NMAI057 – Linear algebra 1

Tutorial 5

Groups and permutations

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Problem 1. Decide and justify, whether the following are groups:

(a) $(\mathbb{Q}, \cdot)$,
(b) $(\mathbb{Q}, -)$,
(c) $(\mathbb{Q} \setminus \{0\}, \circ)$, where for all $a, b \in \mathbb{Q}$, $a \circ b = |ab|$,
(d) $(\mathbb{Q}, \circ)$, where for all $a, b \in \mathbb{Q}$, $a \circ b = \frac{a+b}{2}$,
(e) $(\mathbb{Q}, \circ)$, where for all $a, b \in \mathbb{Q}$, $a \circ b = a + b + 3$,
(f) $(\mathcal{F}, +)$, i.e., the set of all real functions with one variable $\mathcal{F}$ together with the operation of addition of functions,
(g) the set of all rotations around the origin in $\mathbb{R}^2$ together with the operation of function composition,
(h) the set of all translations (shifts) in $\mathbb{R}^2$ together with the operation of function composition.
(i) the set of all matrices in $\mathbb{R}^{n\times n}$ with the operation of matrix multiplication.
(j) the set of all regular matrices in $\mathbb{R}^{n\times n}$ with the operation of matrix multiplication.

Problem 2. Let $(\mathbb{G}, \circ)$ be a group and $x \in \mathbb{G}$. Decide and justify whether $(\mathbb{G}, *)$ is a group with the binary operation $*$ defined for all $a, b \in \mathbb{G}$ as $a * b = a \circ x \circ b$.

Problem 3. Fill the table for binary operation $\circ$ on set $\mathbb{G}$ so that $(\mathbb{G}, \circ)$ is a group with neutral element 0. Justify.

(a)

$\circ$	0	1
0		
1		

(b)

$\circ$	0	1	2
0			
1			
2			

(c)

$\circ$	0
0	

(d)

$\circ$	0	1	2	3
0				
1		0		
2				
3				

Problem 4. Solve "permutation"equation $p \circ x \circ q = \imath$ for p and q.

(a) $p = (6, 4, 1, 5, 3, 2)$, $q = (6, 4, 3, 2, 5, 1)$.
(b) $p = (1, 2, 7, 6, 5, 4, 3, 8, 9)$, $q = (1, 3, 5, 7, 9, 8, 6, 4, 2)$.
(c) $p = (5, 4, 3, 2, 1, 9, 8, 7, 6)$, $q = (8, 6, 4, 2, 1, 3, 5, 7, 9)$
(d) $p = (3, 6, 9, 2, 5, 8, 1, 4, 7)$, $q = (9, 8, 7, 6, 5, 4, 3, 2, 1)$.

Problem 5. Determine the sign of the following permutation

(a) $p = (1, 3, 5, \dots, 2n-1, 2, 4, 6, \dots, 2n)$
(b) $p = (1, 4, 7, \dots, 3n-2, 2, 5, 8, \dots, 3n-1, 3, 6, 9, \dots, 3n)$
(c) $p = (2, 5, 8, \dots, 3n-1, 3, 6, 9, \dots, 3n, 1, 4, 7, \dots, 3n-2)$
(d) $p = (3, 6, 9, \dots, 3n, 2, 5, 8, \dots, 3n-1, 1, 4, 7, \dots, 3n-2)$

Problem 6. Decide and justify whether the following are Abelian (commutative) groups:

(a) The set $\{\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) \mid z \in \mathbb{Z}\}$ together with matrix product.
(b) The set $\{\left(\begin{smallmatrix} a & a \\ a & a \end{smallmatrix}\right) \mid a \in \mathbb{R} \setminus \{0\}\}$ together with matrix product.

Problem 7. Determine graphs, cycles, a factorzation into transpositions, the number of inversions, the sign, and the inverse permutations for the following permutations: p, q and their compositions $q \circ p$ and $p \circ q$.

(Permutations are composed as mappings, i.e. $(q \circ p)(i) = q(p(i))$.)

(a) $p = (6, 4, 1, 5, 3, 2)$, $q = (6, 4, 3, 2, 5, 1)$.
(b) $p = (1, 2, 7, 6, 5, 4, 3, 8, 9)$, $q = (1, 3, 5, 7, 9, 8, 6, 4, 2)$.
(c) $p = (5, 4, 3, 2, 1, 9, 8, 7, 6)$, $q = (8, 6, 4, 2, 1, 3, 5, 7, 9)$.
(d) $p = (3, 6, 9, 2, 5, 8, 1, 4, 7)$, $q = (9, 8, 7, 6, 5, 4, 3, 2, 1)$.

Problem 8. Show four different arguments why the inverse permutatin has the same sign as the original one.

Problem 9. Show that every permutation on n elements can be decomposed into transpositions of form $(1, i)$ for $i \in \{2, \dots, n\}$. Determine a bound of the length of the resulting factorization.

Problem 10. Deretmine powers p^{10} and q^{99} for permutations p a q.

(a) $p = (6, 4, 1, 5, 3, 2)$, $q = (6, 4, 3, 2, 5, 1)$.
(b) $p = (1, 2, 7, 6, 5, 4, 3, 8, 9)$, $q = (1, 3, 5, 7, 9, 8, 6, 4, 2)$.
(c) $p = (5, 4, 3, 2, 1, 9, 8, 7, 6)$, $q = (8, 6, 4, 2, 1, 3, 5, 7, 9)$.
(d) $p = (3, 6, 9, 2, 5, 8, 1, 4, 7)$, $q = (9, 8, 7, 6, 5, 4, 3, 2, 1)$.

Problem 11. Find a permutation on 10 elements s.t. p^i is not the identity (i.e. $p^i \neq \imath$) for all $i = 1, \dots, 29$.

Problem 12. How many permutations on n elements have sign 1, and how many sign -1?