NMAI057 – Linear algebra 1

Tutorial 5

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Problem 1. Establish sufficient conditions for a triangular matrix to be regular.

(Recall that an upper triangular matrix A has arbitrary values on and above the main diagonal, but it is all zero below the diagonal. Formally, for all $i > j$ it holds that $a_{ij} = 0$. Any lower triangular matrix A must satisfy the same condition in the reverse order w.r.t. the main diagonal or, in another words, A^T must be upper triangular.)

Solution:
An upper triangular matrix is almost in a row echelon form. If all the elements on its diagonal are non-zero then they are the pivots and the matrix is regular. If one of the elements on its diagonal is equal to zero then the corresponding column does not contain the pivot and, thus, the matrix is singular.

The situation is similar with respect to lower triangular matrices. That is, a lower triangular matrix is regular if and only if its diagonal does not contain a zero. The justification is straightforward – note that its transpose is an upper triangular matrix and that the transpose operation does not change regularity of a matrix.

Problem 2. Consider the block matrix

$$A = \begin{pmatrix} \alpha & a^T \\ b & C \end{pmatrix},$$

where $\alpha \neq 0$, $a, b \in \mathbb{R}^{n-1}$ and $C \in \mathbb{R}^{(n-1)\times(n-1)}$. Apply to A one iteration of Gaussian elimination and use it to derive a recursive test of regularity.

Solution:
We subtract the $\frac{1}{\alpha}b$-multiple of the first row from the second "block-row" in order to eliminate all the non-zero elements in the first column below the first pivot α. We get

$$\begin{pmatrix} \alpha & a^T \\ b - \alpha\frac{1}{\alpha}b & C - \frac{1}{\alpha}ba^T \end{pmatrix} = \begin{pmatrix} \alpha & a^T \\ o & C - \frac{1}{\alpha}ba^T \end{pmatrix},$$

which is the result of the first step of Gaussian elimination performed on matrix A. As the pivot is non-zero, we can deduce that the matrix A is regular if and only if the matrix $C - \frac{1}{\alpha}ba^T$ is regular. Therefore, we have reduced the problem of deciding whether a square matrix of order n is regular to the problem of deciding the same w.r.t. a square matrix of order $n-1$. And we can recursively proceed with the reduction further.

Problem 3. Find the inverse to the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 10 \end{pmatrix}.$$

Solution:

Using the elementary row operations, we transform the block matrix $(A \mid I_3)$ to its reduced row echelon form:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 3 & 5 & 10 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 3 & 5 & 10 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -0.5 & -0.5 & 0.5 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2.5 & 2.5 & -0.5 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -0.5 & -0.5 & 0.5 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2.5 & 2.5 & -0.5 \\ 0 & 1 & 0 & 2.5 & -0.5 & -0.5 \\ 0 & 0 & 1 & -0.5 & -0.5 & 0.5 \end{array}\right).$$

We get that $A^{-1} = \frac{1}{2}\begin{pmatrix} -5 & 5 & -1 \\ 5 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$.

Problem 4. Invert the matrices of the elementary row operations.

Recall that the matrices representing the elementary row operations are:

(a) Multiplying the i-th row with $\alpha \neq 0$:

$$E_i(\alpha) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \alpha & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

(b) Adding the α-multiple of the j-th row to the i-th row for $i \neq j$:

$$E_{ij}(\alpha) = \begin{array}{c} \\ \\ \\ i \\ \\ \end{array} \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ & \ddots & \ddots & & \vdots \\ & & 1 & \ddots & \vdots \\ & \alpha & & \ddots & 0 \\ & & & & 1 \end{pmatrix}, \quad \text{column } j \text{ under } \alpha$$

(c) Swapping the i-th and j-th row:

$$E_{ij} = \begin{array}{c} i \\ \\ j \end{array} \begin{pmatrix} 0 & 1 \\ & \\ 1 & 0 \end{pmatrix}.$$

Solution:
There are at least two alternative approaches.

1) The first approach is to use the general method for inverting matrices by finding the RREF of an appropriate block matrix. For the first elementary row operation, we get:

$$
\begin{aligned}
(E_i(\alpha) \mid I_n) &= \left(\begin{array}{ccccc|ccccc}
1 & 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & & \vdots & 0 & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \alpha & \ddots & \vdots & \vdots & \ddots & 1 & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 & \vdots & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & 1
\end{array}\right) \sim \\
&\sim \left(\begin{array}{ccccc|ccccc}
1 & 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & & \vdots & 0 & \ddots & \ddots & & \vdots \\
\vdots & \ddots & 1 & \ddots & \vdots & \vdots & \ddots & 1/\alpha & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 & \vdots & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & 1
\end{array}\right) = (I_n \mid E_i(\alpha^{-1})).
\end{aligned}
$$

For the second elementary row operation, we get:

$$
\begin{aligned}
(E_{ij}(\alpha) \mid I_n) &= \left(\begin{array}{ccccc|ccccc}
1 & 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\
 & \ddots & \ddots & & \vdots & 0 & \ddots & \ddots & & \vdots \\
 & & 1 & \ddots & \vdots & \vdots & \ddots & 1 & \ddots & \vdots \\
 & \alpha & & \ddots & 0 & \vdots & & \ddots & \ddots & 0 \\
 & & & & 1 & 0 & \cdots & \cdots & 0 & 1
\end{array}\right) \sim \\
&\sim \left(\begin{array}{ccccc|ccccc}
1 & 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & & \vdots & & \ddots & \ddots & & \vdots \\
\vdots & \ddots & 1 & \ddots & \vdots & & & 1 & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 & & -\alpha & & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1 & & & & & 1
\end{array}\right) = (I_n \mid E_{ij}(-\alpha)).
\end{aligned}
$$

For the third elementary row operation, we get:

$$
\begin{aligned}
(E_{ij} \mid I_n) &= \left(\begin{array}{cc|cc}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{array}\right) \sim \\
&\sim \left(\begin{array}{cc|cc}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{array}\right) = (I_n \mid E_{ij}).
\end{aligned}
$$

Therefore, $E_i(\alpha)^{-1} = E_i(\alpha^{-1})$, $E_{ij}(\alpha)^{-1} = E_{ij}(-\alpha)$ and $E_{ij}^{-1} = E_{ij}$.

2) The second approach exploits our understanding of the matrices representing elementary row operations. The first matrix $E_i(\alpha)$ multiplies the i-th row with some

$\alpha \neq 0$. The inverse operation is dividing the i-th row with α, which can be represented by the matrix $E_i(\alpha^{-1})$. We can verify our solution by checking that, indeed, $E_i(\alpha)E_i(\alpha^{-1}) = I$.

The second matrix $E_{ij}(\alpha)$ adds an α-multiple of the j-th row to the i-th row. The inverse operation is subtracting the α-multiple of the j-th row from the i-th row, which can be represented by the matrix $E_{ij}(-\alpha)$. Again, we can verify that we found the inverse by multiplying the two matrices.

The third matrix E_{ij} swaps the i-th row and the j-th row. The inverse operation is identical, i.e., spapping the i-th row and the j-th row. Thus the matrix E_{ij} is self-inverse.

Problem 5. For $n \in \mathbb{N}$, invert the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}.$$

Solution:
Following the general method, we construct the corresponding block matrix

$$(A \mid I_n) = \left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 2 & 2 & \dots & 2 & 0 & \ddots & \ddots & & \vdots \\ 1 & 2 & 3 & \dots & 3 & \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 1 & 2 & 3 & \dots & n & 0 & \dots & \dots & 0 & 1 \end{array}\right).$$

We subtract the first row from all the remaining rows and we get

$$\left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & -1 & 1 & \ddots & & \vdots \\ 0 & 1 & 2 & \dots & 2 & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 1 & 2 & \dots & n-1 & -1 & 0 & \dots & 0 & 1 \end{array}\right).$$

Notice that in the left block we obtained a matrix of the same structure as A just of smaller order. Thus, we can inductively proceed and after additional $n-2$ steps, we get

$$\left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & -1 & 1 & \ddots & & \vdots \\ 0 & \ddots & 1 & \dots & 1 & 0 & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 & 0 & \dots & 0 & -1 & 1 \end{array}\right).$$

Finally, we subtract the second row from the first and then subtract the third from the second and so on. We get the RREF of the original matrix with the inverse A^{-1} in the right block

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \dots & 0 & 2 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & 1 & & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & \dots & 0 & 1 & 0 & \dots & 0 & -1 & 1 \end{array}\right).$$

Problem 6. Simplify the following expression assuming A, B are regular matrices of the same order:

$$(I - B^T A^{-1})A + (A^T B)^T A^{-1}.$$

Solution:
We use the basic properties of matrix multiplication, transpose, and inverse. We get

$$\begin{aligned}
&(I - B^T A^{-1})A + (A^T B)^T A^{-1} \\
&\qquad = IA - B^T A^{-1} A + (A^T B)^T A^{-1} && \text{[distributivity]} \\
&\qquad = IA - B^T I + (A^T B)^T A^{-1} && \text{[definition of matrix inverse]} \\
&\qquad = A - B^T + (A^T B)^T A^{-1} && \text{[multiplication by } I\text{]} \\
&\qquad = A - B^T + B^T A A^{-1} && \text{[transpose of a product of matrices]} \\
&\qquad = A - B^T + B^T && \text{[definition of matrix inverse]} \\
&\qquad = A\ .
\end{aligned}$$

Thus, the expression simplifies to A.

Problem 7.

(a) Prove that for all $A, B \in \mathbb{R}^{n\times n}$, if A is regular then

$$(ABA^{-1})^k = AB^k A^{-1}.$$

(b) Let $A \in \mathbb{R}^{n\times n}$ be a regular matrix. Find the limit (in case you are not familiar with the formal definition, use the intuitive notion) for

$$\lim_{k\to\infty} AD^k A^{-1}, \quad \text{where} \quad D = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{n} \end{pmatrix},$$

and compute its rank.

(c) Apply the above result to compute the limit for any matrix A with the first column equal $e_1 = (1, 0, \dots, 0)^T$ and the first row equal $e_1^T = (1, 0, \dots, 0)$.

Solution:

(a) We proceed by induction. For $k = 1$, the statement holds since $(ABA^{-1})^1 = AB^1 A^{-1}$.

Now, we show the inductive step. Suppose the statement holds for $k-1$. Therefore, $(ABA^{-1})^{k-1} = AB^{k-1}A^{-1}$, and we can use it to compute

$$\begin{aligned}
(ABA^{-1})^k &= (ABA^{-1})^{k-1}(ABA^{-1}) = (AB^{k-1}A^{-1})(ABA^{-1}) \\
&= AB^{k-1}(A^{-1}A)BA^{-1} = AB^{k-1}BA^{-1} \\
&= AB^kA^{-1}.
\end{aligned}$$

(b) By the previous part of the problem, we have

$$\begin{aligned}
AD^kA^{-1} &= A \begin{pmatrix} 1^k & 0 & \dots & 0 \\ 0 & \frac{1}{2^k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{n^k} \end{pmatrix} A^{-1} \\
&\xrightarrow[k\to\infty]{} A \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix} A^{-1} = A_{*1}(A^{-1})_{1*}\ .
\end{aligned}$$

The matrix is of rank 1 since it is outer product of two vectors.

(c) If $A_{*1} = e_1$ and $A_{1*} = e_1^T$ then the same holds also for the inverse matrix. Specifically, $(A^{-1})_{1*} = e_1^T$. Thus,

$$\lim_{k\to\infty} AD^kA^{-1} = A_{*1}(A^{-1})_{1*} = e_1e_1^T = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}\ .$$