On the foundations of logic and arithmetic

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(*1904*)

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In the last years of the nineteenth century Hilbert provided a satisfactory axiomatization of geometry (*1899*). He then (*1900*) offered a set of axioms for the real numbers and indicated that the question of the consistency of geometry comes down to that of the real-number system. At the Paris International Congress of Mathematicians in 1900, as a natural continuation of this work, he placed the consistency of the real-number system on a list of problems challenging the mathematical world (*1900a*, pp. 264–266). He did not outline any approach, simply stressing that a relative consistency proof seemed out of the question and that, therefore, the problem presented a fundamental difficulty.

Meanwhile the Russell paradox became known, and the question of consistency became more pressing. In 1904, in the paper below, Hilbert presents a first attempt at proving the consistency of arithmetic. In fact, his plan—to show that all the formulas of a certain class possess a certain property (that of being "homogeneous") by showing that the initial formulas have it and the rules transmit it—is the prototype of a device now current in investigations of that nature. Besides the search for a consistency proof the paper offers a critique of the various points of view held at that time on the foundations of arithmetic and introduces the themes that Hilbert is going to develop, modify, or make more precise in his further work in the foundations of mathematics: the reduction of mathematics to a collection of formulas, the extralogical existence of basic objects, like 1, and their combinations, and the construction of logic in parallel with the study of these combinations.

The presentation remains tentative and sketchy. Only many years later (*1917*) will Hilbert come back to the problems of the foundations of mathematics and then present the mature and enriched papers of the twenties (*1922*, *1922a*, *1925*, *1927*). The 1904 paper provides a helpful landmark in the development of Hilbert's conceptions.

The paper was commented upon by Poincaré (*1905*, pp. 17–27; *1908*, pp. 179–191) and Pieri (*1906*). Later commentaries can be found in Bernays (*1935*, pp. 199–200) and Blumenthal (*1935*, p. 422). The paper greatly influenced Julius König's book (*1914*), which in turn inspired von Neumann in his search for a consistency proof of arithmetic (*1927*, footnote 8, p. 22).

An English translation of Hilbert's paper was published (*1905*) in *The monist*, but we have not found it possible to use it. The present translation is by Beverly Woodward, and it is printed here with the kind permission of B. G. Teubner Verlagsgesellschaft, Stuttgart.

While we are essentially in agreement today as to the paths to be taken and the goals to be sought when we are engaged in research into the foundations of geometry, the situation is quite different with regard to the inquiry into the foundations of arithmetic; here investigators still hold a wide variety of sharply conflicting opinions.

In fact, some of the difficulties in the foundations of arithmetic are different in nature from those that had to be overcome when the foundations of geometry were established. In examining the foundations of geometry it was possible for us to leave aside certain difficulties of a purely arithmetic nature; but recourse to another fundamental discipline does not seem to be allowed when the foundations of arithmetic are at issue. The principal difficulties that we encounter when providing a foundation for arithmetic will be brought out most clearly if I submit the points of view of several investigators to a brief critical discussion.

L. Kronecker, as is well known, saw in the notion of the integer the real foundation of arithmetic; he came up with the idea that the integer—and, in fact, the integer as a general notion (parameter value)—is directly and immediately given; this prevented him from recognizing that the notion of integer must and can have a foundation. I would call him a *dogmatist*, to the extent that he accepts the integer with its essential properties as a dogma and does not look further back.

H. Helmholtz represents the standpoint of the *empiricist*; the standpoint of pure experience, however, seems to me to be refuted by the objection that the existence, possible or actual, of an arbitrarily large number can never be derived from experience, that is, through experiment. For even though the number of things that are objects of our experience is large, it still lies below a finite bound.

E. B. Christoffel and all those opponents of Kronecker's who, guided by the correct feeling that without the notion of irrational number the whole of analysis would be condemned to sterility, attempt to save the existence of the irrational number by discovering "positive" properties of this notion or by similar means, I would call *opportunists*. In my opinion they have not succeeded in giving a pertinent refutation of Kronecker's conception.

Among the scholars who have probed more deeply into the essence of the integer I mention the following.

G. Frege sets himself the task of founding the laws of arithmetic by the devices of *logic*, taken in the traditional sense. He has the merit of having correctly recognized the essential properties of the notion of integer as well as the significance of inference by mathematical induction. But, true to his plan, he accepts among other things the fundamental principle that a concept (a set) is defined and immediately usable if only it is determined for every object whether the object is subsumed under the concept or not, and here he imposes no restriction on the notion "every"; he thus exposes himself to precisely the set-theoretic paradoxes that are contained, for example, in the notion of the set of all sets and that show, it seems to me, that the conceptions and means of investigation prevalent in logic, taken in the traditional sense, do not measure up to the rigorous demands that set theory imposes. *Rather, from the very beginning a major goal of the investigations into the notion of number should be to avoid such contradictions and to clarify these paradoxes.*

R. Dedekind clearly recognized the mathematical difficulties encountered when a foundation is sought for the notion of number; for the first time he offered a construction of the theory of integers, and in fact an extremely sagacious one. However, I would call his method *transcendental* insofar as in proving the existence of the infinite he follows a method that, though its fundamental idea is used in a similar way by philosophers, I cannot recognize as practicable or secure because it employs the notion of the totality of all objects, which involves an unavoidable contradiction.

G. Cantor sensed the contradictions just mentioned and expressed this awareness by differentiating between "consistent" and "inconsistent" sets. But, since in my opinion he does not provide a precise criterion for this distinction, I must characterize his conception on this point as one that still leaves latitude for *subjective* judgment and therefore affords no objective certainty.

It is my opinion that all the difficulties touched upon can be overcome and that we can provide a rigorous and completely satisfying foundation for the notion of number, and in fact by a method that I would call *axiomatic* and whose fundamental idea I wish to develop briefly in what follows.

Arithmetic is often considered to be a part of logic, and the traditional fundamental logical notions are usually presupposed when it is a question of establishing a foundation for arithmetic. If we observe attentively, however, we realize that in the traditional exposition of the laws of logic certain fundamental arithmetic notions are already used, for example, the notion of set and, to some extent, also that of number. Thus we find ourselves turning in a circle, and that is why a partly simultaneous development of the laws of logic and of arithmetic is required if paradoxes are to be avoided.

In the brief space of an address I can merely indicate how I conceive of this common construction. I beg to be excused, therefore, if I succeed only in giving you an approximate idea of the direction my researches are taking. In addition, to make myself more easily understood, I shall make more use of ordinary language "in words" and of the laws of logic indirectly expressed in it than would be desirable in an exact construction.

Let an object of our thought be called a *thought-object* ⟦*Gedankending*⟧ or, briefly, an *object* ⟦*Ding*⟧ and let it be denoted by a sign.

We take as a basis of our considerations a first thought-object, 1 (one). We call what we obtain by putting together two, three, or more occurrences of this object, for example,

$$11,\ 111,\ 1111,$$

combinations ⟦*Kombinationen*⟧ of the object 1 with itself; also, any combinations of these combinations, such as

$$(1)(11),\ (11)(11)(11),\ ((11)(11))(11),\ ((111)(1))(1),$$

are again called combinations of the object 1 with itself. The combinations likewise are just called objects and then, to distinguish it, the basic thought-object 1 is called a *simple* object.

We now add a second simple thought-object and denote it by the sign = (equals). Then we form combinations of these two thought-objects, for example,

$$1{=},\ 11{=},\ldots,\ (1)({=}1)({=}{=}{=}),\quad ((11)(1)({=}))({=}{=}),\quad 1{=}1,\quad (11){=}(1)(1).$$

We say that the combination a of the simple objects 1 and = *differs* from the combination b of these objects if the combinations deviate in any way from each other with regard to the mode and order of succession in the combinations or the choice and place of the objects 1 and = themselves, that is, if a and b are not *identical* with each other.

Now we think of the combinations of these two simple objects as falling into two classes, the *class of entities* ⟦die *Klasse der Seienden*⟧ and that *of nonentities* ⟦die *der Nichtseienden*⟧: each object belonging to the class of entities differs from each object belonging to the class of nonentities. Every combination of the two simple objects 1 and = belongs to one of these two classes.

If a is a combination of the two objects 1 and = taken as primitive, then we denote also by a the *proposition* that a belongs to the class of entities and by $\bar{a}$ the *proposition* that a belongs to the class of nonentities. We call a a *true* proposition if a belongs to the class of entities; on the other hand, let $\bar{a}$ be called a *true* proposition if a belongs to the class of nonentities. The propositions a and $\bar{a}$ form a *contradiction*.

The composite ⟦Inbegriff⟧ of two propositions A and B, expressed in signs by

$$A \mid B,$$

and in words by "from A, B follows" or "if A is true, so is B", is also called a proposition; here A is called the *supposition* ⟦*Voraussetzung*⟧ and B the *assertion* ⟦*Behauptung*⟧. Supposition and assertion may themselves in turn consist of several propositions A_1, A_2, or B_1, B_2, B_3, and so forth, and we have in signs

$$A_1 \text{ a. } A_2 \mid B_1 \text{ o. } B_2 \text{ o. } B_3,$$

in words "from A_1 and A_2, B_1 or B_2 or B_3 follows", and so forth.

With the sign o. (or) at our disposal it would be possible to avoid the sign |, since negation has already been introduced; I use it in this address merely in order to follow ordinary language as closely as possible.

We shall understand by $A_1, A_2, \ldots$ the propositions that, briefly stated, result from a proposition $A(x)$ if we take the thought-objects 1 and = and their combinations in place of the "*arbitrary object*" ⟦der "*Willkürlichen*"⟧ x; then we write the propositions A_1 o. A_2 o. $A_3 \ldots$ and A_1 a. A_2 a. $A_3 \ldots$ also as follows: $A(x^{(o)})$, in words "for at least one x", and $A(x^{(a)})$, in words "for every x", respectively; we regard this merely as an abbreviated way of writing.

From the two objects 1 and = taken as primitive we now form the following propositions:

1. $x = x$,
2. $\{x = y \text{ a. } w(x)\} \mid w(y)$.

Here x (in the sense of $x^{(a)}$) means each of the two thought-objects taken as primitive and every combination of them; in 2, y (in the sense of $y^{(a)}$) likewise can be each of these objects and every combination; further, $w(x)$ is an "arbitrary" combination containing the "arbitrary object" x (in the sense of $x^{(a)}$). Proposition 2 reads in words "from $x = y$ and $w(x)$, $w(y)$ follows".

Propositions 1 and 2 form the *definition of the notion* = (equals) and accordingly are also called *axioms*.

If we put the simple objects 1 and = or particular combinations of them in place of the arbitrary objects x and y in Axioms 1 and 2, particular propositions result, which may be called *consequences* [[*Folgerungen*]] of these axioms. We consider a sequence of certain consequences such that the suppositions of the last consequence of the sequence are identical with the assertions of the preceding consequences. If we then take the suppositions of the preceding consequences as supposition, and the assertion of the last consequence as assertion, a new proposition results, which can in turn be called a *consequence* of the axioms. By continuing this deduction process we can obtain further consequences.

We now select from these consequences those that have the simple form of the proposition a (assertion without supposition), and we gather the objects a thus obtained into the class of entities, whereas the objects that differ from these are to belong to the class of nonentities. We recognize that only consequences of the form $\alpha = \alpha$ result from 1 and 2, where α is a combination of the objects 1 and $=$. Axioms 1 and 2 for their part, too, are satisfied with regard to this partition of the objects into the two classes, that is, they are true propositions, and because of this property of Axioms 1 and 2 we say that the notion $=$ (equals) defined by them is a *consistent* notion.

I would like to call attention to the fact that Axioms 1 and 2 do not contain any proposition of the form $\bar{a}$ at all, that is, a proposition according to which a combination is to be found in the class of nonentities. Therefore, we could also satisfy the axioms by including all the combinations of the two simple objects in the class of entities and leaving the class of nonentities empty. But the partition, chosen above, into two classes shows better how we must proceed in subsequent, more difficult, cases.

We now carry the construction of the logical foundations of mathematical thought further by adjoining to the two thought-objects 1 and $=$ the three additional thought-objects $\mathfrak{u}$ (infinite set, infinite [[Unendlich]]), $\mathfrak{f}$ (following [[Folgendes]]), $\mathfrak{f}'$ (accompanying operation [[begleitende Operation]]) and stipulating for them the following axioms:

3. $\mathfrak{f}(\mathfrak{u}x) = \mathfrak{u}(\mathfrak{f}'x)$,
4. $\mathfrak{f}(\mathfrak{u}x) = \mathfrak{f}(\mathfrak{u}y) \,|\, \mathfrak{u}x = \mathfrak{u}y$,
5. $\overline{\mathfrak{f}(\mathfrak{u}x) = \mathfrak{u}1}$.

Here the arbitrary object x (in the sense of $x^{(a)}$) stands for each of the five thought-objects now taken as primitive and every combination of them. The thought-object $\mathfrak{u}$ will be called, simply, *infinite set* and the combination $\mathfrak{u}x$ (for example, $\mathfrak{u}1$, $\mathfrak{u}(11)$, $\mathfrak{u}\mathfrak{f}$) an *element* of this infinite set $\mathfrak{u}$. Axiom 3 then states that each element $\mathfrak{u}x$ is followed by a definite thought-object $\mathfrak{f}(\mathfrak{u}x)$, which is equal to an element of the set $\mathfrak{u}$, namely, the element $\mathfrak{u}(\mathfrak{f}'x)$, that is, which likewise belongs to the set $\mathfrak{u}$. Axiom 4 expresses the fact that, if the same element follows two elements of the set $\mathfrak{u}$, these two elements themselves are equal. According to Axiom 5 there is no element in $\mathfrak{u}$ that is followed by the element $\mathfrak{u}1$; the element $\mathfrak{u}1$ may therefore be called the first element in $\mathfrak{u}$.

We now have to subject Axioms 1–5 to an investigation corresponding to that previously carried out for Axioms 1 and 2; in doing so we must observe that Axioms 1 and 2 now apply to a larger class of objects inasmuch as the arbitrary objects x and y now denote any arbitrary combination of the five simple objects taken as primitive.

We ask again whether certain consequences from Axioms 1–5 form a contradiction or whether, on the contrary, the five thought-objects taken as primitive, 1, =, $\mathfrak{u}$, $\mathfrak{f}$, $\mathfrak{f}'$, and their combinations can be so distributed into the class of entities and the class of nonentities that Axioms 1–5 are satisfied with regard to this partition into two classes, that is, that every consequence of these axioms becomes a true proposition with regard to this partition. To answer this question we note that Axiom 5 is the only one giving rise to propositions of the form $\bar{a}$, that is, to propositions asserting that a combination a of the five thought-objects taken as primitive is to belong to the class of nonentities. Accordingly, propositions that form a contradiction with 5 must certainly be of the form

6. $\mathfrak{f}(\mathfrak{u}x^{(o)}) = \mathfrak{u}1$;

but such a consequence cannot result from Axioms 1–4 in any way.

To see this, we call the equation (that is, the thought-object) $a = b$ a homogeneous equation if a and b are combinations of two simple objects each, likewise if a and b are combinations of three simple objects each or of four, and so forth; for example,

$$(11) = (\mathfrak{f}\mathfrak{u}), \quad (\mathfrak{f}\mathfrak{f} = (\mathfrak{u}\mathfrak{f}'), \quad (\mathfrak{f}11) = (\mathfrak{u}1{=}),$$
$$(\mathfrak{f}1)(\mathfrak{f}1) = (1111), \quad (\mathfrak{f}(\mathfrak{f}\mathfrak{f}'\mathfrak{u})) = (1\mathfrak{u}\mathfrak{u}1),$$
$$((\mathfrak{f}\mathfrak{f})(111)) = ((1)(11)(11)), \quad (\mathfrak{f}\mathfrak{u}111{=}) = (\mathfrak{u}\mathfrak{u}111\mathfrak{u})$$

are called homogeneous equations. From Axioms 1 and 2 alone follow, as we have seen previously, only homogeneous equations, namely, the equations of the form $\alpha = \alpha$. In the same way Axiom 3 yields only homogeneous equations if in it we take any thought-object for x. Likewise Axiom 4 certainly always exhibits a homogeneous equation in the assertion if the assumption is a homogeneous equation, and consequently from Axioms 1–4 only homogeneous equations can appear as consequences. But now equation 6, which after all was the one to be proved, is certainly not a homogeneous equation, since in it we are to take a combination in place of $x^{(o)}$ and the left side thereby becomes a combination of three or more simple objects, while the right side remains a combination of the two simple objects $\mathfrak{u}$ and 1.

The fundamental idea needed for recognizing the truth of my assertion has thus, I believe, been presented. In order to carry out the proof completely we need the notion of finite ordinal number, as well as certain theorems concerning the notion of equinumerousness, which we could in fact easily state and derive at this point; to develop completely the fundamental idea presented here, we must still consider those points of view to which I shall refer briefly at the close of my address (see V below).

Thus we obtain the desired partition if we put all objects a, where a is a consequence of Axioms 1–4, into the class of entities and all objects that differ from these —in particular, the objects $\mathfrak{f}(\mathfrak{u}x) = \mathfrak{u}1$—into the class of nonentities. Having thus established a certain property for the axioms adopted here, we recognize that they never lead to any contradiction at all, and therefore we speak of the thought-objects defined by means of them, $\mathfrak{u}$, $\mathfrak{f}$, and $\mathfrak{f}'$, as *consistent* notions or operations, or as *consistently existing*. So far as the notion of the infinite $\mathfrak{u}$, in particular, is concerned, the assertion of the *existence of the infinite* $\mathfrak{u}$ appears to be justified by the argument outlined above; for it now receives a definite meaning and a content that henceforth is always to be employed.

The considerations just sketched constitute the first case in which a direct proof of consistency has been successfully carried out for axioms, whereas the method of a suitable specialization, or of the construction of examples, which is otherwise customary for such proofs—in geometry in particular—necessarily fails here.

We see that the success of this direct proof here is essentially due to the fact that a proposition of the form $\bar{a}$, that is, a proposition according to which a certain combination is to belong to the class of nonentities, occurs as assertion in only one place, namely, in Axiom 5.

If we translate the well-known axioms for mathematical induction into the language I have chosen, we arrive in a similar way at the consistency of this larger number of axioms, that is, at the proof of the consistent *existence of what we call the smallest infinite*[1] (that is, *of the ordinal type* 1, 2, 3, . . .).

It is not difficult to provide a foundation for the notion of finite ordinal according to the principles adopted above; this can be done on the basis of an axiom stating that every set containing the first element of the ordinal and, whenever any element belongs to it, containing the succeeding one also, must certainly always contain the last element. Here we very easily obtain a proof of the consistency of the axioms by adducing an example, for instance, the number two. The point then is to show that the elements of the finite ordinal can be so ordered that every subset of it has a first and a last element—a fact that we prove by defining a thought-object $<$ through the axiom

$$(x < y \text{ a. } y < z) | x < z$$

and then recognizing the consistency of the axioms obtained when this new axiom is added, provided x, y, and z denote arbitrary elements of the finite ordinal. If we also make use of the fact that the smallest infinite exists, it then follows that for every finite ordinal a still greater one can be found.

The principles that must constitute the standard for the construction and further elaboration of the laws of mathematical thought in the way envisaged here are, briefly, the following.

I. Once arrived at a certain stage in the development of the theory, I may say that a further proposition is true as soon as we recognize that no contradiction results if it is added as an axiom to the propositions previously found true, that is, that it leads to consequences that all are true propositions with regard to a certain partition of the objects into the class of entities and that of nonentities.

II. In the axioms the arbitrary objects—taking the place of the notion "every" or "all" in ordinary logic—represent only those thought-objects and their mutual combinations that at this stage are taken as primitive or are to be newly defined. In the derivation of consequences from the axioms the arbitrary objects that occur in the axioms may therefore be replaced only by such thought-objects and their combinations. We must also duly note that, when a new thought-object is added and taken as primitive, the axioms previously assumed apply to a larger class of objects or must be suitably modified.

III. A set is generally defined as a thought-object m, and the combinations mx are called the elements of the set m, so that—contrary to the usual conception—the

[1] See *Hilbert 1900a*, sec. 2, "Die Widerspruchslosigkeit der arithmetischen Axiome".

notion of element of a set appears only as a subsequent product of the notion of set itself.

Just like the notion "set", the notions "mapping", "transformation", "relation", and "function" are also thought-objects, for which, exactly as was the case above with the notion "infinite", we have to consider suitable axioms and which can then be recognized as consistently existing if the combinations in question can be distributed into the class of entities and that of nonentities.

Point I expresses the creative principle that, in its freest use, justifies us in forming ever new notions, with the sole restriction that we avoid a contradiction. The paradoxes mentioned at the beginning of this address become impossible by virtue of II and III; this holds in particular for the paradox of the set of all sets that do not contain themselves as elements.

To show that the notion of set defined in III agrees to a large extent in content with the ordinary notion of set, I will prove the following theorem:

Let $1, \ldots, \alpha, \ldots, \mathfrak{k}$ be the thought-objects taken as primitive at a certain stage in the development and let $a(\xi)$ be a combination of them containing the arbitrary object ξ; further let $a(\alpha)$ be a true proposition (that is, let $a(\alpha)$ be in the class of entities). Then there certainly exists a thought-object m such that $a(mx)$ represents only true propositions for the arbitrary object x (that is, that $a(mx)$ is always in the class of entities), and conversely, also, every object ξ for which $a(\xi)$ represents a true proposition is equated to a combination $mx^{(o)}$, so that the proposition

$$\xi = mx^{(o)}$$

is true, that is, the objects ξ for which $a(\xi)$ becomes a true proposition form the elements of a set m in the sense of the above definition.

To prove this, we take the following axiom: let m be a thought-object for which the propositions

7. $a(\xi) \,|\, m\xi = \xi,$
8. $\overline{a(\xi)} \,|\, m\xi = \alpha,$

are true; that is, if ξ is an object such that $a(\xi)$ belongs to the class of entities, then $m\xi = \xi$ is to hold, otherwise $m\xi = \alpha$. We add this axiom to the axioms that hold for the objects $1, \ldots, \alpha, \ldots, \mathfrak{k}$, and we then assume that a contradiction thus appears, that is, that for the objects $1, \ldots, \alpha, \ldots, \mathfrak{k}, m$, the propositions

$$p(m) \quad \text{and} \quad \overline{p(m)},$$

say, are at the same time consequences, $p(m)$ being a certain combination of the objects $1, \ldots, \mathfrak{k}, m$. Here 8 means in words the stipulation: $m\xi = \alpha$ if $a(\xi)$ belongs to the class of nonentities. Wherever in $p(m)$ the object m appears in the combination $m\xi$, let us, in accordance with Axioms 7 and 8 and taking 2 into account, replace the combination $m\xi$ by ξ or α; let $q(m)$ (where $q(m)$ now no longer contains the object m in a combination mx) be obtained in this way from $p(m)$; then $q(m)$ would have to be a consequence of the axioms[2] originally posited for $1, \ldots, \alpha, \ldots, \mathfrak{k}$, and therefore would have to remain true even if we took for m any one of these objects, say, the

[2] [[The German text has "dem...Axiome", but the argument seems to call for the plural; the version in *Hilbert 1930a* has the plural.]]

object 1. Since the same considerations also hold for the proposition $\overline{p(m)}$, the contradiction

$$q(1) \quad \text{and} \quad \overline{q(1)}$$

would therefore also exist at the original stage at which the objects $1, \ldots, \alpha, \ldots, \mathfrak{k}$ were taken as primitive; but this cannot be, if we assume that the objects $1, \ldots, \mathfrak{k}$ exist consistently. We must therefore reject our assumption that a contradiction occurs; that is, m exists consistently, which was to be proved.

IV. If we want to investigate a given system of axioms according to the principles above, we must distribute the combinations of the objects taken as primitive into two classes, that of entities and that of nonentities, with the axioms playing the role of prescriptions that the partition must satisfy. The main difficulty will consist in recognizing the possibility of distributing all objects into the two classes, that of entities and that of nonentities. The question whether this distribution is possible is essentially equivalent to the question whether the consequences we can obtain from the axioms by specialization and combination in the sense explained earlier lead to a contradiction or not, *if we still add the familiar modes of logical inference such as*

$$\{(a \mid b) \text{ a. } (\bar{a} \mid b)\} \mid b,$$
$$\{(a \text{ o. } b) \text{ a. } (a \text{ o. } c)\} \mid \{a \text{ o. } (b \text{ a. } c)\}.$$

We can then recognize the consistency of the axioms either by showing how a possible contradiction would already have to have occurred at an earlier stage in the development of the theory or by making the assumption that there is a proof leading from the axioms to a certain contradiction and then showing that such a proof is not possible, because it would itself contain a contradiction. Thus the proof sketched above for the consistent existence of the infinite came down to the recognition that a proof of equation 6 from Axioms 1–4 is not possible.

V. Whenever in the preceding we spoke of *several* thought-objects, of *several* combinations, of *various* kinds of combinations, or of *several* arbitrary objects, a bounded number of such objects was to be understood. Now that we have established the definition of the finite number we are in a position to comprehend the general meaning of this way of speaking. The meaning of the "arbitrary" consequence and of the "differing" of one proposition from all propositions of a certain kind is also now, on the basis of the definition of the finite number (corresponding to the idea of mathematical induction) susceptible of an exact description by means of a recursive procedure. It is in this way that we can carry out completely the proof, sketched above, that the proposition $\mathfrak{f}(\mathfrak{u}x^{(o)}) = \mathfrak{u}1$ differs from every proposition obtained as a consequence of Axioms 1–4 by a finite number of steps; we need only consider the proof itself to be a mathematical object, namely, a finite set whose elements are connected by propositions stating that the proof leads from 1–4 to 6, and we must then show that such a proof contains a contradiction and therefore does not exist consistently in the sense defined by us.

The existence of the totality [[Inbegriff]] of real numbers can be demonstrated in a way similar to that in which the existence of the smallest infinite can be proved; in fact, the axioms for real numbers as I have set them up (*1903*, pp. 24–26) can be expressed by precisely such formulas as the axioms hitherto assumed. In particular, so far as the axiom I called the completeness axiom [[Vollständigkeitsaxiom]] is concerned, it expresses the fact that the totality of real numbers contains, in the sense of a one-to-one correspondence between elements, any other set whose elements satisfy also the axioms that precede; thus considered, the completeness axiom, too, becomes a stipulation expressible by formulas constructed like those above, and the axioms for the totality of real numbers do not differ qualitatively in any respect from, say, the axioms necessary for the definition of the integers. In the recognition of this fact lies, I believe, the real refutation of the conception of the foundations of arithmetic associated with L. Kronecker and characterized at the beginning of my lecture as dogmatic.

In the same way we can show that the fundamental notions of Cantor's set theory, in particular Cantor's alephs, have a consistent existence.