Quaternions

Recall that quaternions are a four-dimensional algebra (that is a vector space with a distributive vector multiplication) $\mathbb{K}$ over real numbers generated by the elements $\{1, \ell, j, k\}$, which satisfy

$$\ell^2 = j^2 = k^2 = \ell jk = -1.$$

First of the imaginary generators is usually denoted as i, but to avoid the confusion caused by identifying with a complex unit, we will use ℓ. Multiplying the equality $\ell jk = -1$ from both sides by k we get $k\ell j = -1$. Similarly, $jk\ell = -1$. Imaginary generators are therefore cyclically interchangeable. Multiplying by only one k we also get $\ell j = k$, and symmetrically $jk = \ell$ and $k\ell = j$. Further, multiplying $\ell j = k$ by ℓ from the left, we get $j = -\ell k$ and similarly $k = -j\ell$ a $kj = -\ell$.The generators are therefore anti-commutative. However, each quaternion obviously commutes with a real number (which is itself a quaternion).

For $q = a + b\ell + cj + dk$ we define the *adjoint element* $q^* = a - b\ell - cj - dk$.

Norm of q is defined as $N(q) := qq^* = a^2 + b^2 + c^2 + d^2 = |q|^2$, where $|q|$ is the Euclidean norm in $\mathbb{R}^4$. The sphere $\mathbb{S}^3$ is therefore naturally identified with unit quaternions $\mathbb{K}_1$ (that is, quaternions of norm one).

We have $(pq)^* = q^*p^*$. This implies $N(pq) = N(p)N(q)$, and unit quaternions form a multiplicative group. Thus, the inverse element of the quaternion q has the form $q^{-1} = q^*/N(q)$, or $q^{-1} = q^*$ for unit quaternions.

Quaternions of the form $b\ell + cj + dk$ are called *imaginary.* Unit imaginary quaternions can be identified with the sphere $\mathbb{S}^2$ and they satisfy $p^2 = -1$ (similarly as the generators), because $p^{-1} = -p$.

We will now show the most important property of quaternions. Conjugation of an imaginary quaternion by any quaternion corresponds to the rotation of three-dimensional space.

Theorem. For $0 \neq q = (r + x\ell + yj + zk) \in \mathbb{K}$, the mapping

$$\rho_q : \mathbb{R}^3 \to \mathbb{R}^3$$
$$(b, c, d) \mapsto (b', c', d')$$

defined by

$$b'\ell + c'j + d'k = q(b\ell + cj + dk)q^{-1}$$

is the rotation around the axis passing through the point (x, y, z) by the angle

$$\omega = 2 \arccos \frac{r}{\sqrt{N(q)}} \,.$$

Proof. Since $qpq^{-1} = (tq)p(tq)^{-1}$ for any real t, we can w.l.o.g. assume that q is a unit quaternion and $qpq^{-1} = qpq^*$.

Conjugation is an automorphism of $\mathbb{K}$. In addition, it is an identity on real numbers, because a real number commutes with any quaternion. Moreover,

$$N(qpq^{-1}) = N(q)N(p)N(q^{-1}) = N(p).$$

Thus, conjugation can be understood as an orthonormal transformation of $\mathbb{R}^4$, preserving the first coordinate. Therefore it is also orthonormal on the orthogonal complement of the first component. Let $q = r + v$, that is, v is the imaginary part of q. Then

$$qvq^* = (r + v)v(r - v) = (r + v)(rv - vv) = (r + v)(r - v)v = N(q)v = v\,.$$

We can see that ρ_q is an isometry with a fixpoint (x, y, z).

Let us write q as

$$q = \cos\frac{\omega}{2} + \sin\frac{\omega}{2}(\ell \sin\vartheta\cos\varphi + j\sin\vartheta\sin\varphi + k\cos\vartheta)\,,$$

where

$$v' = \ell \sin\vartheta\cos\varphi + j\sin\vartheta\sin\varphi + k\cos\vartheta$$

is a unitary imaginary quaternion expressing the axis of rotation using its polar coordinates. Denote

$$\kappa = \cos\frac{\omega}{2} + \sin\frac{\omega}{2}k,$$

which is the case with $v' = k$. Direct calculation of images $\rho_\kappa(\ell)$, $\rho_\kappa(j)$ and $\rho_\kappa(k)$ yields

$$[\rho_\kappa]_{\ell,j,k} = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the theorem holds for this particular case.

Similarly (or from symmetry) we get validity for the cases $v' = \ell$ and $v' = j$, i.e. for rotations around the second and third axes of $\mathbb{R}^3$.

Consider now quaternions

$$q_\varphi = \cos\frac{\varphi}{2} + k\sin\frac{\varphi}{2}\,,$$
$$q_\vartheta = \cos\frac{\vartheta}{2} + j\sin\frac{\vartheta}{2}\,.$$

Their action corresponds to the respective rotations, so

$$q_\varphi q_\vartheta k q_\vartheta^* q_\varphi^* = v'\,,$$

and thus

$$q_\varphi q_\vartheta \kappa q_\vartheta^* q_\varphi^* = q\,.$$

From here we deduce

$$qpq^* = q_\varphi(q_\vartheta(\kappa(q_\vartheta^*(q_\varphi^* p q_\varphi)q_\vartheta)\kappa^*)q_\vartheta^*)q_\varphi^*$$

that is

$$\rho_q = \rho_\varphi \circ \rho_\vartheta \circ \rho_\kappa \circ \rho_\vartheta^{-1} \circ \rho_\varphi^{-1}\,,$$

and ρ_q is the mapping similar to ρ_κ, in other words, it is a rotation by the angle ω with respect to different othonormal basis. In particular

$$[\rho_q]_{\rho_\varphi^* \circ \rho_\vartheta^*(\ell,j,k)} = [\rho_\kappa]_{\ell,j,k}\,.$$

Since we already know the fixpoint of ρ_q the proof is complete. □

Remark: A direct calculation of images $\varphi_q(\ell)$, $\varphi_q(j)$ and $\varphi_q(k)$ yields (for unit q) the matrix

$$[\varphi_q]_{\ell,j,k} = \begin{pmatrix} 1-2(y^2+z^2) & 2(xy-rz) & 2(ry+xz) \\ 2(xy+rz) & 1-2(x^2+z^2) & 2(yz-rx) \\ 2(xz-ry) & 2(rx+yz) & 1-2(x^2+y^2) \end{pmatrix}.$$

We can verify that it is orthogonal with determinant 1.