Concentration risk

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1 Name concentration

Credit risk in a portfolio arises from two sources:

1. **systematic risk** - the effect of changes in macroeconomic and financial market conditions on the performance of borrowers.

2. **idiosyncratic risk** - the effects of risks that are particular to individual borrowers.

The idiosyncratic risk is diversifiable. It is diversified when the largest individual exposures account for a smaller share of the total exposure, i.e. the case of fine-grained portfolio.

One possible approach to modeling the default probability is to set

$$D_n = I\left[r_n < c_n\right], \tag{1}$$

where r_n is the standardized log-return (in period $[0, T]$) on borrower's n asset value. The default corresponds to the situation when the return r_n falls below certain level c_n.

We further assume

$$r_n = \sqrt{\rho_n}\, X + \sqrt{1 - \rho_n}\, \varepsilon_n, \tag{2}$$

where X represents the (single) systematic risk factor, ρ_n corresponds to the correlation between the r_n and the risk factor X. ε_n represents the idiosyncratic component.

We assume that r_n is a random variable with standard normal distribution $N(0, 1)$. From (1) then follows

$$c_n = \Phi^{-1}\left(PD_n\right). \tag{3}$$

We also assume that ε_n and X are independent random variables with standard normal distribution. Then the conditional probability of default for given value of X and the obligor n is

$$\begin{aligned} PD_n(X) &= \mathrm{P}\left(\varepsilon < \frac{\Phi^{-1}\left(PD_n\right) - \sqrt{\rho_n}\, X}{\sqrt{1 - \rho_n}}\right) \\ &= \Phi\left(\frac{\Phi^{-1}\left(PD_n\right) - \sqrt{\rho_n}\, X}{\sqrt{1 - \rho_n}}\right). \end{aligned} \tag{4}$$

Choosing a realization x for the systematic risk factor equal to the qth quantile $\alpha_q(X)$ of X and taking into account, that X is assumed to be normally distributed, we obtain

$$PD_n\left(\alpha_q(X)\right) = \Phi\left(\frac{\Phi^{-1}\left(PD_n\right) - \sqrt{\rho_n}\,\Phi^{-1}(q)}{\sqrt{1-\rho_n}}\right). \tag{5}$$

We are interested in the distribution of the loss ratio

$$L_N = \sum_{n=1}^{N} s_n \ \mathrm{LGD}_n \ D_n. \tag{6}$$

The distribution of L_N can be considered for $N \to \infty$ as the limiting case of so called infinitely-grained portfolio.

We make the following assumptions:

1. Variables

$$U_n = \mathrm{LGD}_n \ D_n, \ \ n = 1, \dots, N, \tag{7}$$

are bounded in $[0, 1]$ and are mutually independent conditional on the systematic factor X.

2.

$$\sum_{n=1}^{N} \mathrm{EAD}_n \to +\infty, \ \text{ for } N \to \infty. \tag{8}$$

3. The largest exposure share is of order $O\left(N^{-\frac{1}{2+\xi}}\right)$, where $\xi > 0$. (Hence it shrinks to 0 as $N \to \infty$.)

From 1) to 3) it can be proved that

$$L_N - \mathrm{E}\left[L_N|X\right] \to 0 \text{ a.s. as } N \to \infty. \tag{9}$$

For simplicity we assume a homogeneous portfolio in the sense that all obligors have the same default probability $PD_n = PD$, $n = 1, \ldots, N$, and we assume that $\mathrm{LGD}_n = 100\%$, $n = 1 \ldots, N$.

Then

$$\begin{aligned} \mathrm{E}\left[L_N|X\right] &= \sum_{n=1}^{N} s_n \ \mathrm{E}\left[D_n|X\right] = \mathrm{PD}(X) \\ &= \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}) - \sqrt{\rho}\, X}{\sqrt{1-\rho}}\right). \end{aligned} \tag{10}$$

Hence,

$$L_N \to \mathrm{PD}(X) \ a.s. \text{ as } N \to \infty, \tag{11}$$

i.e. in an infinitely grained portfolio, the conditional default probability $\mathrm{PD}(X)$ decribes the fraction of defaulted obligors.

For large N we approximate the d.f. of L_N by

$$\begin{aligned} F_{L_N}(x) = \mathrm{P}\left(L_N \le x\right) &\cong \mathrm{P}\left[\mathrm{PD}(X) \le x\right] \\ &= \mathrm{P}\left[-X \le \frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(\mathrm{PD})\right)\right] \\ &= \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(\mathrm{PD})\right)\right), \end{aligned} \tag{12}$$

and we have the density

$$f_{L_N}(x) \cong \frac{\partial\, \mathrm{P}\left[\mathrm{PD}(X) \le x\right]}{\partial x}. \tag{13}$$

If we measure the credit risk of a portfolio by means of value at risk, we need to evaluate

$$\mathrm{VaR}_q\left(L_N\right) = \alpha_q\left(L_N\right).$$

For an infinitely fine-grained portfolio we substitute $\alpha_q\left(L_N\right)$ by $\alpha_q\left(\mathrm{E}\left[L_N|X\right]\right)$.

The difference

$$\alpha_q\left(L_N\right) - \alpha_q\left(\mathrm{E}\left[L_N|X\right]\right) \tag{14}$$

is the adjustment for the effect of undiversified idiosyncratic risk in the portfolio.

An approximation of (14) based on Taylor expansion can be derived as follows:

For $\varepsilon = 1$ we write

$$L_N = \mathrm{E}\left[L_N|X\right] + \varepsilon\ \left(L_N - \mathrm{E}\left[L_N|X\right]\right). \tag{15}$$

We use a Taylor expansion around $\mathrm{E}\left[L_N|X\right]$

$$\begin{aligned}\alpha_q\left(L_N\right) &= \alpha_q\left(\mathrm{E}\left[L_N|X\right] + \varepsilon\ \left(L_N - \mathrm{E}\left[L_N|X\right]\right)\right)\\ &\cong \alpha_q\left(\mathrm{E}\left[L_N|X\right]\right) + \frac{\partial}{\partial\varepsilon}\alpha_q\left(\mathrm{E}\left[L_N|X\right] + \varepsilon\ \left(L_N - \mathrm{E}\left[L_N|X\right]\right)\right)|_{\varepsilon=0}\\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}\alpha_q\left(\mathrm{E}\left[L_N|X\right] + \varepsilon\ \left(L_N - \mathrm{E}\left[L_N|X\right]\right)\right)|_{\varepsilon=0} + O\left(\varepsilon^2\right). \end{aligned} \tag{16}$$

Thus, the approximation of (14) is so called **granularity adjustment**

$$\begin{aligned}\mathrm{GA}_N &= \frac{\partial}{\partial\varepsilon}\alpha_q\left(\mu(X) + \varepsilon\left(L_N - \mu(X)\right)\right)|_{\varepsilon=0}\\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}\alpha_q\left(\mu(X) + \varepsilon\left(L_N - \mu(X)\right)\right)|_{\varepsilon=0}, \end{aligned} \tag{17}$$

where we denote

$$\mu(X) = \mathrm{E}\left[L_N|X\right].$$

It can be shown that the first term on the right-hand side of (17) equals to 0 due to the following lemma:

Lemma. *Let* (Z, Y) *be a bivariate random vector and* $Q(q, \varepsilon) = \alpha_q(Z + \varepsilon\, Y)$ *for some* $\varepsilon > 0$*. Then*

$$\frac{\partial}{\partial \varepsilon} Q(q, \varepsilon) = \mathrm{E}\left[Y|Z + \varepsilon\, Y = Q(q, \varepsilon)\right]. \tag{18}$$

Proof. Denote by $f(z, y)$ the joint density of (Z, Y). We have

$$\mathrm{P}\left[Z + \varepsilon\, Y > Q(q, \varepsilon)\right] = 1 - q \Leftrightarrow \int_{-\infty}^{+\infty} \left(\int_{Q(q,\varepsilon) - \varepsilon\, y}^{+\infty} f(z, y)\, \mathrm{d}\, z \right) \mathrm{d}\, y = 1 - q. \tag{19}$$

Differentiation with respect to ε provides

$$\int_{-\infty}^{+\infty} \left(\frac{\partial Q(q, \varepsilon)}{\partial \varepsilon} - y \right) f\left(Q(q, \varepsilon) - \varepsilon\, y, y\right) \mathrm{d}\, y = 0, \tag{20}$$

from where it follows

$$\begin{aligned} \frac{\partial Q(q, \varepsilon)}{\partial \varepsilon} &= \frac{\int_{-\infty}^{+\infty} y\, f\left(Q(q, \varepsilon) - \varepsilon\, y, y\right) \mathrm{d}\, y}{\int_{-\infty}^{+\infty} f\left(Q(q, \varepsilon) - \varepsilon\, y, y\right) \mathrm{d}\, y} \\ &= \mathrm{E}\left[Y|Z + \varepsilon\, Y = Q(q, \varepsilon)\right]. \end{aligned}$$

□

We set $Z = \mathrm{E}\,[L_N|X]$, $Y = L_N - \mathrm{E}\,[L_N|X]$. From the previous Lemma we obtain

$$\frac{\partial}{\partial \varepsilon}\alpha_q(Z+\varepsilon\,Y)\,|_{\varepsilon=0} = \mathrm{E}\,[Y|Z+\varepsilon\,Y = \alpha_q(Z+\varepsilon\,Y)]\,|_{\varepsilon=0}$$

$$= \mathrm{E}\,[L_N - \mathrm{E}\,[L_N|X]\,|\,\mathrm{E}\,[L_N|X] = \alpha_q(Z)] = 0$$

thanks to the independence of L_N and $L_N - \mathrm{E}\,[L_N|X]$. The granularity adjustment (17) is given by

$$\mathrm{GA}_N = \frac{1}{2}\frac{\partial^2}{\partial \varepsilon^2}\alpha_q\left(\mu(X) + \varepsilon\left(L_N - \mu(X)\right)\right)|_{\varepsilon=0}\,. \tag{21}$$

It can be proved that

$$\mathrm{GA}_N = \frac{-1}{2\,h\left(\alpha_q(X)\right)}\,\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{\sigma^2(x)\,h(x)}{\mu'(x)}\right)|_{x=\alpha_q(X)}\,, \tag{22}$$

where

$$\sigma^2(x) = \mathrm{Var}\,[L_N|X=x]\,,$$

$$\mu(x) = \mathrm{E}\,[L_N|X=x]$$

and $h(x)$ is the density of the systematic risk factor X.

Remark. *When the economic capital corresponding to loss L_N is defined as*

$$\mathrm{EC}\,(L_N) = \alpha_q\,(L_N) - \mathrm{E}\,L_N,$$

by substituting $\alpha_q\,(L_N)$ by $\alpha_q\,(\mathrm{E}\,[L_N|X])$ we obtain the same value of capital, since

$$\mathrm{E}\,L_N = \mathrm{E}\,\mathrm{E}\,[L_N|X]\,.$$

References: E.Lutkebohmert, Concentration Risk in Credit Portfolios. Springer, 2009.