- we will show that: $\boxed{S_\ell(k) = \frac{f_\ell(-k)}{f_\ell(k)}}$ where $f_\ell(k)$ is the Jost function (don't confuse it with scattering ampl.)
for k complex

- $f_\ell(k)$ has simpler analytic properties so it is convenient for analysis
- we will show that $\boxed{f_\ell(k) = 1 + \frac{1}{k}\int_0^\infty dr'\, \hat{h}_\ell^{(+)}(kr')\, U(r')\, \phi_{\ell,k}(r')}$ → so called regular solution... the significance of this solution is that it is an entire function.
from which analytic properties of $f_\ell(k)$ follow

Regular solution $\phi_{\ell,k}(r)$:

- defined by b.c. at $r=0$: $\boxed{\phi_{\ell,k}(r) \xrightarrow[r\to 0]{} \hat{j}_\ell(kr)}$ i.e. it fixes its normalization too! (i.e. the b.c. is not just $\phi_{\ell k}(r=0)\equiv 0$ which leaves in principle some ~~constant~~ proportionality const. undetermined)
- Poincaré theorem: independence of the b.c. on k guarantees that $\phi_{\ell k}(r)$ is entire as function of k.
- we will prove this ~~later~~ for our special case of radial Schr.-eq.
- reminder that $\phi_{\ell,k}(r)$ solves: $\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - U(r) + k^2\right]\phi_{\ell,k}(r) = 0 \cdots\to \phi_{\ell,k}(r) = \phi_{\ell,-k}(r)$ → even function of k

$\boxed{\text{FOR NOW WE ARE STILL AT REAL } k}$

→ $\phi_{\ell k}(r)$ is real ... b.c. is real and the diff. eq. is also real

→ asymptotically $\phi_{\ell k}(r) \xrightarrow[r\to\infty]{} \frac{i}{2}\left[f_\ell^{(-)}(k)\,\hat{h}_\ell^{(-)}(kr) - f_\ell^{(+)}(k)\,\hat{h}_\ell^{(+)}(kr)\right]$

this must be real ⇒ $\boxed{\phi_{\ell k} = \phi_{\ell k}^*}$

$$\phi_{\ell k}^*(r) \longrightarrow -\frac{i}{2}\left[f_\ell^{(-)*}(k)\,\hat{h}_\ell^{(+)}(kr) - f_\ell^{(+)*}(k)\,\hat{h}_\ell^{(-)}(kr)\right] = \frac{i}{2}\left[f_\ell^{(+)*}(k)\,\hat{h}_\ell^{(-)}(kr) - f_\ell^{(-)*}(k)\,\hat{h}_\ell^{(+)}(kr)\right]$$

$$\Rightarrow \boxed{f_\ell^{(-)}(k) = f_\ell^{(+)*}(k)} \longrightarrow \frac{i}{2}\left[f_\ell^{(-)}(k)\,\hat{h}_\ell^{(-)}(kr) - f_\ell^{(-)*}(k)\,\hat{h}_\ell^{(+)}(kr)\right] \equiv \boxed{\frac{i}{2}\left[f_\ell(k)\,\hat{h}_\ell^{(-)}(kr) - f_\ell^*(k)\,\hat{h}_\ell^{(+)}(kr)\right]}$$

$f_\ell(k)$ is the Jost function ... so far for real k only

- the standard solution $u_{\ell k}(r) \sim \hat{j}_\ell(kr)$ is asymptotically: $u_{\ell,k}(r) \xrightarrow[r\to\infty]{} \frac{i}{2}\left[\hat{h}_\ell^{(-)}(kr) - S_\ell(k)\,\hat{h}_\ell^{(+)}(kr)\right]$ $\left(= e^{i\delta_\ell(k)}\sin(kr - \frac{\ell\pi}{2} + \delta_\ell(k))\right)$

we choose normalization now:

$\int_0^\infty dr'\, u_{\ell k}^*(r)\, u_{\ell k'}(r) = \frac{\pi}{2}\delta(k-k')$ ← irrelevant

$$\phi_{\ell k}(r) = f_\ell(k)\cdot\frac{i}{2}\left[\hat{h}_\ell^{(-)}(kr) - \frac{f_\ell^*(k)}{f_\ell(k)}\,\hat{h}_\ell^{(+)}(kr)\right]$$

→ $\boxed{\phi_{\ell,k}(r) = f_\ell(k)\cdot u_{\ell,k}(r)}$ and $\boxed{S_\ell(k) = \frac{f_\ell^*(k)}{f_\ell(k)}}$ ... still for real k

we know:

$$S_\ell(k) = e^{2i\delta_\ell(k)} = \frac{f_\ell^*(k)}{f_\ell(k)} \Rightarrow \left| f_\ell(k) = |f_\ell(k)|\, e^{-i\varphi(k)} \right| = e^{-i2\varphi(k)} \Rightarrow \varphi_\ell = -\delta_\ell(k) \Rightarrow \boxed{f_\ell(k) = |f_\ell(k)|\, e^{+i\delta_\ell(k)}}$$

- the phase-shift is minus the phase of the Jost function

$\Rightarrow$ knowledge of the Jost function is equivalent to solving of the full problem

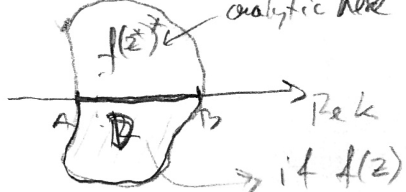
- we have found that $\boxed{S_\ell(k) = \frac{f_\ell^*(k)}{f_\ell(k)} \text{ for real } k\,!}$

- we want to analytically continue this expression to complex k
- assume that $f_\ell(k)$ is analytic (we'll prove this later) $\Rightarrow$ $f_\ell^*(k)$ is not analytic:

analytic function must have a unique derivative: e.g. $f(z) = z \Rightarrow f^*(z) = z^*$,

$$\lim_{z\to z_0} \frac{f^*(z) - f^*(z_0)}{z - z_0} = \frac{z^* - z_0^*}{z - z_0} = \begin{cases} 1 & \text{for } z - z_0 \text{ real} \\ -1 & \text{for } z - z_0 \text{ pure imaginary} \end{cases}$$

However we can make use of Schwarz reflection principle:



$\to$ if $f(z)$ is analytic here (D + real line Re k) and is real on interval AB then $f(z^*)^*$ is analytic on D^*

Therefore we write

$$\boxed{S_\ell(k) = \frac{f_\ell^*(k^*)}{f_\ell(k)} \left(= \frac{f^*(k)}{f_\ell(k)} \text{ for real } k \right)}$$

$\Rightarrow$ $\boxed{\text{to find the region of analyticity of } S_\ell(k) \text{ we need to know the region of analyticity of } f_\ell(k)}$

$\Rightarrow$ we need an equation for $f_\ell(k)$ ... remember that $\phi_{\ell k}(r) \to \frac{i}{2}\left[f_\ell(k)\hat{h}_\ell^{(-)}(kr) - f_\ell^*(k)\hat{h}_\ell^{(+)}(kr) \right]$ so let's study L-S equation for $\phi_{\ell k}(r)$

L-S equation for $\phi_{\ell k}(r)$

- it needs to embed b.c. at $r=0$: $\phi_{\ell k} \xrightarrow{r\to 0} \hat{j}_\ell(kr)$

- we're working in p-w basis so need pw Green's function: starting from

$$G_0^{(+)}(\vec{r},\vec{r}\,') = -(2\pi)^{-3} \lim_{\varepsilon\to 0+} \int \frac{\exp[i\vec{k}\,'\cdot(\vec{r}-\vec{r}\,')]}{(k')^2 - k^2 - i\varepsilon}\, d\vec{k}\,' =$$

expand product of plane waves into two and use pw expansion

$$= G_0^{(+)}(\vec{r},\vec{r}\,') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} G_{\ell,0}^{(+)}(r,r')\, Y_{\ell m}^*(\hat{r}')\, Y_{\ell m}(\hat{r}) \Rightarrow \ldots \boxed{G_{\ell,0}^{(+)}(r,r') = -\frac{1}{k}\hat{j}_\ell(kr_<)\cdot \hat{h}_\ell^{(+)}(kr_>)}$$

$r_< = \min\{r,r'\}$
$r_> = \max\{r,r'\}$

$$\boxed{u_{\ell,k}(r) = \hat{j}_\ell(kr) + \int_0^\infty dr'\, G_{\ell,0}(r,r')\, U(r')\, u_{\ell,k}(r')}$$

3

- we'll rewrite the L-S by splitting the integral $\int_0^\infty dr' = \int_0^r dr' + \int_r^\infty dr'$ (this fixes the form of $G_{\ell,0}$ in each integrand)

$$u_{\ell,k}(r) = \hat{j}_\ell(kr) - \frac{1}{k}\int_0^r dr'\, \hat{j}_\ell(kr')\, \hat{h}_\ell^{(+)}(kr)\, U(r')\cdot u_{\ell,k}(r') - \frac{1}{k}\cdot \underbrace{\int_r^\infty}_{=\int_0^\infty - \int_0^r} dr'\, \hat{j}_\ell(kr)\, \hat{h}_\ell^{(+)}(kr')\, U(r')\, u_{\ell,k}(r') =$$

$$= \underbrace{\left[1 - \frac{1}{k}\int_0^\infty dr'\, \hat{h}_\ell^{(+)}(kr')\, U(r')\, u_{\ell,k}(r')\right]}_{A(k)} \cdot \hat{j}_\ell(kr) + \int_0^r dr' \underbrace{\frac{1}{k}\left[\hat{j}_\ell(kr)\, \hat{h}_\ell^{(+)}(kr') - \hat{j}_\ell(kr')\, \hat{h}_\ell^{(+)}(kr)\right]}_{g_{\ell,k}(r,r')} \cdot U(r')\, u_{\ell,k}(r')$$

$g_{\ell,k}(r,r')$ ... it is not Green's function since $\int dr'$ goes only to $r'=r$!

$$u_{\ell,k}(r) = A(k)\cdot \hat{j}_\ell(kr) + \int_0^r dr'\, g_{\ell,k}(r,r')\, U(r')\, u_{\ell,k}(r')$$

$\hookrightarrow A(k) = 1 - \frac{1}{k}\int_0^\infty dr'\, \hat{h}_\ell^{(+)}(kr')\, U(r')\, u_{\ell,k}(r')$

$\Rightarrow$ we see that the regular solution satisfies

$$\boxed{\phi_{\ell,k}(r) = \hat{j}_\ell(kr) + \int_0^r dr'\, g_{\ell,k}(r,r')\, U(r')\, \phi_{\ell,k}(r')}$$

, indeed for $r \to 0$ $\boxed{\phi_{\ell,k}(r) \to \hat{j}_\ell(kr)}$

we want the Jost function which can be calculated from: $\boxed{\phi_{\ell,k}(r) = f_\ell(k)\cdot u_{\ell,k}(r)}$

$\Rightarrow$ we see that $\boxed{u_{\ell,k}(r) = A\, \phi_{\ell,k}(r)}$ and get: $A = 1 - \frac{1}{k}\int_0^\infty dr'\, \hat{h}_\ell^{(+)}(kr')\, U(r')\, A\, \phi_{\ell,k}(r') \Rightarrow A = \dfrac{1}{1 + \frac{1}{k}\int_0^\infty dr'\, \hat{h}_\ell^{(+)}(kr')\, U(r')\, \phi_{\ell,k}(r')} =$

$= \frac{1}{f_\ell(k)}$

$A = \frac{1}{f_\ell(k)}$

$$\Rightarrow \boxed{f_\ell(k) = 1 + \frac{1}{k}\int_0^\infty dr'\, \hat{h}_\ell^{(+)}(kr')\, U(r')\, \phi_{\ell,k}(r')}$$

explicit equation for Jost function

$\Rightarrow$ we see that analytic properties of $f_\ell(k)$ depend on analytic props. of $\phi_{\ell,k}(r')$

Analytic properties of $\phi_{\ell k}(r)$:

– the integral eq. for $\phi_{\ell k}(r)$ can be solved always by iteration! (similar to Born series... but that is not guaranteed to converge!)

$$\phi_{\ell k}(r) = \sum_n \lambda^n \phi^{(n)}_{\ell k}(r), \quad \phi^{(0)}_{\ell k} = \hat{j}_\ell(kr) \qquad \lambda: \; U(r) \to \lambda U(r)$$

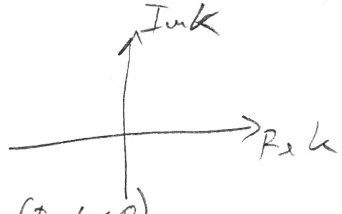
$$\phi^{(n)}_{\ell k}(r) = \int_0 dr'\, g_{\ell k}(r,r')\, U(r')\, \phi^{(n-1)}_{\ell k}(r') \qquad g_{0 k} = \tfrac{1}{k}\sin(k(r-r')) \;\ldots\; \text{for } \ell=0: \; |\sin(kr)| \leq kr$$

it can be shown

$$|\phi_{\ell k}(r)| \leq \left|\sum_n \lambda^n \phi^{(n)}_{\ell k}(r)\right| \leq \sum_n |\phi^{(n-1)}_{\ell k}(r)| \leq B \cdot \left(\frac{|kr|}{1+|kr|}\right)^{\ell+1} \cdot e^{|\mathrm{Im}\, kr|} \cdot \underbrace{\sum_n \frac{(\lambda\alpha)^n}{n!}}_{e^{\alpha}}, \quad \alpha = \int_0^\infty dr\, |U(r) r|$$

$\Rightarrow$ $\phi_{\ell k}(r)$ is entire function of k since the series $\sum_n \frac{(\lambda\alpha)^n}{n!}$ is convergent for all λ and α.

Jost function for complex momenta: (... remember we're interested in $S_\ell(k) = \frac{f_\ell(k^*)^*}{f_\ell(k)}$)



$$f_\ell(k) = 1 + \frac{1}{k}\int_0^\infty dr'\, \hat{h}^{(+)}_\ell(kr')\, U(r')\, \phi_{\ell k}(r')$$

$\Rightarrow$ i.e. what is the region of analyticity of $f_\ell(k)$?

– does this integral converge for $k \in \mathbb{C}$? ... lower k-plane ($\mathrm{Im}\, k < 0$)

$$\hat{h}^{(+)}_\ell(kr') \to e^{i(kr' - \frac{\ell\pi}{2})} \Rightarrow \text{for } k \in \mathbb{C}: \sim e^{-\mathrm{Im}(k) r'} \Rightarrow \text{diverges in } \text{lower } k\text{-plane } (\mathrm{Im}\,k<0)$$

$$\phi_{\ell k}(r') \xrightarrow{r' \to \infty} a\, \hat{h}^{(+)}_\ell(kr') + b \cdot \hat{h}^{(-)}_\ell(kr') \Rightarrow \sim e^{|\mathrm{Im}\,k| r'} \Rightarrow \text{it always diverges}$$

However, in the upper half plane ($\mathrm{Im}\, k > 0$) the divergences cancel ($\hat{h}^{(+)}_\ell \cdot \phi_{\ell k} \sim e^{-\mathrm{Im}\,kr'} e^{+\mathrm{Im}\,kr'} \sim 1$) and the integral converges

in the lower half plane ($\mathrm{Im}\, k < 0$) $\Rightarrow$ $f_\ell(k)$ diverges and is not analytic

$$S_\ell(k) = \frac{f_\ell(k^*)^*}{f_\ell(k)} \overset{\text{Schwarz refl. principle}}{=} \frac{\text{analytic in } \mathrm{Im}\,k<0}{\text{analytic in } \mathrm{Im}\,k>0} \Rightarrow \text{no region of analyticity!}$$

$\Rightarrow$ We need stronger conditions on the potential!

Regions of analyticity of $f_\ell(k)$:

$$f_\ell(k) = 1 + \frac{1}{k}\int^{\infty} dr'\, h_\ell^{(+)}(kr')\, U(r')\, \phi_{\ell k}(r') \rightsquigarrow \underline{\text{divergent}} \text{ in the whole lower complex plane } (\mathrm{Im}\, k < 0)$$

– if we impose more stringent limitations on the potential we can extend this region of analyticity

(1) $\underline{U(r') \text{ is strictly finite range}}$; i.e. $U(r') = 0,\ r' > a$ $\Rightarrow$ $s_\ell(k) = \dfrac{f_\ell^*(k^*)}{f_\ell(k)}$ is also $\underline{\text{entire}}$

$\Rightarrow \int_0^{\infty} dr' [\dots]$ always converges $\Rightarrow$ $\underline{f_\ell(k) \text{ is entire}}$

(2) $U(r')$ falls off exponentially: $U(r') = \mathcal{O}(e^{-\mu r'})$ (e.g. Yukawa $\dfrac{e^{-\mu r}}{r}$)

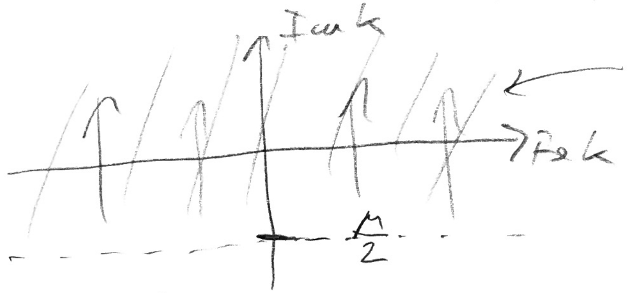
the integrand in the limit $r' \to \infty$: $\underbrace{e^{|\mathrm{Im}\,k| r'}}_{\phi_{\ell k}(r')}\ \underbrace{e^{-\mathrm{Im}(k) r'}}_{h_\ell^{(+)}}\ \underbrace{e^{-\mu r'}}_{U(r')}$

$\Rightarrow$ the integral converges if the combined exponent is negative:

lower complex plane: $|\mathrm{Im}\,k| = -\mathrm{Im}\,k$

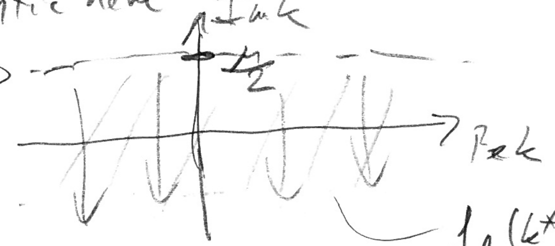
$$|\mathrm{Im}\,k| - \mathrm{Im}\,k - \mu < 0 \Leftrightarrow -2\,\mathrm{Im}(k) - \mu < 0$$

$$\Rightarrow \boxed{\mathrm{Im}(k) > -\frac{\mu}{2}}$$



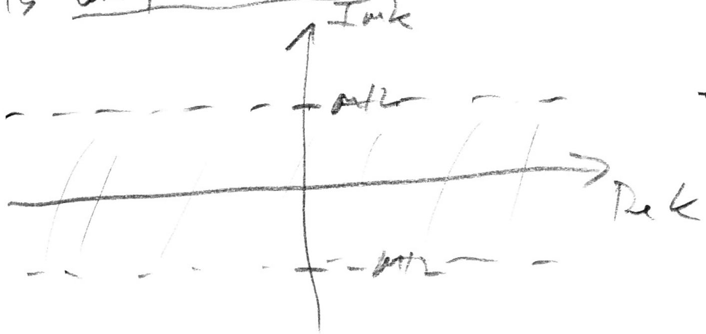
$f_\ell(k)$ is analytic here

Schwarz refl. princ. $\Rightarrow$



$f_\ell(k^*)^*$ is analytic here

$\Rightarrow$ $s_\ell(k) = \dfrac{f_\ell(k^*)^*}{f_\ell(k)}$ is $\underline{\text{analytic in the strip:}}$



$\Rightarrow$ long-range potentials ($\mu \sim 0$) have worse analytic properties

Finally we notice that for real k: $\boxed{f(-k) = f(k)^*}$

Proof: $f_l(k) = 1 + \frac{1}{k}\int_0^\infty dr'\, \hat{h}_l^{(+)}(kr')\, U(r')\, \phi_{l,k}(r')$

it would seem that $\phi_{l,k} = \phi_{l,-k}$ since the diff. eq. dep. only on k^2 but we have the b.c. too

$\nearrow \phi_{l,-k} \xrightarrow{r\to 0} \hat{j}_l(-kr') \sim \frac{(-kr')^{l+1}}{(2l+1)!!} \Rightarrow \phi_{l,-k}(r') = (-1)^{l+1}\phi_{l,k}(r')$

put $k \to -k$: $f_l(-k) = 1 - \frac{1}{k}\int_0^\infty dr'\, \underbrace{\hat{h}_l^{(+)}(-kr')}\, U(r')\, \phi_{l,-k}(r) = 1 + \frac{1}{k}\int_0^\infty dr'\, [\hat{h}_l^{(+)}(kr')]^*\, U(r')\, \phi_{l,k}(r') = f_l^*(k)$

$\sim e^{i(-kr-\frac{l\pi}{2})} = e^{-i(kr-\frac{l\pi}{2})}\underbrace{e^{-i\pi l}}_{(-1)^l} = [\hat{h}_l^{(+)}(kr)]^*(-1)^l$

$\underbrace{f_l(-k)} = f_l^*(k)$

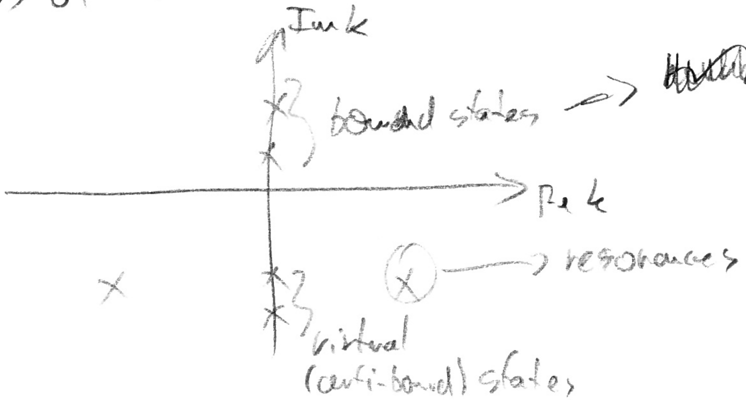
analytic in $\mathrm{Im}\,k < 0$ (due to θk) $\Rightarrow$ we can replace $f_l^*(k)$ by $f_l^*(k^*)$ which is analytic if $f_l(-k)$ is! (... and it is)

$\boxed{f_l(-k) = f_l^*(k^*)}$

$\Rightarrow$ $\boxed{S_l(k) = \frac{f_l^*(k^*)}{f_l(k)} = \frac{f_l(-k)}{f_l(k)}}$

$\Rightarrow$ if $f_l(k)$ has a zero at k_0 then it has a zero at $-k_0^*$ $\left[f_l(\underbrace{-k}_{k_0}) = f_l^*(\underbrace{k^*}_{-k_0^*}) \right]$

$\Rightarrow$ poles of S-matrix come in pairs



bound states $\Rightarrow$ ~~[illegible]~~ must lie here since e^{ikr} for $\mathrm{Im}\,k > 0$ corresponds to a normalizable state $\Rightarrow$ real eigenvalue and H is hermitian ~~[illegible]~~ $\frac{(i\,\mathrm{Im}\,k)^2}{2} = -\frac{|\mathrm{Im}\,k|^2}{2}$

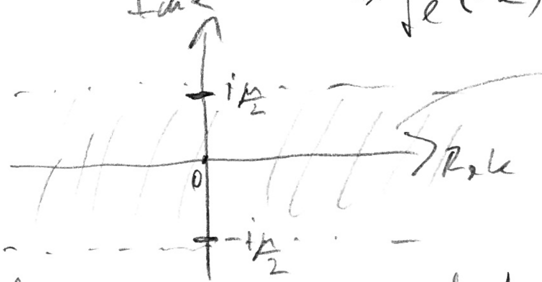
- We have $S_\ell(k) = \dfrac{f_\ell(-k)}{f_\ell(k)}$ $\begin{matrix}\longrightarrow \text{analytic for } \mathrm{Im}\,k<0 \\ \longrightarrow \text{analytic for } \mathrm{Im}\,k>0\end{matrix}$

- if we put more stringent requirements on $U(r)$ we can make $S_\ell(k)$ analytic

(1) Short-range $U(r)$ ... $f_\ell(\pm k)$ is entire and so is $S_\ell(k)$

(2) $U(r) \sim O(e^{-\mu r})$ ... $f_\ell(+k)$ is analytic in $\mathrm{Im}\,k > -\frac{\mu}{2}$

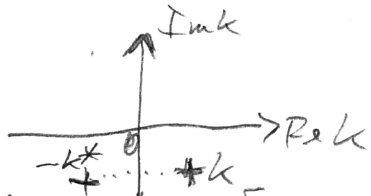
... $f_\ell(-k)$ is analytic in $\mathrm{Im}\,k < \frac{\mu}{2}$



Poles of $S_\ell(k)$ and zeros of $f_\ell(k)$:

- for $f_\ell(k) = 0$ the S-matrix has a pole

- using the property: $f_\ell(-k) = f_\ell(k^*)^* \longrightarrow f_\ell(k) = [f_\ell(-k^*)]^* \Rightarrow$ zeros of $f_\ell(k)$ appear in pairs



Regular solution at the pole:

$$\phi_{\ell k} \xrightarrow{r\to\infty} \underbrace{f_\ell(k)}_{=0} h_\ell^{(-)}(kr) - \underbrace{f_\ell^*(k^*)}_{=f_\ell(-k)} \hat{h}_\ell^{(+)}(kr) \sim e^{ikr} \to \boxed{\text{Siegert state}}$$

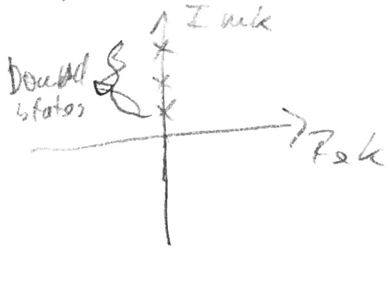
analytic continuation of $f_\ell^*(k)$

(1) Zeros in UHP ($\mathrm{Im}\,k>0$):

in this case $\phi_{\ell k} \sim e^{+i\mathrm{Re}k\cdot r} \cdot e^{-\mathrm{Im}k\cdot r} \Rightarrow$ normalizable wavefunction

$\Rightarrow$ Since $\phi_{\ell k}$ is an eigenstate (solution of Schr. eq.), normalizable and the Hamiltonian is hermitian the eigenvalue ($E = \frac{k^2}{2m}$) must be real $\Rightarrow$ $\mathrm{Re}\,k = 0$ and k must be purely imaginary.

$\Rightarrow$ poles in the upper half plane lie on the imaginary axis and correspond to the bound states!



$\phi = \frac{i}{2}\left[f_\ell(k)\, \hat{h}_\ell^{(-)}(kr) - f_\ell^*(k)\, \hat{h}_\ell^{(+)}(kr) \right]$, $r \to \infty$

if $f_\ell(k) = 0$ for $k \in \mathbb{R} \Rightarrow f_\ell^*(k) = 0 \Rightarrow \phi = 0$ asymptotically

However, we can write ϕ also in terms of Jost solutions $\chi_\pm(kr) \xrightarrow{} h_\ell^{(\pm)}$ ~~$\exp(\pm ikr)$~~, $r \to \infty$ … these have both incoming, outgoing waves at $r \to 0$ (are not regular at origin)

for all $\underline{r}$ we can now write: ($k \in \mathbb{R}$ where $\chi_\pm$ exist and don't diverge)

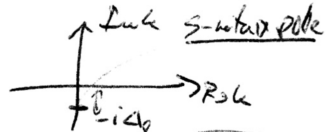
$$\phi = \frac{i}{2}\left[f_\ell(k)\, \chi_-(kr) - f_\ell^*(k)\, \chi_+(kr) \right]$$

$\chi_- \to h_\ell^{(-)}(kr)$, $\chi_+ \to h_\ell^{(+)}(kr)$

$\Rightarrow$ if $f_\ell(k) = 0$ for real k then $f_\ell^*(k) = 0$ too and $\underline{\phi = 0 \text{ everywhere}}$

However $\phi \xrightarrow[r\to 0]{} \hat{j}_\ell(kr) \to \frac{(kr)^{\ell+1}}{(2\ell+1)!!}$ so ϕ cannot be zero except at the origin perhaps !

- we discussed before the low-energy behavior of the solutions (phase-shifts) and considered solutions $\frac{\hat{j}_\ell(kr)}{k^{\ell+1}} \xrightarrow{r\to 0} \frac{r^{\ell+1}}{(2\ell+1)!!}$
- we have seen that $u_\ell \underset{r\to\infty}{\sim} r^{\ell+1} - \frac{\alpha_\ell}{r^\ell}$, where α_ℓ is the scattering length $\Rightarrow$



$\boxed{\alpha_\ell = \frac{1}{\Delta_\ell}}$

for $\underline{\ell > 0}$: $\alpha_\ell \to \infty$ $(\Delta \to 0)$ $\Leftrightarrow$ $u_\ell \sim r^{-\ell}$ … normalizable $\Rightarrow$ there can be bound states at threshold

for $\underline{\ell = 0}$: $\alpha_0 \to \infty$ $(\Delta \to 0)$ $\Leftrightarrow$ $u_\ell \sim r$ $\Rightarrow$ not normalizable $\Rightarrow$ no bound states at threshold

$$\boxed{\begin{aligned} n_\ell \pi &= \delta_\ell(0) - \delta_\ell(\infty) \\ (n_0 + \tfrac{1}{2})\pi &= \delta_0(0) - \delta_0(\infty) \end{aligned}}$$

$\to$ we will prove the case without zeros at threshold: $\boxed{n_\ell \pi = \delta_\ell(0) - \delta_\ell(\infty)}$

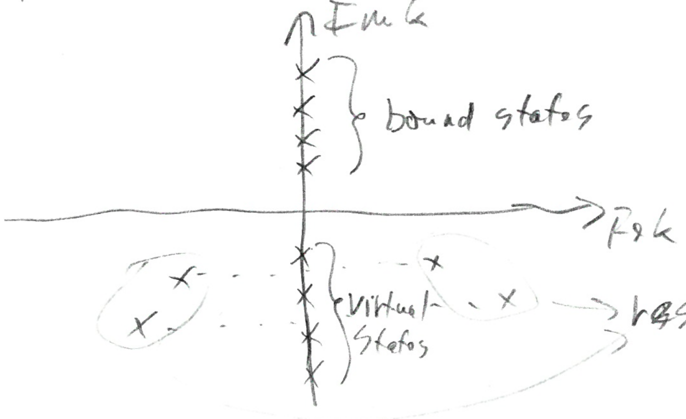
$f_\ell(k) \xrightarrow[|k|\to\infty]{} 1$ … from bounds on $h_\ell^{(\pm)}\phi$
$(\mathrm{Im}\, k \to 0)$

(1)

② Zeros in the lower half-plane

$e^{ikr} \sim e^{i\,\mathrm{Re}k\,r}\, e^{|\mathrm{Im}k|\cdot r}$ → diverges, i.e. not normalizable so eigenvalue does not have to be real

⇒ poles can lie on the imaginary axis and off-axis:



→ resonances (poles appear in pairs due to $f_l(k) = f_l(-k^*)^*$)

Virtual states: $\mathrm{Re}k = 0$, $\mathrm{Im}k < 0$

Resonances: $\mathrm{Re}k \neq 0$, $\mathrm{Im}k < 0$

We have shown that on the real axis $S_l(k)$ is unitary ($S = e^{2i\delta(k)}$) so we can write:

$$S_l(k) = e^{2i\delta_l(k)} = e^{2i(\delta_0(k) + \delta_{res}(k))} = e^{2i\delta_0(k)} \cdot \underbrace{\frac{E - E_r - \frac{1}{2}i\Gamma}{E - E_r + \frac{1}{2}i\Gamma}}_{\text{unitary and has a pole}},$$

slowly varying background

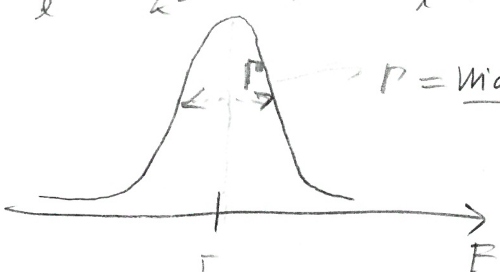
Where $E = \frac{k^2}{2m} = \underbrace{E_r}_{>0} - \frac{1}{2}i\underbrace{\Gamma}_{>0}$

assuming that $\delta_0 = 0$:

$e^{2i\delta_r} = \frac{E - E_r - \frac{1}{2}i\Gamma}{E - E_r + \frac{1}{2}i\Gamma}$ $\sigma = \frac{\pi}{k^2}|e^{2i\delta_l(k)} - 1|^2$

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l(k) = \frac{\pi}{k^2}(2l+1)\frac{\frac{1}{4}\Gamma^2}{(E-E_r)^2 + \frac{1}{4}\Gamma^2}$$ Breit-Wigner form

⇒



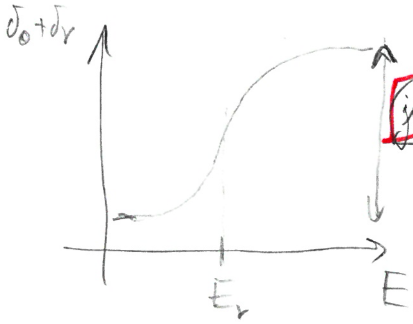
Γ = width of the peak at half-maximum

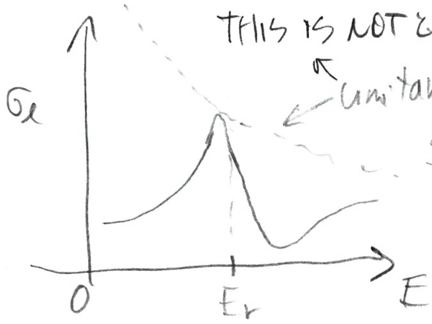
$\frac{(\ )\cdot[\ ]^*}{[\ ]\cdot[\ ]^*}$ and separate the real and imag. parts

— similarly equating: $e^{2i\delta_r} = \frac{E - E_r - \frac{1}{2}i\Gamma}{E - E_r + \frac{1}{2}i\Gamma}$ ⇒ we get

Breit-Wigner form

$$\boxed{\delta_r = \mathrm{Arctan}\left[\frac{\frac{1}{2}\Gamma}{E_r - E}\right]}$$





THIS IS NOT CONSISTENT WITH THE FORMULA ABOVE ?

unitarity limit (touched when $E = E_r$)

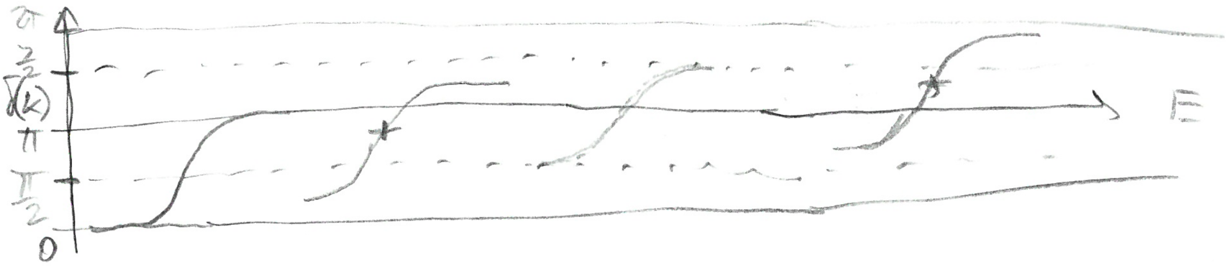
$\sin^2\delta_l = 1$

$\sigma_l = \frac{4\pi}{k^2}(2l+1)$

– shapes of res cross-sections depends on δ_0:

$\delta = \delta_0 + \delta_{res}$

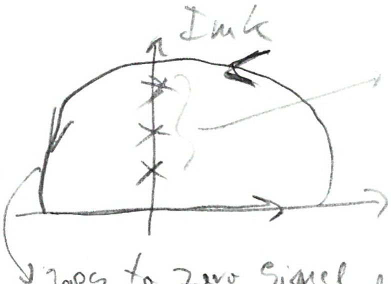
interference of scattering with/without resonance formation (we observe only asymptotic states)



<u>Levinson's theorem</u>: $\delta_\ell(0) - \delta_\ell(\infty) = \pi n_\ell$, where n_ℓ is the number of bound states with ang. mom. ℓ

<u>Proof</u>:

$I = \oint dk \frac{f_\ell'(k)}{f_\ell(k)}$ on the contour:



$f_\ell = 1 + \int \ldots$, $k \to \infty$

$f_\ell(k) \xrightarrow{k\to\infty} 1 \Rightarrow$

$f_\ell'(k) \to 0$

$I = 2\pi i\, n_\ell \Rightarrow$ number of bound states

On the real axis: $\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_0^\infty = \int_0^\infty (\ldots) - (\ldots)$

$$I = \int_{-\infty}^{\infty} dk \frac{f_\ell'(k)}{f_\ell(k)} = \int_0^\infty dk \left(\frac{f_\ell'(k)}{f_\ell(k)} - \frac{f_\ell'(-k)}{f_\ell(-k)} \right) = \ln f_\ell(+\infty) - \ln f_\ell(0+) +$$

$$+ \ln f_\ell(0) - \ln f_\ell(-\infty) = \left| f_\ell(k) = |f_\ell(k)| e^{-i\delta_\ell(k)},\ f_\ell(-k) = f_\ell(k)^* = |f_\ell| \cdot e^{+i\delta(k)} \right| =$$

$$= 2i\left(\delta_\ell(0) - \delta_\ell(\infty)\right)$$

$$\Rightarrow I/2i = \boxed{\delta_\ell(0) - \delta_\ell(\infty) = \pi \cdot n_\ell}$$

– in case of <u>s-wave bound states</u> at $k=0$ we have to add $\frac{\pi}{2} \Rightarrow$

[there are no b.s. at $k=0$ for $\ell \ne 0$]

$$\delta_\ell(0) - \delta_\ell(\infty) = \pi\left(n_\ell + \frac{1}{2}\right)$$

$\hookrightarrow$ number of b.s. <u>not at zero energy</u>

Riemann surface:

– we want to use energy: $k = \sqrt{2mE} = 2m\,|E|^{1/2} \cdot \exp\left[i\frac{\varphi}{2}\right]$

$\Rightarrow$ the momentum becomes double-valued function

(one revolution around origin brings me to the same point

but $k \to -k$ (for initial $k \in \mathbb{R}$) !

$\Rightarrow$ Solution: we define $k(E)$ on the Riemann surface

Riemann-surface of Energy:

two sheets: Physical sheet: $\mathrm{Im}\,k > 0$ $\left(\varphi_E \in [0, 2\pi] \to E = \frac{k^2}{2m}\right)$

– bound states are here

unphysical sheet: virtual states, resonances $\mathrm{Im}\,k < 0$

$\left(\varphi_E \in [2\pi, 4\pi]\right)$

thanks to $E = \frac{k^2}{2m}$ one half-plane in momentum is mapped to a full plane in Energy!