Complex unitary spaces

Complex unitary space of dimension n is a vector space $\mathbb{C}^n$ with scalar product. If $\alpha = a + bi$, $a, b \in \mathbb{R}$ we will denote the α^* number associated with α, i.e. $a - bi$.

Recall that the scalar product, which we will denote for a moment by the symbol $\odot$, is the mapping $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ satisfying the following relations:

- $u \odot (v + w) = u \odot v + u \odot w$;
- $u \odot (\alpha v) = \alpha(u \odot v)$;
- $v \odot u = (u \odot v)^*$;
- $u \odot u > 0$ pro $u \neq 0$.

The fourth condition silently assumes that $u \odot u$ is a real number, which is guaranteed by the third condition. The most important thing to realize is that the conditions yield $(\alpha u) \odot v = \alpha^*(u \odot v)$, so the scalar product **is not** linear in the first component. However, it is linear in the second component.

The *Hilbert space* of the dimension n, denoted by the symbol $\mathbb{H}_n$, is actually the n-dimensional complex unitary space. The difference between the terms unitary space and Hilbert space is given by the additional condition that Hilbert space must be complete with respect to the norm defined by the scalar product. However, this condition is always fulfilled for finite dimensional spaces, and therefore both concepts coincide on the finite dimension.

The fact that the variable u indicates an element of a vector space is sometimes referred to as $\vec{u}$. We will use the notation introduced by Dirac, common in quantum physics, which denotes the vector space element by the symbol $|u\rangle$.

As we have already said, the scalar product is linear in the second component, i.e. the mapping $\tilde{u} : \mathbb{C}^n \to \mathbb{C}$ given by the formula $\tilde{u}(v) = u \odot v$, is a linear form, or linear mapping from vector space to the field (or, equivalently, to one-dimensional vector space). Linear forms themselves form a vector space called *dual space*. Because it is, in matrix notation, a line vector space from $\mathbb{C}^n$, the dual space is isomorphic to $\mathbb{C}^n$, in which, by convention, we use column vectors. The dual vector $\tilde{u}$ to the vector u is written in Dirac notation as $\langle u|$. The origin of this notation is that the scalar product $u \odot v$ can now be written as $\langle u|v\rangle$ after omitting the $\odot$ sign, which is a notation commonly used for scalar product. The English word for the parentheses, *bracket*, gave rise to the designation *bra* -vector for elements $\langle u|$ of the dual space and *ket* -vector for elements $|v\rangle$ of the original space.

In finite-dimensional space, we are used to write vectors as n-tuples using their coordinates with respect to the chosen base. It is worth noting that in the case of an arithmetic vector space, such as $\mathbb{C}^n$, and with the choice of the canonical base $K = (\mathbf{e}_1, \dots, \mathbf{e}_n)$, a n-tuple understood as a vector is the same as a n-tuple understood as coordinates with respect to K. Formally,

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right\}_K .$$

The scalar product is easily expressed using coordinates in the orthonormal basis, i.e. in the basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ satisfying $\langle \mathbf{b}_i | \mathbf{b}_j \rangle = \delta_{ij}$. Then for

$$u = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

we have

$$\langle u | = (a_1^*, \dots, a_n^*).$$

We can write this as $\langle u | = (|u\rangle^*)^{\mathrm{T}}$, which can be abbreviated as $\langle u | = |u\rangle^\dagger$.

We also have

$$\langle u | u \rangle = \sum_{i=1}^{n} |a_i|^2 .$$

Recall that the scalar product allows you to define the norm of the vector $\|u\|$ as $\sqrt{\langle u|u\rangle}$. Note that $|\alpha| = \|\alpha\|$, if we understand α once as a complex number and once as a one-dimensional vector.

Spectral properties of linear operators

Linear mappings (or homomorphisms) of vector spaces are also called (especially in physics) *linear operators*. Each operator $\varphi : \mathbb{C}^m \to \mathbb{C}^n$ can, as is well known, be represented as the multiplication by a matrix A of size $n \times m$. This matrix is given by the choice of bases M and N of spaces $\mathbb{C}^m$ and $\mathbb{C}^n$ and we have

$$A \cdot \{|u\rangle\}_M = \{\varphi|u\rangle\}_N.$$

It follows from the previous notation why we will understand vectors $\mathbb{C}^n$ as **columns, not rows**: it is more natural due to the convention that we multiply the vector by the matrix of the operator from the left. We are interested in matrices precisely because they are (along with multiplication) linear operators. So when we talk about a matrix, we mean the corresponding operator. Therefore, we will usually write $A|u\rangle$, instead of $A \cdot \{|u\rangle\}_M$.

For a operator φ we define the *adjoint* operator $\varphi^\dagger$ by the relation

$$\langle \varphi^\dagger(u) | v \rangle = \langle u | \varphi(v) \rangle,$$

where $\varphi^\dagger(u)$ is an abbreviation for $\varphi^\dagger|u\rangle$ for clarity, and $\varphi(v)$ for $\varphi|v\rangle$. This notation may be a bit confusing from a formal point of view (which physicians usually don't care so much about), but without Dirac's notation we can write it as

$$\varphi^\dagger(u) \odot v = u \odot \varphi(v).$$

It is not difficult to verify that in the matrix notation of the operator, the symbol $\dagger$ has the usual meaning of the Hermite-associated matrix (transposed and complex conjugated), which we already used above in the characterization of $\langle u|$. Especially in the context of quantum mechanics, the Hermite-associated matrix is simply called the *adjoint* matrix (although this term is often used in linear algebra for a matrix defined by subdeterminants).

The *eigenvalues* and *eigenvectors* are decisive for the properties of operators. The eigenvectors (which by definition are non-zero) determine one-dimensional subspaces that are mapped on themselves by the operator (they are therefore an invariant of the mapping). Therefore, if $|u\rangle$ is the eigenvector of an operator φ, then

$$\varphi|u\rangle = \lambda \cdot |u\rangle,$$

where λ is a complex number, called the eigenvalue corresponding to the (linear space spanned by the) vector $|u\rangle$. The set of eigenvalues is called the *spectrum* of the operator. The following list characterizes matrices of operators that have some nice spectral properties.

- The matrix A is *diagonalizable* if there exists a regular (i.e. invertible) matrix P such that $P^{-1}AP$ is diagonal. This occurs iff there is a basis of eigenvectors of the operator A. The matrix P is the matrix of the transition from the canonical basis to the basis of eigenvectors.
- The matrix A is called *normal* if the equality
$$AA^\dagger = A^\dagger A,$$
holds, that is, if the mapping commutes with its adjoint mapping. One of the most important theorems of the complex linear algebra is the **theorem on spectral decomposition of normal operators**, which says that an operator is normal if and only its eigenvectors form an orthonormal basis (since eigenvectors are given up to a scalar factor, it would be more accurate to say that it forms an orthogonal basis, which, however, can be converted into an orthonormal one by normalization). There are two important subclasses of normal matrices:
  - The matrix A is *Hermitian*, or *self-adjoint*, if
  $$A = A^\dagger.$$
  Hermitian matrices are obviously normal and have real eigenvalues.
  - The matrix U is called *unitary* if it preserves the scalar product. This is true when
  $$U^\dagger U = E,$$
  which is clearly shown by Dirac's notation:
  $$\langle u|v\rangle = \langle u|U^\dagger U|v\rangle.$$
  The equality $U^\dagger U = E$ also shows that the columns (rows) of the matrix U form an orthonormal basis. The unitary matrices are obviously normal.

  The theorem on spectral decomposition of normal operators can now also be formulated so that the operator is normal iff it is unitarily diagonalizable, i.e. when the corresponding transition matrix is unitary. This must be true because both the initial, i.e. canonical, and target bases of the eigenvectors are orthonormal. (The canonical base is orthonormal by definition; in other words, by convention, we always write operators in the base that is orthonormal in the given unitary space.)

Dirac notation provides an elegant notation for the projection operators P_v on the selected vector v. We have:
$$P_v = |v\rangle\langle v|.$$
The product of the arithmetic form (i.e. of the expression in coordinates) of the vectors $|v\rangle$ and $\langle v|$ in this order is a square matrix. That this is a projection operator can be seen from the formula
$$P_v|u\rangle = |v\rangle\langle v|u\rangle$$
and from the fact that the scalar product $\langle v|u\rangle$ determines the size of the projection of the vector u on the vector v.

It is also easy to see that each normal operator can be written as a linear combination of projections on its own vectors v_1, v_2, $\ldots$, v_n (forming an orthonormal basis). So

$$A = \sum_{i=1}^{n} a_i |v_i\rangle\langle v_i|,$$

where a_i is the eigenvalue of the corresponding vector v_i.

This also allows us to extend standard functions of complex numbers to operators. If $f : \mathbb{C} \to \mathbb{C}$ is a function, then $f(A)$ means the operator

$$f(A) = \sum_{i=1}^{n} f(a_i) |v_i\rangle\langle v_i|.$$