Constructive negation, implication, and co-implication

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Dedicated to Dimiter Vakarelov
on the occasion of his 70th anniversary

ABSTRACT. *In this paper, a family of paraconsistent propositional logics with constructive negation, constructive implication, and constructive co-implication is introduced. Although some fragments of these logics are known from the literature and although these logics emerge quite naturally, it seems that none of them has been considered so far. A relational possible worlds semantics as well as sound and complete display sequent calculi for the logics under consideration are presented.*

KEYWORDS: *constructive logic, connexive logic, constructive negation, constructive implication, constructive co-implication.*

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1. Introduction

It is sometimes said that classical logic admits of a constructive interpretation if it is assumed that every proposition is decidable, but this does not imply that classical logic *is* constructive, and although classical logic has been called a (new) constructive logic by Girard (Girard, 1991), there seems to be a broad agreement among logicians that classical logic is *not* constructive. But what is a constructive logic? Sometimes the term ‘constructive logic’ is used as a synonym for ‘intuitionistic logic’. However, logics other than intuitionistic logic have also been said to be constructive, like, for instance, Johansson’s minimal logic, Heyting-Brouwer logic, or David Nelsons’s logics with strong negation. Whereas there exists *the* system of classical propositional and predicate logic, it is far from clear whether there exists exactly one system of constructive logic. In a situation where there are no clear, agreed-upon, individually necessary and jointly sufficient conditions for the constructiveness of a logical system, it seems quite difficult or next to pointless to designate one particular logic as *the* correct constructive logic. Nevertheless, for some reasons certain logics may still be regarded as constructive logics.

1.1. *Positive constructive propositional logics*

It is well known that the implicational fragments of intuitionistic and classical logic differ, as Peirce's law $((A \to B) \to A) \to A$ is classically but not intuitionistically valid, and it seems that there is a consensus among logicians that, among other things, the failure of Peirce's Law indicates that

$(*)$ Intuitionistic implicational logic is a constructive logic.

Intuitionistic logic and classical logic (understood as consequence relations) share their conjunction-disjunction fragment, and the constructiveness of this fragment appears to be uncontroversial.[1] In their disjunction-negation fragments, however, intuitionistic and classical logic differ. In particular, intuitionistic logic enjoys a constructive feature which classical logic fails to have in its disjunction-negation fragment, the *disjunction property*: If a disjunction $(A \vee B)$ is provable, then A is provable or B is provable. In classical logic $p \vee \sim p$ is provable, but neither the atomic formula p nor its classical negation $\sim p$ is provable. Moreover, the conjunction-negation fragment of intuitionistic logic lacks a constructive feature which Nelsons's constructive logics enjoy, namely the *constructible falsity property*: If $\sim(A \wedge B)$ is provable, then $\sim A$ is provable or $\sim B$ is provable. In intuitionistic logic $\sim(p \wedge \sim p)$ is provable, but neither the literal $\sim p$ nor its negation $\sim\sim p$ is provable. Still, there appears to be an agreement among logicians that

$(**)$ Positive intuitionistic propositional logic, IPL^+, is a constructive logic.

This view is supported by the observation that IPL^+ is a fragment not only of intuitionistic logic, but also of Johansson's minimal logic, Heyting-Brouwer logic, and Nelsons's logics with strong negation.

Heyting-Brouwer logic adds to intuitionistic logic a binary connective which is a natural companion to implication and which is often called co-implication. Whereas intuitionistic implication, $\to$, is the residual of conjunction in IPL^+ in the sense that

$$A \wedge B \vdash C \text{ iff } A \vdash B \to C, \qquad (1)$$

1. Gödel (Gödel, 1933) noticed that intuitionistic and classical propositional logic understood as sets of formulas share their conjunction-negation fragment.

co-implication (also referred to as 'pseudo-difference'), $-\!\!<$ [2], is the residual of disjunction in the $\{\wedge, \vee, -\!\!<\}$-fragment of Heyting-Brouwer logic:

$$C \vdash A \vee B \text{ iff } C -\!\!< B \vdash A.^{3} \tag{2}$$

Let us refer to the $\{\wedge, \vee, -\!\!<\}$-fragment of Heyting-Brouwer logic as HB^+. Is HB^+ a constructive logic?[4] To justify $(**)$, one may point to the so-called proof (alias Brouwer-Heyting-Kolmogorov) interpretation of IPL^+, see, for instance (van Dalen, 2004). According to this interpretation, a (canonical) proof of an implication $A \to B$ is a construction that transforms any proof of A into a proof of B, and a proof of a conjunction $A \wedge B$ is a pair (π_1, π_2) consisting of a proof π_1 of A and a proof π_2 of B. A proof of a disjunction $A \vee B$ is a pair (i, π) such that $i = 0$ and π is a proof of A or $i = 1$ and π is a proof of B. One can then show that positive propositional intuitionistic logic is sound with respect to its proof interpretation: For every formula A provable (derivable from the empty set) in IPL^+, there exists a construction of A. That is, one possible criterion for the constructiveness of a logic is its correctness with respect to an interpretation in terms of canonical proofs.[5] For HB^+ we may consider dual proofs: *reductiones ad absurdum*. According to this interpretation, a (canonical) reductio ad absurdum of a co-implication $B -\!\!< A$ is a construction that transforms any reductio of A into a reductio of B. A reductio of a disjunction $A \vee B$ is a pair (π_1, π_2) consisting of a reductio π_1 of A and a reductio π_2 of B. A reductio of a conjunction $A \wedge B$ is a pair (i, π) such that $i = 0$ and π is a reductio of A or $i = 1$ and π is a reductio of B. One can then show that HB^+ is sound with respect to its dual proof interpretation: For every formula A reducible to absurdity (formula A from which the empty set can be derived) in HB^+, there exists a construction of A, see the Appendix (Section A). In view of this observation, we draw the conclusion that

2. I will use the symbol for co-implication suggested in (Goré, 2000). The more familiar symbol used, for example, in (Rauszer, 1980) is $\dot{-}$. As Goré (Goré, 2000) explains, the left-right symmetry of the more familiar symbol hides the asymmetry of the pseudo-difference operation. In $C -\!\!< B$, C is in a positive position and B in a negative position. This becomes clear, for instance, if the Boolean understanding of $B \to C$ as $\sim B \vee C$ is analogously applied to co-implication by reading $C -\!\!< B$ as $C \wedge \sim B$. Wolter (Wolter, 1998) uses $\varphi \stackrel{\smile}{\to} \psi$ instead of $\psi -\!\!< \varphi$. $C -\!\!< B$ may be read as "B co-implies C" or as "C excludes B".

Co-implication has been thoroughly investigated by Cecylia Rauszer (Rauszer, 1974), (Rauszer, 1977), (Rauszer, 1980), who added co-implication (and co-negation, see below) to intuitionistic logic to obtain Heyting-Brouwer logic. See also (Urbas, 1996), (Goré, 2000), (Buisman *et al.*, 2007), and the references therein.

3. Classical implication is the residual of conjunction in classical logic. One may therefore ask whether there exists a purely co-implicational formula which stands to the result of dropping implication and intuitionistic negation from Heyting-Brouwer logic as Peirce's Law stands to intuitionistic logic. This co-implicative analogue of Peirce's Law is stated in Section 3.

4. In Heyting-Brouwer logic, intuitionistic negation and the co-negation of Heyting-Brouwer logic can be defined using $\to$ and $-\!\!<$, see Observation 4. The addition of $-\!\!<$ to IPL^+ allows one to define intuitionistic negation, and the addition of $\to$ to HB^+ allows one to define co-negation.

5. It turns out that for logics with strong negation *disproofs* naturally enter the picture in addition to proofs, see (López-Escobar, 1972), (Wansing, 1993).

$(***)$ HB^+ is a constructive logic.

1.2. *Adding strong negation*

The result of adding $\prec$ to IPL^+ (alias the result of adding $\rightarrow$ to HB^+) is propositional Heyting-Brouwer logic HB (also called bi-intuitionistic logic (Goré, 2000) or subtractive logic (Crolard, 2001)). As we will see, in this logic, intuitionistic negation and co-negation are definable. Is HB a constructive logic? The status not only of classical negation but also of intuitionistic negation and co-negation as a constructive connective is contentious. The addition of classical negation to the $\{\wedge, \vee\}$-fragment of intuitionistic logic results in a failure of the desirable disjunction property, and so does the addition of co-negation (see Section 4), whereas the addition of intuitionistic negation results in a failure of the desirable constructible falsity property. Also, intuitionistic negation has been criticized, because it does not express the idea of direct falsification. An intuitionistically negated formula $\sim A$ is verified at a possible world (alias state) s in an intuitionistic Kripke model iff at every state related to s by the pre-order of the model, A fails to be verified. There is no way of falsifying A at s in the sense of verifying the negation of A by considering just s. In Nelson's logics with strong negation (see, among many other sources, (Almukdad *et al.*, 1984), (Dunn, 2000), (Gurevich, 1977), (Kamide, 2002), (Kamide, 2006), (Nelson, 1949), (Odintsov, 2003), (Odintsov, 2008), (Odintsov *et al.*, 2004) (Routley, 1974), (Thomason, 1969), (Vakarelov, 1977), (Vakarelov, 2005), (Vakarelov, 2006), (Wansing, 1993), (Wansing, 2001), (Wansing, 2005b)) the situation is different. In the relational semantics of these logics, support of truth and support of falsity conditions are stated separately. A state s supports the truth of an atom p iff p is verified at s, and s supports the falsity of p iff p is falsified at s. Verification and falsification of atomic formulas may vary from model to model. Strong negation is interpreted as leading from the support of truth to the support of falsity, and vice versa: A state s supports the truth (falsity) of $\sim A$ iff s supports the falsity (truth) of A. In the relational semantics of intuitionistic logic and HB, only verification conditions are specified for all kinds of formulas. If in addition to verification falsification is acknowledged as a semantic category in its own right, and if falsity is expressed in the object language by a unary negation operation, then the separate consideration of support of falsity conditions for all kinds of formulas leads to separate support of truth conditions for all kinds of negated formulas. This may well be interpreted as a constructive treatment of negation. The following question thus arises:

> What are the correct support of truth conditions for negated complex formulas? (Or, equivalently, what are the support of falsity conditions for complex formulas?)

In intuitionistic logic the double-negation elimination law $\sim\sim A \rightarrow A$ and the DeMorgan law $\sim(A \wedge B) \rightarrow (\sim A \vee \sim B)$ fail to be valid. If one considers intuitionistic logic as the correct system of constructive logic, these failures indicate that the double negation law and the above DeMorgan law are not constructively valid. But we have already seen that the constructive nature of intuitionistic negation is doubtful. If one is not prejudiced by the assumption that intuitionistic logic is the correct constructive logic, then nothing stands in the way of accepting both double negation laws and all the familiar DeMorgan laws. And indeed, in Nelson's constructive logics with strong negation, all these principles are valid. The view that a situation supports the falsity of a conjunction $(A \wedge B)$ (disjunction $(A \vee B)$) iff it supports the falsity of A or (and) it supports the falsity of B seems to be deeply rooted in our intuitive understanding of conjunction, disjunction, truth, and falsity. Moreover, if negation as falsity is a bridge from support of truth to support of falsity, and vice versa, then there is no way around both double negation laws.

The picture is less clear when we consider the support of truth conditions of negated implications, and it gets more complicated when we at the same time consider the support of truth conditions for negated co-implications. On the classical understanding of negated implications, a formula $\sim(A \to B)$ is true iff A is true and B is *not true*. On the intuitionistic reading, $\sim(A \to B)$ is verified at a state s iff for every 'later' state t (every possible expansion t of s), there is a state t' later than t such that A is verified at t', whereas B is not verified at t'. In Nelsons's logic, support of falsity of a formula A (support of truth of $\sim A$) is always a matter determined at the state of evaluation, and a state s supports the truth of $\sim(A \to B)$ iff s supports the truth of A and s supports the *falsity* of B. According to this view, $\sim(A \to B)$ is equivalent to $(A \wedge \sim B)$, where $\sim$ expresses falsity and not the absence of truth. Since the support of truth and the support of falsity are persistent along a model's pre-order, a state s supports the truth of $\sim(A \to B)$ iff every possible expansion t of s supports the truth of A and the falsity of B. If the semantics is set up such that the equivalence

$$\sim(A \to B) \leftrightarrow (A \wedge \sim B) \tag{3}$$

is valid (and atomic formulas may not only be neither verified nor falsified at some state but also both verified and falsified at some state), we obtain Nelson's constructive four-valued propositional logic N4 in the co-implication-free language $\{\sim, \wedge, \vee, \to\}$.[6]

This is not the end of the story concerning the language $\{\sim, \wedge, \vee, \to\}$, however, because another understanding of the relation between implication and negation has been proposed already since ancient times. It turns out that a slight modification of the support of truth conditions for negated implications leads from N4 to a system of *connexive* logic in which the support of falsity of implications is *not* interpreted as falsification at the world of evaluation, see (Wansing, 2006) for a survey and references. Connexive logics have a standard logical vocabulary but contain certain non-theorems of classical logic as theorems. Since classical propositional logic is Post-complete, any additional axiom in its language gives rise to the trivial system, so that any non-trivial system of connexive logic in this vocabulary must leave out some theorems of

6. Note that in N4 a truth constant $\top$ can be defined as $p \to p$ for some atom p, but no falsity constant $\bot$. Odintsov (Odintsov, 2008) investigates extensions of the system $\mathsf{N4}^{\bot}$ in the language $\{\sim, \wedge, \vee, \to, \bot\}$, which is axiomatized by adding the formulas $\bot \to A$ and $A \to \sim\bot$ to the axioms of N4.

classical logic. Among the characteristic theorems of connexive logics are *Aristotle's Theses*:

$$\sim(\sim A \to A),\ \sim(A \to \sim A), \tag{4}$$

and *Boethius' Theses*

$$(A \to B) \to \sim(A \to \sim B),\ (A \to \sim B) \to \sim(A \to B) \tag{5}$$

which are not theorems of classical logic. A connective $\to$ that satisfies the above theses is sometimes said to be a connexive implication.

1.3. *Motivations of connexive logic*

Since connexive logic is not a well-established area of non-classical logic, we will briefly look at motivations of it. In addition to an appeal to certain intuitions about meaning connections between the antecedent and the succedent of valid implications, there exist at least two motivating ideas for connexive logic. The first comes from Aristotle's syllogistic. It is well known that the syllogistic contains inferences that are not classically valid under the standard translation into predicate logic. One of the most prominent examples is the inference from 'Every P is Q' to 'Some Ps are Qs':

$$\forall x(P(x) \to Q(x)) \vdash \exists x(P(x) \wedge Q(x)) \tag{6}$$

Normally, we do not quantify over the empty set. If we assume that the interpretation of P is empty, there is hardly any reason to assume that every P is Q, but if the interpretation of P is non-empty, (6) is a valid inference. Inference (6) cannot be consistently added as a rule to a proof system for classical predicate logic, as is obvious from the following instance of (6):

$$\forall x((P(x) \wedge \sim P(x)) \to Q(x)) \vdash \exists x((P(x) \wedge \sim P(x)) \wedge Q(x)) \tag{7}$$

The premise of (7) is classically valid, whereas the conclusion is classically unsatisfiable. Now, in classical logic, inference (6) is interchangeable with

$$\forall x(P(x) \to Q(x)) \vdash \exists x \sim(P(x) \to \sim Q(x)). \tag{8}$$

Storrs McCall (McCall, 1967) pointed out that in a system of connexive logic (8) is a valid inference. This is especially perspicuous in the quantified connexive logic QC introduced in (Wansing, 2005a), because there

$$\sim(A \to B) \leftrightarrow (A \to \sim B) \tag{9}$$

is an axiom. One might therefore suggest to translate statements of the form 'Some Ps are Qs' not as $\exists x(P(x) \wedge Q(x))$ but as $\exists x \sim(P(x) \to \sim Q(x))$, which in the system QC is equivalent to $\exists x(P(x) \to Q(x))$.

Another motivation comes from Categorial Grammar, see (Wansing, 2007). In the Lambek Calculus, there are two implications, $\backslash$ and $/$, which are the residuals of a non-commutative, so-called multiplicative (intensional) conjunction, $\cdot$. In one version of the calculus, $\cdot$ is assumed to be associative; in another version, it is non-associative. The formulas of the Lambek Calculus stand for syntactic types, and a derivability statement (sequent) $x \vdash y$ is to be understood as 'every expression of type x is also of type y'. An expression e is of type $x \setminus y$ iff for every expression e' of type x, the string $e'e$ is of type y, and e is of type y/x iff for every expression e' of type x, the string ee' is of type y. A transitive verb like `loves`, for example, may be syntactically typed as $((n \setminus s)/n)$, because it combines with any name of syntactic type n from the right to an expression of type $(n \setminus s)$ that looks to the left for a name to result in an expression of type s, a sentence. It then makes sense to introduce a negation $\sim x$ to designate the class of expressions that are definitely not of type x. An expression e is of type $\sim(x \setminus y)$ iff for every expression e' of type x, the string $e'e$ is of type $\sim y$, and e is of type $\sim(y/x)$ iff for every expression e' of type x, the string ee' is of type $\sim y$. These definitions validate the sequents $\sim(x \setminus y) \vdash x \setminus \sim y$, $\sim(y/x) \vdash (\sim y/x)$, and their converses. The expression `loves Mary`, for example, is of type $\sim(n \setminus (n \setminus s))$, because in combination with any name from the left it results in an expression which is definitely not an intransitive verb, namely in a sentence. Clearly, the suggested reading of $\sim$ also justifies the double negation laws. As a result of these considerations, we obtain directional versions Boethius' Theses (as sequents) such as:[7]

$$(x \setminus y) \vdash \sim(x \setminus \sim y). \tag{10}$$

1.4. *Completing the picture*

Not only the equivalences (3) and (9) are serious candidates for expressing the support of truth conditions for negated implications. If we think of the classical understanding of a co-implication $(A \multimap\!\!\prec B)$ as $(A \wedge \sim B)$, the following equivalence must also be taken into account:

$$\sim(A \rightarrow B) \leftrightarrow (A \multimap\!\!\prec B), \tag{11}$$

and classical DeMorgan duality then suggests yet another equivalence:[8]

$$\sim(A \rightarrow B) \leftrightarrow (\sim B \multimap\!\!\prec \sim A) \tag{12}$$

Eventually, we have to specify the support of truth conditions for constructively negated co-implications. In analogy to what we have done for negated implications, we may consider the classical (or rather Nelson-like) reading of negated co-implications, the connexive understanding of negated co-implications, the reading of negated co-implications as implications, and the understanding of negated co-implications as

7. In Categorial Grammar, the left-hand side of a sequent may not be empty, because the empty string has no syntactic type.
8. This equivalence was pointed out to me by Greg Restall.

contraposed implications. Altogether, this range of readings will give us *sixteen* systems of constructive propositional logic. For want of a better terminology and notation, in Table 1 the characteristic equivalences in question are listed as equivalences I_1 – I_4 and C_1 – C_4. For convenience, the constructive propositional logics in the language $\{\wedge, \vee, \rightarrow, \prec, \sim\}$ that differ from each other only with respect to validating a certain pair of these equivalences (one from the I-equivalences and one from the C-equivalences) will be referred to as systems (I_i, C_j), $i, j \in \{1, 2, 3, 4\}$.

Table 1. *Constructively negated implications and co-implications*

I_1	$\sim(A \rightarrow B)$	$\leftrightarrow$	$(A \wedge \sim B)$	negated implication, classical reading
I_2	$\sim(A \rightarrow B)$	$\leftrightarrow$	$(A \rightarrow \sim B)$	negated implication, connexive reading
I_3	$\sim(A \rightarrow B)$	$\leftrightarrow$	$(A \prec B)$	negated implication as co-implication
I_4	$\sim(A \rightarrow B)$	$\leftrightarrow$	$(\sim B \prec \sim A)$	negated implication as contraposed co-impl.
C_1	$\sim(A \prec B)$	$\leftrightarrow$	$(\sim A \vee B)$	negated co-implication, classical reading
C_2	$\sim(A \prec B)$	$\leftrightarrow$	$(\sim A \prec B)$	negated co-implication, connexive reading
C_3	$\sim(A \prec B)$	$\leftrightarrow$	$(A \rightarrow B)$	negated co-implication as implication
C_4	$\sim(A \prec B)$	$\leftrightarrow$	$(\sim B \rightarrow \sim A)$	negated co-implication as contraposed impl.

2. Syntax and relational semantics

We will consider a propositional language $\mathcal{L}$ defined in Backus–Naur form as follows:

$$\begin{aligned} \text{atomic formulas:} \quad & p \in \textit{Atom} \\ \text{formulas:} \quad & A \in \textit{Form(Atom)} \\ A ::= \quad & p \mid \sim A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid (A \prec A). \end{aligned}$$

The intended reading of the logical operations is familiar, except, possibly, for the less well-known connective $\prec$: $\sim$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\prec$ (co-implication). The language without $\prec$ is the language of intuitionistic propositional logic, IPL, of David Nelson's propositional logics with strong negation, and of connexive propositional logic (if we do *not* use distinct symbols for the 'corresponding' connectives from distinct logics). In $\mathcal{L}$, where both $\rightarrow$ and $\prec$ are present, two distinct unary negation connectives can be defined: intuitionistic negation, which we now denote as $\neg$, and co-negation, $-$. We will focus, however, on the single *primitive* strong negation $\sim$. Equivalence, $\leftrightarrow$, is defined as usual, and co-equivalence, $\succ\!\prec$, is defined as expected, by setting $A \succ\!\prec B := (A \prec B) \vee (B \prec A)$.

In this section, we will introduce the sixteen constructive logics (I_i, C_j), $i, j \in \{1, 2, 3, 4\}$, semantically. Since all these logics are interpreted in models based on

(Kripke) frames, the semantic presentation admits of a transparent comparison between the logics under consideration.

DEFINITION 1. — *A frame is a pre-order $\langle I, \leq \rangle$. Intuitively, I is a non-empty set of information states, and $\leq$ is a reflexive transitive binary relation of possible expansion of states on I.*

Instead of $w \leq w'$, we also write $w' \geq w$.

DEFINITION 2. — *A model is a structure $\langle I, \leq, v^+, v^- \rangle$, where $\langle I, \leq \rangle$ is a frame and v^+ (v^-) is a function that maps every $p \in Atom$ to a subset of I (namely the states that support the truth (falsity) of p). It is assumed that the functions v^+ and v^- satisfy the following persistence conditions for atoms: if $w \leq w'$, then $w \in v^+(p)$ implies $w' \in v^+(p)$; if $w \leq w'$, then $w \in v^-(p)$ implies $w' \in v^-(p)$. The relations $\mathcal{M}, w \models^+ A$ ('state w supports the truth of $\mathcal{L}$-formula A in model $\mathcal{M}$') and $\mathcal{M}, w \models^- A$ ('state w supports the falsity of $\mathcal{L}$-formula A in model $\mathcal{M}$') are inductively defined as follows:*

$\mathcal{M}, w \models^+ p$	*iff*	$w \in v^+(p)$
$\mathcal{M}, w \models^- p$	*iff*	$w \in v^-(p)$
$\mathcal{M}, w \models^+ \sim A$	*iff*	$\mathcal{M}, w \models^- A$
$\mathcal{M}, w \models^- \sim A$	*iff*	$\mathcal{M}, w \models^+ A$
$\mathcal{M}, w \models^+ (A \wedge B)$	*iff*	$\mathcal{M}, w \models^+ A$ *and* $\mathcal{M}, w \models^+ B$
$\mathcal{M}, w \models^- (A \wedge B)$	*iff*	$\mathcal{M}, w \models^- A$ *or* $\mathcal{M}, w \models^- B$
$\mathcal{M}, w \models^+ (A \vee B)$	*iff*	$\mathcal{M}, w \models^+ A$ *or* $\mathcal{M}, w \models^+ B$
$\mathcal{M}, w \models^- (A \vee B)$	*iff*	$\mathcal{M}, w \models^- A$ *and* $\mathcal{M}, w \models^- B$
$\mathcal{M}, w \models^+ (A \rightarrow B)$	*iff*	*for every* $w' \geq w : \mathcal{M}, w' \not\models^+ A$ *or* $\mathcal{M}, w' \models^+ B$
$\mathcal{M}, w \models^+ (A \prec B)$	*iff*	*there exists* $w' \leq w : \mathcal{M}, w' \models^+ A$ *and* $\mathcal{M}, w' \not\models^+ B$

where $\mathcal{M}, w \not\models^+ A$ is the classical negation of $\mathcal{M}, w \models^+ A$.

In Table 2, we list the support of falsity conditions corresponding to the equivalences I_1 – I_4 and C_1 – C_4 from Table 1. No matter which equivalences we choose, support of truth and support of falsity is persistent for arbitrary formulas.

OBSERVATION 3 (PERSISTENCE). — For every $\mathcal{L}$-formula A, model $\langle I, \leq, v^+, v^- \rangle$, and $w, w' \in I$: if $w \leq w'$, then $w \in v^+(A)$ implies $w' \in v^+(A)$; if $w \leq w'$, then $w \in v^-(A)$ implies $w' \in v^-(A)$. □

We can make the following simple but important observation concerning the expressive power of the logics we are about to define.

OBSERVATION 4. — Let p be a certain atomic formula, let $\top := p \rightarrow p$, and let $\bot := p \prec p$. For every model $\mathcal{M}$ and every state w from $\mathcal{M}$, $\mathcal{M}, w \models^+ \top$ and $\mathcal{M}, w \not\models^+ \bot$. Thus, we can define the co-negation '$-$' of Heyting-Brouwer logic by setting $-A := \top \prec A$ and intuitionistic negation $\neg$, by setting $\neg A := A \rightarrow \bot$. □

Table 2. *Support of falsity conditions for negated implications and co-implications*

cI_1	$\mathcal{M}, w \models^- (A \to B)$	iff	$\mathcal{M}, w \models^+ A$ and $\mathcal{M}, w \models^- B$
cI_2	$\mathcal{M}, w \models^- (A \to B)$	iff	for every $w' \geq w : \mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^- B$
cI_3	$\mathcal{M}, w \models^- (A \to B)$	iff	there is $w' \leq w : \mathcal{M}, w' \models^+ A$ and $\mathcal{M}, w' \not\models^+ B$
cI_4	$\mathcal{M}, w \models^- (A \to B)$	iff	there is $w' \leq w : \mathcal{M}, w' \not\models^- A$ and $\mathcal{M}, w' \models^- B$
cC_1	$\mathcal{M}, w \models^- (A \prec B)$	iff	$\mathcal{M}, w \models^- A$ or $\mathcal{M}, w \models^+ B$
cC_2	$\mathcal{M}, w \models^- (A \prec B)$	iff	there is $w' \leq w : \mathcal{M}, w' \models^- A$ and $\mathcal{M}, w' \not\models^+ B$
cC_3	$\mathcal{M}, w \models^- (A \prec B)$	iff	for every $w' \geq w : \mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^+ B$
cC_4	$\mathcal{M}, w \models^- (A \prec B)$	iff	for every $w' \geq w : \mathcal{M}, w' \models^- A$ or $\mathcal{M}, w' \not\models^- B$

The support of truth clause for co-negation then is:

$$\mathcal{M}, w \models^+ -A \quad \text{iff} \quad \text{there exists } w' \leq w \text{ and } \mathcal{M}, w' \not\models^+ A,$$

whereas the support of truth conditions for intuitionistic negation are the familiar ones:

$$\mathcal{M}, w \models^+ \neg A \quad \text{iff} \quad \text{for every } w' \geq w,\ \mathcal{M}, w' \not\models^+ A.$$

Note that if $\mathcal{M} = \langle I, \leq \rangle$ is a frame, v is a function from *Atom* to subsets of I, and $\mathcal{M}, w \models A$ is defined exactly as $\mathcal{M}, w \models^+ A$, except that $\mathcal{M}, w \models p$ iff $w \in v(p)$, then $\langle I, \leq, v \rangle$ is a model for HB. The logic HB, understood as a set of formulas, is the set of all $\sim$-free $\mathcal{L}$-formulas A such that for every model $\mathcal{M} = \langle I, \leq, v \rangle$, and every $w \in I$, $\mathcal{M}, w \models A$.

DEFINITION 5. — *The logics (I_i, C_j) are defined as the triples $(\mathcal{L}, \models^+_{I_i,C_j}, \models^-_{I_i,C_j})$, where the entailment relations $\models^+_{I_i,C_j}$, $\models^-_{I_i,C_j} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ are defined as follows: $\Delta \models^+_{I_i,C_j} \Gamma$ iff for every model $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ defined with clauses cI_i and cC_j and every $w \in I$, if $\mathcal{M}, w \models^+ A$ for every $A \in \Delta$, then $\mathcal{M}, w \models^+ B$ for some $B \in \Gamma$, and*
$\Delta \models -_{I_i,C_j} \Gamma$ iff for every model $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ defined with clauses cI_i and cC_j and every $w \in I$, if $\mathcal{M}, w \models^- A$ for every $A \in \Gamma$, then $\mathcal{M}, w \models^- B$ for some $B \in \Delta$.
For singleton sets $\{A\}$ and $\{B\}$, we write $A \models^+_{I_i,C_j} B$ ($A \models^-_{I_i,C_j} B$) instead of $\{A\} \models^+_{I_i,C_j} \{B\}$ ($\{A\} \models^-_{I_i,C_j} \{B\}$). If the context is clear, we shall sometimes omit the subscript ${}_{I_i,C_j}$.

This definition of a logic as comprising *two* entailment relations instead of just one is unusual but not at all unnatural, see, for instance, (Shramko *et al.*, 2005), (Wansing *et al.*, 2008a), (Wansing *et al.*, 2008b). The set of all constructively false sentences is not the complement of the set of all constructively true sentences, and we can make the following observation.

OBSERVATION 6. — If $(I_i, C_j) \neq (I_4, C_4)$, then $\models^+_{I_i,C_j} \neq \models^-_{I_i,C_j}$. □

PROOF. — For every logic (I_i, C_j), it holds that $(p \wedge (p \rightarrow q)) \models^+ q$. However, for no logic (I_1, C_j) and for no logic (I_3, C_j), we have $\sim q \models^+ \sim(p \wedge (p \rightarrow q))$. To see this, a one-element countermodel suffices, where the following holds for the single state w: $w \in v^-(q)$, $w \notin v^+(p)$, and $w \notin v^-(p)$. Other counterexamples work for the logics (I_2, C_j) and (I_4, C_1) – (I_4, C_3). For every logic (I_i, C_j), it holds that $p \models^+ (q \rightarrow p)$. But in a singleton model where $w \notin v^-(p)$ and $w \notin v^+(q)$, we have $w \models^+ (q \rightarrow \sim p)$ and $w \not\models^+ \sim p$, which shows for the logics (I_2, C_j) that $\sim(q \rightarrow p) \not\models^+ \sim p$. For every logic (I_i, C_j), it holds that $r \wedge (r \rightarrow (p \prec p)) \models^+ q$. Consider a singleton model with $w \models^- q$, $w \not\models^- r$, $w \not\models^- p$, and $w \not\models^+ p$. This model shows that $\sim q \not\models^+ \sim r \vee ((\sim p \vee p) \prec \sim r)$ in the case of logic (I_4, C_1) and that $\sim p \not\models^+ \sim r \vee ((\sim p \prec p) \prec \sim r)$ in the case of logic (I_4, C_2). In (I_4, C_3) we have $\sim (p \prec q) \models^+ p \rightarrow q$. A singleton model in which $w \not\models^+ p$, $w \models^- q$ and $w \not\models^- p$ shows that $\sim q \prec \sim p \not\models^+ p \prec q$. ■

We do not require that for atomic formulas p, $v^+(p) \cap v^-(p) = \varnothing$. Therefore, the logics under consideration are *paraconsistent*. Neither is it the case that for any formula B, $\{p, \sim p\} \models^+_{I_i,C_j} B$ nor is it the case that $B \models^-_{I_i,C_j} \{p, \sim p\}$.[9]

The next observation on negation normal forms will be used in the completeness proof of Section 3. A formula is in *negation normal form* if it contains $\sim$ only in front of atoms. The following translations ρ_{I_i,C_j} send every formula A to a formula in negation normal form, where $p \in Atom$ and $\odot \in \{\vee, \wedge, \rightarrow, \prec\}$:

$$
\begin{array}{lcl}
\rho_{I_i,C_j}(p) & = & p \\
\rho_{I_i,C_j}(\sim p) & = & \sim p \\
\rho_{I_i,C_j}(\sim\sim A) & = & \rho_{I_i,C_j}(A) \\
\rho_{I_i,C_j}(A \odot B) & = & \rho_{I_i,C_j}(A) \odot \rho_{I_i,C_j}(B) \\
\rho_{I_i,C_j}(\sim(A \vee B)) & = & \rho_{I_i,C_j}(\sim A) \wedge \rho_{I_i,C_j}(\sim B) \\
\rho_{I_i,C_j}(\sim(A \wedge B)) & = & \rho_{I_i,C_j}(\sim A) \vee \rho_{I_i,C_j}(\sim B) \\
\rho_{I_1,C_j}(\sim(A \rightarrow B)) & = & \rho_{I_1,C_j}(A) \wedge \rho_{I_1,C_j}(\sim B) \\
\rho_{I_2,C_j}(\sim(A \rightarrow B)) & = & \rho_{I_2,C_j}(A) \rightarrow \rho_{I_2,C_j}(\sim B) \\
\rho_{I_3,C_j}(\sim(A \rightarrow B)) & = & \rho_{I_3,C_j}(A) \prec \rho_{I_3,C_j}(B) \\
\rho_{I_4,C_j}(\sim(A \rightarrow B)) & = & \rho_{I_4,C_j}(\sim B) \prec \rho_{I_4,C_j}(\sim A) \\
\rho_{I_i,C_1}(\sim(A \prec B)) & = & \rho_{I_i,C_1}(\sim A) \vee \rho_{I_i,C_1}(B) \\
\rho_{I_i,C_2}(\sim(A \prec B)) & = & \rho_{I_i,C_2}(\sim A) \prec \rho_{I_i,C_2}(B) \\
\rho_{I_i,C_3}(\sim(A \prec B)) & = & \rho_{I_i,C_3}(A) \rightarrow \rho_{I_i,C_3}(B) \\
\rho_{I_i,C_4}(\sim(A \prec B)) & = & \rho_{I_i,C_4}(\sim B) \rightarrow \rho_{I_i,C_4}(\sim A)
\end{array}
$$

LEMMA 7. — *For every formula A, $\rho_{I_i,C_j}(A)$ is in negation normal form and $A \models^+_{I_i,C_j} \rho_{I_i,C_j}(A)$, $\rho_{I_i,C_j}(A) \models^+_{I_i,C_j} A$, $A \models^-_{I_i,C_j} \rho_{I_i,C_j}(A)$, $\rho_{I_i,C_j}(A) \models^-_{I_i,C_j} A$.*

9. Co-negation is, of course, also a paraconsistent negation, see (Urbas, 1996), (Brunner *et al.*, 2005), whereas intuitionistic negation is 'paracomplete'.

3. Display calculi

Developing a proof system for logics with both intuitionistic implication *and* co-implication encounters some problems. The standard sequent calculus for intuitionistic logic is asymmetric; it uses sequents with multiple antecedents and (at most) single conclusions in order to avoid the provability of Peirce's Law. If one admits symmetric sequents (with multiple antecedents and succedents) and just adds the natural and obvious sequent rules for introducing co-implication (in the style of Gentzen's sequent calculus for classical logic, LK), namely:

$$\frac{\Gamma, B \vdash A, \Delta}{\Gamma, (B \mathrel{-\!\!<} A) \vdash \Delta} \qquad \frac{\Gamma \vdash B, \Delta \quad \Sigma, A \vdash \Pi}{\Sigma, \Gamma \vdash (B \mathrel{-\!\!<} A), \Delta, \Pi} \tag{13}$$

one can not only prove Peirce's Law, but also a sequent which contains just one co-implicative formula and is an analogue of the sequent expressing the provability of Peirce's Law:[10]

$$\dfrac{\dfrac{A \vdash A \qquad \dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A, B \mathrel{-\!\!<} A \vdash \varnothing}}{\dfrac{A, A \vdash A \mathrel{-\!\!<} (B \mathrel{-\!\!<} A)}{A \vdash A \mathrel{-\!\!<} (B \mathrel{-\!\!<} A)}}}{A \mathrel{-\!\!<} (A \mathrel{-\!\!<} (B \mathrel{-\!\!<} A)) \vdash \varnothing}$$

The formula $A \mathrel{-\!\!<} (A \mathrel{-\!\!<} (B \mathrel{-\!\!<} A))$ may be called Peirce's Co-Law.

The sequent calculus for Heyting-Brouwer logic in (Crolard, 2001) uses single-conclusion sequents but imposes a 'singleton on the left' constraint on the left introduction rule for co-implication (and a 'singleton on the right' constraint on the right introduction rule for implication). This sequent calculus is thus asymmetric, but it does not enjoy cut-elimination. Nor does the sequent system for HB in (Rauszer, 1974) allow cut-elimination. A counterexample due to T. Uustalu is presented in (Buisman *et al.*, 2007). These problems can be overcome in display logic.[11] We will employ this

10. The corresponding proof of Peirce's Law in the multiple-conclusion sequent calculus is:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A, B}}{\varnothing \vdash (A \to B), A} \qquad A \vdash A}{\dfrac{(A \to B) \to A \vdash A, A}{(A \to B) \to A \vdash A}}}{\varnothing \vdash ((A \to B) \to A) \to A}$$

11. Buisman and Goré (Buisman *et al.*, 2007) have presented a non-standard cut-free sequent calculus for Heyting-Brouwer logic. In this calculus, the sequent rule for implications in succedent position of a sequent and the rule for co-implications in antecedent position of a sequent

very general and flexible sequent-style proof-theoretical framework and present display sequent calculi for the logics (I_i, C_j), which add strong negation $\sim$ to HB. We may then apply a very general cut-elimination theorem stating that every 'properly displayable' logic enjoys cut-elimination, a theorem due to Nuel Belnap (Belnap, 1982), see Theorem 17.

One fundamental idea of display calculi is to exploit the fact that certain logical operations are residuated pairs to specify rules for introducing these operations on the left and the right side of the derivability sign $\vdash$, that is, in antecedent and in succedent position. Moreover, it is characteristic of display logic to associate a single structural connective $\diamond$ in the language of sequents with a pair $(\diamond_1, \diamond_2)$ of connectives from the logical object language, so that in antecedent position $\diamond$ is interpreted as $\diamond_1$ and in succedent position as $\diamond_2$. The left introduction rule for $\diamond_1$ and the right introduction rule for $\diamond_2$ may then be stated as follows:

$$\frac{A \diamond B \vdash X}{A \diamond_1 B \vdash X} \qquad\qquad \frac{X \vdash A \diamond B}{X \vdash A \diamond_2 B},$$

where X is a structure, a term in the language of sequents. The connectives $\diamond_1$ and $\diamond_2$ may be said to be Gentzen duals of each other.

A cut-free sound and complete display calculus for Heyting-Brouwer logic has been presented by Goré (Goré, 2000). In this section, I will develop a variant of Goré's system and extend it by rules for constructively negated formulas. Whereas Goré treats the pair of commutative operations $\wedge$ and $\vee$ as Gentzen duals and the non-commutative operations $\rightarrow$ and $\prec$, we here will treat the residuated pairs $(\wedge, \rightarrow)$ and $(\prec, \vee)$ as pairs of Gentzen duals.

In Gentzen's sequents, the comma, ',', may bee seen as a context sensitive structural connective to be understood as conjunction in antecedent position and as disjunction in succedent position of a sequent. In our display calculi, we will use the binary operations $\circ$ and $\bullet$ as structural connectives. In antecedent position, $\circ$ is to be interpreted as conjunction and in succedent position as implication. In antecedent position, $\bullet$ is to be read as co-implication and in succedent position as disjunction. A sequent is an expression of the shape $X \vdash Y$, where X and Y are structures. We also assume the empty structure $\mathbf{I}$, and the set of structures is defined in the obvious way as follows:

$$\begin{array}{rl} \text{formulas:} & A \in \textit{Form(Atom)} \\ \text{structures} & X \in \textit{Struc(Form)} \\ X ::= & A \mid \mathbf{I} \mid (X \circ X) \mid (X \bullet X). \end{array}$$

The intuitive interpretation of the structural connectives justifies certain 'display postulates' (dp) (we omit outer brackets):

---

come with side conditions on variables for families of sets of formulas. Two other, cut-free sequent calculi for Heyting-Brouwer logic are presented in (Goré *et al.*, 2008). The first calculus is intermediate between display calculi and standard sequent systems. From this system a variant is defined, which is amenable to automated proof-search.

$$\frac{\dfrac{Y \vdash X \circ Z}{X \circ Y \vdash Z}}{X \vdash Y \circ Z} \qquad \frac{\dfrac{X \vdash Y \circ Z}{X \circ Y \vdash Z}}{Y \vdash X \circ Z} \qquad \frac{\dfrac{X \bullet Z \vdash Y}{X \vdash Y \bullet Z}}{X \bullet Y \vdash Z} \qquad \frac{\dfrac{X \bullet Y \vdash Z}{X \vdash Y \bullet Z}}{X \bullet Z \vdash Y}$$

Moreover, we assume certain rules $(\mathbf{I}r)$ which govern the empty structure:

$$\frac{\dfrac{X \circ \mathbf{I} \vdash Y}{X \vdash Y}}{\mathbf{I} \circ X \vdash Y} \qquad \frac{\dfrac{\mathbf{I} \circ X \vdash Y}{X \vdash Y}}{X \circ \mathbf{I} \vdash Y} \qquad \frac{\dfrac{X \vdash Y \bullet \mathbf{I}}{X \vdash Y}}{X \vdash \mathbf{I} \bullet Y} \qquad \frac{\dfrac{X \vdash \mathbf{I} \bullet Y}{X \vdash Y}}{X \vdash Y \bullet \mathbf{I}}$$

certain 'logical' structural rules:

$$\frac{}{p \vdash p}\ (id) \qquad \frac{}{\sim p \vdash \sim p}\ (id\sim) \qquad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y}\ (cut)$$

and versions of the standard structural rules from ordinary Gentzen calculi for classical logic, monotonicity (alias thinning or weakening), exchange (alias permutation), and contraction, together with associativity (presented in Table 3). Note that the failure of left (right) monotonicity for $\bullet$ ($\circ$) blocks the provability of Peirce's Co-Law (Peirce's Law).

Table 3. *Structural sequent rules*

$$\frac{X \vdash Y}{X \vdash Y \bullet Z}\ (rm) \qquad \frac{X \vdash Y}{X \circ Z \vdash Y}\ (lm)$$

$$\frac{X \vdash Y \bullet Z}{X \vdash Z \bullet Y}\ (re) \qquad \frac{X \circ Z \vdash Y}{Z \circ X \vdash Y}\ (le)$$

$$\frac{X \vdash Y \bullet Y}{X \vdash Y}\ (rc) \qquad \frac{X \circ X \vdash Y}{X \vdash Y}\ (lc)$$

$$\frac{X \vdash (Y \bullet Z) \bullet X'}{X \vdash Y \bullet (Z \bullet X')}\ (ra) \qquad \frac{(X \circ Y) \circ Z \vdash X'}{X \circ (Y \circ Z) \vdash X'}\ (la)$$

The display sequent calculi $\delta(I_i, C_j)$, $i, j \in \{1, 2, 3, 4\}$, for the constructive logics (I_i, C_j) share these rules and the introduction rules stated in Table 4. The particular display calculus $\delta(I_i, C_j)$ then is the proof system obtained by adding the rules rI_i and rC_j from Table 5.

A derivation of a sequent s from a set of sequents $\{\mathsf{s}_1, \ldots, \mathsf{s}_n\}$ in $\delta(I_i, C_j)$ is defined as a tree with root s such that every leaf is an instantiation of (id), $(id\sim)$, or a

Table 4. *Introduction rules shared by all logics (I_i, C_j)*

$\dfrac{X \vdash A \quad Y \vdash B}{X \circ Y \vdash (A \wedge B)}\ (\vdash \wedge)$	$\dfrac{A \circ B \vdash X}{(A \wedge B) \vdash X}\ (\wedge \vdash)$
$\dfrac{X \vdash A \bullet B}{X \vdash (A \vee B)}\ (\vdash \vee)$	$\dfrac{A \vdash X \quad B \vdash Y}{(A \vee B) \vdash X \bullet Y}\ (\vee \vdash)$
$\dfrac{X \vdash A \circ B}{X \vdash (A \to B)}\ (\vdash \to)$	$\dfrac{X \vdash A \quad B \vdash Y}{(A \to B) \vdash X \circ Y}\ (\to \vdash)$
$\dfrac{X \vdash B \quad A \vdash Y}{X \bullet Y \vdash B {-\!\!\!<} A}\ (\vdash {-\!\!\!<})$	$\dfrac{B \bullet A \vdash X}{B {-\!\!\!<} A \vdash X}\ ({-\!\!\!<} \vdash)$
$\dfrac{X \vdash {\sim}A \bullet {\sim}B}{X \vdash {\sim}(A \wedge B)}\ (\vdash {\sim}\wedge)$	$\dfrac{{\sim}A \vdash X \quad {\sim}B \vdash Y}{{\sim}(A \wedge B) \vdash X \bullet Y}\ ({\sim}\wedge \vdash)$
$\dfrac{X \vdash {\sim}A \quad Y \vdash {\sim}B}{X \circ Y \vdash {\sim}(A \vee B)}\ (\vdash {\sim}\vee)$	$\dfrac{{\sim}A \circ {\sim}B \vdash X}{{\sim}(A \vee B) \vdash X}\ ({\sim}\vee \vdash)$
$\dfrac{X \vdash A}{X \vdash {\sim}{\sim}A}\ (\vdash {\sim}{\sim})$	$\dfrac{A \vdash X}{{\sim}{\sim}A \vdash X}\ ({\sim}{\sim} \vdash)$

sequent from $\{\mathsf{s}_1, \ldots, \mathsf{s}_n\}$, and every other node is obtained by an application of one of the remaining rules. A proof of a sequent s in $\delta(I_i, C_j)$ is a derivation of s from $\varnothing$. Sequents s and s' are said to be interderivable iff s is derivable from $\{\mathsf{s}'\}$ and s' is derivable from $\{\mathsf{s}\}$.

Two sequents s and s' are said to be structurally equivalent if they are interderivable by means of display postulates only. It is characteristic for display calculi that any substructure of a given sequent s may be displayed as the entire antecedent or succedent of a structurally equivalent sequent s'.

If $\mathsf{s} = X \vdash Y$ is a sequent, then the displayed occurrence of X (Y) is an antecedent (succedent) part of s. If an occurrence of $(Z \circ W)$ is an antecedent part of s, then the displayed occurrences of Z and W are antecedent parts of s. If an occurrence of $(Z \bullet W)$ is an antecedent part of s, then the displayed occurrence of Z (W) is an antecedent (succedent) part of s. If an occurrence of $(Z \circ W)$ is a succedent part of s, then the displayed occurrence of Z (W) is an antecedent (succedent) part of s. If an occurrence of $(Z \bullet W)$ is a succedent part of s, then the displayed occurrences of Z and W are succedent parts of s.

THEOREM 8 (CF. (BELNAP 1982)). — *For every sequent s and every antecedent (succedent) part X of s, there exists a sequent s' structurally equivalent to s such that X is the entire antecedent (succedent) of s'.*

Table 5. *Sequent rules for negated implications and co-implications*

rI_1	$\dfrac{X \vdash A \quad Y \vdash {\sim} B}{X \circ Y \vdash {\sim}(A \to B)}$	$\dfrac{A \circ {\sim} B \vdash X}{{\sim}(A \to B) \vdash X}$
rI_2	$\dfrac{X \vdash A \circ {\sim} B}{X \vdash {\sim}(A \to B)}$	$\dfrac{X \vdash A \quad {\sim} B \vdash Y}{{\sim}(A \to B) \vdash X \circ Y}$
rI_3	$\dfrac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash {\sim}(A \to B)}$	$\dfrac{A \bullet B \vdash X}{{\sim}(A \to B) \vdash X}$
rI_4	$\dfrac{X \vdash {\sim} B \quad {\sim} A \vdash Y}{X \bullet Y \vdash {\sim}(A \to B)}$	$\dfrac{{\sim} B \bullet {\sim} A \vdash X}{{\sim}(A \to B) \vdash X}$
rC_1	$\dfrac{X \vdash {\sim} A \bullet B}{X \vdash {\sim}(A \mathbin{-\!\!<} B)}$	$\dfrac{{\sim} A \vdash X \quad B \vdash Y}{{\sim}(A \mathbin{-\!\!<} B) \vdash X \bullet Y}$
rC_2	$\dfrac{X \vdash {\sim} A \quad B \vdash Y}{X \bullet Y \vdash {\sim}(A \mathbin{-\!\!<} B)}$	$\dfrac{{\sim} A \bullet B \vdash X}{{\sim}(A \mathbin{-\!\!<} B) \vdash X}$
rC_3	$\dfrac{X \vdash A \circ B}{X \vdash {\sim}(A \mathbin{-\!\!<} B)}$	$\dfrac{Y \vdash A \quad B \vdash X}{{\sim}(A \mathbin{-\!\!<} B) \vdash Y \circ X}$
rC_4	$\dfrac{X \vdash {\sim} B \circ {\sim} A}{X \vdash {\sim}(A \mathbin{-\!\!<} B)}$	$\dfrac{Y \vdash {\sim} B \quad {\sim} A \vdash X}{{\sim}(A \mathbin{-\!\!<} B) \vdash Y \circ X}$

OBSERVATION 9. — For every $\mathcal{L}$-formula A and every display calculus $\delta(I_i, C_j)$, $A \vdash A$ is provable (and hence $\mathbf{I} \vdash A \to A$ and $A \mathbin{-\!\!<} A \vdash \mathbf{I}$ are provable). □

PROOF. — The proof is by induction on the number of occurrences of connectives in A. We here display two cases for $\delta(I_4, C_3)$:

$$\dfrac{\dfrac{{\sim} B \vdash {\sim} B \quad {\sim} A \vdash {\sim} A}{{\sim} B \bullet {\sim} A \vdash {\sim}(A \to B)}}{{\sim}(A \to B) \vdash {\sim}(A \to B)} \qquad \dfrac{\dfrac{A \vdash A \quad B \vdash B}{{\sim}(A \mathbin{-\!\!<} B) \vdash A \circ B}}{{\sim}(A \mathbin{-\!\!<} B) \vdash {\sim}(A \mathbin{-\!\!<} B)}$$

The remaining cases are equally simple. ■

One can define translations τ_1 and τ_2 from structures into formulas such that these translations reflect the intuitive, context-sensitive interpretation of the structural connectives: τ_1 translates structures which are antecedent parts of a sequent, whereas τ_2 translates structures which are succedent parts of a sequent.

DEFINITION 10. — *The translations* τ_1 *and* τ_2 *from structures into formulas are inductively defined as follows, where* A *is a formula and* p *is a certain atom:*

$$\begin{array}{rclrcl} \tau_1(A) & = & A & \tau_2(A) & = & A \\ \tau_1(\mathbf{I}) & = & p \to p & \tau_2(\mathbf{I}) & = & p \prec p \\ \tau_1(X \circ Y) & = & \tau_1(X) \wedge \tau_1(Y) & \tau_2(X \circ Y) & = & \tau_1(X) \to \tau_2(Y) \\ \tau_1(X \bullet Y) & = & \tau_1(X) \prec \tau_2(Y) & \tau_2(X \bullet Y) & = & \tau_2(X) \vee \tau_2(Y) \end{array}$$

THEOREM 11 (SOUNDNESS). — *(1) If the sequent* $X \vdash Y$ *is provable in* $\delta(I_i, C_j)$*, then* $\tau_1(X) \models^+_{I_i,C_j} \tau_2(Y)$*. (2) If* $X \vdash Y$ *is provable in* $\delta(I_i, C_j)$*, then* $\sim\tau_2(Y) \models^-_{I_i,C_j} \sim\tau_1(X)$*.*

PROOF. — (1) can be proved by induction on derivations in the display calculi $\delta(I_i, C_j)$. We present here just two cases and omit some subscripts. (a) rC_2 right-hand side of $\vdash$. Suppose (*) $\tau_1(X) \models^+ \tau_2(\sim A)$ and $\tau_1(B) \models^+ \tau_2(Y)$. To show: $\tau_1(X \bullet Y) \models^+ \tau_2(\sim(A \prec B))$. $\tau_1(X \bullet Y) = \tau_1(X) \prec \tau_2(Y)$. Let $w \models^+ \tau_1(X) \prec \tau_2(Y)$. Then $\exists w'$ with $w' \leq w$, $w' \models^+ \tau_1(X)$, and $w' \not\models^+ \tau_2(Y)$. By (*), $w' \models^+ \tau_2(\sim A)$ (i.e., $w' \models^- A$) and $w' \not\models^+ \tau_1(B)$. Thus, $w \models^+ \tau_2(\sim(A \prec B))$. (b) rC_4 left-hand side of $\vdash$. Suppose (*) $\tau_1(Y) \models^+ \tau_2(\sim B)$ and $\tau_1(\sim A) \models^+ \tau_2(X)$. To show: $\tau_1(\sim(A \prec B)) \models^+ \tau_2(Y \circ X)$. $\tau_2(Y \circ X) = \tau_1(Y) \to \tau_2(X)$. Let $w \models^+ \tau_1(\sim(A \prec B))$. Then, by cC_4, $\forall w' \geq w$: $w' \models^- A$ or $w' \not\models^- B$. By (*), $\forall w' \geq w$: $w' \models^+ \tau_2(X)$ or $w' \not\models^+ \tau_1(Y)$. Thus, $w \models^+ \tau_1(Y) \to \tau_2(X)$. (2) follows from (1), the definition of $\models^-_{I_i,C_j}$ and the fact that $w \models^+ \sim A$ iff $w \models^- A$. (Indeed, the succedents of the two claims are equivalent.) ■

In order to prove completeness, we will apply some lemmata. We add to the language $\mathcal{L}$ for every atomic formula p a new atom p^* to obtain the language $\mathcal{L}^*$. If A is an $\mathcal{L}$-formula, $(A)^*$ is the result of replacing every strongly negated atom $\sim p$ in A by p^*.

LEMMA 12. — *For every* $\mathcal{L}$*-formula* A*, if* $\varnothing \models^+_{I_i,C_j} A$*, then* $(\rho_{I_i,C_j}(A))^*$ *is valid in* HB*.*

PROOF. — Let $\varnothing \models^+_{I_i,C_j} A$. By Lemma 7, this is the case iff $\varnothing \models^+_{I_i,C_j} \rho_{I_i,C_j}(A)$. If $(\rho_{I_i,C_j}(A))^*$ is not valid in HB, then there is a model $\mathcal{M} = \langle I, \leq, v \rangle$ and $w \in I$ with $\mathcal{M}, w \not\models (\rho_{I_i,C_j}(A))^*$. Define the structure $\mathcal{M}' = \langle I', \leq', v^+, v^- \rangle$ by setting $I' := I$, $\leq' := \leq$, $v^+ := v$ and $w \in v^-(p)$ iff $w \in v(p^*)$, for every atomic $\mathcal{L}$-formula p. Clearly, $\mathcal{M}'$ is a model. By induction on $\mathcal{L}$-formulas A, one can show that $\mathcal{M}, w \not\models (\rho_{I_i,C_j}(A))^*$ iff $\mathcal{M}', w \not\models^+ \rho_{I_i,C_j}(A)$, which contradicts $\varnothing \models^+_{I_i,C_j} \rho_{I_i,C_j}(A)$. ■

LEMMA 13. — *For every* $\sim$*-free* $\mathcal{L}$*-formula* A*, if* A *is provable in* HB*, then* $\mathbf{I} \vdash A$ *is provable in* $\delta(I_i, C_j)$ *without using any sequent rules for strongly negated formulas.*

PROOF. — It is enough to show that the axiom schemata for HB stated in (Rauszer, 1974, p. 24) and (Rauszer, 1980, p. 18) are provable in $\delta(I_i, C_j)$ and that modus ponens and the rule

$$\frac{A}{\neg - A}$$

preserve provability in $\delta(I_i, C_j)$ without making appeal to sequent rules for strongly negated formulas. For the latter and for Axiom (A_{11}) from (Rauszer, 1980), for example, see:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{I} \vdash A}{\mathbf{I} \vdash A \bullet (p \prec p)}}{\mathbf{I} \circ (p \to p) \vdash A \bullet (p \prec p)}}{(p \to p) \vdash A \bullet (p \prec p)}}{((p \to p) \bullet A) \vdash (p \prec p)}}{((p \to p) \prec A) \vdash (p \prec p)}}{\mathbf{I} \circ ((p \to p) \prec A) \vdash (p \prec p)}}{\mathbf{I} \vdash ((p \to p) \prec A) \circ (p \prec p)}}{\mathbf{I} \vdash ((p \to p) \prec A) \to (p \prec p)}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A \bullet B \vdash (A \prec B)}}{A \vdash B \bullet (A \prec B)}}{A \circ \mathbf{I} \vdash B \bullet (A \prec B)}}{A \circ \mathbf{I} \vdash B \vee (A \prec B)}}{\mathbf{I} \vdash A \circ (B \vee (A \prec B))}}{\mathbf{I} \vdash A \to (B \vee (A \prec B))}$$

■

LEMMA 14. — *For every $\mathcal{L}$-formula A, the sequents $A \vdash \rho_{I_i,C_j}(A)$ and $\rho_{I_i,C_j}(A) \vdash A$ are provable in $\delta(I_i, C_j)$.*

LEMMA 15. — *Every sequent $X \vdash \tau_1(X)$ and $\tau_2(X) \vdash X$ is provable in $\delta(I_i, C_j)$, for all $i, j \in \{1, 2, 3, 4\}$.*

PROOF. — By simultaneous induction on X. For instance, we have:

$$\dfrac{X \vdash \tau_1(X) \qquad \tau_2(Y) \vdash Y}{X \bullet Y \vdash \tau_1(X) \prec \tau_2(Y)}$$

■

THEOREM 16 (COMPLETENESS). — *(1) If $\rho_{I_i,C_j}(\tau_1(X)) \models^+_{I_i,C_j} \rho_{I_i,C_j}(\tau_2(Y))$, then $X \vdash Y$ is provable in $\delta(I_i, C_j)$. (2) If $\rho_{I_i,C_j}(\sim\tau_2(Y)) \models^-_{I_i,C_j} \rho_{I_i,C_j}(\sim\tau_1(X))$, then $X \vdash Y$ is provable in $\delta(I_i, C_j)$.*

PROOF. — (1) Suppose $\rho_{I_i,C_j}(\tau_1(X)) \models^+_{I_i,C_j} \rho_{I_i,C_j}(\tau_2(Y))$. Then,

$$\varnothing \models^+_{I_i,C_j} \rho_{I_i,C_j}(\tau_1(X)) \to \rho_{I_i,C_j}(\tau_2(Y)).$$

Using Lemma 12, we obtain that $(\rho_{I_i,C_j}(\tau_1(X)))^* \to (\rho_{I_i,C_j}(\tau_2(Y)))^*$ is valid in HB. By completeness of Rauszer's axiomatization of HB, it follows that

$$(\rho_{I_i,C_j}(\tau_1(X)))^* \to (\rho_{I_i,C_j}(\tau_2(Y)))^*$$

is provable in this axiom system. By Lemma 13, we obtain a proof of the sequent $\mathbf{I} \vdash (\rho_{I_i,C_j}(\tau_1(X)))^* \to (\rho_{I_i,C_j}(\tau_2(Y)))^*$. By applying (cut) to this sequent and the provable sequent

$$(\rho_{I_i,C_j}(\tau_1(X)))^* \to (\rho_{I_i,C_j}(\tau_2(Y)))^* \vdash (\rho_{I_i,C_j}(\tau_1(X)))^* \circ (\rho_{I_i,C_j}(\tau_2(Y)))^*,$$

we may see that $(\rho_{I_i,C_j}(\tau_1(X)))^* \vdash (\rho_{I_i,C_j}(\tau_2(Y)))^*$ is provable in $\delta(I_i, C_j)$. Since $(\rho_{I_i,C_j}(\tau_1(X)))^* \vdash (\rho_{I_i,C_j}(\tau_2(Y)))^*$ is provable without any appeal to sequent rules for strongly negated formulas, the sequent $\rho_{I_i,C_j}(\tau_1(X)) \vdash \rho_{I_i,C_j}(\tau_2(Y))$ is provable in $\delta(I_i, C_j)$, and then, by Lemma 14, $\tau_1(X) \vdash \tau_2(Y)$ is provable in $\delta(I_i, C_j)$. Finally, by Lemma 15, $X \vdash Y$ is provable in $\delta(I_i, C_j)$. (2): Obvious. ■

Belnap (Belnap, 1982) presents a very general cut-elimination theorem covering all 'properly displayable' logics, which are logics satisfying a number of conditions (C1) – (C8). Condition (C8) is the requirement of eliminability of principal cuts, i.e., applications of (cut) in which the two premise sequents have been obtained by introducing the main connective of the cut-formula A. The display calculi $\delta(I_i, C_j)$ do not satisfy condition (C1), which says that each formula which is a constituent of some premise of a sequent rule is a subformula of the conclusion sequent. We may note, however, that $X \vdash Y$ is provable in $\delta(I_i, C_j)$ iff $(\rho_{Ii,C_j}(\tau_1(X)))^* \vdash (\rho_{Ii,C_j}(\tau_2(Y)))^*$ is provable in $\delta(I_i, C_j)$ without any appeal to rules involving $\sim$. Let $\delta(I_i, C_j)^+$ denote the result of dropping all sequent rules exhibiting $\sim$ from $\delta(I_i, C_j)$.

THEOREM 17. — *If* $X \vdash Y$ *is provable in system* $\delta(I_i, C_j)$*, then* $(\rho_{Ii,C_j}(\tau_1(X)))^* \vdash (\rho_{Ii,C_j}(\tau_2(Y)))^*$ *is provable in* $\delta(I_i, C_j)^+$ *without any applications of* (cut).

PROOF. — The system $\delta(I_i, C_j)^+$ satisfies Belnap's conditions (C1)–(C8). The principal cut-elimination step for $\mathrel{-\!\!\!\prec}$ is:

$$\dfrac{\dfrac{X \vdash B \quad A \vdash Y}{X \bullet Y \vdash B \mathrel{-\!\!\!\prec} A} \qquad \dfrac{B \bullet A \vdash Z}{B \mathrel{-\!\!\!\prec} A \vdash Z}}{X \bullet Y \vdash Z}$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{X \vdash B \quad \dfrac{B \bullet A \vdash Z}{B \vdash A \bullet Z}}{X \vdash A \bullet Z}}{X \bullet Z \vdash A} \qquad A \vdash Y}{X \bullet Z \vdash Y}}{X \vdash Z \bullet Y}}{X \bullet Y \vdash Z}$$

■

4. Summary

We noted above that intuitionistic logic enjoys the disjunction property but does not enjoy the constructible falsity property with respect to intuitionistic negation. In Heyting-Brouwer logic, the disjunction property fails. If we take co-negation as primitive, the disjunction property already fails in the $\{-, \wedge, \vee, \mathrel{-\!\!\!\prec}\}$-fragment of HB (alias dual intuitionistic logic), since for every atom p, $p \vee -p$ is valid, but obviously neither p nor $-p$ is valid. However,

OBSERVATION 18. — If $-(A \wedge B)$ is valid in HB, then so are $-A$ or $-B$. □

PROOF. — By 'gluing' of models. Suppose there are models $\mathcal{M}_1$ and $\mathcal{M}_2$ and states w_1, w_2 with $\mathcal{M}_1, w_1 \not\models -A$ and $\mathcal{M}_2, w_2 \not\models -B$. We add a new state w such that no atom is verified at w and consider the relation $\leq'$, which is the reflexive, transitive closure of $\leq \cup \{\langle w, w_1\rangle, \langle w, w_2\rangle\}$. The resulting structure is a model, and at w it verifies the valid $-(A \wedge B)$, which contradicts the fact that $\mathcal{M}_1, w_1 \not\models -A$ and $\mathcal{M}_2, w_2 \not\models -B$. ■

We may now summarize our results. We have motivated and defined the sixteen logics (I_i, C_j), $i, j \in \{1, 2, 3, 4\}$,[12] which comprise both intuitionistic implication and co-implication. These logics enrich the combination of the constructive logics IPL^+ and HB^+ by a strong negation operation $\sim$, which may be regarded as a constructive negation. Its conservative addition to IPL^+ in the systems of Nelson does not lead to a violation of the disjunction property and gives rise to the constructible falsity property. The logics (I_i, C_j) may be viewed as constructive logics, if one is not disturbed by the fact that these logics fail to enjoy the constructible falsity property for the definable intuitionistic negation and the disjunction property for the definable co-negation. The constructiveness of the logics (I_i, C_j) would have to be further justified by showing them correct with respect to an interpretation in terms of canonical proofs, dual proofs, *disproofs*, and *dual disproofs*, where a disproof (dual disproof) of A is a derivation of $\sim A$ from the empty set (derivation of the empty set from $\sim A$). Moreover, we have presented strongly sound and complete display sequent calculi for the logics (I_i, C_j).

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12. In this paper, we do not consider logics which combine co-implication and strong negation, but in which implication is absent. Among these logics, we can find a 'dual' of $\mathsf{N4}$, see also (Kamide, 2003).

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A. Appendix

We refer to the result of dropping the sequent rules for $\rightarrow$ from $(I_i, C_j)^+$ as $\delta\mathsf{HB}^+$. $\delta\mathsf{HB}^+$ is a display sequent calculus for HB^+ in the language $\{\wedge, \vee, \prec\}$. If $X \vdash Y$ is provable in $\delta\mathsf{HB}^+$, then it follows from Theorem 11 that $\tau_1(X)$ entails $\tau_2(Y)$ in HB^+; the converse follows by Theorem 16. Since the structural connective $\circ$ is interpreted as implication in succedent position, the proof of Theorem 19 refers to *both* proofs and dual proofs. In particular, we must say what is a canonical reductio (dual proof) of an implication $(A \rightarrow B)$, namely a pair (π_1, π_2), where π_1 is a proof of A and π_2 is a reductio of B.[13] Moreover, we require that for no formula A, there exists both a proof and a reductio.

THEOREM 19. — *If $A \vdash \mathbf{I}$ is provable in $\delta\mathsf{HB}^+$, then there exists a construction π which is a reductio ad absurdum of A.*

PROOF. — We prove a more general claim by induction on proofs in $\delta\mathsf{HB}^+$, namely: If $X \vdash Y$ is provable in $\delta\mathsf{HB}^+$, then there exists a construction π such that $\pi(\pi')$ is a reductio ad absurdum of $\tau_1(X)$ whenever π' is a reductio ad absurdum of $\tau_2(Y)$. Note that any reductio of $\tau_2(\mathbf{I}) = (p \prec p)$ is the identity function.

The cases of the rules $(\prec \vdash)$, $(\wedge \vdash)$, and $(\vdash \vee)$ are trivial.

$(\vdash \prec)$: Suppose $\pi_1({\pi_1}')$ is a reductio of $\tau_1(X)$ whenever ${\pi_1}'$ is a reductio of B, and $\pi_2({\pi_2}')$ is a reductio of A whenever ${\pi_2}'$ is a reductio of $\tau_2(Y)$. We define a construction π^* such that $\pi^*(\pi^{*\prime})$ is a reductio of $\tau_1(X) \prec \tau_2(Y)$ whenever $\pi^{*\prime}$ is a reductio of $B \prec A$. Let $\pi^{*\prime}$ be a reductio of $B \prec A$. Then for every reductio θ of A, $\pi^{*\prime}(\theta)$ is a reductio of B. Therefore, $\pi^{*\prime}(\pi_2)$ is a reductio of $B \prec \tau_2(Y)$, and $\pi^* := \pi_1(\pi^{*\prime}(\pi_2))$ is a reductio of $\tau_1(X) \prec \tau_2(Y)$.

$(\vdash \wedge)$: Suppose $\pi_1({\pi_1}')$ is a reductio of $\tau_1(X)$ whenever ${\pi_1}'$ is a reductio of A, and $\pi_2({\pi_2}')$ is a reductio of $\tau_1(Y)$ whenever ${\pi_2}'$ is a reductio of B. We define a construction π^* such that $\pi^*(\pi^{*\prime})$ is a reductio of $\tau_1(X) \wedge \tau_1(Y)$ whenever $\pi^{*\prime}$ is a reductio of $A \wedge B$. Let $\pi^{*\prime}$ be a reductio of $A \wedge B$. Then $\pi^{*\prime}$ is a pair (i, π), such that $i = 0$ and π is reductio of A or $i = 1$ and π is a reductio of B. Clearly, $\pi^* = (0, \pi_1(\pi))$ or $\pi^* = (1, \pi_2(\pi))$ is a reductio of $\tau_1(X) \wedge \tau_1(Y)$.

The display postulate: $X \bullet Z \vdash Y \,/ X \vdash Y \bullet Z$: Suppose $\pi(\pi')$ is a reductio of $\tau_1(X \bullet Z)$ $(= \tau_1(X) \prec \tau_2(Z))$ whenever π' is a reductio of $\tau_2(Y)$. We define a construction π^* such that $\pi^*(\pi^{*\prime})$ is a reductio of $\tau_1(X)$ whenever $\pi^{*\prime}$ is a reductio of $\tau_2(Y \bullet Z)$ $(= \tau_2(Y) \vee \tau_2(Z))$. Thus, let $\pi^{*\prime} = (\pi_1, \pi_2)$, where π_1 is a reductio of $\tau_2(Y)$ and π_2 is a reductio of $\tau_2(Z)$. Then $\pi(\pi_1)$ is a reductio of $\tau_1(X) \prec \tau_2(Z)$, and $\pi(\pi_1)(\pi_2)$ is a reductio of $\tau_1(X)$.

The display postulate: $X \circ Y \vdash Z \,/ X \vdash Y \circ Z$: Suppose $\pi(\pi')$ is a reductio of $\tau_1(X) \wedge \tau_1(Y)$ whenever π' is a reductio of $\tau_2(Z)$. That is, $\pi(\pi')$ is a pair (i, π'') such that $i = 0$ and π'' is a reductio of $\tau_1(X)$ or $i = 1$ and π'' is a reductio of $\tau_1(Y)$. Suppose π^* is a pair (π_1, π_2), where π_1 is a proof of $\tau_1(Y)$ and π_2 is a reductio of $\tau_2(Z)$. Then $\pi(\pi_2)$ is a pair $(0, \pi'')$ and π'' is a reductio of $\tau_1(X)$. That $\pi(\pi_2)$ is a pair $(1, \pi'')$ where π'' is a reductio of $\tau_1(Y)$ is impossible, because π_1 is a proof of

13. A proof of $(A \prec B)$ then is a reductio of $(A \rightarrow B)$.

$\tau_1(Y)$.
The structural rule (lm): $X \vdash Y \;/ X \circ Z \vdash Y$. Suppose $\pi(\pi')$ is a reductio of $\tau_1(X)$ whenever π' is a reductio of $\tau_2(Y)$. Let π^* be a reductio of $\tau_2(Y)$. Then $(0, \pi(\pi^*))$ is a reductio of $\tau_1(X) \wedge \tau_1(Z)$.
The remaining cases are left to the reader. ■