Bodové odhady

(1) $U(0,\vartheta)$ — neznáme $\qquad \bar{t}_n \doteq ET = \frac{\vartheta}{2}$: $\frac{0+\vartheta}{2}$

vidíme

 → zaznamenáme, že čas $T \sim U(0,\vartheta)$ budík zazvoní $\qquad$ čas v sec. $T \in \mathbb{R}$

---→ zopakujeme $10\times$ $\quad (n = 10)$

→ čísla $t_1, t_2, \ldots, t_{10}$ $\quad$ (realizace náh. výběru $T_1, \ldots, T_{10} \sim U(0,\vartheta)$ nez.)

a) odhad ϑ: $\max(t_1, \ldots, t_n) =: \hat{\vartheta}_1$

číslo ... realizace odhadu $\hat{\vartheta}_1$, n.v.

b) $\bar{t}_n = \frac{t_1 + \ldots + t_n}{n}$

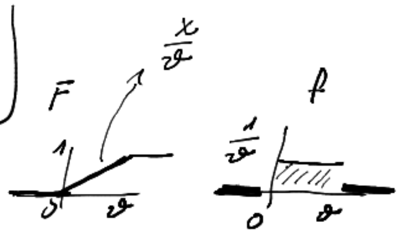
2. odhad ϑ: $2\bar{t}_n =: \hat{\vartheta}_2$ $\qquad 1, 1, 10 \rightsquigarrow 4 \rightsquigarrow 8$

c) $\hat{\vartheta}_3 := \max(\hat{\vartheta}_1, \hat{\vartheta}_2)$

→ neparametrický model "nevíme nic"

→ parametrický model
- → $U(0, \vartheta)$
- → $Pois(\vartheta)$ ---- # emailů, # dětí, ⋮
- → $N(\ldots)$

Bodové odhady

1) $X_1, X_2, \dots, X_n \sim \underline{U(0,\vartheta)}$ 

$\to$ metoda moment.

a) $\underline{m_1(\vartheta)} = \mathbb{E}X_1 = \frac{\vartheta}{2}$ $\quad \leftarrow \frac{0+\vartheta}{2}$

nestranný; konzist. odhad $\quad \mathbb{E}X_1 = \frac{\vartheta}{2}$ parametr, neznámé číslo

$\widehat{m_1(\vartheta)} = \frac{X_1 + \dots + X_n}{n} = \overline{X}_n$

náh. veličiny

$\hat{\Theta}_2 := 2\,\overline{X}_n$

funkce $X_1, \dots, X_n$

$\to$ metoda max. věr.

b) $L(x_1, \dots, x_n; \vartheta) = \underline{\prod_{i=1}^{n} f(x_i; \vartheta)} = \left(\frac{1}{\vartheta}\right)^n$ $\quad = 0$ pokud naopak

$\underline{f_{X_1 \dots X_n}(x_1, \dots, x_n; \vartheta)}$

max. $\frac{1}{\vartheta^n}$

$\vartheta \geq \underline{\max\{X_1, \dots, X_n\}}$

$\hat{\Theta}_1 := \max\{X_1, \dots, X_n\}$

funkce $X_1, \dots, X_n$

c) výběr. průměr, stř. hodn.

$\mathbb{E}\hat{\Theta}_{n,2} = 2\,\mathbb{E}\overline{X}_n = 2\,\mathbb{E}X_1 = \vartheta$ --- funguje vždy, když použijeme M.M.

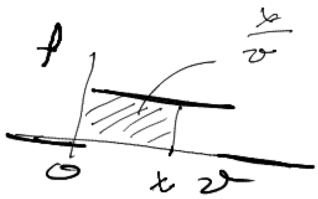
$\mathbb{E}\hat{\Theta}_{n,1}$

$\overline{X}_n = \frac{X_1 + \dots + X_n}{n}$

$\mathbb{E}\overline{X}_n = \frac{\mathbb{E}X_1 + \dots + \mathbb{E}X_n}{n}$

linearita stř. hodn.

$M = \max\{X_1, \dots, X_n\}$



$F_M(x) = P(M \leq x)$ (vlastnost max)

$= P(X_1 \leq x \;\&\; X_2 \leq x \;\&\; \dots \;\&\; X_n \leq x)$

$= P(X_1 \leq x) \cdot P(X_2 \leq x) \dots$ (nez.)

$= \left(\frac{x}{\vartheta}\right)^n$ pro $x \in (0, \vartheta)$, jinak 0

$f_M(x) = F_M'(x) = \frac{n}{\vartheta} \cdot \left(\frac{x}{\vartheta}\right)^{n-1}$ $\quad [0, \vartheta]$

$\mathbb{E}M = \int_0^{\vartheta} x \cdot f_M(x) = \int_0^{\vartheta} \frac{n}{\vartheta^n} x^n = \frac{n}{\vartheta^n} \cdot \left[\frac{x^{n+1}}{n+1}\right]_0^{\vartheta} = \frac{n}{n+1}\vartheta$

$\hat{\Theta}_{n,1} = M$ není nestranný odhad! ale je asympt. nestr.

$\hat{\Theta}_{n,3} := \frac{n+1}{n} \cdot M = \frac{n+1}{n} \max\{X_1, \dots, X_n\}$ je nestr. odh.

$\mathbb{E}(\hat{\Theta}_{n,3}) = \frac{n+1}{n} \cdot \mathbb{E}M = \vartheta$

$\to \hat{\Theta}_{n,2}$ je konzist. dle ZVČ $\quad \overline{X}_n \xrightarrow{P} \frac{\vartheta}{2} = \mathbb{E}X_1$

$\overline{X}_n \xrightarrow{P} \frac{\vartheta}{2}$ $\overset{\text{DEF}}{\iff}$ $P\left(\left|\overline{X}_n - \frac{\vartheta}{2}\right| > \varepsilon\right) \longrightarrow 0$ $\forall \varepsilon > 0$

$\Downarrow$ $\qquad\qquad$ $\parallel$

$P\left(\left|2\overline{X}_n - \vartheta\right| > 2\varepsilon\right) \longrightarrow 0$

$2\overline{X}_n \xrightarrow{P} \vartheta$

$\longrightarrow$ $\hat{\Theta}_{n,1}$ je konzistentní: $M = \max(X_1, \dots, X_n) \xrightarrow{P} \vartheta$ $\quad \le \vartheta$

$$P(|M-\vartheta| > \varepsilon) \xrightarrow{n\to\infty} 0 \ ?$$

$X_1, \dots, X_n \in [0, \vartheta]$

$\le \vartheta$

$$P(M < \vartheta - \varepsilon) = P(M \le \vartheta - \varepsilon) \quad \text{(díky spojitosti)}$$

$M - \vartheta > \varepsilon$ nebo $M - \vartheta < -\varepsilon$

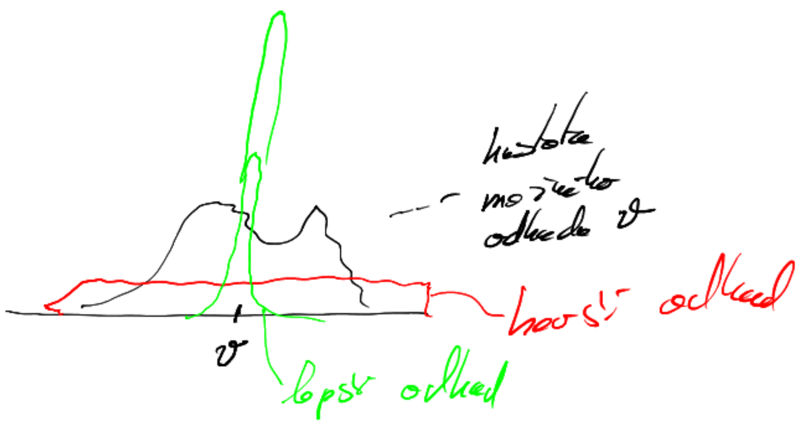
$M > \vartheta + \varepsilon$ nebo $M < \vartheta - \varepsilon$

nejde

$$= F_M(\vartheta - \varepsilon)$$

$$= \left(\underbrace{\frac{\vartheta - \varepsilon}{\vartheta}}_{<1}\right)^n \xrightarrow{n\to\infty} 0$$

$\longrightarrow$ $\hat{\Theta}_{n,3}$ je konzist.



d) $MSE = \mathbb{E}\left((\hat{\vartheta} - \vartheta)^2\right) = \text{bias}^2 + \text{var}\,\hat{\vartheta}$

def. odhad spr. hodn. věta

$\boxed{E\hat{\theta} - \vartheta =: \text{bias}}$

$\hat{\theta}_{n,2}$

$\text{var}\, 2\bar{X}_n = \text{var}\, \frac{2}{n}(X_1 + \dots + X_n) = \frac{4}{n^2}(\text{var}\, X_1 + \dots + \text{var}\, X_n)$ $\quad U(0,\vartheta)$

$MSE = \frac{4}{n^2} \cdot n \cdot \frac{\vartheta^2}{12} = \frac{\vartheta^2}{3n}$

$\text{var}\, M = \mathbb{E}(M^2) - \mathbb{E}(M)^2 = \vartheta^2\left(\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2\right) = \vartheta^2 \frac{n(n+1)^2 - (n+2)n^2}{(n+2)(n+1)^2}$

$\int_0^\vartheta x^2 f_M(x) = \int_0^\vartheta x^2 \frac{n}{\vartheta}\left(\frac{x}{\vartheta}\right)^{n-1} = \frac{n}{\vartheta^n} \int_0^\vartheta x^{n+1} = \frac{n}{\vartheta^n}\left[\frac{x^{n+2}}{n+2}\right]_0^\vartheta$

$= \frac{n}{\vartheta^n} \frac{\vartheta^{n+2}}{n+2} = \frac{n \cdot \vartheta^2}{n+2}$

$M \ \hat{\vartheta} \quad \hat{\vartheta} \quad \vartheta$

$\vartheta^2 \cdot \frac{n}{(n+2)(n+1)^2} \leq \frac{\vartheta^2}{n^2}$

$\text{bias}(M) = \mathbb{E}M - \vartheta = \frac{n}{n+1}\vartheta - \vartheta$

$MSE = \frac{\vartheta^2}{n^2} + \underbrace{\left(\frac{n}{n+1}\vartheta - \vartheta\right)^2}_{\frac{\vartheta}{n+1}\ (\text{bias})^2} = \frac{\vartheta^2}{n^2} + \frac{\vartheta^2}{(n+1)^2} = \frac{2\vartheta^2}{n^2}$ ✓

ϑ^*: bias = 0

$\text{var}\, \vartheta^* = \frac{(n+1)^2}{n^2} \cdot \vartheta^2 \cdot \frac{n}{(n+2)(n+1)^2} = \boxed{\frac{\vartheta^2}{n(n+2)}}$

$P(|\vartheta^* - \vartheta| \geq \varepsilon) \leq \frac{\text{var}\,\vartheta^*}{\varepsilon^2} = \frac{\vartheta^2/\varepsilon^2}{n(n+2)} \to 0$

konzistentní odhad

② $X_1, \dots, X_n \sim Exp(\lambda)$ / $Exp^*(\vartheta)$ $\quad \vartheta = \frac{1}{\lambda}$

a) $m_1(\lambda) = \frac{1}{\lambda} = \vartheta$ $\quad \hat{\lambda} = \frac{n}{X_1 + \dots + X_n}$ $\quad \boxed{\hat{\vartheta} = \frac{X_1 + \dots + X_n}{n}}$

$\widehat{m_1(\lambda)} = \frac{X_1 + \dots + X_n}{n} = \bar{X}_n$ $\quad \boxed{f(x) = \lambda e^{-\lambda x}}$ $\quad F = 1 - e^{-\lambda x}$ $\quad \boxed{1 - e^{-\frac{x}{\vartheta}}}$

$F(x) = P(X \le x) =$

b) $L(x_1, \dots, x_n; \lambda) = \prod_i \lambda e^{-\lambda x_i}$
$x_1, \dots, x_n > 0$

$\ell(\qquad) = \sum_i (\log(\lambda) - \lambda x_i)$

max $\ell(\vartheta)$: $\quad \ell'(\lambda) = \sum_i \left(\frac{1}{\lambda} - x_i\right) = 0$ $\quad \boxed{\hat{\vartheta} = \frac{\sum x_i}{n} = \bar{X}_n}$

$\sum x_i = \frac{n}{\lambda} = n\vartheta$ $\quad \hat{\lambda} = \frac{n}{\sum x_i}$

$E\hat{\vartheta} = E\bar{X}_n = EX_1 = \vartheta$ $\quad E\hat{\lambda} = E\frac{n}{\sum X_i}$

MSE: var $\hat{\vartheta}$ = var $\bar{X}_n = \frac{1}{n} \cdot$ var $X_1 = \frac{1}{n} \cdot \frac{1}{\lambda^2}$
$= \boxed{\frac{1}{n} \cdot \vartheta^2}$

odhad ϑ je nestranný – dle ZVČ konzistentní!

(3) $X_1, \dots, X_n \sim Geom(p)$

MM $\mathbb{E}X_i = m_1(p) = \frac{1}{p} \longrightarrow \boxed{\hat{p} = \frac{1}{\overline{X}_n}}$

$\widehat{m_1(p)} = \overline{X}_n$

$L(x_1, \dots, x_n; p) = - - - - P(X_1 = x_1, \dots, X_n = x_n)$

MLE $L = \prod_{i=1}^{n} (1-p)^{x_i - 1} p > 0$

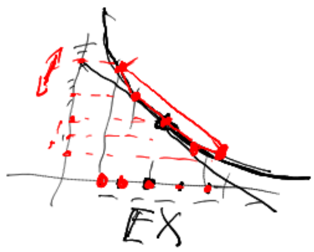
$\ell(p) = \sum \left[(x_i - 1) \log(1-p) + \log p \right]$

$\ell'(p) = \sum \left(\frac{-x_i}{1-p} + \frac{1}{1-p} + \frac{1}{p} \right) = 0$

$\sum x_i = n + n \cdot \frac{1-p}{p} = \frac{n}{p} \rightarrow \hat{p} := \frac{n}{\sum_{i=1}^{n} x_i}$

$= \frac{1}{\overline{X}_n}$

$\mathbb{E} \frac{1}{\hat{p}} = \mathbb{E}\overline{X}_1 = \frac{1}{p}$



$\mathbb{E}\hat{p} = \mathbb{E}\frac{1}{\overline{X}_n} > \frac{1}{\mathbb{E}\overline{X}_n} = \frac{1}{1/p}$

$g(x) = \frac{1}{x}$ Jensen

$X = \overline{X}_n$

$g(\mathbb{E}X) \leq \mathbb{E}g(X)$

$\mathbb{E}\hat{p} > = p$

$=$ jen pro X konst. nebo g lineární

$\mathbb{E}\hat{p} - p = O\left(\frac{1}{n}\right)$

asymptot. nestranný

$\mathbb{E}\frac{1}{X} \neq \frac{1}{\mathbb{E}X} = \dots$

(4) $X_1, \ldots, X_n \sim Exp(\lambda)$

$$p = P(X_i > 1) = e^{-\lambda}$$

var 1 : spočteme $\hat{\lambda} = \frac{1}{\overline{X_n}}$, zkusíme $\hat{p} = e^{-\hat{\lambda}}$

var 2 : $Y_i = \begin{cases} 1 & X_i > 1 \\ 0 & X_i \leq 1 \end{cases}$ $\quad Y_i \sim Bern(p)$

odhad $\hat{\hat{p}} = \overline{Y_n}$

$$= \frac{\# i : X_i > 1}{n}$$