NMR topic II: modal logics - duality and applications

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Boolean algebras and classical propositional logic CPC

Boolean algebras

$A = (A, \wedge, \vee, \neg, \top, \bot)$ is a BA, where

- $\wedge, \vee$ are associative, commutative, idempotent, absorptive and distribute over each other
- $\top$ is identity of $\wedge$, $\bot$ of $\vee$
- $\neg$ satisfies double negation law, de Morgan and complementation laws

Examples

- $\mathbf{2} = (\{0,1\}, \min, \max, \neg, 1, 0)$
- Powerset algebras: $PX = (PX, \cap, \cup, -, X, \emptyset)$
- Lindenbaum-Tarski algebra of $\mathscr{L}(At)$ of CPC:

$$\mathsf{L} = (\{[\varphi] \mid \varphi \in \mathscr{L}(At)\}, \wedge, \vee, \neg, [\top], [\bot])$$

Algebraic completeness of CPC

Language $\mathscr{L}(At)$ of CPC over a fixed set At:

$$\varphi := p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid \top \mid \bot$$

where moreover $\varphi \to \psi := \neg\varphi \vee \psi, \varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Take your favourite axiomatization of CPC and define a congruence

$$\varphi \equiv \psi \quad \text{IFF} \quad \vdash \varphi \leftrightarrow \psi$$

Lindenbaum-Tarski algebra of $\mathscr{L}(At)$

$$\mathsf{L} = (\{[\varphi]_\equiv \mid \varphi \in \mathscr{L}(At)\}, \wedge, \vee, \neg, [\top], [\bot])$$

$$[\varphi] \wedge [\psi] = [\varphi \wedge \psi] \qquad \neg[\varphi] = [\neg\varphi]$$

$$[\varphi] \vee [\psi] = [\varphi \vee \psi]$$

Algebraic completeness of CPC

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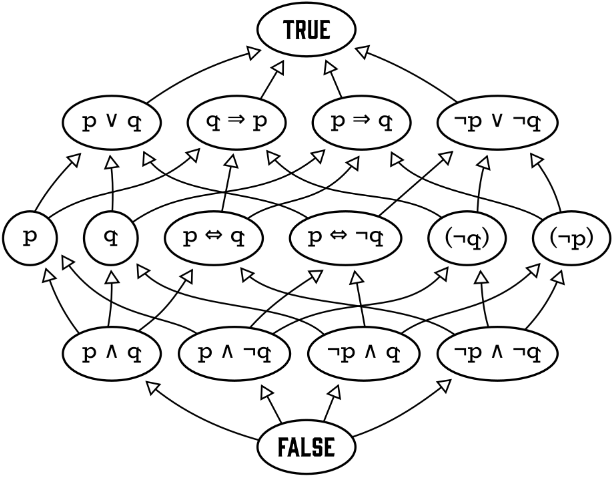
Take your favourite axiomatization of CPC, a set of formulas Γ, and define a congruence relation

$$\varphi \equiv_\Gamma \psi \;\text{ IFF }\; \Gamma \vdash \varphi \leftrightarrow \psi$$

Lindenbaum-Tarski algebra of Γ

$$\mathsf{L}_\Gamma = (\{[\varphi]_\Gamma \mid \varphi \in \mathscr{L}(At)\}, \wedge, \vee, \neg, [\top], [\bot])$$

Free BA of two generators $\{p, q\}$



Algebraic completeness of CPC

Observe a few things first:

- $\varphi \vdash \psi$ IFF $\vdash \varphi \to \psi$ IFF $[\varphi] \leq [\psi]$
- $\varphi \dashv\vdash \psi$ IFF $\vdash \varphi \leftrightarrow \psi$ IFF $[\varphi] = [\psi]$
- $\vdash \varphi$ IFF $\vdash \varphi \leftrightarrow \top$ IFF $[\varphi] = [\top]$
- proper filters correspond to consistent theories, ultrafilters correspond to maximal consistent theories
- any valuation $v : \mathscr{L}(At) \longrightarrow \mathbf{2}$ is indeed a homomorphism of BA, and thus corresponds to an ultrafilter $\{[\varphi] \mid v(\varphi) = 1\}$

Completeness w.r.t. BA

Assume $\nvdash \varphi$, then $[\varphi] \neq [\top]$, and we have a canonical valuation $v(\varphi) = [\varphi]$ in L refuting φ.

Algebraic completeness of CPC

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Strong completeness w.r.t. BA

Assume $\Gamma \nvdash \varphi$, then $[\varphi] \neq [\top]$, and we have a canonical valuation $v(\varphi) = [\varphi]$ in L_Γ refuting φ.

Completeness w.r.t. **2**

Strong completeness w.r.t. BA

Assume $\Gamma \nvdash \varphi$, then $[\varphi] \neq [\top]$, and we have a canonical valuation $v(\varphi) = [\varphi]$ in L_Γ refuting φ.

Observe:

1. $\{[\top]\}$ is a proper filter on L_Γ
2. as $[\top] \nleq [\varphi]$, there is an ultrafilter F extending $\{[\top]\}$ and $[\varphi] \notin F$.
3. Thus F corresponds to a homomorphism $v_F : \mathsf{L}_\Gamma \longrightarrow \mathbf{2}$ defined as

$$v_F([\psi]) = 1 \text{ IFF } [\psi] \in F$$

4. Now compose $v \circ v_F$ to obtain a valuation satisfying all formulas in Γ and refuting φ in **2**.

CPC via Kripke semantics

Language $\mathscr{L}(At)$ of CPC over a fixed set At:

$$\varphi := p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid \top \mid \bot$$

Models

A nonempty set W of possible worlds, a valuation $V : At \longrightarrow PW$

$$\begin{array}{rcl} w \Vdash p \equiv w \in V(p) & & |p| = V(p) \\ w \Vdash \neg\varphi \equiv w \nVdash \varphi & & |\neg\varphi| = W - |\varphi| \\ w \Vdash \varphi \wedge \psi \equiv w \Vdash \varphi \text{ and } w \Vdash \psi & & |\varphi \wedge \psi| = |\varphi| \cap |\psi| \\ w \Vdash \varphi \vee \psi \equiv w \Vdash \varphi \text{ or } w \Vdash \psi & & |\varphi \wedge \psi| = |\varphi| \cup |\psi| \end{array}$$

Observe: $\Gamma_w = \{\varphi \mid w \Vdash \varphi\}$ is a maximal consistent theory, i.e. it corresponds to an ultrafilter on L and to a two-valued valuation.

Yet another completeness proof of CPC

Canonical model of CPC

$W_c = \{\Gamma \mid \Gamma \text{ a max. cons. theory}\}, \quad V_c(p) = \{\Gamma \mid p \in \Gamma\}$

1. prove that for each $\varphi : V_c(p) = \{\Gamma \mid p \in \Gamma\}$ (truth lemma)
2. If $\Delta \nvdash \varphi$, then $\Delta \cup \{\neg\varphi\}$ is consistent, and therefore there is a max. cons. theory $\Gamma \in W_c$ with $\Gamma \supseteq \Delta \cup \{\neg\varphi\}$, thus satisfying all formulas in Δ and refuting φ.

A duality

1. For each set W, $[W, 2] = 2^W = (PW, \cap, \cup, -, W, \emptyset)$ is a BA
2. For each BA $\mathbb{A}$, the set of boolean homomorphisms $(\mathbb{A}, \mathbf{2})$ is (isom. to) the set of ultrafilters on $\mathbb{A}$.

Modal logic via Kripke semantics

Language $\mathscr{L}_\Box(At)$ of CPC over a fixed set At:

$$\varphi := p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid \Box\varphi \mid \top \mid \bot$$

where moreover $\Diamond\varphi := \neg\Box\neg\varphi$

Frames and Models

A frame $(W, R), R \subseteq W \times W$, plus a valuation $V : At \longrightarrow PW$

$$w \Vdash p \equiv w \in V(p) \qquad |p| = V(p)$$

$$w \Vdash \neg\varphi \equiv w \nVdash \varphi \qquad |\neg\varphi| = W - |\varphi|$$

$$w \Vdash \varphi \wedge \psi \equiv w \Vdash \varphi \text{ and } w \Vdash \psi \qquad |\varphi \wedge \psi| = |\varphi| \cap |\psi|$$

$$w \Vdash \varphi \vee \psi \equiv w \Vdash \varphi \text{ or } w \Vdash \psi \qquad |\varphi \wedge \psi| = |\varphi| \cup |\psi|$$

$$w \Vdash \Box\varphi \equiv \forall u(wRu \to u \Vdash \varphi) \qquad |\Box\varphi| = \{w \mid R[w] \subseteq |\varphi|\}$$

$$w \Vdash \Diamond\varphi \equiv \exists u(wRu \wedge u \Vdash \varphi) \qquad |\Diamond\varphi| = \{w \mid R[w] \cap |\varphi| \neq \emptyset\}$$

Canonical completeness proof of K

Canonical model of K

$W_c^K = \{\Gamma \mid \Gamma \text{ a max. cons. theory in K}\}, \quad V_c(p) = \{\Gamma \mid p \in \Gamma\}$

1. prove that for each modal $\varphi : V_c(p) = \{\Gamma \mid p \in \Gamma\}$ (truth lemma)
2. If $\Delta \nvdash_K \varphi$, then $\Delta \cup \{\neg\varphi\}$ is consistent, and therefore there is a max. cons. theory $\Gamma \in W_c^K$ with $\Gamma \supseteq \Delta \cup \{\neg\varphi\}$, thus satisfying all formulas in Δ and refuting φ.

A (lifted) duality?

1. For each frame (W, R), $2^W = (PW, \cap, \cup, -, W, \emptyset, \Box)$ with $\Box Y = \{w \mid R[w] \subseteq Y\}$ is a BAO.
2. For each BA $\mathbb{A}$, the set of boolean homomorphisms $(\mathbb{A}, \mathbf{2})$ (isom. to the set of ultrafilters on $\mathbb{A}$) can be equipped with an R.

Normal modal logics

- n.m.l. are logics (in the modal language) containing K, and closed under MP and Nec rules
- e.g. logics of certain classes of Kripke frames
- e.g. axiomatic extensions of K

	modal axiom	frame condition	
T	$\Box\varphi \to \varphi$	$\forall x\ xRx$	reflexivity
D	$\Box\varphi \to \Diamond\varphi$	$\forall x \exists y\ xRy$	seriality
4	$\Box\varphi \to \Box\Box\varphi$	$\forall x, y, z\ xRy \wedge yRz \to xRz$	tranzitivity
5	$\Diamond\varphi \to \Box\Diamond\varphi$	$\forall x, y, z\ xRy \wedge xRz \to yRz$	euclideanness
B	$\varphi \to \Box\Diamond\varphi$	$\forall x, y\ xRy \to yRx$	symmetry

Modal algebras (of K)

Boolean algebras with operators

$A = (A, \wedge, \vee, \neg, \Box, \top, \bot)$ is a BAO, if $(A, \wedge, \vee, \neg, \top, \bot)$ is a Boolean algebra, and

$$\Box(a \wedge b) = \Box a \wedge \Box b \qquad \Box\top = \top.$$

Homomorphisms of BAO

$h : A \longrightarrow B$ is a homomorphism of BAO, if it is a boolean homomorphism and

$$h(\Box_A a) = \Box_B h(a).$$

Notice:

1. BAO is a variety - equationally defined class of algebras.
2. Formula algebra $\mathscr{L}_\Box(At)$ factorized by provable equivalence in a normal modal logic is a BAO.
3. Recall notions of filters and ultrafilters of BA, and ultrafilter theorem. Boolean homs from A to $\mathbf{2}$, $(A, \mathbf{2})$, correspond to ultrafilters on A.

Frames

Kripke frames

$F = (W, R)$ is a frame if $W \neq \emptyset$ and $R \subseteq W \times W$.

Frame (bounded) morphisms

$f : F_1 \longrightarrow F_2$ is a frame morphism, iff

1. xR_1y implies $f(x)R_2f(y)$
2. $f(x)R_2w$ implies $\exists y(f(y) = w \wedge xR_1y)$

Recall:

1. Morphisms preserve frame validity of modal formulas:

$$F_1, x \Vdash \varphi \;\longrightarrow\; F_2, f(x) \Vdash \varphi.$$

2. Identities are frame morphisms, and frame morphisms compose.

Stone Duality - BA and sets

The dual picture:

$$\mathsf{Set}^{op} \underset{Pred}{\overset{Stone}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{BA}$$

1. $Pred : X \mapsto [X, 2]$. The **predicate algebra** of X is the Boolean algebra of **subsets** of X: $(PX, \cap, \cup, -)$.
2. $Stone : A \mapsto (A, \mathbf{2})$. The **Stone set** of A are **ultrafilters** on A.

On morphisms:

1. For $f : X_2 \longrightarrow X_1$ define $Pred(f) : P(X_1) \longrightarrow P(X_2)$ as $Y_1 \mapsto f^{-1}[Y_1]$.
2. For $h : A \longrightarrow B$ define[a] $Stone(h) : Stone(B) \longrightarrow Stone(A)$ as $u_B \mapsto h^{-1}[u_B]$.

[a] Prove that h^{-1} maps ultrafilters on B to ultrafilters on A.

Duality

BAO and frames

The dual picture[a]:

$$\mathsf{Fr}^{op} \underset{Pred^{\#}}{\overset{Stone^{\#}}{\underset{?}{\leftrightarrows}}} \mathsf{BAO}$$

1. $Pred^{\#}$ from $Pred : F \mapsto [F, 2]$.
   The **complex algebra**[b] of F is based on the BA of **subsets** of $\mathbb{F}$,

   $$\Box X = \{y \mid yRz \longrightarrow z \in X\}.$$

2. $Stone^{\#}$ from $Stone : A \mapsto (A, \mathbf{2})$.
   The **canonical frame** of A is based on **ultrafilters** on A, related by

   $$uRv \equiv \forall a \in A(\Box a \in u \longrightarrow a \in v).$$

[a] The book denotes $Pred F$ as F^{+}, and $Stone A$ as A_{+}.
[b] Prove this is a BAO.

BAO and frames

The dual picture:

$$\mathsf{Fr}^{op} \underset{Pred^{\#}}{\overset{Stone^{\#}}{\underset{?}{\leftrightarrows}}} \mathsf{BAO}$$

On morphisms:

1. For $f : F_2 \longrightarrow F_1$ define[a] $Pred^{\#}(f) : Pred\, F_1 \longrightarrow Pred\, F_2$ as
$$Y_1 \mapsto f^{-1}[Y_1].$$
2. For $h : A \longrightarrow B$ define[b] $Stone^{\#}(h) : Stone\, B \longrightarrow Stone\, A$ as
$$u_B \mapsto h^{-1}[u_B].$$

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[a] Prove this is a BAO homomorphism.
[b] Prove this is a frame morphism.

Canonical extension of an algebra

$Pred^{\#}Stone^{\#}A$ is the **canonical extension** of A.

$$A \rightarrowtail Pred^{\#}Stone^{\#}A$$

mapping $a \mapsto \hat{a} = \{u \mid a \in u\}$. The fact that this is an embedding encompasses completeness:

$$a \not\leq b \text{ IFF } \hat{a} \not\subseteq \hat{b} \text{ IFF } \exists u(a \in u \wedge b \notin u).$$

Notice if A is the formula BAO factorized by provable equivalence in a n.m. logic L (Lindenbaum-Tarski algebra of L), then

1. Ultrafilters on A are MCS (i.e. complete consistent theories),
2. $Stone^{\#}A$ is the canonical frame of L, $Pred^{\#}Stone^{\#}A$ its complex algebra,
3. the embedding above[1], provides the basic step for completeness, the fact it is a BAO homomorphism encompasses Truth lemma.

[1] Jónsson-Tarski theorem: see section 5.3 of the book Modal Logic.

Ultrafilter extension of a frame

$Stone^{\#}Pred^{\#}F$ is the **ultrafilter extension** of F.

$$F \overset{??}{\rightarrowtail} Stone^{\#}Pred^{\#}F$$

where $x \mapsto \{U \mid x \in U\}$[a].

[a] Prove this is an ultrafilter.

Notice:

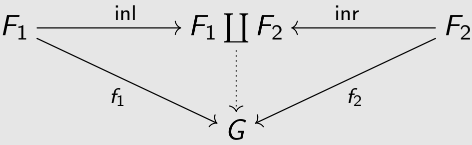
1. The above mapping is in general **not** a frame morphism.
2. However, it **reflects** frame validity of formulas[2]

$$Stone^{\#}Pred^{\#}F \Vdash \varphi \quad \text{then} \quad F \Vdash \varphi.$$

[2] Prove this. See Corollary 3.16 and Proposition 2.59 in the book Modal Logic.

Disjoint unions of frames

Coproducts of frames



Notice:

1. $F_1 \coprod F_2 = (X_1 \cupdot X_2, R_1 \cupdot R_2)$, inl, inr are injective frame morphisms (inclusions)[3].
2. $Pred^{\#}(\coprod_{i \in I} F_i) \cong \prod_{i \in I}(Pred^{\#} F_i)$[4].
3. All $F_i \Vdash \varphi$ IFF $\coprod_{i \in I} F_i \Vdash \varphi$

[3] Make sure you can prove they are injective frame morphisms.

[4] Provide this isomorphism (in BAO), see Theorem 5.48 in the book Modal Logic.

(Generated) subframes

We say that F_1 is (isomorphic to) a subframe of F_2

$$F_1 \overset{f}{\rightarrowtail} F_2$$

if f is an **injective** frame morphism.

Generated subframes

For F and its subset X, we define the X-generated subframe F_X as the smallest subframe containing X and closed under finite iterations of R[a].

[a] Show that the inclusion is indeed injective frame morphism.

Notice:

1. If $F_2 \Vdash \varphi$ then $F_1 \Vdash \varphi$[a].

[a] Prove this.

Images of frames

We say that F_2 is a morphic image of F_1

$$F_1 \xrightarrow{f} \!\!\!\!\rightarrow F_2$$

if f is a surjective frame morphism.

Notice:

1. If $F_1 \Vdash \varphi$ then $F_2 \Vdash \varphi$[a].
2. Each frame is a morphic image of the disjoint union of its point-generated subframes[b].

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[a] Prove this.
[b] Prove this.

From the dual picture:

1. If $F_1 \xrightarrow{f} \twoheadrightarrow F_2$ then $Pred^{\#} F_2 \rightarrowtail \xrightarrow{Pred^{\#}(f)} Pred^{\#} F_1$
2. If $F_1 \rightarrowtail \xrightarrow{f} F_2$ then $Pred^{\#} F_2 \xrightarrow{Pred^{\#}(f)} \twoheadrightarrow Pred^{\#} F_1$
3. If $A_1 \xrightarrow{h} \twoheadrightarrow A_2$ then $Stone^{\#} A_2 \rightarrowtail \xrightarrow{Stone^{\#}(h)} Stone^{\#} A_1$
4. If $A_1 \rightarrowtail \xrightarrow{h} A_2$ then $Stone^{\#} A_2 \xrightarrow{Stone^{\#}(h)} \twoheadrightarrow Stone^{\#} A_1$ [a]

[a] Prove these. See Theorem 5.47 in the book Modal logic.

The definability theorem

Goldblatt-Thomason Theorem for classes of Kripke frames[a]

[a] Theorem 5.54 of the book Modal Logic.

Suppose $\mathbb{C}$ is a class of frames closed under the ultrafilter extensions ($F \in \mathbb{C}$ implies that $Stone^{\#} Pred^{\#} F \in \mathbb{C}$). Then the following are equivalent:

1. $\mathbb{C}$ is modally definable.
2. $\mathbb{C}$ has the following closure properties:
    1. If F_1 is in $\mathbb{C}$, $f : F_1 \longrightarrow F_2$ is surjective, then F_2 is in $\mathbb{C}$. ($\mathbb{C}$ closed under morphic images.)
    2. If F_2 is in $\mathbb{C}$, $f : F_1 \longrightarrow F_2$ is injective, then F_1 is in $\mathbb{C}$. ($\mathbb{C}$ closed under (generated) subframes.)
    3. If F_i for all $i \in I$ are in $\mathbb{C}$, then $\coprod_{i \in I} F_i$ is in $\mathbb{C}$. ($\mathbb{C}$ closed under disjoint unions.)
    4. If $Stone^{\#} Pred^{\#} F$ is in $\mathbb{C}$, then F is in $\mathbb{C}$. ($\mathbb{C}$ **reflects** ultrafilter extensions.)

A proof of the theorem (using Birkhoff's theorem and duality)

If $\mathbb{C}$ is modally definable, then it satisfies the closure properties (routine observation that mentioned morphisms preserve frame validity of formulas[a]).

For the interesting direction, we will show that, given the closure properties, the logic of $\mathbb{C}$ defines $\mathbb{C}$:

1. Assume F satisfies **the logic of** $\mathbb{C}$ ($\{\varphi \mid \mathbb{C} \vDash \varphi\}$). Then $Pred^{\#} F$ satisfies the corresponding equational theory of **the variety generated by the complex algebras of** $\mathbb{C}$ ($\{\varphi \approx \top \mid \mathbb{C} \vDash \varphi\}$).
2. Therefore $Pred^{\#} F$ is in $HSP(Pred^{\#}[\mathbb{C}])$, meaning there is B:
3. In BAO: $Pred^{\#}(F) \twoheadleftarrow B \rightarrowtail \prod(Pred^{\#} F_i) \cong Pred^{\#} \coprod F_i$ with all $F_i \in \mathbb{C}$.
4. In Fr: $Stone^{\#} Pred^{\#}(F) \rightarrowtail Stone^{\#} B \twoheadleftarrow Stone^{\#} Pred^{\#} \coprod F_i$ by which $Stone^{\#} Pred^{\#}(F) \in \mathbb{C}$, and therefore $F \in \mathbb{C}$.

[a] See Proposition 5.53. The item 4 requires some thinking - see Corollary 3.16 and Proposition 2.59 in the book Modal Logic.

Remark: a model-theoretic proof of the theorem[5]

- To prove that the logic of $\mathbb{C}$ defines $\mathbb{C}$. Assume $\mathbb{C}$ is closed under ultraproducts and assume F validates the logic of the class $\mathbb{C}$. Assume w.l.o.g. that F is point-generated by w.
- Put $At_F = \{p_Y \mid Y \in Pred^\# F\}$, and generate language $\mathscr{L}(At)_F$. Consider F with the obvious valuation as the model $\mathscr{M}$. Define $\Delta = \{\alpha \mid \mathscr{M}, w \Vdash \alpha\}$.
- Each $\Delta' \subseteq_\omega \Delta$ is satisfiable in $\mathbb{C}$, w.l.o.g. in a point-generated frame (model). (If not, $\neg \bigwedge \Delta'$ would be in the logic of $\mathbb{C}$ - a contradiction.)
- Therefore Δ is satisfiable in $\mathbb{C}$, w.l.o.g. in a point-generated frame (model) - in some ultraproduct of the frames in $\mathbb{C}$ obtained above. Consider a countably saturated ultrapower $\mathscr{N}$ of this model, with a frame G in $\mathbb{C}$.
- Show that $G \longrightarrow\!\!\!\!\!\rightarrow Stone^\# Pred^\# F$, and conclude that F in $\mathbb{C}$.

[5] See Section 3.8 of the book Modal Logic.