TENSOR PRODUCT AND QUANTUM REGISTERS

Tensor product of Hilbert spaces U and V, is the vector space of bilinear forms on the Cartesian product of dual spaces. That is

$$U \otimes V = \{f \mid f : U^\dagger \times V^\dagger \to \mathbb{C}, f \text{ bilinear}\}.$$

We associate with each pair $(|u\rangle, |v\rangle) \in U \times V$ an element of the tensor product, denoted $|u\rangle \otimes |v\rangle$, defined by

$$|u\rangle \otimes |v\rangle : (\langle u'|, \langle v'|) \mapsto \langle u'|u\rangle\langle v'|v\rangle.$$

By abuse of notation and terminology, the resulting mapping

$$\begin{aligned} U \times V &\to U \otimes V \\ (|u\rangle, |v\rangle) &\mapsto |u\rangle \otimes |v\rangle, \end{aligned}$$

is also called a tensor product (of vectors). This mapping is bilinear, in the sense that:

$$\begin{aligned} |w\rangle \otimes (|u\rangle + |v\rangle) &= |w\rangle \otimes |u\rangle + |w\rangle \otimes |v\rangle; \\ (|u\rangle + |v\rangle) \otimes |w\rangle &= |u\rangle \otimes |w\rangle + |v\rangle \otimes |w\rangle; \\ (\alpha|u\rangle) \otimes |v\rangle = |u\rangle \otimes (\alpha|v\rangle) &= \alpha(|u\rangle \otimes |v\rangle). \end{aligned}$$

Note however that it is not a linear mapping between the two vector spaces, that is, $(|u\rangle + |u'\rangle) \otimes (|v\rangle + |v'\rangle)$ is in general not equal to $|u\rangle \otimes |v\rangle + |u'\rangle \otimes |v'\rangle$.

We often shorten the tensor product of vectors $|u\rangle \otimes |v\rangle$ to $|u\rangle|v\rangle$ or even (especially for base vectors) to $|uv\rangle$.

Let $n = \dim U$ and $m = \dim V$, and let $|\mathbf{b}_i\rangle$, $i = 1, \ldots, n$ and $|\mathbf{c}_i\rangle$, $i = 1, \ldots, m$ be some bases of U and V respectively. Since the bilinear forms from the definition of the tensor product can be seen as matrices, or, more algebraically, they are determined by their values on pairs $(|\mathbf{b}_i\rangle, |\mathbf{c}_j\rangle)$, it is easy to see that the dimension of $U \otimes V$ is nm, and that vectors $|\mathbf{b}_i\rangle \otimes |\mathbf{c}_j\rangle$, denoted by the above convention also as $|\mathbf{b}_i\mathbf{c}_j\rangle$, form its basis, and for the tensor product of vectors $|u\rangle \in U$ and $|v\rangle \in V$ we get

$$|u\rangle \otimes |v\rangle = \left(\sum_i \alpha_i|\mathbf{b}_i\rangle\right) \otimes \left(\sum_j \beta_j|\mathbf{c}_j\rangle\right) = \sum_{i,j} \alpha_i\beta_j|\mathbf{b}_i\mathbf{c}_j\rangle.$$

Finally, we make the tensor product of Hilbert spaces a Hilbert space by defining the scalar product on $U \otimes V$ by saying that vectors $(|\mathbf{b}_i\mathbf{c}_j\rangle)_{i,j}$ form an orthonormal basis of $U \otimes V$ if $(|\mathbf{b}_i\rangle)_i$ and $(|\mathbf{c}_j\rangle)_j$ form orthonormal bases of U and V resp. We than get

$$\langle u \otimes v|u' \otimes v'\rangle = \langle u|u'\rangle\langle v|v'\rangle$$

which can be seen as an alternative definition of the scalar product of tensor products, independent of the choice of bases.

Note that "most" elements $U \otimes V$ are not tensor products $|u\rangle \otimes |v\rangle$. These are only the bilinear forms whose matrix is of rank one.

The definition of tensor product is extended on more than two Hilbert spaces in an obvious way: the tensor product of Hilbert spaces U_i, $i = 1, 2, \ldots, n$, is the vector space of multilinear forms on the Cartesian product of dual spaces. That is

$$U_1 \otimes U_2 \otimes \cdots \otimes U_n = \{f \mid f : U_1^\dagger \times U_2^\dagger \times \cdots \times U_n^\dagger \to \mathbb{C}, f \text{ multilinear}\}.$$

We also in a natural way identify $(U \otimes V) \otimes W$, $U \otimes (V \otimes W)$ and $U \otimes V \otimes W$, and consider the tensor product (of both spaces and vectors) as associative.

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In quantum informatics, mainly products of qubits, so-called *quantum registers*, are used. A quantum register $\mathbb{H}_2^{\otimes n}$ of n qubits has basis $|0\rangle \otimes \cdots \otimes |0\rangle$, $|0\rangle \otimes \cdots \otimes |1\rangle$, $\ldots$, $|1\rangle \otimes \cdots \otimes |1\rangle$, which according to the above convention can be shortened to $|0 \cdots 0\rangle$, $|0 \cdots 1\rangle$, $\ldots$, $|1 \cdots 1\rangle$. If we now understand zeros and ones as digits of binary notation, we get two different bases of size 2^n: one is the basis of the space $\mathbb{H}_2^{\otimes n}$, the other of the space $\mathbb{H}_{2^n}$. We thus obtain a natural tensor decomposition of the basis $|0\rangle$, $|1\rangle$, $\ldots$, $|2^n - 1\rangle$.

As noted above, the space $U \otimes V$ also contains vectors that cannot be written as a tensor product of vectors from the original spaces. For example, the state $|00\rangle + |11\rangle$ is indecomposable; we have

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

and it is easy to see that no a, b, c, d satisfy $c = bd = 1$ and $ad = bc = 0$. It is crucial for quantum phenomena that such *entangled* states of two or more systems are physically possible, the corresponding systems can even be spatially quite distant (e.g. by sending two entangled photons in different directions). The fact that spatially discontinuous particles can form a single system is called *nonlocal* character of quantum mechanics.

We can also make tensor products of operators. If $A : U_1 \to U_2$ and $B : V_1 \to V_2$ are two operators, their tensor product is a linear mapping $A \otimes B : U_1 \otimes V_1 \to U_2 \otimes V_2$ defined by their values on the generating set of decomposable vectors as follows:

$$(A \otimes B)(|u\rangle \otimes |v\rangle) = (A|u\rangle) \otimes (B|v\rangle).$$

From the above properties of the scalar and tensor product, it is not difficult to verify that the tensor product of unitary operators is again unitary. The matrix of the operator $A \otimes B$ of the type $mp \times nq$ arises from the matrices A of the type $m \times n$ and B of the type $p \times q$ using the so-called Kronecker product, which is given as follows:

$$A \otimes B = \begin{pmatrix} a_{1,1}B & \ldots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \ldots & a_{m,n}B \end{pmatrix}$$

For instance, for

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

we get

$$H^{\otimes 2} = \frac{1}{2} \left(\begin{array}{|cc|cc|} \hline 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ \hline \end{array} \right).$$

The operator H is called the *Hadamard* operator and we will later encounter its tensor powers. Let's see what the tensor power $H^{\otimes n}$ looks like. Its matrix is a square of size $2^n \times 2^n$ and if we factor out the coefficient $\left(\frac{1}{\sqrt{2}}\right)^n$, we get a matrix with entries 1 and -1. Let's index the rows and columns with the numbers $0, 1, \ldots, 2^n - 1$ and look at the sign of the element $(H^{\otimes n})_{i,j}$. We can take advantage of the fact that the j-th column is a vector $H^{\otimes n}|j\rangle$. If we write j in binary, we get a tensor decomposition

$$H^{\otimes n}|j\rangle = H^{\otimes n}|j_1 j_2 \cdots j_n\rangle = H^{\otimes n}|j_1\rangle|j_2\rangle \cdots |j_n\rangle =$$
$$= \bigotimes_{k=1}^{n} H|j_k\rangle = \left(\frac{1}{\sqrt{2}}\right)^n \bigotimes_{k=1}^{n}(|0\rangle + (-1)^{j_k}|1\rangle).$$

Multiplying out the last expression we find the required sign as a coefficient for the vector $|i\rangle = |i_1 i_2 \cdots i_n\rangle$. Minus signs are contributed to the product by the vectors $|i_k\rangle$ for which $i_k = j_k = 1$. Indeed, that's exactly when we take $|1\rangle$ from k-th expanded parentheses (because $i_k = 1$) and at the same time this $|1\rangle$ has a coefficient of -1 (because $j_k = 1$). Hence we have

$$(H^{\otimes n})_{i,j} = \left(\frac{1}{\sqrt{2}}\right)^n (-1)^{i_1 j_1 + i_2 j_2 + \cdots + i_n j_n} = \left(\frac{1}{\sqrt{2}}\right)^n (-1)^{i \cdot j},$$

where $i \cdot j$ denotes the dot product of the vectors of the binary expansion of digits i and j, i.e. the sum of $i_1 j_1 + i_2 j_2 + \cdots + i_n j_n$.

Note that if we understand the scalar product $\langle u|v\rangle$ as the application of the linear form $\langle u|$ on $|v\rangle$, the definition of the scalar product $\langle u_1 \otimes v_1 | u_2 \otimes v_2\rangle$ corresponds to the tensor product of the forms $\langle u_1|$ and $\langle u_2|$. Similarly, the vector of coordinates of the tensor product $|u\rangle \otimes |v\rangle$ in the basis $(|\mathbf{b}_i c_j\rangle)_{i,j}$ is a Kronecker product of vectors of coordinates in bases $(|\mathbf{b}_i\rangle)_i$ an $(|\mathbf{c}_j\rangle)_j$ of $|u\rangle$ and $|v\rangle$ respectively.

Note also that for endomorphisms A and B, the eigenvectors of the endomorphism $A \otimes B$ are vectors $|\mathbf{b}_i \otimes \mathbf{c}_j\rangle$ with eigenvalues $\lambda_i \cdot \kappa_j$ where $|\mathbf{b}_i\rangle$ is the eigenvector of A with eigenvalue λ_i and $|\mathbf{c}_j\rangle$ is the eigenvector of B with eigenvalue κ_j. Similarly, $|\mathbf{b}_i \otimes \mathbf{c}_j\rangle$ is an eigenvector of the endomorphism $A \otimes I + I \otimes B$ with eigenvalue $\lambda_i + \kappa_j$ (where I denotes identical operators of the appropriate size). These relations provide a handy proof of the commutative algebra fact that the integral elements of a ring form a ring.