v}$	$\hat{E}$	$2\hat{C}_5$	$2\hat{C}_5^2$	$5\hat{\sigma}_v$		$\alpha = 72^\circ$
$A_1$	1	1	1	1	$z$	$x^2 + y^2; z^2$
$A_2$	1	1	1	-1	$R_z$	
$E_1$	2	$2\cos \alpha$	$2\cos 2\alpha$	0	$(x, y); (R_x, R_y)$	$(xz, yz)$
$E_2$	2	$2\cos 2\alpha$	$2\cos \alpha$	0		$(x^2 - y^2, xy)$

$C_{6v}$	$\hat{E}$	$2\hat{C}_6$	$2\hat{C}_3$	$\hat{C}_2$	$3\hat{\sigma}_v$	$3\hat{\sigma}_d$	
$A_1$	1	1	1	1	1	1	$z$ $x^2 + y^2; z^2$
$A_2$	1	1	1	1	-1	-1	$R_z$
$B_1$	1	-1	1	-1	1	-1	
$B_2$	1	-1	1	-1	-1	1	
$E_1$	2	1	-1	-2	0	0	$(x, y); (R_x, R_y)$ $(xz, yz)$
$E_2$	2	-1	-1	2	0	0	$(x^2 - y^2, xy)$

$C_{2h}$	$\hat{E}$	$\hat{C}_2$	$\hat{i}$	$\hat{\sigma}_h$		
$A_g$	1	1	1	1	$R_z$	$x^2; y^2; z^2; xy$
$B_g$	1	-1	1	-1	$R_x; R_y$	$xz; yz$
$A_u$	1	1	-1	-1	$z$	
$B_u$	1	-1	-1	1	$x; y$	

$C_{3h}$	$\hat{E}$	$\hat{C}_3$	$\hat{C}_3^2$	$\hat{\sigma}_h$	$\hat{S}_3$	$\hat{S}_3^5$		$\epsilon = \exp(2\pi i/3)$
$A'$	1	1	1	1	1	1	$R_z$	$x^2 + y^2; z^2$
$E'$	$\left\{ \begin{array}{ccc} 1 & \epsilon & \epsilon^* \\ 1 & \epsilon^* & \epsilon \end{array} \right\}$	1	1	$\epsilon$	$\epsilon^*$	$\left. \begin{array}{l} (x, y) \\ z \end{array} \right\}$	$(x^2 - y^2, xy)$	
$A''$		1	1	-1	-1			-1
$E''$	$\left\{ \begin{array}{ccc} 1 & \epsilon & \epsilon^* \\ 1 & \epsilon^* & \epsilon \end{array} \right\}$	-1	-1	$-\epsilon$	$-\epsilon^*$	$\left. \begin{array}{l} (R_x, R_y) \\ (xz, yz) \end{array} \right\}$		

$C_{4h}$	$\hat{E}$	$\hat{C}_4$	$\hat{C}_2$	$\hat{C}_4^3$	$\hat{i}$	$\hat{S}_4^3$	$\hat{\sigma}_h$	$\hat{S}_4$	
$A_g$	1	1	1	1	1	1	1	1	$R_z$ $x^2 + y^2; z^2$
$B_g$	1	-1	1	-1	1	-1	1	-1	$x^2 - y^2; xy$
$E_g$	$\left\{ \begin{array}{ccc} 1 & i & -1 \\ 1 & -i & -1 \end{array} \right\}$	-1	-1	1	$i$	-1	-1	$\left. \begin{array}{l} (R_x, R_y) \\ z \end{array} \right\}$	$(xz, xy)$
$A_u$		1	1	1	1	-1	-1		
$B_u$	1	-1	1	-1	-1	1	-1	1	
$E_u$	$\left\{ \begin{array}{ccc} 1 & i & -1 \\ 1 & -i & -1 \end{array} \right\}$	-1	-1	-1	$-i$	1	1	$\left. \begin{array}{l} (x, y) \end{array} \right\}$	

$C_{5h}$	$\hat{E}$	$\hat{C}_5$	$\hat{C}_5^2$	$\hat{C}_5^3$	$\hat{C}_5^4$	$\hat{\sigma}_h$	$\hat{S}_5$	$\hat{S}_5^7$	$\hat{S}_5^3$	$\hat{S}_5^9$	$\varepsilon = \exp(2\pi i/5)$		
$A'$	1	1	1	1	1	1	1	1	1	1		$R_z$	$x^2 + y^2; z^2$
$E'_1$	1	$\varepsilon$	$\varepsilon^2$	$\varepsilon^{2*}$	$\varepsilon^*$	1	$\varepsilon$	$\varepsilon^2$	$\varepsilon^{2*}$	$\varepsilon^*$	}	$(x, y)$	
$E'_2$		$\varepsilon^*$	$\varepsilon^{2*}$	$\varepsilon$	$\varepsilon^2$	$\varepsilon^*$	$\varepsilon$	$\varepsilon^2$	$\varepsilon^{2*}$	$\varepsilon$		$(x^2 - y^2; xy)$	
$A''$	1	1	1	1	1	-1	-1	-1	-1	-1		$z$	
$E''_1$	1	$\varepsilon$	$\varepsilon^2$	$\varepsilon^{2*}$	$\varepsilon^*$	-1	$-\varepsilon$	$-\varepsilon^2$	$-\varepsilon^{2*}$	$-\varepsilon^*$	}	$(R_x, R_y)$	$(xz, yz)$
$E''_2$		$\varepsilon^*$	$\varepsilon^{2*}$	$\varepsilon$	$\varepsilon^2$	$\varepsilon^*$	-1	$-\varepsilon^*$	$-\varepsilon^2$	$-\varepsilon$		$-\varepsilon^{2*}$	
	1	$\varepsilon^2$	$\varepsilon$	$\varepsilon^*$	$\varepsilon^2$	-1	$-\varepsilon^2$	$-\varepsilon^*$	$-\varepsilon$	$-\varepsilon^{2*}$			
	1	$\varepsilon^{2*}$	$\varepsilon$	$\varepsilon^*$	$\varepsilon^2$	-1	$-\varepsilon^{2*}$	$-\varepsilon$	$-\varepsilon^*$	$-\varepsilon^2$			

$C_{6h}$	$\hat{E}$	$\hat{C}_6$	$\hat{C}_3$	$\hat{C}_2$	$\hat{C}_3^2$	$\hat{C}_6^5$	$\hat{i}$	$\hat{S}_3^5$	$\hat{S}_6^5$	$\hat{\sigma}_h$	$\hat{S}_6$	$\hat{S}_3$	$\alpha = \exp(2\pi i/6)$		
$A_g$	1	1	1	1	1	1	1	1	1	1	1	1		$R_z$	$x^2 + y^2; z^2$
$B_g$	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1			
$E_{1g}$	1	$\varepsilon$	$-\varepsilon^*$	-1	$-\varepsilon$	$\varepsilon^*$	1	$\varepsilon$	$-\varepsilon^*$	-1	$-\varepsilon$	$\varepsilon^*$	}	$(R_x, R_y)$	$(xz, yz)$
$E_{2g}$		$\varepsilon^*$	$-\varepsilon$	-1	$-\varepsilon^*$	$\varepsilon$	1	$\varepsilon^*$	$-\varepsilon$	-1	$-\varepsilon^*$	$\varepsilon$		$(x^2 - y^2, xy)$	
	1	$-\varepsilon^*$	$-\varepsilon$	1	$-\varepsilon^*$	$-\varepsilon$	1	$-\varepsilon^*$	$-\varepsilon$	1	$-\varepsilon^*$	$-\varepsilon$			
	1	$-\varepsilon$	$-\varepsilon^*$	1	$-\varepsilon$	$-\varepsilon^*$	1	$-\varepsilon$	$-\varepsilon^*$	1	$-\varepsilon$	$-\varepsilon^*$		$z$	
$A_u$	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1			
$B_u$	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1			
$E_{1u}$	1	$\varepsilon$	$-\varepsilon^*$	-1	$-\varepsilon$	$\varepsilon^*$	-1	$-\varepsilon$	$\varepsilon^*$	1	$\varepsilon$	$-\varepsilon^*$	}	$(x, y)$	
$E_{2u}$		$\varepsilon^*$	$-\varepsilon$	-1	$-\varepsilon^*$	$\varepsilon$	-1	$-\varepsilon^*$	$\varepsilon$	1	$\varepsilon^*$	$-\varepsilon$			
	1	$-\varepsilon^*$	$-\varepsilon$	1	$-\varepsilon^*$	$-\varepsilon$	-1	$\varepsilon^*$	$\varepsilon$	-1	$\varepsilon^*$	$\varepsilon$			
	1	$-\varepsilon$	$-\varepsilon^*$	1	$-\varepsilon$	$-\varepsilon^*$	-1	$\varepsilon$	$\varepsilon^*$	-1	$\varepsilon$	$\varepsilon^*$			

$D_{2h} = V_h$	$\hat{E}$	$\hat{C}_2(z)$	$\hat{C}_2(y)$	$\hat{C}_2(x)$	$\hat{i}$	$\hat{\sigma}(xy)$	$\hat{\sigma}(xz)$	$\hat{\sigma}(yz)$		
$A_g$	1	1	1	1	1	1	1	1		$x^2; y^2; z^2$
$B_{1g}$	1	1	-1	-1	1	1	-1	-1	$R_z$	$xy$
$B_{2g}$	1	-1	1	-1	1	-1	1	-1	$R_y$	$xz$
$B_{3g}$	1	-1	-1	1	1	-1	-1	1	$R_x$	$yz$
$A_u$	1	1	1	1	-1	-1	-1	-1		
$B_{1u}$	1	1	-1	-1	-1	-1	1	1		$z$
$B_{2u}$	1	-1	1	-1	-1	1	-1	1		$y$
$B_{3u}$	1	-1	-1	1	-1	1	1	-1		$x$

$D_{3h}$	$\hat{E}$	$2\hat{C}_3$	$3\hat{C}_2$	$\hat{\sigma}_h$	$2\hat{S}_3$	$3\hat{\sigma}_v$			
$A'_1$	1	1	1	1	1	1			$x^2 + y^2; z^2$
$A'_2$	1	1	-1	1	1	-1	$R_z$		
$E'$	2	-1	0	2	-1	0	$(x, y)$		$(x^2 - y^2, xy)$
$A''_1$	1	1	1	-1	-1	-1			
$A''_2$	1	1	-1	-1	-1	1	$z$		
$E''$	2	-1	0	-2	1	0	$(R_x, R_y)$		$(xz, yz)$

$D_{4h}$	$\hat{E}$	$2\hat{C}_4$	$\hat{C}_2$	$2\hat{C}'_2$	$2\hat{C}''_2$	$\hat{i}$	$2\hat{S}_4$	$\hat{\sigma}_h$	$2\hat{\sigma}_v$	$2\hat{\sigma}_d$		
$A_{1g}$	1	1	1	1	1	1	1	1	1	1		$x^2 + y^2; z^2$
$A_{2g}$	1	1	1	-1	-1	1	1	1	-1	-1	$R_z$	
$B_{1g}$	1	-1	1	1	-1	1	-1	1	1	-1		$x^2 - y^2$
$B_{2g}$	1	-1	1	-1	1	1	-1	1	-1	1		$xy$
$E_g$	2	0	-2	0	0	2	0	-2	0	0	$(R_x, R_y)$	$(xz, yz)$
$A_{1u}$	1	1	1	1	1	-1	-1	-1	-1	-1		
$A_{2u}$	1	1	1	-1	-1	-1	-1	-1	1	1		$z$
$B_{1u}$	1	-1	1	1	-1	-1	1	-1	-1	1		
$B_{2u}$	1	-1	1	-1	1	-1	1	-1	1	-1		
$E_u$	2	0	-2	0	0	-2	0	2	0	0		$(x, y)$

$D_{5h}$	$\hat{E}$	$2\hat{C}_5$	$2\hat{C}_5^2$	$5\hat{C}_2$	$\hat{\sigma}_h$	$2\hat{S}_5$	$2\hat{S}_5^3$	$5\hat{\sigma}_v$		$\alpha = 72^\circ$
$A_1'$	1	1	1	1	1	1	1	1		$x^2 + y^2; z^2$
$A_2'$	1	1	1	-1	1	1	1	-1	$R_z$	
$E_1'$	2	$2 \cos \alpha$	$2 \cos 2\alpha$	0	2	$2 \cos \alpha$	$2 \cos 2\alpha$	0	$(x, y)$	
$E_2'$	2	$2 \cos 2\alpha$	$2 \cos \alpha$	0	2	$2 \cos 2\alpha$	$2 \cos \alpha$	0		$(x^2 - y^2, xy)$
$A_1''$	1	1	1	1	-1	-1	-1	-1		
$A_2''$	1	1	1	-1	-1	-1	-1	1	$z$	
$E_1''$	2	$2 \cos \alpha$	$2 \cos 2\alpha$	0	-2	$-2 \cos \alpha$	$-2 \cos 2\alpha$	0	$(R_x, R_y)$	$(xz, yz)$
$E_2''$	2	$2 \cos 2\alpha$	$2 \cos \alpha$	0	-2	$-2 \cos 2\alpha$	$-2 \cos \alpha$	0		

$D_{6h}$	$\hat{E}$	$2\hat{C}_6$	$2\hat{C}_3$	$\hat{C}_2$	$3\hat{C}_2'$	$3\hat{C}_2''$	$\hat{i}$	$2\hat{S}_3$	$2\hat{S}_6$	$\hat{\sigma}_h$	$3\hat{\sigma}_d$	$3\hat{\sigma}_v$		
$A_{1g}$	1	1	1	1	1	1	1	1	1	1	1	1		$x^2 + y^2; z^2$
$A_{2g}$	1	1	1	1	-1	-1	1	1	1	1	-1	-1	$R_z$	
$B_{1g}$	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1		
$B_{2g}$	1	-1	1	-1	-1	1	1	-1	1	-1	-1	1		
$E_{1g}$	2	1	-1	-2	0	0	2	1	-1	-2	0	0	$(R_x, R_y)$	$(xz, yz)$
$E_{2g}$	2	-1	-1	2	0	0	2	-1	-1	2	0	0		$(x^2 - y^2, xy)$
$A_{1u}$	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1		
$A_{2u}$	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	$z$	
$B_{1u}$	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1		
$B_{2u}$	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1		
$E_{1u}$	2	1	-1	-2	0	0	-2	-1	1	2	0	0	$(x, y)$	
$E_{2u}$	2	-1	-1	2	0	0	-2	1	1	-2	0	0		

$D_{2d} = V_d$	$\hat{E}$	$2\hat{S}_4$	$\hat{C}_2$	$2\hat{C}'_2$	$2\hat{\sigma}_d$		
$A_1$	1	1	1	1	1		$x^2 + y^2; z^2$
$A_2$	1	1	1	-1	-1	$R_x$	
$B_1$	1	-1	1	1	-1		$x^2 - y^2$
$B_2$	1	-1	1	-1	1	$z$	$xy$
$E$	2	0	-2	0	0	$(x, y); (R_x, R_y)$	$(xz, yz)$

$D_{3d}$	$\hat{E}$	$2\hat{C}_3$	$3\hat{C}'_2$	$\hat{i}$	$2\hat{S}_6$	$3\hat{\sigma}_d$		
$A_{1g}$	1	1	1	1	1	1		$x^2 + y^2; z^2$
$A_{2g}$	1	1	-1	1	1	-1	$R_x$	
$E_g$	2	-1	0	2	-1	0	$(R_x, R_y)$	$(x^2 - y^2, xy); (xz, yz)$
$A_{1u}$	1	1	1	-1	-1	-1		
$A_{2u}$	1	1	-1	-1	-1	1	$z$	
$E_u$	2	-1	0	-2	1	0	$(x, y)$	

$D_{4d}$	$\hat{E}$	$2\hat{S}_8$	$2\hat{C}_4$	$2\hat{S}_8^3$	$\hat{C}_2$	$4\hat{C}'_2$	$4\hat{\sigma}_d$		
$A_1$	1	1	1	1	1	1	1		$x^2 + y^2; z^2$
$A_2$	1	1	1	1	1	-1	-1	$R_x$	
$B_1$	1	-1	1	-1	1	1	-1		
$B_2$	1	-1	1	-1	1	-1	1	$z$	
$E_1$	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	0	0	$(x, y)$	
$E_2$	2	0	-2	0	2	0	0		$(x^2 - y^2, xy)$
$E_3$	2	$-\sqrt{2}$	0	$\sqrt{2}$	-2	0	0	$(R_x, R_y)$	$(xz, yz)$

$D_{5d}$	$\hat{E}$	$2\hat{C}_5$	$2\hat{C}_5^2$	$5\hat{C}_2'$	$\hat{i}$	$2\hat{S}_{10}^3$	$2\hat{S}_{10}$	$5\hat{\sigma}_d$		$\alpha = 72^\circ$
$A_{1g}$	1	1	1	1	1	1	1	1		$x^2 + y^2; z^2$
$A_{2g}$	1	1	1	-1	1	1	1	-1	$R_z$	
$E_{1g}$	2	$2 \cos \alpha$	$2 \cos 2\alpha$	0	2	$2 \cos \alpha$	$2 \cos 2\alpha$	0	$(R_x, R_y)$	$(xz, yz)$
$E_{2g}$	2	$2 \cos 2\alpha$	$2 \cos \alpha$	0	2	$2 \cos 2\alpha$	$2 \cos \alpha$	0		$(x^2 - y^2, xy)$
$A_{1u}$	1	1	1	1	-1	-1	-1	-1		
$A_{2u}$	1	1	1	-1	-1	-1	-1	1	$z$	
$E_{1u}$	2	$2 \cos \alpha$	$2 \cos 2\alpha$	0	-2	$-2 \cos \alpha$	$-2 \cos 2\alpha$	0	$(x, y)$	
$E_{2u}$	2	$2 \cos 2\alpha$	$2 \cos \alpha$	0	-2	$-2 \cos 2\alpha$	$-2 \cos \alpha$	0		

$D_{6d}$	$\hat{E}$	$2\hat{S}_{12}$	$2\hat{C}_6$	$2\hat{S}_4$	$2\hat{C}_3$	$2\hat{S}_{12}^5$	$\hat{C}_2$	$6\hat{C}_2'$	$6\hat{\sigma}_d$		
$A_1$	1	1	1	1	1	1	1	1	1		$x^2 + y^2; z^2$
$A_2$	1	1	1	1	1	1	1	-1	-1	$R_z$	
$B_1$	1	-1	1	-1	1	-1	1	1	-1		
$B_2$	1	-1	1	-1	1	-1	1	-1	1	$z$	
$E_1$	2	$\sqrt{3}$	1	0	-1	$-\sqrt{3}$	-2	0	0	$(x, y)$	
$E_2$	2	1	-1	-2	-1	1	2	0	0		$(x^2 - y^2, xy)$
$E_3$	2	0	-2	0	2	0	-2	0	0		
$E_4$	2	-1	-1	2	-1	-1	2	0	0		
$E_5$	2	$-\sqrt{3}$	1	0	-1	$\sqrt{3}$	-2	0	0	$(R_x, R_y)$	$(xz, yz)$

$S_4$	$\hat{E}$	$\hat{S}_4$	$\hat{C}_2$	$\hat{S}_4^3$		
$A$	1	1	1	1	$R_z$	$x^2 + y^2; z^2$
$B$	1	-1	1	-1	$z$	$x^2 - y^2; xy$
$E$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$	$\left\{ \begin{array}{l} i \\ -i \end{array} \right.$	$\left\{ \begin{array}{l} -1 \\ -1 \end{array} \right.$	$\left\{ \begin{array}{l} -i \\ i \end{array} \right.$	$(x, y); (R_x, R_y)$	$(xz, yz)$

$S_6$	$\hat{E}$	$\hat{C}_3$	$\hat{C}_3^2$	$\hat{i}$	$\hat{S}_6^5$	$\hat{S}_6$	$\epsilon = \exp(2\pi i/3)$
$A_g$	1	1	1	1	1	1	$R_z$ $x^2 + y^2; z^2$
$E_g$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$	$\epsilon$	$\epsilon^*$	1	$\epsilon$	$\epsilon^*$	$(R_x, R_y)$ $(x^2 - y^2, xy); (xz, yz)$
$A_u$		1	1	1	-1	-1	
$E_u$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$	$\epsilon$	$\epsilon^*$	-1	$-\epsilon$	$-\epsilon^*$	$z$ $(x, y)$
		1	$\epsilon^*$	$\epsilon$	-1	$-\epsilon^*$	

$S_8$	$\hat{E}$	$\hat{S}_8$	$\hat{C}_4$	$\hat{S}_8^3$	$\hat{C}_2$	$\hat{S}_8^5$	$\hat{C}_4^3$	$\hat{S}_8^7$	$\epsilon = \exp(2\pi i/8)$
$A$	1	1	1	1	1	1	1	1	$R_z$ $x^2 + y^2; z^2$
$B$	1	-1	1	-1	1	-1	1	-1	$z$
$E_1$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$	$\epsilon$	$i$	$-\epsilon^*$	-1	$-\epsilon$	$-i$	$\epsilon^*$	$(x, y); (R_x, R_y)$
		1	$\epsilon^*$	$-i$	$-\epsilon$	-1	$-\epsilon^*$	$i$	
$E_2$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$	$i$	-1	$-i$	1	$i$	-1	$-i$	$(x^2 - y^2, xy)$
		1	$-i$	-1	$i$	1	$-i$	-1	
$E_3$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$	$-\epsilon^*$	$-i$	$\epsilon$	-1	$\epsilon^*$	$i$	$-\epsilon$	$(xz, yz)$
		1	$-\epsilon$	$i$	$\epsilon^*$	-1	$\epsilon$	$-i$	

$T_d$	$\hat{E}$	$8\hat{C}_3$	$3\hat{C}_2$	$6\hat{S}_4$	$6\hat{\sigma}_d$	
$A_1$	1	1	1	1	1	$x^2 + y^2 + z^2$
$A_2$	1	1	1	-1	-1	
$E$	2	-1	2	0	0	$(2z^2 - x^2 - y^2, x^2 - y^2)$
$T_1(F_1)$	3	0	-1	1	-1	$(R_x, R_y, R_z)$
$T_2(F_2)$	3	0	-1	-1	1	$(x, y, z)$ $(xy, xz, yz)$



$D_{\infty h}$	$\hat{E}$	$2\hat{C}(\phi)$	...	$\infty\hat{\sigma}_v$	$\hat{i}$	$2\hat{S}(\phi)$	...	$\infty\hat{C}'_2$		
$\Sigma_g^+$	1	1	...	1	1	1	...	1		$x^2 + y^2; z^2$
$\Sigma_g^-$	1	1	...	-1	1	1	...	-1	$R_z$	
$\Pi_g$	2	$2\cos\phi$	...	0	2	$-2\cos\phi$	...	0	$(R_x, R_y)$	$(xz, yz)$
$\Delta_g$	2	$2\cos 2\phi$	...	0	2	$2\cos 2\phi$	...	0		$(x^2 - y^2, xy)$
...	...	...	...	...	...	...	...	...		
$\Sigma_u^+$	1	1	...	1	-1	-1	...	-1	$z$	
$\Sigma_u^-$	1	1	...	-1	-1	-1	...	1		
$\Pi_u$	2	$2\cos\phi$	...	0	-2	$2\cos\phi$	...	0	$(x, y)$	
$\Delta_u$	2	$2\cos 2\phi$	...	0	-2	$-2\cos 2\phi$	...	0		
...	...	...	...	...	...	...	...	...		

$I_h$	$\hat{E}$	$12\hat{C}_5$	$12\hat{C}_5^2$	$20\hat{C}_3$	$15\hat{C}_2$	$\hat{i}$	$12\hat{S}_{10}$	$12\hat{S}_{10}^3$	$20\hat{S}_6$	$15\hat{\sigma}$		
$A_g$	1	1	1	1	1	1	1	1	1	1		$x^2 + y^2 + z^2$
$T_{1g}$	3	$2\cos\frac{\pi}{5}$	$2\cos\frac{3\pi}{5}$	0	-1	3	$2\cos\frac{3\pi}{5}$	$2\cos\frac{\pi}{5}$	0	-1	$(R_x, R_y, R_z)$	
$T_{2g}$	3	$2\cos\frac{3\pi}{5}$	$2\cos\frac{\pi}{5}$	0	-1	3	$2\cos\frac{\pi}{5}$	$2\cos\frac{3\pi}{5}$	0	-1		
$G_g$	4	-1	-1	1	0	4	-1	-1	1	0		
$H_g$	5	0	0	-1	1	5	0	0	-1	1		$(2z^2 - x^2 - y^2, x^2 - y^2, xy, yz, xz)$
$A_u$	1	1	1	1	1	-1	-1	-1	-1	-1		
$T_{1u}$	3	$2\cos\frac{\pi}{5}$	$2\cos\frac{3\pi}{5}$	0	-1	-3	$-2\cos\frac{3\pi}{5}$	$-2\cos\frac{\pi}{5}$	0	1	$(x, y, z)$	
$T_{2u}$	3	$2\cos\frac{3\pi}{5}$	$2\cos\frac{\pi}{5}$	0	-1	-3	$-2\cos\frac{\pi}{5}$	$-2\cos\frac{3\pi}{5}$	0	1		
$G_u$	4	-1	-1	1	0	-4	1	1	-1	0		
$H_u$	5	0	0	-1	1	-5	0	0	1	-1		

# Appendix C

## Direct Product Tables

The antisymmetric product is in brackets. Since the tables are symmetric about the principal diagonal, the part below the diagonal is omitted.

$C_s$	$A'$	$A''$		
$A'$	$A'$	$A''$		
$A''$		$A'$		
$C_i$	$A_g$	$A_u$		
$A_g$	$A_g$	$A_u$		
$A_u$		$A_g$		
$C_2, C_{2h}^1$	$A$	$B$		
$A$	$A$	$B$		
$B$		$A$		
$C_{2v}$	$A_1$	$A_2$	$B_1$	$B_2$
$A_1$	$A_1$	$A_2$	$B_1$	$B_2$
$A_2$		$A_1$	$B_2$	$B_1$
$B_1$			$A_1$	$A_2$
$B_2$				$A_1$
$D_2, D_{2h}^1$	$A$	$B_1$	$B_2$	$B_3$
$A$	$A$	$B_1$	$B_2$	$B_3$
$B_1$		$A$	$B_3$	$B_2$
$B_2$			$A$	$B_1$
$B_3$				$A$

After Harris, D. C. and Bertolucci, M. D., *Symmetry and Spectroscopy*, Dover, New York, 1989.

$C_3, C_{3h}, {}^2S_6^1$	A	E				
A	A	E				
E	[A] + A + E					
$C_{3v}, D_3, D_{3d}, {}^1D_{3h}^2$	A <sub>1</sub>	A <sub>2</sub>	E			
A <sub>1</sub>	A <sub>1</sub>	A <sub>2</sub>	E			
A <sub>2</sub>		A <sub>1</sub>	E			
E	A <sub>1</sub> + [A <sub>2</sub> ] + E					
$C_4, C_{4h}, {}^1S_4$	A	B	E			
A	A	B	E			
B		A	E			
E	[A] + A + 2B					
$C_{4v}, D_4, D_{2d}, D_{4h}^1$	A <sub>1</sub>	A <sub>2</sub>	B <sub>1</sub>	B <sub>2</sub>	E	
A <sub>1</sub>	A <sub>1</sub>	A <sub>2</sub>	B <sub>1</sub>	B <sub>2</sub>	E	
A <sub>2</sub>		A <sub>1</sub>	B <sub>2</sub>	B <sub>1</sub>	E	
B <sub>1</sub>			A <sub>1</sub>	A <sub>2</sub>	E	
B <sub>2</sub>				A <sub>1</sub>	E	
E	A <sub>1</sub> + [A <sub>2</sub> ] + B <sub>1</sub> + B <sub>2</sub>					
$C_5, C_{5h}^2$	A	E <sub>1</sub>	E <sub>2</sub>			
A	A	E <sub>1</sub>	E <sub>2</sub>			
E <sub>1</sub>	[A] + A + E <sub>2</sub>		E <sub>1</sub> + E <sub>2</sub>			
E <sub>2</sub>	[A] + A + E <sub>1</sub>					
$C_{5v}, D_5, D_{5d}, {}^1D_{5h}^2$	A <sub>1</sub>	A <sub>2</sub>	E <sub>1</sub>	E <sub>2</sub>		
A <sub>1</sub>	A <sub>1</sub>	A <sub>2</sub>	E <sub>1</sub>	E <sub>2</sub>		
A <sub>2</sub>		A <sub>1</sub>	E <sub>1</sub>	E <sub>2</sub>		
E <sub>1</sub>			A <sub>1</sub> + [A <sub>2</sub> ] + E <sub>2</sub>		E <sub>1</sub> + E <sub>2</sub>	
E <sub>2</sub>	A <sub>1</sub> + [A <sub>2</sub> ] + E <sub>1</sub>					
$C_6, C_{6h}^1$	A	B	E <sub>1</sub>	E <sub>2</sub>		
A	A	B	E <sub>1</sub>	E <sub>2</sub>		
B		A	E <sub>2</sub>	E <sub>1</sub>		
E <sub>1</sub>			[A] + A + E <sub>2</sub>		2B + E <sub>1</sub>	
E <sub>2</sub>	[A] + A + E <sub>2</sub>					
$C_{6v}, D_6, D_{6h}^1$	A <sub>1</sub>	A <sub>2</sub>	B <sub>1</sub>	B <sub>2</sub>	E <sub>1</sub>	E <sub>2</sub>
A <sub>1</sub>	A <sub>1</sub>	A <sub>2</sub>	B <sub>1</sub>	B <sub>2</sub>	E <sub>1</sub>	E <sub>2</sub>
A <sub>2</sub>		A <sub>1</sub>	B <sub>2</sub>	B <sub>1</sub>	E <sub>1</sub>	E <sub>2</sub>
B <sub>1</sub>			A <sub>1</sub>	A <sub>2</sub>	E <sub>2</sub>	E <sub>1</sub>
B <sub>2</sub>				A <sub>1</sub>	E <sub>2</sub>	E <sub>1</sub>
E <sub>1</sub>					A <sub>1</sub> + [A <sub>2</sub> ] + E <sub>2</sub>	
E <sub>2</sub>						B <sub>1</sub> + B <sub>2</sub> + E <sub>1</sub>
						A <sub>1</sub> + [A <sub>2</sub> ] + E <sub>2</sub>

$D_{6d}$	$A_1$	$A_2$	$B_1$	$B_2$	$E_1$	$E_2$	$E_3$	$E_4$	$E_5$
$A_1$	$A_1$	$A_2$	$B_1$	$B_2$	$E_1$	$E_2$	$E_3$	$E_4$	$E_5$
$A_2$		$A_1$	$B_2$	$B_1$	$E_1$	$E_2$	$E_3$	$E_4$	$E_5$
$B_1$			$A_1$	$A_2$	$E_5$	$E_4$	$E_3$	$E_2$	$E_1$
$B_2$				$A_1$	$E_5$	$E_4$	$E_3$	$E_2$	$E_1$
$E_1$					$A_1 + [A_2] + E_2$	$E_1 + E_3$	$E_2 + E_4$	$E_3 + E_5$	$B_1 + B_2 + E_4$
$E_2$						$A_1 + [A_2] + E_4$	$E_1 + E_5$	$B_1 + B_2 + E_2$	$E_3 + E_5$
$E_3$							$A_1 + [A_2] + B_1 + B_2$	$E_1 + E_5$	$E_2 + E_4$
$E_4$								$A_1 + [A_2] + E_4$	$E_1 + E_3$
$E_5$									$A_1 + [A_2] + E_2$

$O, O_h, {}^1 T_d$	$A_1$	$A_2$	$E$	$T_1$	$T_2$
$A_1$	$A_1$	$A_2$	$E$	$T_1$	$T_2$
$A_2$		$A_1$	$E$	$T_2$	$T_1$
$E$			$A_1 + [A_2] + E$	$T_1 + T_2$	$T_1 + T_2$
$T_1$				$A_1 + E + [T_1] + T_2$	$A_2 + E + T_1 + T_2$
$T_2$					$A_1 + E + [T_1] + T_2$

$C_{\infty v}, D_{\infty h}^1$	$\Sigma^+$	$\Sigma^-$	$\Pi$	$\Delta$	$\Phi$	$\Gamma$	...
$\Sigma^+$	$\Sigma^+$	$\Sigma^-$	$\Pi$	$\Delta$	$\Phi$	$\Gamma$	
$\Sigma^-$		$\Sigma^+$	$\Pi$	$\Delta$	$\Phi$	$\Gamma$	
$\Pi$			$\Sigma^+ + [\Sigma^-] + \Delta$	$\Pi + \Phi$	$\Delta + \Gamma$	$\Phi + H$	
$\Delta$				$\Sigma^+ + [\Sigma^-] + \Gamma$	$\Pi + H$	$\Delta + I$	
$\Phi$					$\Sigma^+ + [\Sigma^-] + I$	$\Pi + K$	
$\Gamma$						$\Sigma^+ + [\Sigma^-] + \Lambda$	
...							

$I_h^1$	$A$	$T_1$	$T_2$	$G$	$H$
$A$	$A$	$T_1$	$T_2$	$G$	$H$
$T_1$		$A + [T_1] + H$	$G + H$	$T_2 + G + H$	$T_1 + T_2 + G + H$
$T_2$			$A + [T_2] + H$	$T_1 + G + H$	$T_1 + T_2 + G + H$
$G$				$A + [T_1] + [T_2] + G + H$	$T_1 + T_2 + G + 2H$
$H$					$A + [T_1] + [T_2] + [G] + G + 2H$
$K, K_h^1$	$S$	$P$	$D$	$F$	$\dots$
$S$	$S$	$P$	$D$	$F$	
$P$		$S + [P] + D$	$P + D + F$	$D + F + G$	
$D$			$S + [P] + D + [F] + G$	$P + D + F + G + H$	
$F$				$S + [P] + D + [F] + G + [H] + I$	
$\dots$					

<sup>1</sup>Add the  $g$ - $u$  selection rules, viz.,  $g \times g = g; g \times u = u; u \times u = g$ .

<sup>2</sup>Add the prime-double prime selection rules, viz., ' $\times$ ' = ' '; ' $\times$ ' = " ; " $\times$ " = ' .

## Appendix D

# Introductory Textbooks

In addition to more specialized monographs, there are a number of textbooks that cover the entire field of molecular spectroscopy at an introductory level. Unfortunately many of them are out of print and copies are difficult to obtain.

Atkins, P. W. and Friedman, K. S., *Molecular Quantum Mechanics*, 3rd ed., Oxford University Press, Oxford, 1999.

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# Figure Acknowledgments

- Figure 1.2. After Figure 0-1 of Harris and Bertolucci.
- Figure 1.5. After a figure provided by S. C. Liao and Dr. J. F. Ogilvie.
- Figure 1.10. After Figure 2.1 of Letokhov and Chebotayev.
- Figure 1.11. After Figure 2.30 of Demtröder.
- Figure 1.21. After Figure 2.18 of Svelto.
- Figure 2.11. After Table 3-7.1 of Bishop.
- Figure 3.5. After Figure 5-2.3 of Bishop.
- Figure 5.1. Courtesy of G. Herzberg and I. Dabrowski.
- Figure 5.13. After Figure 28 of Herzberg, *Atoms*.
- Figure 5.14. After Figure 27 of Herzberg, *Atoms*.
- Figure 5.15. After Figure 32 of Herzberg, *Atoms*.
- Figure 6.15. Reproduced, with permission, from Fleming, J. W. and Chamberlain, J., *Infrared Phys.* **14**, 277 (1974).
- Figure 6.19. Courtesy of Professor H. W. Kroto, University of Sussex.
- Figure 6.22. After Figure 27(a) of Herzberg *IR and Raman Spectra*.
- Figure 6.27. After Figure 3.1 of Zare.
- Figure 6.32. Courtesy of Professor H. W. Kroto, University of Sussex.
- Figure 7.5. Courtesy of Professor R. J. Le Roy, University of Waterloo.
- Figure 7.10. After Figure 1 from Le Roy, R. J., in *Molecular Spectroscopy: A Specialist Periodical Report*, Vol. 1, Barrow, R. F., Long, D. A., and Millen, D. J., editors, Chemical Society, London, 1973.
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- Figure 7.35. Reproduced, with permission, from Frum, C. I., Engleman, R., and Bernath, P. F., *J. Chem. Phys.* **95**, 1435 (1991).

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- Figure 7.40. After Figure 91 of Herzberg *IR and Raman Spectra*.
- Figure 7.42. After Figure 122 of Herzberg *IR and Raman Spectra*.
- Figure 7.47. After Figure 32 of Herzberg *IR and Raman Spectra*.
- Figure 7.48 and 7.49. After Figure 116 of Herzberg *IR and Raman Spectra*.
- Figure 7.50. After Figure 117 of Herzberg *IR and Raman Spectra*.
- Figure 7.51. After Figure 118 of Herzberg *IR and Raman Spectra*.
- Figure 7.52. After Figure 99 of Herzberg *Polyatomics*.
- Figure 7.54. After Figure 41 of Herzberg *IR and Raman Spectra*.
- Figure 7.56. After Figure 137 of Herzberg *IR and Raman Spectra*.
- Figure 7.58. After Figure 60 of Herzberg *IR and Raman Spectra*.
- Figure 7.62. After Figure 5.64 of Hollas.
- Figure 8.4. After Figure 3.3 of Long.
- Figure 8.5. After Figure 3.6 of Long.
- Figure 8.7, 8.8, and 8.10. Courtesy of R. Bartholomew and Prof. D. Irish, University of Waterloo.
- Figure 8.14. Reproduced, with permission, from Hollas, Figure 4.33(a).
- Figure 8.16. Reproduced, with permission, from Bendtsen, J., *J. Raman Spectrosc.* **2**, 133 (1974).
- Figure 9.2. After Figure 4.27 of DeKock and Gray.
- Figure 9.5. After Figure 195 of Herzberg *Diatomics*.
- Figure 9.6. Reproduced, with permission, from Tyte, D. C. and Nicholls, R. W., *Identification Atlas of Molecular Spectra*, Vol. 2, 1964.
- Figure 9.9. Courtesy of Professor R. J. Le Roy, University of Waterloo.
- Figure 9.14. Reproduced, with permission, from Pianalto, F. S., Brazier, C. R., O'Brien, L. C., and Bernath, P. F., *J. Mol. Spectrosc.* **132**, 80 (1988).
- Figure 10.2. After Figure 5-7 of DeKock and Gray.
- Figure 10.3. After Figure 5-4 of DeKock and Gray.
- Figure 10.4. After Figure 6.63 of Hollas.
- Figure 10.5. After Figure 3 in Chapter 7 of Gimarc.
- Figure 10.10. After Figure 5-37 in DeKock and Gray.
- Figure 10.11. Courtesy of C. Chuaqui and Prof. R. J. Le Roy, University of Waterloo.

- Figure 10.12. After Figure 6.92 of Hollas.
- Figure 10.13. After Figure 7.13 of Struve.
- Figure 10.14. After Figure 11 of Herzberg *Polyatomics*.
- Figure 10.17. After Figure 5-19 of Harris and Bertolucci.
- Figure 10.17. Courtesy of Professor D. Klapstein, St. Francis Xavier University.
- Figure 10.19. After Figure 45 in Herzberg *Polyatomics*.
- Figure 10.20. Courtesy of G. Herzberg and I. Dabrowski.
- Figure 10.21. After Figure 10.21 of King.
- Figure 10.22. After Figure 10.22 of King.
- Figure 10.23. Reproduced, with permission, from Miller, R. G. and Lee, E. K. C., *Chem. Phys. Lett.* **33**, 104 (1975).
- Figure 10.24. After Figure 3 of Chapter 5 of Gimarc.
- Figure 10.25. After Figure 2 of Jensen, P. and Bunker, P. R., *J. Mol. Spectrosc.* **94**, 114 (1982).

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