Definition of projective planes. A *projective plane* consists of a set of points $\mathcal{P}$ and set of lines $\mathcal{L}$. Each line is a subset of $\mathcal{P}$. The following holds:

(1) $\forall x, y \in \mathcal{P}: \ x \neq y \ \Rightarrow \ \exists! \ell \in \mathcal{L}$ such that $x, y \in \ell$;
(2) $\forall \ell, m \in \mathcal{L} : \ell \neq m \ \Rightarrow \ \exists! x \in \mathcal{P}$ such that $x \in m \cap \ell$; and
(3) $\exists x_i \in \mathcal{P}; 1 \leq i \leq 4$, such that whenever $1 \leq i < j \leq 4$, then $x_i \neq x_j$ and $x_k \notin \ell$ if $\ell \in \mathcal{L}$, $\{x_i, x_j\} \subseteq \ell$ and $k \in \{1,2,3,4\} \setminus \{i,j\}$.

Planes incident to the same line are called *collinear*. Point (3) may be thus rephrased by saying that there exist four points no three of which are collinear.

Remarks:

(a) One of $\exists!$ may be replaced by $\exists$, but not both of them (to get a counterexample double a point);
(b) point (3) is needed to avoid a situation when there is one long line each point of which is incident to a 2-point line, with all of the 2-point lines meeting in a common point that is not incident to the long line.

Facts which are easy to verify:

(i) All lines are of the same size (cardinality). In the finite case it is usual to say that the plane is of *order* n if the lines consist of $n+1$ points;
(ii) in the finite case each point is incident to $n+1$ lines and the overall number of both points and lines is equal to n^2+n+1;
(iii) the least possible order is 2. Each projective plane of order 2 is isomorphic to the Fano plane.
(iv) Let $x_1, \dots, x_4$ be points no three of which are collinear. For $1 \leq i < j \leq 4$ denote by $\ell_{i,j}$ the line passing through x_i and x_j. There are six such lines from which there may chosen four such that no three of them meet in a common point (e.g., $\ell_{1,2}$, $\ell_{2,3}$, $\ell_{3,4}$ and $\ell_{4,1}$.

Write $xI\ell$ if $x \in \mathcal{P}$ belongs to $\ell \in \mathcal{L}$. This may be also expressed by saying that *x is incident to ℓ* or that *ℓ is incident to x*.

Consider I as subset of $\mathcal{P} \times \mathcal{L}$. Another way how to interpret I is to regard it as a bipartite graph with partitions $\mathcal{L}$ and $\mathcal{P}$. Axioms of projective plane may be expressed in this form:

(1) $\forall x, y \in \mathcal{P} : x \neq y \Rightarrow \exists! \ell \in \mathcal{L}$ with $xI\ell$ and $yI\ell$;
(2) $\forall \ell, m \in \mathcal{L} : \ell \neq m \Rightarrow \exists! x \in \mathcal{P}$ with $xI\ell$ and xIm;
(3) $\exists x_1, \dots, x_4 \in \mathcal{P}$ such that if $1 \leq i < j \leq 4$, then $x_i \neq x_j$, and $x_k I \ell$ is not true if $x_i I \ell$, $x_j I \ell$ and $k \in \{1,2,3,4\} \setminus \{i,j\}$.

Because of point (iv) above the following statement is clear:
If $(\mathcal{P}, \mathcal{L}, I)$ fulfills (1–3), then $(\mathcal{L}, \mathcal{P}, J)$ fulfills (1–3) too, with $\ell J x \ \Leftrightarrow \ xI\ell$.

This defines the *dual projective plane*. A *dual line* thus consists of all lines passing through a given point, while a *dual point* is a line.

Perspectivities. A *collineation* is an automorphism of a projective plane. In the classical seting this is a permutation ψ of $\mathcal{P}$ that fulfils condition

$$\ell \text{ is a line} \Leftrightarrow \psi(\ell) \text{ is a line.}$$

In the setting of incidence geometry a collineation is a pair (α, β) such that α permutes $\mathcal{P}$, β permutes $\mathcal{L}$ and

$$xI\ell \ \Leftrightarrow \ \alpha(x) I \beta(\ell).$$

It is easy to see that both definitions are equivalent. Note that (α, β) is completely determined by α, and that (α, β) is a collineation in the sense of incidence geometry if and only if α is a collineation in the classical sense.

The notion of collineation is also being used to express an isomorphism of two distinct projective planes. Some notions defined here for automorphisms may be straightforwardly generalized to isomorphisms. The same is true for statements involving automorphisms.

A mapping ψ is called *collinear* if any set of collinear points is mapped upon a set of collinear points. This is the same as saying that any three distinct collinear points are mapped upon collinear points.

It is easy to see that:

- A permutation ψ is a collineation if and only if both ψ and ψ^{-1} are collinear; while
- in the finite case ψ is a collineation if and only if ψ is collinear.

Let ψ be a collineation. A point x is said to be a *center* of ψ if $\psi(\ell) = \ell$ whenever $xI\ell$. A line is said to be an *axis* of ψ is $\psi(x) = x$ whenever $xI\ell$.

By these definitions, the notions of center and axis are dual. (This means that x is a center of ψ if and only if x is an axis of ψ in the dual plane.)

This is easy:

- A collineation with two centers is trivial (i.e., the identity mapping); a collineation with two axes is trivial.
- If ℓ is an axis of a collineation ψ and $x \notin \ell$ is a point fixed by ψ, then x is a center of ψ.
- If x is a center of a collineation ψ and ℓ is a line fixed by ψ such that $x \notin \ell$, then ℓ is an axis of ψ.

Proposition. *A collineation posseses an axis if and only if it possesses a center.*

Proof. Because of duality it suffices to prove the implication that assumes the existence of an axis. As remarked above, if ψ fixes a point not upon the axis ℓ, then the point is a center. Suppose that ψ moves every point that is not incident to ℓ. For $x \notin \ell$ denote by p_x the line connecting x and $\psi(x)$. It is clear that $\psi(p_x) = p_x$. Suppose that there exist $x, y \notin \ell$ such that $p_x \neq p_y$ and the intersection of p_x and p_y is not upon ℓ. Such an intersection is fixed by ψ, and hence it is a center of ψ. The remaining possibility is that if $x, y \notin \ell$, then p_x and p_y meet upon ℓ. It is clear that in such a case all p_x meet at the same point of ℓ. That point is the center of ψ. □

A collineation with center and axis is called a *perspectivity.* If c is the center and a is the axis, then the perspectivity is also called a (c, a)-collineation.

Proposition. *Let c be a point and a a line. If none of points x and y is incident to a, and both are distinct from c, then there exists at most one (c, a)-collineation ψ such that $\psi(x) = y$.*

Proof. Let ψ be such a collineation and let $z \notin a$ be a point such that z, c and x are not collinear. Thus $s \neq \ell$, where s is the line connecting z and c, and ℓ is the line connecting z and x. Denote by u the intersection of ℓ and a, and note that $\psi(\ell)$ is the line connecting u and y. Since $\psi(s) = s$, the image of z is equal to the intersection of s and $\psi(\ell)$. This gives $\psi(z)$ for every z that is not upon the line t that connects c and x. Considering a pair $(z, \psi(z))$ in the same manner provides images for points upon t. □

A (c, a)-collineation is called an *elation* if $c \in a$, and a *homology* if $c \notin a$. Each nontrivial perspectivity is thus either an elation or a homology.

The projective plane is called (c, a)-*transitive* if for any $x, y \notin a \cup \{c\}$ such that x, y and c are collinear there exists a (c, a)-collineation ψ in which $\psi(x) = y$.

Lemma. *Let x and c be distinct points, and let a be a line. If for any point y such that x, y and c are collinear, $y \notin a \cup \{x, c\}$, there exists a (c, a)-collineation in which $\psi(x) = y$, then the projective plane is (c, a)-transitive.*

Proof. Let x', y' and c be three distinct collinear points, with $x', y' \notin a$. Let ψ be the collineation that maps x upon the intersection y of ℓ' and s, where ℓ connects x and x', ℓ' connects y' and $\ell \cap a$, and s connects x and c. Denote by ψ the (c, a)-collineation with $\psi(x) = y$. Then $\psi(\ell) = \ell'$. Thus $\psi(x') = y'$, since y' is the intersection of ℓ' and s', where the latter line connects x' and c. □

It is immediately clear that *all (c, a)-collineations form a group.*

Elations. The first step is to observe that *elations ψ_i, $i \in \{1, 2\}$, with the same axis a and with centers $c_1 \neq c_2$ commute, and that their composition yields an elation with axis a and a center $c_3 \notin \{c_1, c_2\}$.*

Proof. The collineation $\psi_1\psi_2$ has to be a perspectivity since it possesses an axis. Let c be its center. If $c \notin a$, then $\psi_2(c) \neq c$ is upon the line connecting c and c_2, and $\psi_1\psi_2(c)$ upon the line connecting $\psi_2(c)$ and c_1. This line is distinct from the line connecting c and c_1. Hence $\psi_1\psi_2(c) \neq c$. That is a contradiction. Therefore $c \in a$. All (c, a)-collineations form a group. Hence $c \notin \{c_1, c_2\}$.

What remains is to show the commutativity. Choose $x \notin a$. We shall show that $\psi_2\psi_1(x) = \psi_1\psi_2(x)$. Denote by s_i the connection of x and c_i, and put $x_i = \psi_i(x)$. Denote by ℓ_i the connection of c_i and x_j, $j \neq i$. Then $\psi_i(\ell_i) = \ell_i = \psi_j(s_i)$. Since x_i is equal to the intersection of s_i and ℓ_j, $\psi_j(x_i)$ is at the intersecion of ℓ_i and ℓ_j. Hence $\psi_j(x_i) = \psi_i(x_j)$. □

All elations with an axis a thus form a group.

Theorem (Baer). *Suppose that c_i, $i \in \{1, 2\}$ are distinct points upon a line a. If the group of (c_i, a)-collineations is nontrivial for both $i \in \{1, 2\}$, then the group of elations of a is commutative. Furthermore, either all nontrivial elements of this group are of infinite order, or they are of the same prime order.*

Proof. Let ψ and ψ' be two nontrivial elations with axis a. If they have distinct centers, then they commute. Suppose that c_1 is a center for both of them. Choose nontrivial elation ψ_2 with center c_2. Then $\psi\psi_2 = \psi_2\psi$ is an elation with center $c_3 \notin \{c_1, c_2\}$. Hence $\psi'\psi\psi_2 = \psi\psi_2\psi' = \psi\psi'\psi_2$, and thus $\psi'\psi = \psi\psi'$.

Suppose now that there exists a nontrivial elation of finite order. Then there exists a nontrivial elation of a prime order p, say ψ_1, with center c_1. Let ψ_2 be an elation with center $c_2 \neq c_1$. Then $\psi_3 = \psi_1\psi_2$ is with center $c_3 \notin \{c_1, c_2\}$. Then $\psi_2^p = \psi_3^p$ is an elation, for which both c_2 and c_3 yield a center. This means that ψ_2^p is the identity. The same argument may be then used to show that every nontrivial elation with center c_1 is of order p too. □

A line a is called a *translation line* if it is (c, a)-transitive for each $c \in a$. Note that this definition may be alternatively expressed by saying that elations with axis a are transitive on all points which are not upon a. (If they are transitive, they are sharply transitive—in other words elations act regularly upon the set of all points that are not incident to a.)

Theorem. *Let a be a line with points $c_1 \neq c_2$. If the plane is (c_i, a)-transitive for both $i \in \{1, 2\}$, then a is a translation line.*

Proof. It suffices to mimick the proof of commutativity. Choose a point c_3 on a that is distinct from both c_1 and c_2. Fix a point $x \notin a$, and a point $y \notin a$, $x \neq y$, such that x, y and c_3 are collinear. Denote by s_i the line connecting x and c_i, and by ℓ_i

the line connecting c_i and y. Furthermore, let x_i be intersection of s_i and ℓ_j, $j \neq i$. Choose a (c_i, a)-collineation ψ such that $\psi_i(x) = x$. Then $\psi_2\psi_1(x) = y = \psi_1\psi_2(x)$. The collineation $\psi = \psi_1\psi_2$ is an elation that maps x upon y. The center of this elation is thus equal to c_3. □

A point c is said to be a *translation point* if the plane is (c, a)-transitive for any line a passing through c. Notions of a translation line and a translation point are dual. Amongs others, this means that a point c is a translation point if and only if the perspectivities with center c are transitive on lines that do not pass through c. Furthermore, we may state:

Dual facts. *Let a_i, $i \in \{1, 2\}$, be distinct lines passing through a point c. If the group of (c, a_i)-collineations is nontrivial for both $i \in \{1, 2\}$, then all elations with center c form a group that is commutative. If the plane is (c, a_i)-transitive for both $i \in \{1, 2\}$, then it is c is a translation point*

Let us state explicitly the following trivial observation:

Lemma. *Let α be a collineation, a a line and c a point. The mapping $\psi \mapsto \alpha\psi\alpha^{-1}$ induces an isomorphism that maps the group of (c, a)-collineations upon the group of $(\alpha(c), \alpha(a))$-collineations.*

Corollary. *Let α be a collineation, a a line and c a point. The plane is (c, a)-transitive if and only if it is $(\alpha(c), \alpha(a))$-transitive. The line a is a translation line if and only if it is $\alpha(a)$ is a translation line. The point c is translation point if and only if $\alpha(c)$ is a translation point.*

A projective plane in which each line is a translation line is called a *Moufang plane.*

Lines which meet in a common point are called *concurrent.*

Proposition. *A projective plane with three noncurrent translation lines is Moufang.*

Proof. Choose two of these lines and consider the point c in which they meet. As proved above (dual facts) this is a translation point. The group of perspectivities with center c is transitive upon the lines that do not pass through c. If one of them a translation line, all of them are translations lines. □

It may be proved that two translations lines suffice for the plane to be Moufang. Furthermore, finite Moufang planes are desarguesian (see below). This means that if a finite plane is not constructed from a finite field in the usual way, then the plane possesses at most one translation line and at most one translation point.

Desarguesian planes. Triples of points (x_1, x_2, x_3) and (y_1, y_2, y_3) are said to be *centrally perspective* if the following holds:

(1) Points x_1, x_2 and x_3 are not collinear.
(2) Points y_1, y_2 and y_3 are not collinear.
(3) $x_i \neq y_i$ if $1 \leq i \leq 3$.
(4) Let s_i be the line that connects x_i and y_i, $1 \leq i \leq 3$. If $j \in \{1, 2, 3\}$ and $j \neq i$, then $x_j \notin s_i$ and $y_j \notin s_i$.
(5) There exists a point c at which the lines s_1, s_2 and s_3 meet.

Triples of points (x_1, x_2, x_3) and (y_1, y_2, y_3) are said to be *axially perspective* if (1)–(4) are true and, in addition, the following holds:

(5') Let m_i be the line that connects x_j and x_k, and n_i the line that connects y_j and y_k, where $\{i, j, k\} = \{1, 2, 3\}$. Then there exists a line a such that m_i and n_i meet on a for each $i \in \{1, 2, 3\}$.

Triples (x_1, x_2, x_3) and (y_1, y_2, y_3) are often called triangles. The notation chosen here emphasis that a certain correspondence between vertices of triangles is assumed.

Let us observe the duality induced by (1)–(4). Under these assumptions it is immediate to see that

(1*) Lines m_1, m_2 and m_3 are not concurrent.
(2*) Lines n_1, n_2 and n_3 are not concurrent.
(3*) $m_i \neq n_i$ if $1 \leq i \leq 3$.
(4*) Let z_i be the point at which m_i and n_i meet, $i \in \{1, 2, 3\}$. If $j \in \{1, 2, 3\}$ and $j \neq i$, then $z_i \notin m_j$ and $z_i \notin n_j$.

It is now clear that (m_1, m_2, m_3) and (n_1, n_2, n_3) are triples of dual points (lines) that fulfil (1)–(4) in the dual projective plane. The dual configuration is centrally perspective if and only if the initial configuration is axially perspective, and the dual configuration is axially perspective if and only if the initial configuration is centrally perspective..

Say that the plane is (c, a)-*desarguesian* if centrally perspective triples with center c are axially perspective whenever a is the line that connects z_2 and z_2.

Proposition. *Let a be a line and c a point. The plane is (c, a)-transitive if and only if it is (c, a)-desarguesian.*

Proof. Suppose first the (c, a)-transitivity. Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be centrally perspective triples with center c such that z_2 and z_3 is upon a. Now, z_3 is the intersection of m_3, which passes through x_1 and x_2, and of n_3, which passes through y_1 and y_2. Let ψ be the (c, a)-collineation which maps x_1 upon y_1. Then ψ maps x_2 upon y_2. Since z_2 of m_2, which goes through x_1 and x_3, and n_2, which goes through y_1 and y_3, there has to be $\psi(x_3) = y_3$ as well. This implies that m_1 is mapped upon n_1. Since ψ fixes the intersection of m_1 and a, this intersection has to be upon n_1. Therefore $z_1 \in a$.

Suppose now that the plane is (c, a)-desarguesian. Choose x_1 and y_1 in such a way that none of them belongs to $a \cup \{c\}$, and that $x_1 \neq y_1$. Our goal is to show that there exists a (c, a)-collineation that sends x_1 upon y_1. To this end define first partial mapping ψ_1 that fixes each point of $a \cup \{c\}$ and is also defined at each $x_2 \notin s_1 \cup a$, where s_1 connects x_1 and c. For such an x_2 consider the line m_3 that connects x_1 and x_2, and denote by z_3 the intersection of m_3 and a. Denote by n_3 the line connecting y_1 and z_3, and set $\psi_1(x_2) = y_2$, where y_2 is the intersection of n_3 and s_2, where s_2 connects x_2 and c.

Suppose now that $x_2 \notin s_1 \cup a$ is fixed. We shall define a partial mapping ψ_2 in a way which mimicks the definition of ψ_1. This means that ψ_2 sends $x_3 \notin s_2 \cup a$ upon point y_3, which is the intersection of s_3 and n_1, where s_3 connects c and x_3, and n_1 connects y_2 and z_1, with z_1 being the intersection of a and m_1, which is the line that connects x_2 and x_3.

The next aim is to show that $\psi_2(x_3) = \psi_1(x_3)$ if $x_3 \notin s_1$. This is clear if $x_3 \in \{c\} \cup a$. Assume $x_3 \notin \{c\} \cup a$. If $x_3 \in m_3$, then $m_1 = m_3$, $z_1 = z_3$, and thus $y_3 = \psi_1(x_3)$ if $x_3 \neq x_1$, and $y_1 = \psi_2(x_1)$. Assume $x_3 \notin m_3$. Then (x_1, x_2, x_3) and (y_1, y_2, y_3) are centrally perspective triples. Hence a carries the point z_2, which is the intersection of m_2 and n_2, where m_2 connects x_1 and x_3, and n_2 connects y_2 and y_3. This means that $\psi_1(x_3) = y_3$.

Since ψ_1 and ψ_2 agree at all points where they are defined, their union determines a mapping ψ that permutes points of the plane. It is clear that the definition of ψ is independent of the choice of $x_2 \notin s_1 \cup a$. It remains to show that ψ is a collineation, i.e., that $\psi(\ell) = \ell$ for any line ℓ. This is obvious if $c \in \ell$ or $\ell = a$. Suppose that none of that is true. If $x_1, x_2 \in \ell$, $x_2 \neq x_1$, then $\ell = m_3$ is mapped upon n_3. If $x_1 \notin \ell$, choose $x_2, x_3 \in \ell \setminus a$, $x_2 \neq x_3$. Then $\ell = m_1$ is mapped upon n_1. $\square$

Corollary. *For a projective plane the following is equivalent:*

(i) *The plane is (c, a)-desarguesian for each point c and line a.*
(ii) *The plane is (c, a)-transitive for each point c and line a.*

Planes which fulfil the condition of the preceding statement are called *desarguesian.*