Letter to Keferstein

RICHARD DEDEKIND

(*1890a*)

Hans Keferstein, *Oberlehrer* in Hamburg, published a paper (*1890*) on the notion of number in which he commented on Frege's (*1884*) and Dedekind's (*1888*) books on the subject. His comments on Dedekind's work, although not entirely negative, included a number of suggestions for amending the text that revealed his lack of real understanding of some fundamental points, for example, the equivalence of two sets. Dedekind felt obliged to answer with an essay (*1890*) in which he showed how pointless the "corrections" were. He sent it to Keferstein on 9 February 1890, with a letter in which he suggested that the Hamburg Mathematical Society, in whose yearly *Mitteilungen* Keferstein's paper had appeared, publish either the essay or, should Keferstein realize that his suggestions were based upon misunderstandings, a declaration to that effect.

Dedekind's essay dealt with three points. The first was an objection of Keferstein's to Dedekind's proof that there exists an infinite set. This proof has often been criticized (see, for instance, below, p. 131); but Keferstein's objection rested upon a wrong argument, a plain confusion of the equivalence relation between sets with their identity, and Dedekind had no difficulty in answering him. The second point was Keferstein's claim that he had found two conflicting definitions of infinite sets in the book; Dedekind pointed out that one was in fact merely a stylistic variant of the other. The third point was the substitution by Keferstein of a new definition of simply infinite sets for that given by Dedekind (*1888*, art. 71). Keferstein's purpose was to avoid the notion of chain, and his proposal amounted in effect to the abandonment of mathematical induction; Dedekind showed that this proposal would bar the possibility of providing an adequate foundation for the theory of natural numbers.

On 14 February 1890 Keferstein acknowledged receipt of the essay, announcing that at the next meeting of the Society he would propose its publication, that he was confident that the proposition would be accepted, that, moreover, he did not consider his criticisms, especially the third one, as mere misunderstandings on his part and would return to them in case the essay should be published.

On 27 February 1890 Dedekind sent to Keferstein a long letter that is a brilliant presentation of the development of his ideas on the notion of natural number. In it he tried to show that his assumptions had not been haphazardly chosen and that each one of them had a profound justification. This is especially true, Dedekind insisted, of the notion of chain, which Keferstein wanted to eliminate. Professor Hao Wang published (*1957*) an English translation of a major part of the letter, with commentaries. The text below is a translation of the whole letter.

On 19 March 1890 Keferstein thanked

Dedekind for the letter and asked his permission to use it in a lecture before the Hamburg Mathematical Society. Dedekind granted this permission in his next letter, dated 1 April 1890, adding a few lines of explanation on the notion of chain.

On 17 November 1890, as publication of the yearly volume of the *Mitteilungen* of the Society was drawing near, Keferstein wrote an "Erwiderung" (*1890b*) that was to follow Dedekind's essay (*1890*) in the volume. But on 19 December 1890 he had to inform Dedekind that the editorial board of the Society had declined to publish Dedekind's essay, as well as Keferstein's rejoinder, the reason invoked being the lack of space and the fact that Dedekind's reply was longer than Keferstein's original criticism of Dedekind's book. Keferstein also announced his intention of publishing in the Society's coming yearly volume a note withdrawing his proposed "corrections" to Dedekind's work. The note appeared in volume 3 of the *Mitteilungen* (p. 31), published in February 1891; it consisted of a few lines incorporated in a report of the 11 October 1890 meeting of the Society.

On 23 December 1890 Dedekind wrote his last letter to Keferstein, acknowledging receipt of his returned manuscript as well as of a copy of Keferstein's "Erwiderung". He expressed his regrets that, although the polemic and the correspondence had taken so much of his time, Keferstein's reply still contained many misunderstandings.

Dedekind's time and efforts were, however, not wasted at all. The controversy produced the letter below, which remains a masterly presentation of his ideas.

Dedekind's essay and the Dedekind–Keferstein correspondence are preserved in the Niedersächsische Staats- und Universitätsbibliothek in Göttingen. They come from the Dedekind estate. Keferstein's letters are the originals, as received by Dedekind; Dedekind's letters and his essay, as well as Keferstein's "Erwiderung", are clean copies in Dedekind's hand. Dedekind's letter of 27 February 1890 is reproduced below with the kind permission of the Library (where it has the classmark: Göttingen, UB, Cod. Ms. Nachlass Dedekind, 13).

Stefan Bauer-Mengelberg translated the parts of the letter omitted from Professor Wang's paper and introduced some changes into the text of Professor Wang's translation. Permission to make use of that translation was granted by Professor Wang and *The journal of symbolic logic*.

My dear Doctor,

I should like to express my sincerest thanks for your kind letter of the 14th of this month and for your willingness to publish my reply. But I would ask you not to rush anything in this matter and to come to a decision only after you have once more carefully read and thoroughly considered the most important definitions and proofs in my essay on numbers, if you have the time. For I think that most probably you will then be converted on all points to my conception and to my treatment of the subject; and this is just what I should value most, since I am convinced that you really have a deep interest in the matter.

In order to further this rapprochement wherever possible, I should like to ask you to lend your attention to the following train of thought, which constitutes the genesis of my essay. How did my essay come to be written? Certainly not in one day; rather, it is a synthesis constructed after protracted labor, based upon a prior analysis of the sequence of natural numbers just as it presents itself, in experience, so to speak, for our consideration. What are the mutually independent fundamental properties of the sequence N, that is, those properties that are not derivable from one another but from which all others follow? And how should we divest these properties of their specifically arithmetic character so that they are subsumed under more general notions and under activities of the understanding *without* which no thinking is possible at all but *with* which a foundation is provided for the reliability and completeness of proofs and for the construction of consistent notions and definitions?

When the problem is posed in this way, one is, I believe, forced to accept the following facts:

(1) The number sequence N is a *system* of individuals, or elements, called numbers. This leads to the general consideration of systems as such (§ 1 of my essay).

(2) The elements of the system N stand in a certain relation to one another; a certain order obtains, which consists, to begin with, in the fact that to each definite number n there corresponds a definite number n', the succeeding, or next greater, number. This leads to the consideration of the general notion of a *mapping* φ of a system (§ 2), and since the image $\varphi(n)$ of every number n is again a *number*, n', and therefore $\varphi(N)$ is a part of N, we are here concerned with the mapping φ of a system N *into itself*, of which we must therefore make a general investigation (§ 4).

(3) Distinct numbers a and b are succeeded by distinct numbers a' and b'; the mapping φ, therefore, has the property of distinctness, or *similarity*[1] (§ 3).

(4) Not every number is a successor n'; in other words, $\varphi(N)$ is a proper part of N. This (together with the preceding) is what makes the number sequence N infinite (§ 5).

(5) And, in particular, the number 1 is the *only* number that does not lie in $\varphi(N)$. Thus we have listed the facts that you (p. 124, ll. 8–14) regard as the complete characterization of an ordered, simply infinite system N.

(6) I have shown in my reply (III),[2] however, that these facts are still far from being adequate for completely characterizing the nature of the number sequence N. All these facts would hold also for every system S that, besides the number sequence N, contained a system T, of arbitrary additional elements t, to which the mapping φ could always be extended while remaining similar and satisfying $\varphi(T) = T$. But such a system S is obviously something quite different from our number sequence N, and I could so choose it that scarcely a single theorem of arithmetic would be preserved in it. What, then, must we add to the facts above in order to cleanse our system S again of such alien intruders t as disturb every vestige of order and to restrict it to N? This was one of the most difficult points of my analysis and its mastery required lengthy reflection. If one presupposes knowledge of the sequence N of natural numbers and, accordingly, allows himself the use of the language of arithmetic, then, of course, he has an easy time of it. He need only say: an element n belongs to the sequence N if and only if, starting with the element 1 and counting on and on steadfastly, that is, going through a finite number of iterations of the mapping φ (see the end of article 131 in my essay), I actually reach the element n at some time; by this procedure, however, I shall never reach an element t outside of the sequence N. But this way of characterizing the distinction between those elements t that are to be ejected from S and those elements n that alone are to remain is surely quite useless for our purpose; it would, after all, contain the most pernicious and obvious kind of vicious

[1] [[See footnote 3, p. 93 above.]]

[2] [[This refers to sec. III in *Dedekind 1890*; see introductory note.]]

circle. The mere words "finally get there at some time", of course, will not do either; they would be of no more use than, say, the words "karam sipo tatura", which I invent at this instant without giving them any clearly defined meaning. Thus, how can I, without presupposing any arithmetic knowledge, give an unambiguous conceptual foundation to the distinction between the elements n and the elements t? Merely through consideration of the *chains* (articles 37 and 44 of my essay), and yet, by means of these, completely! If I wanted to avoid my technical expression "chain" I would say: an element n of S belongs to the sequence N if and only if n is an element of *every* part K of S that possesses the following two properties: (i) the element 1 belongs to K and (ii) the image $\varphi(K)$ is a part of K. In my technical language: N is the intersection [[Gemeinheit]] 1_0, or $\varphi_0(1)$, of all those chains K (in S) to which the element 1 belongs. Only now is the sequence N characterized completely. In passing I would like to make the following remark on this point. Frege's *Begriffsschrift* and *Grundlagen der Arithmetik* came into my possession for the first time for a brief period last summer (1889), and I noted with pleasure that his way of defining the non-immediate succession of an element upon another in a sequence agrees in *essence* with my notion of chain (articles 37 and 44); only, one must not be put off by his somewhat inconvenient terminology.

(7) After the essential nature of the simply infinite system, whose abstract type is the number sequence N, had been recognized in my analysis (articles 71 and 73), the question arose: does such a system *exist* at all in the realm of our ideas? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not perhaps contain internal contradictions. Hence the need for such proofs (articles 66 and 72 of my essay).

(8) After this, too, had been settled, there was the question: does what has been said so far also contain a *method of proof* sufficient to establish, in full generality, propositions that are supposed to hold for *all* numbers n? Yes! The famous method of proof by induction rests upon the secure foundation of the notion of chain (articles 59, 60, and 80 of my essay).

(9) Finally, is it possible also to set up the *definitions* of numerical operations consistently for *all* numbers n? Yes! This is in fact accomplished by the theorem of article 126 of my essay.

Thus the analysis was completed and the synthesis could begin; but this still caused me trouble enough! Indeed the reader of my essay does not have an easy task either; apart from sound common sense, it requires very strong determination to work everything through completely.

I shall now turn to some parts of your paper that I did not mention in my recent reply [[*1890*]] because they are not as important; but perhaps my remarks about them will contribute something more to the clarification of the issue.

(*a*) P. 121, l. 19.[3] Why the mention of a *part* here? I later (article 161 of my essay) ascribe a *number* [[*Anzahl*]] to each *finite* system and to no other.

(*b*) P. 122, l. 8.[4] Here we have a confusion between *mapping* and *map*; instead of

[3] [[Keferstein had written: "In fact he [[Dedekind]] later ascribes each number to a certain part [[Teil]] of such a system...".]]

[4] [[Keferstein had written: "... to the mapping φ of S we can match an inverse mapping $\bar{\varphi}(S')$ ...".]]

"mapping $\bar{\varphi}(S')$" it should be "mapping $\bar{\varphi}$ of the system S'". Not $\bar{\varphi}(S')$ but $\bar{\varphi}$ is a *mapping* (the mapping cartographer) [[*Abbildung* (der abbildende Maler)]], which generates the *map* $\bar{\varphi}(S') = S$ from the *system* S' (the original). Such confusions can become quite dangerous in our investigations.

(*c*) P. 123, ll. 1–2.[5] These words might perhaps apply to Frege, but they certainly do not apply to me. I define the *number* [[*Zahl*]] 1 as the basic element of the number sequence without any ambiguity in articles 71 and 73, and, just as unambiguously, I arrive at the *number* [[*Anzahl*]] 1 in the theorem of article 164 as a consequence of the general definition in article 161. Nothing further *may* be added to this at all if the matter is not to be muddled.

(*d*) P. 123, ll. 29–31.[6] The preceding remark, (*c*), has already taken care of this. And how would the greater reliability and the lesser prolixity shape up in *actual fact*?

(*e*) P. 124, ll. 21–24.[7] The meaning of these lines (as well as of the preceding and subsequent ones) is not quite clear to me. Do they perhaps express the desire that my definition of the number sequence N and of the way in which the element n' follows the element n be propped up, if possible, by an *intuitive* sequence? If so, I would resist that with the utmost determination, since the danger would immediately arise that from such an intuition we might perhaps unconsciously also take as self-evident theorems that must rather be derived quite abstractly from the logical definition of N. If I *call* (article 73) n' the element *following* n, that is only a new *technical expression* by means of which I merely bring some variety into my *language*; this language would sound even more monotonous and repelling if I had to deny myself this variety and were always to call n' only the *map* $\varphi(n)$ of n. But one expression is to *mean* exactly the same as the other.

(*f*) P. 124, l. 33—p. 125, l. 7.[8] The word "merely" [["lediglich"]], taken from the third line of my definition in article 73, is obviously meant to indicate the sole *restriction* to which the word "entirely" [["gänzlich"]], which occurs just before, is subject;[9]

[5] [[Keferstein had written: "In our opinion, both Frege (*1884*, pp. 89–90) and Dedekind, who incidentally derives the notion of cardinal number only from the previously defined notion of ordinal number (*1888*, pp. 21 [[article 73]] and 54 [[article 161]]), have, when all is said and done, introduced the notion of the number 1 without an adequate definition".]]

[6] [[Keferstein had written: ". . . especially since, by the previous introduction of the number 1, the latter [[Dedekind]] seems not only to gain in reliability but also to lose in prolixity".]]

[7] [[Keferstein had written: "Since Dedekind does not emphasize this fact [[that N can be regarded as a sequence in which $\varphi(n) = n'$ immediately follows n]], the notions of sequence and of succession in a sequence turn up in an *apparently* abrupt way in the definition of ordinal numbers that comes at that point".]]

[8] [[Keferstein had written: "When the above comments are properly taken into account, there remains in these propositions at most one point that could give offense, namely, the demand that we *entirely* disregard the particular character of the elements and retain merely their distinguishability, since objects remain distinguishable, after all, only if they still exhibit differences. If we strike out the words 'ihre Unterscheidbarkeit festhält und nur' [[see footnote 9]], however, the difficulty vanishes, since the relations in which the elements are put with one another by the ordering mapping φ are conceived by precisely a pure mental activity that remains completely independent of the particular character of the objects toward which it is directed".]]

[9] [[The German text to which Dedekind refers reads: "Wenn man bei der Betrachtung eines einfach unendlichen, durch eine Abbildung φ geordneten Systems N von der besonderen Beschaffenheit der Elemente gänzlich absieht, lediglich ihre Unterscheidbarkeit festhält und nur die Beziehungen auffaßt, in die sie durch die ordnende Abbildung φ zueinander gesetzt sind, so heißen diese Elemente *natürliche Zahlen* oder *Ordinalzahlen* oder auch schlechthin *Zahlen*, und das Grundelement 1 heißt die *Grundzahl* der *Zahlenreihe N*". (*1888*, art. 73).]]

if one were to remove this restriction—if, in other words, the word "entirely" were to assume its full meaning—then we would lose the distinguishability of the elements, which, after all, is indispensable for the notion of the simply infinite system. This "merely", therefore, does not seem at all superfluous to me, but necessary. I do not understand how it could arouse any objection.

Repeating the wish I expressed at the beginning and begging you to excuse the thoroughness of my discussion, I remain with kindest regards

Yours very truly,

R. DEDEKIND

27 February 1890
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