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S.I.: RECENT DEVELOPMENTS IN FINANCIAL MODELING AND RISK MANAGEMENT

A decision-dependent randomness stochastic program for asset–liability management model with a pricing decision

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Abstract
In this study, we present a stochastic programming asset–liability management model which deals with decision-dependent randomness. The model focuses on a pricing problem and the subsequent asset–liability management problem describing the typical life of a consumer loan. Such problems are frequently tackled by many companies, including multinationals. When doing so, they must consider numerous factors. These factors include the possibility of their customer rejecting the loan, the possibility of the customer defaulting on the loan and the possibility of prepayment. The randomness associated with these factors have a clear relationship with the offered interest rate of the loan which is the company's decision and thus, induces decision-dependent randomness. Another important factor, which plays a major role for liabilities, is the price of money in the market. This is determined by the market interest rates. We captured their evolution in the form of a scenario tree. In summary, we formulated a non-linear, multi-stage stochastic program with decision-dependent randomness, which spanned the lifetime of a typical consumer loan. Its solution showed us the optimal decisions that the company should make. In addition, we performed a sensitivity analysis demonstrating the results of the model for various parameter settings that described different types of customers. Finally, we discuss the losses caused if companies do not act in the optimal way.

Keywords Stochastic programming · Decision-dependent randomness · Asset–liability management

Mathematics Subject Classification 90-04 · 90C15 · 90B50

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1 Introduction

Stochastic programming is a quickly developing area of mathematical optimization that is applied in various fields, especially the area of financial planning and control. One such application occurred in 1986 when Kusy and Ziemba published their famous paper dealing with a bank asset–liability model (Kusy and Ziemba 1986). Two years later, Dempster and Ireland (1988) introduced a different model that focused on the immunisation of liabilities and for the first time took into account the risks that are associated with financial problems. Carino et al. (1994) presented the very first asset–liability model for an insurance company, which was later followed by many others. There have also been several applications focused on insurance companies (Hoyland 1998; Pliska and Ye 2007; Broeders et al. 2009) and pension funds (Dert 1995; Consigli and Dempster 1998; Geyer and Ziemba 2008; Dupačová and Polívka 2009); these are the two types of financial institutions where stochastic programming was used most often.

These works, however, do not exhaust the potential of applications for stochastic programming in asset–liability management. For example, Consigli (2008) investigated the problem of an individual investor who needed to undertake investment decisions and manage his consumption. Kopa et al. (2018) analysed the effect of new, modern risk constraints on the optimal solution as they applied a second-order stochastic dominance constraint. An important aspect of the modelling exercise is to focus on the accuracy of model formulation. Vitali et al. (2017) and Moriggia et al. (2019) paid a lot of attention to this aspect, considering a variety of investment possibilities. Recently, Consigli et al. (2019) studied optimal decisions from the household point of view and Zapletal et al. (2019) investigated optimal policies in emission management of a steel company.

So far, in all the stochastic programming models of asset–liability management that we have seen, authors look for optimal decisions of a single agent (insurance company, pension fund, etc) in the market. Such market is always considered large enough to justify assuming independence of evolving market prices from agent's strategies, and thus risk sources are treated as exogenous. In our work, we focus on a stochastic programming formulation of an asset–liability management problem describing a consumer loan that a company gives to its customer. This acknowledges a relationship between the two parties. In this situation, the customer's behaviour is affected by the company's actions. This phenomenon is known as decision-dependent/endogenous uncertainty/randomness within the stochastic programming; it has not yet been considered in stochastic programming models of asset–liability management. The formulation and the analysis of such a model is the main contribution of this article.

The issue of endogenous uncertainty was first addressed by Pflug (1990) who investigated a general Markovian process in which states depended on the decision parameter of the model. In the following years, more models with decision-dependent randomness have emerged; these can be generally divided into two types. In the first type, decisions taken in the program help to determine uncertainty in the model. This case was tackled, for example, by Jonsbraten et al. (1998), Goel and Grossmann (2004) and Tarhan et al. (2009) who all described models for finding the optimal strategy in an offshore oil field development plan. The general idea was that companies could decide to run exploratory analyses on potential oil fields, obtain a better estimate of how much oil is available and determine where it is more profitable to set up a plant. In the second type, as is the case of our problem, decisions affect scenario probabilities. This type of endogenous uncertainty was thoroughly investigated, for example, in network flows problems (Ahmed 2000; Held and Woodruff 2005; Vishwanath et al. 2004).

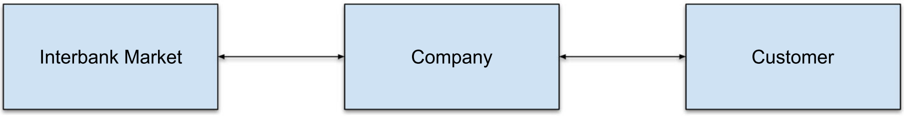

Fig. 1 Economic agents which enter the optimization model

Under this setting, the probabilities of scenarios depend directly on decisions taken and they can be also considered as variables in the program.

In financial problems, the decision-dependent uncertainty is typically observed within a bilateral relationship as in our case. One party acts as a price setter and its pricing decision affects the demand for some good of the second party. In mathematical terms, this can be translated to observing a change in the underlying probability distribution of the demand (endogenous randomness). Apart from demand distribution, probabilities of some subsequent events such as default and prepayment can also be affected. Other decisions, such as marketing decisions or branding investments, also indirectly affect the company's books and lead to decision-dependent uncertainty.

In this paper, we analyse a program describing the lifecycle of a loan which a company provides to an individual customer. We present the model formulation in detail in Sect. 2. Among other things, we also discuss what decisions can be taken and how they affect the uncertainties the company faces. The main decision is the offered interest rate which affects the probabilities of random events (accepting the loan, prepayment, default). We describe scenarios of all random processes, their implementation into the program and all constraints which are part of the model formulation. Thereafter, in Sect. 3, we show the results of this program for one parameter's setting to illustrate the optimal decisions and how the loan value would evolve depending on the company decisions. Furthermore, we discuss the effect of the customer's properties on the model solution, especially on the offered interest rate and the expected value of the loan. We also mention the losses which are incurred by the company if it does not behave in the optimal way. Our findings are then summarized and concluded in Sect. 4.

2 Model formulation

2.1 Model environment and objective

First, let us describe the situation and the time-frame on which the optimization model is built. We consider three economic agents:

– A credit institution operating in the interbank market,
– A company, or lending entity within this problem framework, that will borrow money in the interbank market and give loans to consumers,
– Customers, or borrowing agents, who through the loan will be in condition to finalize the purchase of a good.

We illustrate the relationship of these three agents in Fig. 1. The individual customer approaches the company and asks for a loan. On the other hand, the company borrows money from the financial market to obtain sufficient funds.

The primary objective of this paper is to construct and analyse an optimization model (often called a program) which a company can use to determine its decisions. These decisions shall be taken so the company maximises the expected value of the loan (the expected profit) at the agreed loan maturity. More specifically, at every point in time, we define the term *value of the loan* as the sum of cash and present value of assets and liabilities resulting from the loan contract with the customer (see, later, Eq. (14)). Consequently, all the loans from the market which are used to finance the consumer loan are also included. The value of the loan changes over time. This depends on the market interest rate evolution and the customer's behaviour (possible loan default or prepayment). These factors are, however, not known a-priori, so they are treated as random.

Next, let us highlight where the main difficulty in the asset–liability stochastic programming model arises. Usually, random effects are assumed to be exogenous. In other words, no decisions taken by the decision maker in the optimization model affect the uncertain elements entering the model. However, this is not the case in this situation. Here, the company's initial decision is to offer the customer a loan with a fixed interest rate. This decision directly affects the probability of the customer accepting the loan offer. This is a random event (from the company's point of view), and it is endogenous—dependent on the initial decision. The probability of the customer prepaying or defaulting also depends on the offered interest rate. The strength of this relationship and how this endogenous uncertainty is dealt with is discussed in detail in Sect. 2.2.

In such loan operations, a common practice is for the lending agencies to borrow in the market so that they can match asset and liability cash flows from the start of the loan. However, Rusy and Kopa (2019) showed that such a strategy is not optimal as there are alternative strategies which have better risk properties and higher profitability. For this reason, we give numerous options to the company on how to form and optimize its liability side. The fact that we jointly optimize the initial interest rate decision and the company's borrowings allows us to consider the different probabilities of cash flows that stem from the decision-dependent randomness and adjust the company's borrowing strategy accordingly. This joint optimization of the pricing problem and the consequent asset–liability problem is a key feature of the model and the main contribution of this paper. It provides us with the optimal decisions corresponding to the genuine nature of the problem.

In this work, we restrict ourselves to the simplest loan type—a non-collateralized consumer loan with fixed maturity and fixed interest rate. However, this framework can easily be extended to a variety of other problems which combine pricing decision of some good together with subsequent action depending on the demand for the good. There, we have a decision-dependent randomness between the price and the demand. The framework of stochastic programming is then flexible enough to take into account numerous features which are connected to special properties of the good. From the financial perspective, other products which could be modelled are, for example, mortgages or pension/building savings. There, the structure of the resulting actions and cash flows is more complex, but the general idea is the same.

We assume that a customer comes to the company and wants to borrow N_0 amount of money for a period of T months.[1] Afterwards, the company offers the customer an interest rate r for such a loan. We assume this is the only cost the company charges the customer. Next, the customer decides whether to accept or reject the offer. If the loan is agreed upon, we proceed further to model the life of the consumer loan, which is repaid regularly each month $t, t = 1, \dots, T$ by equal instalments.

[1] Note that usually the principal N_0 is set by the company. However, as it enters the model only as a *scale* parameter (multiplier of the objective function), we treat it as fixed—for example, determined by a risk management unit of the company.

The multi-stage stochastic optimization program will, however, consist only from $K+1 \leq T+1$, $K \in \mathbb{N}$ stages, $0 = t_0 < \ldots < t_K = T$. At these times, scenarios of other random quantities (interest rate evolution, customer's loan default or prepayment) will be observed and the company will be able to make decisions. In other times, no decisions are made; only instalments are paid. The decisions will define cash flows between the company and the market and form the liability side of the company. In this paper, we use time index $t, t \in \{0, \ldots, T\}$ to denote months within the duration of the loan; index $k, k \in \{0, \ldots, K\}$ and times t_k denote decision stages of the optimization program. We have $\{t_k, k = 0, \ldots, K\} \subset \{0, \ldots, T\}$. In some cases, indices i, j are also used to iterate over the set of decision times.

2.2 Random elements, scenarios and decision-dependent randomness

In this section, we will describe four random elements which are part of the life of the loan in the model. They are represented by the following questions:

- Will the customer accept the loan?
- If yes, will he afterwards prepay the loan?
- Will he default on the loan?
- What will be the evolution of market interest rates?

The first three elements describe uncertain customer behaviour. They are all considered endogenous as they depend on the interest rate decision. The fourth element, which is considered exogenous, captures the evolution of prices in the financial market.

2.2.1 Probability of accepting the loan

Once the customer is offered a loan with a specific interest rate, there is the possibility that he will either accept or reject it. The probability that the customer accepts the loan offer and enters into a contract with interest rate r is denoted by $p(r)$. This function is customer-specific. Here, multiple customer-dependent factors can play a role, for example the customer's knowledge of market conditions or alternative offers from other market participants. We assume that the relationship $p(r)$ is estimated by logistic regression, where one of the regressors is the interest rate offered to the customer. However, in general, any functional form which describes the desired relationship could be used. We employ a function:

$$p(r) = \frac{\exp\{b_1(b_0 - r)\}}{1 + \exp\{b_1(b_0 - r)\}} = \frac{1}{1 + \exp\{-b_1(b_0 - r)\}}, \tag{1}$$

where b_0 and b_1 are [customer-dependent] parameters. Parameter b_0 represents the rate at which the customer is indifferent to accepting or rejecting the loan. We will call this *midrate* and it holds that $p(b_0) = 0.5$. The second parameter, b_1, expresses the customer's sensitivity to interest rates as it captures the effect on the probability of accepting the loan when we deviate from midrate by a certain amount. This can be quantified exactly by the usual interpretation of logistic regression models. When $b_1 = 100$, an increase of 1% in the offered rate r implies a decrease in the odds ratio (accept/reject) by a factor of $e^{-b_1/100} = e^{-1}$.

For a given client, one can estimate the value of midrate and sensitivity by a simple logistic regression. Consider a dataset where the response is a successful or unsuccessful offer while regressors are the offered interest rate and customer's properties. Then, a model can be formulated, such that the first order terms of customer's properties define the midrate and covariate interactions with the interest rate specify the sensitivity relationship. The fitted model together with covariates of new customer would generate estimates of parameters b_0 and b_1.

This random event (accepting/rejecting the loan) is realized before issuing the loan. If the customer's decision is to reject, no cash-flows take place and the value of the loan is 0. On the other hand, if the decision is to accept, then the loan is issued, the company makes an initial decision on how to borrow at the market and cash-flows are exchanged.

The quantity $p(r)$ captures the essence of endogeneity of this random event. It links the company's decision to the random variable's distribution.

2.2.2 Interest rate evolution

Next, we introduce market interest rates which express the cost of money at the market. We denote y_t^τ to be the annualized, risk-free interest rate with time-to-maturity τ at time t. For $t > 0$, this quantity is random. It is considered exogenous because the company is not thought to be able to affect its evolution. We also denote m_t^τ to be the rates for which the company can borrow from a market participant—a bank. We define: $m_t^\tau = y_t^\tau + m(\tau)$.

Quantity $m(\tau)$ represents the spread between the risk-free rates and the rates which the bank charges the company. This is fixed over time and can be interpreted as the mark-up of the bank. In other words, we assume that the company has a contract with the bank regarding floating rate borrowing. In the numerical part, the values of mark-up $m(\tau)$ were set as follows. We defined $m(0) = 0.0048$, $m(2) = 0.0096$ and $m(5) = 0.0132$, where τ is in years. Values in between were obtained by linear interpolation.

The risk-free rates y_t^τ are modelled by the Hull–White model formulated by Hull and White (1990), which belongs to the class of one-factor short-rate models introduced by Vasicek (1977). It is defined by the following stochastic partial differential equation:

$$dr_t = (\theta(t) - \alpha r_t)dt + \sigma dW_t,$$

where r_t stands for short-rate and W_t is the standard Brownian motion. Parameter α stands for the mean reversion factor and σ summarizes the volatility of the short-rate. Finally, $\theta(t)$ is set such that the observed market prices are fitted perfectly, i.e:

$$\theta(t) = \frac{\partial f^M(0,u)}{\partial u}\Big|_{u=t} + \alpha f^M(0,t) + \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right),$$

where $f^M(0,t)$ is the market instantaneous forward rate at time 0 for time t. We employ the usual starting condition $r_0 = f^M(0,0)$ and calculate the yields y_t^τ via the formulas for zero-coupon bond prices (Brigo and Mercurio 2001). Thanks to the fact that the Hull–White model uses exogenous information in the form of the observed market yield curve, predictions of yields based on this model are close to market expectations. The calibration of the model's parameters was inspired by Chen and Scott (1993) who estimated the Cox–Ingersoll–Ross model of Cox et al. (1985) by the maximum likelihood method on observed yields. For this estimation, we used the daily PRIBOR rates observed on the Czech market from 28th June 2015 to 1st March 2018. The estimated values of the parameters were $\hat{\alpha} = 0.1346, \hat{\sigma} = 0.006427$. Such an estimation procedure is built on numerous model assumptions. The normality of short-rate movements and their autocorrelation require particular attention. From *acf* and *pacf* plots, we concluded that no autocorrelation is present, which is a consequence of the fact that the Hull–White model uses the observed market curve for future predictions. For testing normality, we used the Shapiro–Wilk test, which gave us a p value of 0.00545. At the usual significance level, we would reject the normality hypothesis. This is mainly due to 4 outliers, which correspond to jumps that are a little larger than one would expect in normal distributions. However, because prediction based on this model looked reasonable, we decided to accept and use it in the scenario generation procedure.

In our model, we capture the interest rate evolution in the form of a regular scenario tree, often seen in multi-stage stochastic programming, such that in each stage every node has the same number of successors. Moreover, in such a tree, all scenarios are equiprobable. We will denote $S_k, k \in \{0, \ldots, K\}$ as the set of nodes of the interest-rate tree in a decision stage t_k and $a_i(s_k)$ as the time t_i ancestor of a node $s_k \in S_k, 0 \leq i < k$. We will also denote $y_{t_k}^{\tau}(s_k)$ as the risk-free interest rate and $m_{t_k}^{\tau}(s_k) = y_{t_k}^{\tau}(s_k) + m(\tau)$ as the rates for the company at time t_k with time to maturity τ in the scenario node $s_k \in S_k$.

The scenarios of the short-rate were chosen to be the quantiles of the model-implied distribution (conditioned on the observed value of the short-rate in the ancestor node). We derived the term structure $y_{t_k}^{\tau}(s_k)$ and discount factors $P(t_k, t_k+\tau; s_k)$, $\tau > 0$ from the short-rate based on well-known formulas for zero-coupon bond prices implied by the Hull–White model (see, for example, Brigo and Mercurio (2001) for more details).

2.2.3 Probability of default and prepayment

The final two types of randomness which enter the model are loan prepayments and customer defaults. We treat these effects as endogenous, as they are closely connected to the offered rate of the loan. This, however, introduces another decision-dependent randomness into the model, which is present in all stages of the multi-stage program. First, let us describe why we need to take these effects into account in the model.

Customer defaults are generally considered to be the biggest risk factor affecting the profitability of a loan. This is simply because it can happen that the customer becomes unable to fulfil his commitments. Under such an event, the company loses not only the interest rate charged to the customer but also a part of its principal. Such a proportion is called *loss given default* and we include it in the model as a fixed parameter, denoted as lgd and set to a value of 0.5. If the customer defaults at time t_k, the company is modelled to receive $(1 - \text{lgd})$ times the remaining principal as the recovered part of the loan. Scenarios of customer defaults will be added to the model. Thereafter, we assign them probabilities so they match our initial assumption about hazard rates $h(t_k, r)$—the probability of default at time t_k given that the loan with interest rate r survived up to time t_{k-1}.

Prepayment means that the customer repays more money than it was scheduled in the original contract. Usually, such a prepayment comes from one of the two following reasons:

– The customer has spare money which he can afford to use for loan prepayment.
– The customer finds a cheaper loan and he refinances it.

The first reason is unconnected to the decision variables or random quantities in the problem. On the other hand, the second reason is closely related to the interest rate of the loan. Naturally, if the price of the loan is too high, the customer is more likely to look for cheaper options at the market and, thus, more likely to repay the loan earlier. Therefore, this random effect is also endogenous. Similar to customer defaults, we add scenarios of loan prepayments into the model and assign them probabilities to match our assumptions about the hazard rate $g(t_k, r)$ of prepayment at time t_k of the loan with interest rate r. By loan prepayment, we mean only full prepayment of the loan, so when it occurs, the remaining principal is repaid to the company.

Next, let us describe how we determine the hazard rates for default $h(t_k, r)$ and prepayment $g(t_k, r)$. We formulate both the default and the prepayment model on the following ideas:

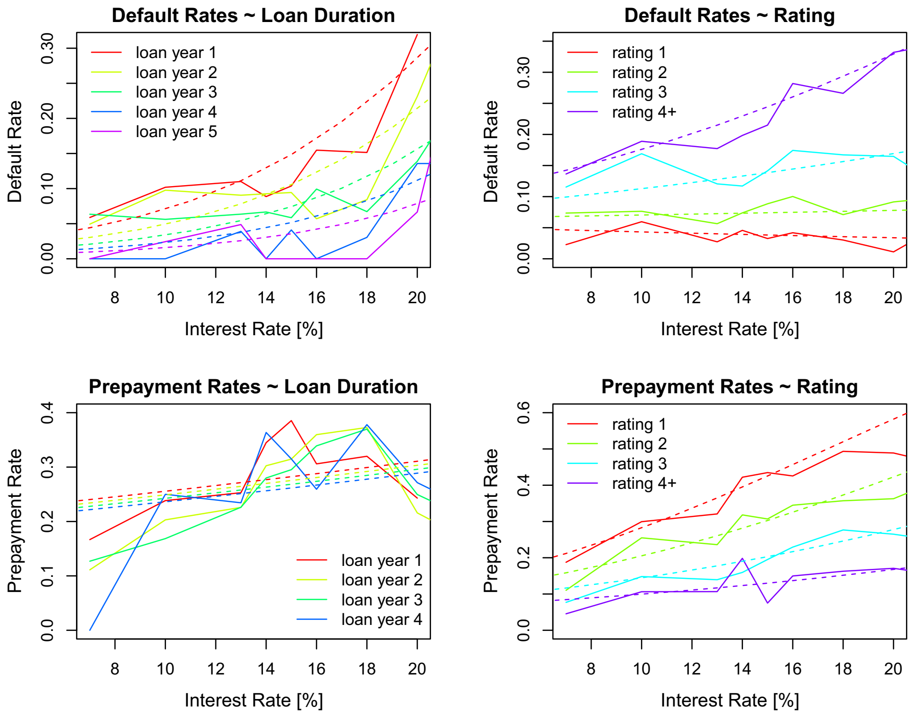

Fig. 2 Prepayment and default rates and fitted probabilities in analysed dataset. Dashed lines show the fitted logistic model

– The probability depends on the interest rate, time to maturity of the loan and the initial rating of the customer.
– There might be an interaction between the interest rate and the rating of the customer.

From there, we formulate a logistic regression model:

$$\text{prob}(t_k, r, \rho) = \frac{\exp(\eta)}{1 + \exp(\eta)}, \qquad \eta = \beta_0 + \beta_1 r + \beta_2 \rho + \beta_3 t_k + \beta_4 \rho \cdot r, \tag{2}$$

where $\rho = \{1, 2, 3, 4\}$ denotes rating of the customer[2] and $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$ are parameters. Symbol $\text{prob}(t_k, r, \rho)$ stands for the probability that a loan with interest rate r of a customer with rating ρ will be defaulted/prepaid by the time t_k, given that it survived till time t_{k-1}.

We fitted the model to real market data of a Czech company operating in the industry and tested whether models in (2) could be used to capture the relationship. The data consisted of all [17 554] consumer loans of the company with maturity between 5 and 6 years active at one point in time. We had the initial rating of the customer (which was valid when the loan was agreed) and we observed whether the loans were repaid or defaulted in the following year. The observed and fitted default and prepayment probabilities are summarized graphically in Fig. 2, while the estimated parameters are shown in Table 1. For better readability and interpretation, values of interest rate r are thought to be in percent. To obtain values of

[2] We should note that we considered four different ratings of a customer, from the best rating (1) up to the worst rating (4). A few customers with a rating worse than 4 on the usual scale $1 - 8$ were assigned rating 4 for this analysis.

Table 1 Estimated parameters of models for probability of default and prepayment together with McFadden R^2

Model	Symbol	R^2	β_0	β_1	β_2	β_3	β_4
Prepayment	$g(t_k, r)$	0.053	-1.93^{***}	0.18^{***}	-0.17^{*}	-0.21^{***}	-0.028^{***}
Default	$h(t_k, r)$	0.106	-2.93^{***}	-0.033	$0.20^{\cdot}$	-0.22^{***}	0.031^{***}

Asterisks denote a statistical significance of the coefficient: *** denotes a p value smaller than 0.001, * smaller than 0.05, and $\cdot$ smaller than 0.1

$g(t_k, r)$ and $h(t_k, r)$, one needs to use a given value of rating ρ. The corresponding intercept and coefficients for interest rate and time will be denoted $\beta_0^g, \beta_1^g, \beta_3^g$ and $\beta_0^h, \beta_1^h, \beta_3^h$ respectively.

Let us briefly comment on the interpretation of this model. The coefficient estimates are very much in-line with our expectations. The parameter estimates show that defaults occur in early stages of the loan duration and that there is a strong relationship between interest rate and default for customers with bad rating. On the other hand, for prepayment, we can see that customers with good rating have high prepayment rate for high interest rate loans. This interaction also has reasonable interpretation. In some of the plots, the fitted values seem to be far from the observed quantities. This is partly due to few observations in these areas (for example, low- and high-interest rate). Finally, we would like to stress that this analysis was performed in order to get a reasonable (real) estimate of the relationship between interest rates and default/prepayment probabilities. We could have described the econometric relationship in more detail. However, this will increase model complexity and we might lose its computational tractability.

Now, let us discuss how we implement the scenarios of loan prepayments and customer defaults into the program. First, we already have $|S_K|$ interest rate scenarios from the initial decision period to the maturity of the loan. Take one scenario as fixed and on every node/stage, loan prepayment or customer default can occur. Note that during the life of the loan, only one customer default or loan prepayment can take place. Hence, each program scenario can be defined as a pairing of the interest rate scenario and the event specification. Events are formulated to be loan prepayments or customer defaults at any stage $t_k \neq t_0$. For example, default at stage t_2 is considered to be one event. This implies, that we have in total $2 \cdot K$ events. We should also mention that loan prepayment at time $t_K = T$ corresponds to the loan being repaid at the initially agreed time.

Mathematically speaking, we define set E as a set of all possible events. For event $e \in E$, we define $t(e)$ to be the time when the event occurs, $d(e)$ an indicator function which is 1 when the event is customer default and 0 when the event is full prepayment. So, for example, for the event that the customer defaults at stage t_2, we have $t(e) = t_2, d(e) = 1$. These functions are important for determining scenario probabilities as in (3) (calculated from conditional probabilities of default and prepayment) and for the specification of the budget equation given in (11). The scenario of a program is then uniquely defined by a pair $(s, e), s \in S_K, e \in E$.

What is important to realize under this parametrisation is that at time t_k, we cannot distinguish between two program scenarios $(s, e_1), (s, e_2), s \in S_K, e_1, e_2 \in E$ such that $t_k < \min\{t(e_1), t(e_2)\}$. This is simply because by time t_k, we do not observe the nature of event e. Such a property will lead to the inclusion of non-anticipativity constraints into the program.

2.2.4 The probabilities of scenarios

In the final part of this section, we will comment on how we calculate probabilities of scenarios $p(s, e, r), s \in S_K, e \in E$ where r is the initial interest rate decision. These are obtained iteratively by the multiplication of hazard rates in each node—a probability that one reached this node multiplied by the probability that from this node, one moves to its child. For event $e \in E$ and time $t_k \leq t(e)$, we use hazard rate $h(t_k, r)$ if default occurs: $\mathbb{I}_{[t(e)=t_k]} = 1, d(e) = 1$, hazard rate $g(t_k, r)$ if prepayment occurs: $\mathbb{I}_{[t(e)=t_k]} = 1, d(e) = 0$ and finally hazard rate $1 - h(t_k, r) - g(t_k, r)$ if the event is not yet observed: $\mathbb{I}_{[t(e)>t_k]} = 1$. Otherwise, when $\mathbb{I}_{[t(e)<t_k]} = 1$, probabilities are distributed equally depending only on the branching of the interest rate tree given by $|S_{k-1}|/|S_k|$. The formula is as follows:

$$
\begin{aligned}
p(s, e, r) = \prod_{k=1}^{K} \Big(& \mathbb{I}_{[t(e)=t_k]} d(e) h(t_k, r) + \mathbb{I}_{[t(e)=t_k]} \big(1 - d(e)\big) g\big(t_k, r\big) \\
& + \mathbb{I}_{[t_k<t(e)]} \big(1 - h(t_k, r) - g(t_k, r)\big) + \mathbb{I}_{[t_k>t(e)]} \Big) \big(|S_{k-1}|/|S_k|\big), \\
& s \in S_K, e \in E.
\end{aligned} \tag{3}
$$

We require that $\forall k \in 1, \ldots, K$ it holds that $h(t_k, r) \geq 0, g(t_k, r) \geq 0$ and also that $\big(1 - h(t_k, r) - g(t_k, r)\big) \geq 0$. Moreover, in the last stage, we must have $g(t_K, r) = 1 - h(t_K, r), \forall s \in S_K$. Under such conditions, the probabilities $p(s, e, r)$ of scenarios are non-negative and sum up to one for all values of interest rate decision r.

The Eq. (3) provides another link between the decision variable r and scenario probability. First, we have a model measuring the effect of the interest rate decision on the probability of moving from each node to its successor as in (2). Then, we calculate the probability of each scenario by multiplication of the hazard rates. Here we use Bayesian conditional probabilities and the Bayes' theorem. This captures the effect of the decision-dependent randomness in defaults and prepayments into the multi-stage program.

2.3 Stochastic programming model formulation

In this section, we will complete the formulation of the asset–liability stochastic programming model. We have already introduced the interest rate decision r and the probability of accepting the loan $p(r)$. We described the time structure of the model—we have months $t, t \in \{0, \ldots, T\}$, where T is the maturity of the loan, and decision times $t_k, k \in \{0, \ldots, K\}$. See Sect. 2.1 for more details.

Scenarios are defined as a pair $(s, e), s \in S_K, e \in E$, where s captures the interest rate evolution and e the event which realizes on the side of the customer. For a scenario (s, e) and offered interest rate r we have the scenario's probability $p(s, e, r)$, see Sect. 2.2.

In the next part, we will describe the evolution of the principal of the loan (Sect. 2.3.1) and define how the company can borrow and lend money in the financial market (Sect. 2.3.2) at the prevailing interest rates. Apart from defining which decisions can be made by the company, we will also derive quantities (such as income at given time and scenario) corresponding to the cash-flows between the company and the market.

In Sect. 2.3.3, we introduce the current account Eq. (11), which links the company's cash-flows with both the customer and the market. Other constraints, such as the non-anticipativity constraints, are also introduced. All the decisions, quantities and equations are then summarized in Sect. 2.3.4, where the entire model is presented in the compact form.

2.3.1 Loan instalment and principal

First, we calculate the value of a single instalment π. This is constant for the duration of the loan. It depends on the interest rate decision r and it is calculated as in Eq. (4). Next, we denote $N_t, t = 1, \ldots, T$, the principal which stays on the account after payment of the instalment in month t. The interest credited from month $t-1$ to t is equal to $N_{t-1} \cdot (r/12)$, while the amortization is $\pi - N_{t-1} \cdot (r/12)$. We can also determine the value of N_1 and other principal amounts $N_t, t = 1, \ldots, T$.

$$N_0 = \sum_{t=1}^{T} \frac{\pi}{(1+r/12)^t} \quad \Rightarrow \quad \pi = N_0(r/12)\Big(1-(1+r/12)^{-T}\Big)^{-1}, \tag{4}$$

$$N_t = N_0 \left(1 - \frac{(1+r/12)^t - 1}{(1+r/12)^T - 1}\right), \quad t = 1, \ldots, T. \tag{5}$$

The principal behaves as we calculated only in the case when the loan is repaid in the way as agreed at the beginning of the contract. In case of a default of the customer or a full prepayment of the loan, the evolution is different. If default occurs at time t_k, the company is thought to receive $\text{lgd} \cdot N_{t_{k-1}}$ at time t_k, while no cash-flows are exchanged between the consumer and the company in months between decision stages t_{k-1} and t_k. When prepayment occurs, the company receives the remaining principal N_{t_k}.

2.3.2 Cost of financing the loan

Another aspect which needs to be considered is the cost of financing such a loan. To obtain sufficient funds, the company could enter exactly the same contract with the market, a practice often seen in the industry. Rusy and Kopa (2019) showed that such a simple approach is not efficient. Hence, we go beyond it and give the company many possibilities to form its liability side, so the company can find the optimal financing strategy. Consider now two time instances t_i and t_j, $t_i < t_j \leq T_K$ of the program. At time t_i, in each node of every scenario (s_i, e), $s_i \in S_i$, $e \in E$, $i = 0, \ldots, K-1$, the company will have two possibilities of borrowing money from the market. It could borrow from t_i to t_j and repay the money monthly with regular instalments (typically funded by the loan with matched dates) or reimburse all costs at the expiry. We will denote such amounts as $u_{t_i,t_j}(s_i, e)$ and $v_{t_i,t_j}(s_i, e)$ respectively. We can calculate the amount $u_{t_i,t_j}(t; s_i, e)$ repaid at time t, $t_i < t \leq t_j$ from a loan $u_{t_i,t_j}(s_i, e)$ as

$$u_{t_i,t_j}(t; s_i, e) = \frac{u_{t_i,t_j}(s_i, e)}{\sum_{\tau=1}^{t_j-t_i} \left(1 + m_{t_i}^{\tau}(s_i)/12\right)^{-\tau}}, \qquad t = t_i + 1, \ldots, t_j. \tag{6}$$

On the contrary, at time t_j, the company pays back $v_{t_i,t_j}(t_j; s_i, e)$ such that

$$v_{t_i,t_j}(t_j; s_i, e) = v_{t_i,t_j}(s_i, e) \left(1 + m_{t_i}^{t_j-t_i}(s_i)/12\right)^{t_j-t_i}. \tag{7}$$

We also make it possible for the company to invest spare money and gain interest. Such an opportunity may arise, for example, when the client unexpectedly prepays the loan. We denote $w_{t_i,t_j}(s_i, e)$ as the amount of money lent to others for the market risk-free yield $y_{t_i}^{t_j-t_i}(s_i)$. This money will be repaid at time t_j, as the company would receive amount $w_{t_i,t_j}(t_j; s_i, e)$:

$$w_{t_i,t_j}(t_j; s_i, e) = w_{t_i,t_j}(s_i, e) \left(1 + y_{t_i}^{t_j-t_i}(s_i)/12\right)^{t_j-t_i}. \tag{8}$$

These decisions allow the company to freely initiate numerous contracts. However, we still require the company to meet all obligations from all previous decisions—i.e. there is no option for prepayment on the company's side. This is summarized in the following equations, which express both the amount of the company's income (I) and expenditure (J) at time instance $t_k, k = 0, \ldots, K$ and node (s_k, e), $s_k \in S_k$, $e \in E$ from the financial market. These are:

$$I_{t_k}(s_k, e) = \sum_{t_i : t_i < t_k} w_{t_i,t_k}(t_k; a_i(s_k), e) + \sum_{t_j : t_k < t_j} u_{t_k,t_j}(s_k, e) + \sum_{t_j : t_k < t_j} v_{t_k,t_j}(s_k, e), \tag{9}$$

$$J_{t_k}(s_k, e) = \sum_{t_j : t_k < t_j} w_{t_k,t_j}(s_k, e) + \sum_{\substack{t, t_i, t_j : \\ t_i \le t_{k-1} < t \le t_k \le t_j}} u_{t_i,t_j}(t; a_i(s_k), e) + \sum_{t_i : t_i < t_k} v_{t_i,t_k}(t_k; a_i(s_k), e). \tag{10}$$

In the income Eq. (9), we sum the money returned from loans provided to other institutions operating in the interbank market maturing at time t_k with the inflows from loans provided to the company by the market in that scenario. In the expenditures Eq. (10), the company needs to pay instalments from loans provided by the market in previous times and also pay the money lent to the market in the given scenario.

The mismatch between the assets and liabilities could cause duration gaps in the optimal solution. Such a portfolio composition may be considered risky and volatile. This property can be controlled by introducing various risk constraints restricting the space of the decision vector in the model. This is, however, a well-studied area of stochastic programming and it goes beyond the aims and objectives of this paper.

2.3.3 Constraints and objective function

We continue with the specification of the remaining constraints implemented in the model. Let us denote $B_{t_k}(s_k, e)$ as the amount of money the company has in its account immediately after time t_k in scenario (s_k, e) and C_{t_k} as the company's operating costs of the loan from time t_k to t_{k+1}. We have:

$$\begin{aligned} B_{t_k}(s_k, e) = {} & B_{t_{k-1}}(a_{k-1}(s_k), e) - C_{t_k} + I_{t_k}(s_k, e) - J_{t_k}(s_k, e) - \mathbb{I}_{[k=0]} N_0 \\ & + \mathbb{I}_{[t_k < t(e)]}(t_k - t_{k-1})\pi + \mathbb{I}_{[t_k = t(e)]} d(e) \cdot \mathrm{lgd} \cdot N_{t_{k-1}} \\ & + \mathbb{I}_{[t_k = t(e)]}(1 - d(e)) \left((t_k - t_{k-1})\pi + N_{t_k}\right), \end{aligned} \tag{11}$$

for $k = 0, \ldots, K$, where $B_{t_{-1}} = 0$, and also $C_{t_K} = 0$. The relationship on the first line of (11) expresses the initial exchange of the principal and the cash-flows between the bank and the company. On the second and the third line, the amount of funds the company receives from the customer in different stages under the scenario $e \in E$ is described. The indicator functions mean the same as described in Sect. 2.2.4.

The definition of the company's cash account brings us to a very natural survival condition such that the company's cash account must not be lower than 0. We require this only in stages from $0, \ldots, K-1$ as, in the last stage, the loan is concluded and we look at the final balance, its performance through its life and asses its profitability. We require:

$$B_{t_k}(s_k, e) \ge 0, \qquad e \in E, s_k \in S_k, \quad k = 0, \ldots, K-1. \tag{12}$$

Next, we move to the cash-flows which take place between decision stages. The company has to make sure it has enough money to cover its expenditures up to the next decision stage. Such a liquidity constraint can be implemented only by checking whether the company has enough funds in the month before the next decision stage as all the cash-flows in between the decision stages are the same every month. The constraint is as follows:

$$0 \leq B_{t_k}(s_k, e) - \sum_{\substack{t, t_i, t_j: \\ t_i \leq t_k < t < t_{k+1} \leq t_j}} u_{t_i, t_j}(t; a_i(s_k), e) + \mathbb{I}_{[t_k < t(e)]}(t_{k+1} - t_k - 1)\pi,$$

$$k = 0, \ldots, K-1. \quad (13)$$

The next step is to express the value of the loan in each node. Such a value is calculated as the sum of discounted cash-flows from loans running at the given time. Let us denote $P(t_k, t_l; s_k)$ as the discount factor from time t_k to time t_l at a node $s_k \in S_k$. For easier formulation, we divide payments between assets $A_{t_k}(s_k, e)$ and liabilities $L_{t_k}(s_k, e)$. These can be calculated as follows:

$$A_{t_k}(s_k, e) = \sum_{\substack{t_i, t_j: \\ t_i \leq t_k < t_j}} P(t_k, t_j; s_k) w_{t_i, t_j}(t_j; a_i(s_k), e) + \mathbb{I}_{[t_k < t(e)]} \sum_{t: t_k < t \leq T} P(t_k, t; s_k)\pi,$$

$$L_{t_k}(s_k, e) = \sum_{\substack{t, t_i, t_j: \\ t_i \leq t_k < t \leq t_j}} P(t_k, t; s_k) u_{t_i, t_j}(t; a_i(s_k), e) + \sum_{\substack{t_i, t_j: \\ t_i \leq t_k < t_j}} P(t_k, t_j; s_k) v_{t_i, t_j}(t_j; a_i(s_k), e).$$

Assets are calculated as the sum of discounted cash-flows stemming from loans provided to the market and payments the customer is yet to make—before the event e is observed. For liabilities, we total all instalments which the company is yet to make. The difference between values of assets and liabilities (the asset–liability gap) together with the amount of money in the current account $B_{t_k}(s_k, e)$ gives us the value of the portfolio $V_{t_k}(s_k, e)$ at node (s_k, e). This reads as:

$$V_{t_k}(s_k, e) = B_{t_k}(s_k, e) + A_{t_k}(s_k, e) - L_{t_k}(s_k, e). \quad (14)$$

In the final stage, when we have no running contracts, this turns into the net income—the total loan profit. This leads us to the formulation of the objective function $f(r, u, v, w)$, which expresses the value of the loan at the final time horizon. We have:

$$f(r, u, v, w) = p(r) \cdot \sum_{s_K \in S_K, e \in E} p(s_K, e, r) V_{t_K}(s_K, e), \quad (15)$$

where variables u, v, w symbolically stand for the sets of decision variables as defined above. In (15), we weigh each scenario according to its probability $p(s_K, e, r)$ and we multiply the entire sum by the probability $p(r)$ that the loan is agreed to.

To complete the model formulation, we need to specify the final set of constraints. This will consist of the already-mentioned non-anticipativity constraints, as we need to make sure that the decisions of the company in times and scenarios where the event has not yet been observed are the same. We impose:

$$\begin{gathered} u_{t_i, t_j}(s_i, e_1) = u_{t_i, t_j}(s_i, e_2), \quad v_{t_i, t_j}(s_i, e_1) = v_{t_i, t_j}(s_i, e_2), \\ w_{t_i, t_j}(s_i, e_1) = w_{t_i, t_j}(s_i, e_2), \\ \forall s_i \in S_i, e_1, e_2 \in E : t_i < \min\{t(e_1), t(e_2)\}, t_j > t_i, i = 0, \ldots, K. \end{gathered} \quad (16)$$

Finally, we would like to comment on model limitations. The model is specifically designed to capture the relationship between an individual person and the lending company. For other borrower-lender relationships, the model would need to be slightly adjusted. For example, for peer-to-peer lending, the liability side would need to be formulated differently, as individual people usually do not have the borrowing options that are specified in the model. On the other hand, the borrowing of small companies also has its specifics. Furthermore, individual approaches and a more detailed analysis of the relevant data are necessary, especially for larger loans.

2.3.4 Stochastic programming model formulation

Let us now summarize the decision variables and present the complete program. We have months $t, t \in \{0, \dots, T\}$ and decision times $t_k, k \in \{0, \dots, K\}$. In the first decision stage t_0, the company offers a loan to the customer. If it is accepted, the company borrows money at the market at observed (known) interest rates. In the next decision stages $(t_1, \dots t_{K-1})$, we model the evolution of interest rates by the interest rate tree, where at decision time t_k we have a set of nodes S_k. Finally, in the last stage t_K, no decisions are made as we only evaluate the final position at the end of the loan contract. Time t_i ancestor of a node $s_k, i < k$, is denoted as $a_i(s_k)$. Set S_K denotes all final-stage nodes and, hence, also denotes interest rate scenarios. In these future stages $(t_k, k > 0)$, the rate of the consumer loan does not change. We only evaluate all contracts of the company, which can also initiate new contracts at a price determined by the interest rates in the given scenario. Set E describes all events which can happen to the customer. We have functions $d(e)$ identifying whether it is default or prepayment and $t(e)$ specifying at which decision stage it is observed. These are important mainly for calculation of scenario probabilities and specification of cash flows between the company and the customer in different stages. Finally, the pair $(s_K, e), s_K \in S_K, e \in E$ denotes the program scenario. We have introduced four types of decisions over which we optimize:

- r—the interest rate decision

For times $t_i, t_j, i < j \in 0, \dots, K$ and each node $(s_i, e), s_i \in S_i, e \in E$:

- $u_{t_i,t_j}(s_i, e)$—amount borrowed at t_i, repaid monthly with maturity t_j
- $v_{t_i,t_j}(s_i, e)$—amount borrowed at t_i, repaid in total at time t_j
- $w_{t_i,t_j}(s_i, e)$—amount lent at t_i, repaid in total at time t_j

We also defined the following quantities:

- b_0, b_1—midrate and interest rate sensitivity of the customer
- $p(r)$—probability of accepting the loan by the customer
- $h(t_k, r)$—hazard rate for default of the customer at t_k with rate r
- $\beta_0^h, \beta_1^h, \beta_3^h$—coefficients for logistic regression model for $h(t_k, r)$
- $g(t_k, r)$—hazard rate for loan prepayment at t_k with rate r
- $\beta_0^g, \beta_1^g, \beta_3^g$—coefficients for logistic regression model for $g(t_k, r)$
- $p(s_K, e, r)$—probability of scenario (s_K, e), if loan with interest r is agreed
- π—instalment of the consumer loan
- N_t—the principal remaining at time t from the consumer loan
- y_t^τ, m_t^τ—risk-free rates and rates the company pays for loans at the market at time t with time-to-maturity τ
- $u_{t_i,t_j}(t; s_i, e)$—amount repaid at time t from loan $u_{t_i,t_j}(s_i, e)$
- $v_{t_i,t_j}(t_j; s_i, e)$—amount repaid at t_j from loan $v_{t_i,t_j}(s_i, e)$
- $w_{t_i,t_j}(t_j; s_i, e)$—amount repaid at t_j from loan $w_{t_i,t_j}(s_i, e)$
- $I_{t_k}(s_k, e)$—income on the market side at t_k, node (s_k, e)
- $J_{t_k}(s_k, e)$—expenditures on the market side at t_k, node (s_k, e)
- C_{t_k}—operating costs of the loan from time t_k to time t_{k+1}

- $B_{t_k}(s_k, e)$—amount on current account at t_k, node (s_k, e)
- $A_{t_k}(s_k, e)$—value of assets at t_k, node (s_k, e)
- $L_{t_k}(s_k, e)$—value of liabilities at t_k, node (s_k, e)
- $V_{t_k}(s_k, e)$—value of the loan at t_k, node (s_k, e)

From here, we formulate the asset–liability multi-stage stochastic program as:

$$
\begin{aligned}
&\max_{r,u,v,w} \quad p(r) \cdot \sum_{s_K \in S_K, e \in E} p(s_K, e, r) V_{t_K}(s_K, e),\\
&\text{s.t.} \;\; p(r) = \frac{1}{1 + \exp\{-b_1(b_0 - r)\}},\\
&h(t_k, r) = \frac{1}{1 + \exp\{-(\beta_0^h + \beta_1^h r + \beta_3^h t_k)\}},\\
&g(t_k, r) = \frac{1}{1 + \exp\{-(\beta_0^g + \beta_1^g r + \beta_3^g t_k)\}}, \qquad k = 1, \ldots, K,\\
&p(s_K, e, r) = \prod_{k=1}^{K} \Big(\mathbb{I}_{[t(e)=t_k]} d(e) h(t_k, r) + \mathbb{I}_{[t(e)=t_k]} \big(1 - d(e)\big) g\big(t_k, r\big)\\
&\qquad + \mathbb{I}_{[t_k < t(e)]} \big(1 - h(t_k, r) - g(t_k, r)\big) + \mathbb{I}_{[t_k > t(e)]} \Big) \big(|S_{k-1}|/|S_k|\big),\\
&\qquad\quad s_K \in S_K, e \in E,\\
&\pi = N_0 (r/12) \Big(1 - (1 + r/12)^{-T}\Big)^{-1},\\
&N_t = N_0 \left(1 - \frac{(1 + r/12)^t - 1}{(1 + r/12)^T - 1}\right), \qquad t = 1, \ldots, T,\\
&u_{t_i,t_j}(t; s_i, e) = \frac{u_{t_i,t_j}(s_i, e)}{\sum_{\tau=1}^{t_j - t_i} \big(1 + m_{t_i}^{\tau}(s_i)/12\big)^{-\tau}}, \qquad t = t_i + 1, \ldots, t_j,\\
&v_{t_i,t_j}(t_j; s_i, e) = v_{t_i,t_j}(s_i, e) \left(1 + m_{t_i}^{t_j - t_i}(s_i)/12\right)^{t_j - t_i},\\
&w_{t_i,t_j}(t_j; s_i, e) = w_{t_i,t_j} \left(1 + y_{t_i}^{t_j - t_i}(s_i)/12\right)^{t_j - t_i}, \qquad i, j = 0, \ldots, K, i < j,\\
&I_{t_k}(s_k, e) = \sum_{t_i : t_i < t_k} w_{t_i,t_k}(t_k; a_i(s_k), e) + \sum_{t_j : t_k < t_j} u_{t_k,t_j}(s_k, e) + \sum_{t_j : t_k < t_j} v_{t_k,t_j}(s_k, e),\\
&J_{t_k}(s_k, e) = \sum_{t_j : t_k < t_j} w_{t_k,t_j}(s_k, e) + \sum_{\substack{t, t_i, t_j :\\ t_i \le t_{k-1} < t \le t_k \le t_j}} u_{t_i,t_j}(t; a_i(s_k), e) + \sum_{t_i : t_i < t_k} v_{t_i,t_k}(t_k; a_i(s_k), e),\\
&B_{t_k}(s_k, e) = B_{t_{k-1}}(a_{k-1}(s_k), e) - C_{t_k} + I_{t_k}(s_k, e) - J_{t_k}(s_k, e) - \mathbb{I}_{[k=0]} N_0\\
&\qquad + \mathbb{I}_{[t_k < t(e)]} (t_k - t_{k-1}) \pi + \mathbb{I}_{[t_k = t(e)]} d(e) \cdot \mathrm{lgd} \cdot N_{t_{k-1}}\\
&\qquad + \mathbb{I}_{[t_k = t(e)]} (1 - d(e)) \left((t_k - t_{k-1}) \pi + N_{t_k}\right),\\
&A_{t_k}(s_k, e) = \sum_{\substack{t_i, t_j :\\ t_i \le t_k < t_j}} P(t_k, t_j; s_k) w_{t_i,t_j}(t_j; a_i(s_k), e) + \mathbb{I}_{[t_k < t(e)]} \sum_{t : t_k < t \le T} P(t_k, t; s_k) \pi,\\
&L_{t_k}(s_k, e) = \sum_{\substack{t, t_i, t_j :\\ t_i \le t_k < t \le t_j}} P(t_k, t; s_k) u_{t_i,t_j}(t; a_i(s_k), e) + \sum_{\substack{t_i, t_j :\\ t_i \le t_k < t_j}} P(t_k, t_j; s_k) v_{t_i,t_j}(t_j; a_i(s_k), e),\\
&V_{t_k}(s_k, e) = B_{t_k}(s_k, e) + A_{t_k}(s_k, e) - L_{t_k}(s_k, e), \qquad k = 0, \ldots, K, s_k \in S_k, e \in E,
\end{aligned}
$$

$$B_{t_k}(s_k, e) \geq \sum_{\substack{t, t_i, t_j: \\ t_i \leq t_k < t < t_{k+1} \leq t_j}} u_{t_i, t_j}(t; a_i(s_k), e) - \mathbb{I}_{[t_k < t(e)]}(t_{k+1} - t_k - 1)\pi,$$

$$B_{t_k}(s_k, e) \geq 0, \qquad k = 0, \ldots, K-1, s_k \in S_k, e \in E,$$

$$u_{t_i, t_j}(s_i, e_1) = u_{t_i, t_j}(s_i, e_2), \qquad v_{t_i, t_j}(s_i, e_1) = v_{t_i, t_j}(s_i, e_2),$$

$$w_{t_i, t_j}(s_i, e_1) = w_{t_i, t_j}(s_i, e_2),$$

$$i, j = 0, \ldots, K, i < j, s_i \in S_i, e_1, e_2 \in E : t_i < \min\{t(e_1), t(e_2)\}.$$

Note especially the first four equations in the model definition, which capture the endogeneity in the random variables induced by the initial interest rate decision r. There is another non-linearity in the program in the equations for π and N_t. The other equations form a usual asset–liability multi-stage stochastic program.

3 Numerical results

In this section, we present the results of the model. We focus on how decision-dependent uncertainty and the parameters associated with it affect the model solution, especially the interest rate decision. We also discuss the losses connected to offering a non-optimal interest rate for the loan.

For this model, we set the notional to be $N_0 = 50{,}000$ CZK with maturity $T = 5$ years. We set decision stages to be at the end of each year ($K = 5$). The branching of the interest rate tree was chosen to be $5 - 4 - 3 - 2 - 1$, leading to $|S_K| = 120$ interest rate scenarios. There is no branching to the final stage, as no decision is made there and we only evaluate the final value of a loan. Given that we have 5 "future" decision stages and that, in each stage, we can have default or prepayment, we have a total of $|E| = 2 \cdot K = 10$ events leading up to $|S_K| \cdot |E| = 120 \cdot 10 = 1200$ scenarios.

The program was written in GAMS and solved by CONOPT3 on a standard laptop (Intel Core i5 2.60 GHz, 8GB RAM). Scenario generation and results' analysis were performed in R. The model itself had 58690 variables and 45619 constraints with 322783 Jacobian elements, 4989 of which were non-linear. The Hessian of the Lagrangian had 1 element on the diagonal, 4250 elements below the diagonal and 3052 non-linear variables.

3.1 Model solution

Here, we will give a detailed description of the solution and its properties for a single model. For this purpose, we chose parameter values as midrate $b_0 = 0.14$, interest rate sensitivity $b_1 = 100$ and rating $\rho = 2$. The optimal solution of the program was to offer the customer a rate $r = 12.24\%$, with the probability of accepting the loan as $p(r) = 0.853$. The optimal borrowing and lending strategy of the company was to close only one-year loans. If spare money is available, then the company should lend to the market for the longest period possible (until the final time horizon). That is due to interest rate tree having relatively constant expected future rates and also because the shorter the loan, the cheaper. If the rates were, for example, increasing, the company would tend to close longer loans with the market. The optimal value of the program showed that expected profit from the loan was 7392 CZK on the considered 50000 CZK loan. This approximates to an annual gain of 0.028 CZK per 1 CZK borrowed. However, note that this depends greatly on the characteristics of the customer.

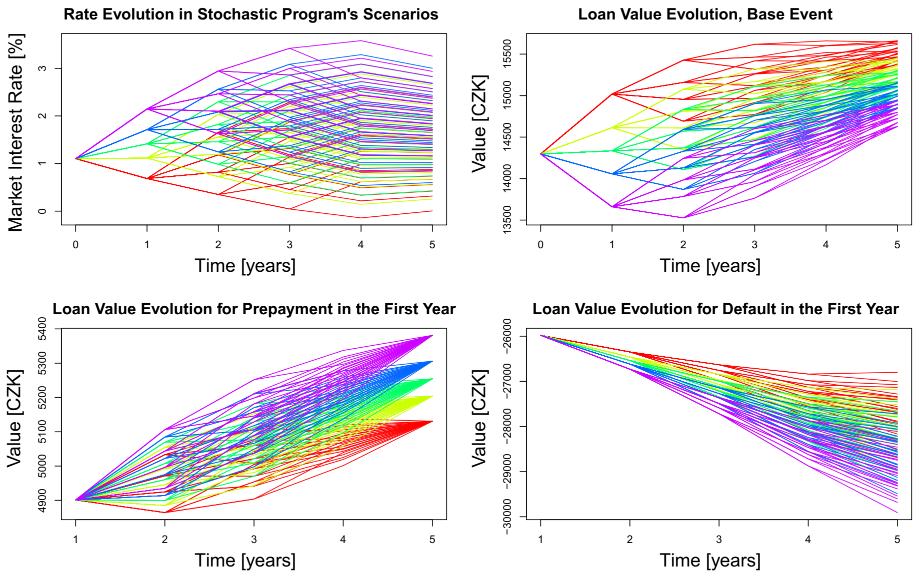

Fig. 3 Interest rate and the optimal loan value evolution in time

To analyse the results, we investigated loan performance in different interest rate scenarios for different events. First, we divided interest rate scenarios into five groups—*purple, blue, green, yellow* and *red*—depending on their first-stage node (see top-left figure in Fig. 3). This will help us to illustrate the dependency of the optimal loan value on interest rate evolution. We looked into performance when the customer complied with the original terms of the loan. Because of the one-year borrowing strategy, the company profits on an interest rate decrease and it loses money when the interest rate increases. However, it is also able to use a *high* (purple) interest rate environment to compensate for the initial loss by lending money earned from the loan in order to earn high interest in latter stages of the loan. The bottom figures in Fig. 3 show the performance of a loan when it is fully repaid/defaulted at the end of the first year. One can see that a high interest rate environment is preferred for loan prepayment because the company can then reinvest money for higher interest. It is the complete opposite for the case of customer defaults. Then, the company is required to borrow additional money to finance its liabilities and this costs more in an environment with higher interest rates.

We also shortly investigated sensitivity on the mark-up $m(\tau)$, which the market charges the company. Our hypothesis was that increasing the mark-up increases the costs of the loan for the company and hence, "not accepting" the loan is relatively less expensive. This should lead to higher interest rate decision. However, the company would not increase the interest rate by the same margin as the client would be discouraged from entering the loan. This was confirmed by the program, as when we set the mark-up twice the analysed value, the objective value of the program decreased to 6725 CZK and the optimal offered interest rate was 12.34% with $p(r) = 0.840$.

A question arises about what the added value of the inclusion of decision-dependent randomness is. First, note that it is not possible to formulate the model without decision-dependent randomness in the probability of accepting the loan, as this would not really make sense. Therefore, we have solved a model without decision-dependent randomness in default and prepayment. There, the results depend heavily on the strength of the relationship between the offered interest rate and the default by the customer. For our particular case, the objective value of the non-decision-dependent randomness model was 7348 CZK, which means a difference of 44 CZK. However, for ratings 3 and 4, the difference was 2209 and 5635 CZK respectively. From this, we can conclude that it is advantageous to reflect the relationship, especially in cases where the initial decision has a greater impact on the random variable.

The framework we have introduced allows us to take into account all the elements of the life of a consumer loan (customer properties, event probabilities, interest rate evolution, etc.), assess their costs depending on the company's decisions and then select the optimal ones. Division of this joint optimization into sub-problems is a simplification which does not produce accurate results. For example, if the company would be forced to replicate the customer's loan at the market, then, it would pay more interest than it would be required for the case of customer's prepayment. This higher cost of prepayment would make the company conservative and cause them to offer rate lower than the optimal one.

3.2 Sensitivity analysis

In this section, we will investigate the behaviour of the optimal decisions and optimal value of the model when we modify the characteristics of the customer. Parameters ρ and b_1 are of the main interest, as they capture the decision-dependent randomness in the model. We will look into how the offered interest rate depends on the probability distribution of accepting the loan. Moreover, we will analyse how much the company loses when it makes a wrong decision regarding the offered interest rate. This will be studied for all possible values of the rating.

We consider rating $\rho = \{1, 2, 3, 4\}$, which defines the probability of default and prepayment as given in (2) and (3). Then, we specify the probability of accepting the loan by parameters midrate b_0 and interest rate sensitivity b_1, as described in Sect. 2.2.1. Interest rate sensitivity is the parameter which captures the strength of decision-dependent randomness in the probability of accepting the loan. We see that a higher value of b_1 implies that the customer is more sensitive—he has good information about current market conditions and any deviation from the midrate means either a large increase or a large decrease in the probability of accepting the loan. Midrates are considered to be from a sequence $\{0.1, 0.12, 0.14, 0.16, 0.18\}$, while sensitivities will take on values of $\{25, 35, 50, 75, 100, 125, 150, 175, 200\}$.

In Figs. 4 and 7, 9, 11 and 13 in Sect. 5 of the Appendix, we show the results of the model for each midrate, each interest rate sensitivity and each rating listed above. The figures report the results of runs of the program with a common midrate and consist of four plots. First, in the top-left one, we show logistic curves which are generated by the pair b_0 and b_1. There, one can note that all the curves intersect at one point—the common midrate with 50% probability. Second, in the top-right plot, we show how the optimal interest rate varies for different ratings across all sensitivity values. Finally, in the bottom two figures, the loan probability (probability that the customer accepts the offered interest rate) and the objective value of the program as given by the optimal solution are shown.

From there, we can observe several properties of the optimization problem:

- It is always more profitable to have a consumer with a better rating.
- A consumer with a better rating gets better rates from the company.
- The higher the interest rate sensitivity b_1, the higher the loan probability. This is simply because it is less costly to ensure the customer has a greater probability of accepting the loan.

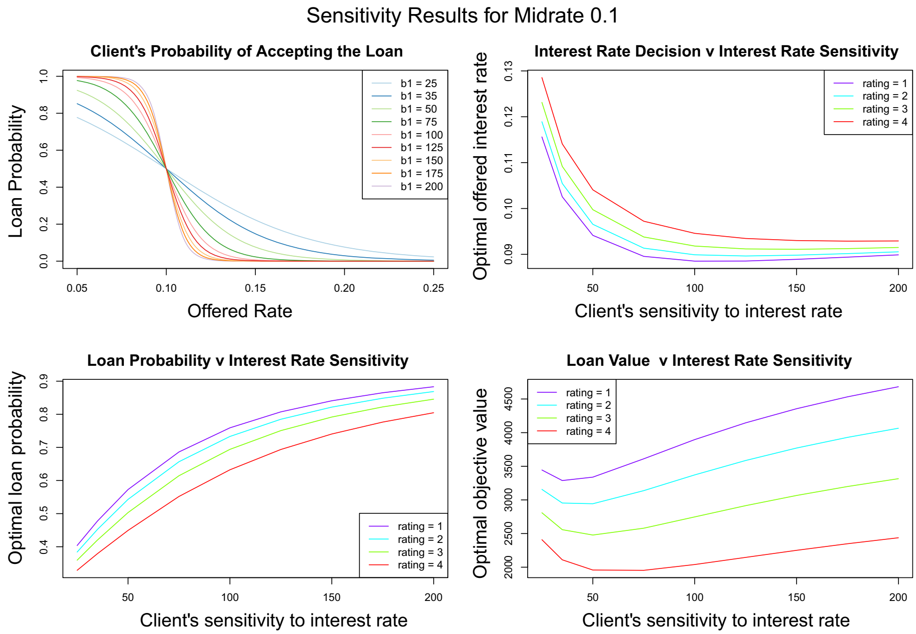

Fig. 4 Sensitivity analysis results for midrate 0.1

– The objective function is not monotone in b_1. For some settings (see Fig. 4 for midrate 0.1), the company makes money on the fact that the customer is willing to accept higher rates.

We can see that all these findings have real application which is required for any practical use of the model. It is clear that the shape of the "probability of accepting the loan" curve is absolutely crucial for the model to produce the most realistic results. We believe that the set of curves we chose approximate most of the options which can practically happen. In the end, only local properties of the curve around the "almost optimal rate" are what matters most.

We also investigated how important it is for the company to set the interest rate correctly. In other words, how much the company loses when it offers a non-optimal interest rate to the customer. To answer this question, we present Fig. 5, which depicts the dependence of the objective value of the program when fixing the interest rate r and also the interest rate sensitivity b_1 on certain values. The figure is given for a customer with midrate $b_0 = 0.1$ and rating 2. Figures for other midrates are presented in Sect. 5 of the Appendix.

The conclusion which we obtain from Fig. 5 is that the difference between objective values of the optimal solution and a solution with fixed r depends greatly on the interest rate sensitivity of the customer. It is apparent that when the customer is sensitive (large b_1) it is extremely important to "hit" the optimal rate with the offer, otherwise the company loses a significant amount of money. What can also be seen from Fig. 5 is that it is more costly to offer a higher interest rate than a lower interest rate compared to the optimal rate. The potentially missed opportunity on a loan has greater impact on the objective than smaller revenue from a loan with a lower interest rate.

In order to quantify the loss which is incurred by offering a non-optimal interest rate, we present Table 2, 3, 4, 5 and 6. Here, we show for each considered customer, the loss incurred

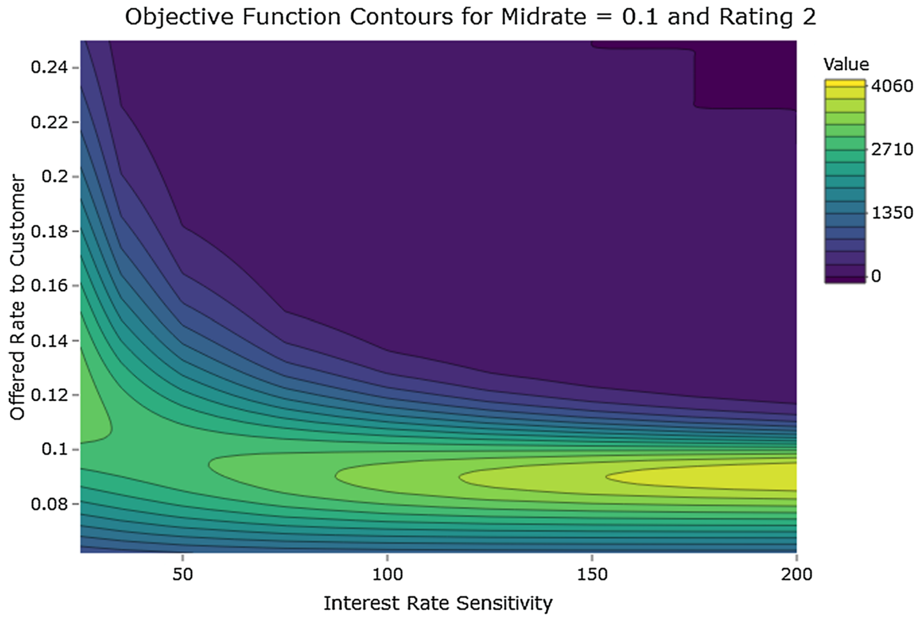

Fig. 5 Contour plot constructed from objective values of the program when fixing offered interest rate r for different values of b_1

Table 2 Differences in objective function for ±1% change in offered rate against the optimal value for midrate 10%

Rating →	b_1 ↓	Absolute (CZK)				Relative (%)			
		1	2	3	4	1	2	3	4
−1%	25	66	62	57	52	1.9	2	2	2.2
	35	107	102	95	86	3.2	3.4	3.7	4.1
	50	177	169	158	142	5.3	5.8	6.4	7.3
	75	296	286	272	250	8.2	9.1	10.5	12.8
	100	401	391	377	353	10.3	11.6	13.7	17.3
	125	489	481	467	445	11.8	13.4	16	20.7
	150	564	557	544	522	13	14.8	17.7	23.2
	175	628	621	608	588	13.9	15.8	19	25
	200	683	676	663	643	14.6	16.6	20	26.4
+1%	25	57	54	50	44	1.7	1.7	1.8	1.8
	35	95	89	81	70	2.9	3	3.2	3.3
	50	163	151	136	117	4.9	5.1	5.5	6
	75	304	282	252	211	8.4	9	9.8	10.8
	100	472	439	392	327	12.1	13	14.3	16.1
	125	666	619	552	460	16.1	17.3	19	21.4
	150	881	819	730	605	20.2	21.7	23.8	26.9
	175	1117	1037	923	761	24.6	26.4	28.9	32.4
	200	1372	1270	1128	927	29.3	31.2	34	38

Table 3 Differences in objective function for ±1% change in offered rate against the optimal value for midrate 12%

Rating →	b_1 ↓	Absolute (CZK)				Relative (%)			
		1	2	3	4	1	2	3	4
−1%	25	75	72	68	63	1.6	1.7	1.8	1.9
	35	125	120	112	103	2.7	2.8	3	3.3
	50	200	195	186	173	4.1	4.4	4.8	5.4
	75	319	314	304	289	5.9	6.4	7.2	8.4
	100	423	417	406	391	7.2	7.9	8.9	10.5
	125	511	504	494	478	8.2	9	10.2	12.1
	150	586	580	568	552	9	9.8	11.1	13.3
	175	649	642	631	614	9.6	10.5	11.9	14.2
	200	703	697	685	668	10.1	11.1	12.6	14.9
+1%	25	69	65	61	55	1.5	1.5	1.6	1.7
	35	115	109	101	91	2.5	2.6	2.7	2.9
	50	200	190	176	157	4.1	4.3	4.6	4.9
	75	370	353	328	294	6.8	7.2	7.8	8.5
	100	569	544	509	458	9.7	10.3	11.1	12.3
	125	797	764	713	643	12.8	13.6	14.7	16.3
	150	1053	1008	943	849	16.2	17.1	18.5	20.4
	175	1341	1282	1196	1076	19.9	21	22.6	24.9
	200	1655	1581	1472	1321	23.9	25.2	27	29.6

by missing the optimal interest rate by ±1%—i.e. by ±100 bps. From an initial glance, we can learn that the losses increase with the value of interest rate sensitivity. This means the more sensitive a customer is, the more careful the company should be with its offer. We can also see the effect of the rating. Here, absolute costs are greater with a better rating. On the contrary, relative costs increase with lower ratings. This is due to the smaller objective value of a loan for worse customers. The tables also confirm that it is generally better to offer lower rates than higher rates, which is something we explored in the previous paragraph. We also compare losses across midrates. Here, we see that they become larger with increasing midrate in absolute values, but the opposite is true in relative terms. This imbalance is due to the absolute change of 1% which is applied. This has a stronger relative effect for lower midrates. However, the fact that we "play" for more money for higher midrates implies a greater absolute effect there.

The relationship between the loss incurred to the company and the distance of the offered interest rate from the optimal value can be exploited in more detail in Fig. 6. Here, one can see the effect of rating, interest rate sensitivity, midrate and the distance of the offered rate from the optimal decision. We can draw a similar conclusion as that from the numbers in Table 2; it is more costly to offer a higher interest rate than a lower one and the loss depends largely on the interest rate sensitivity of the customer. Moreover, we can see that the loss appears to be concave in distance from the optimal interest rate decision, meaning the loss increases with increasing rate when moving away from the optimal decision.

In summary, we investigated the effect of customer's properties such as the expected offered rate, interest rate sensitivity and credit quality on a loan provided by a company.

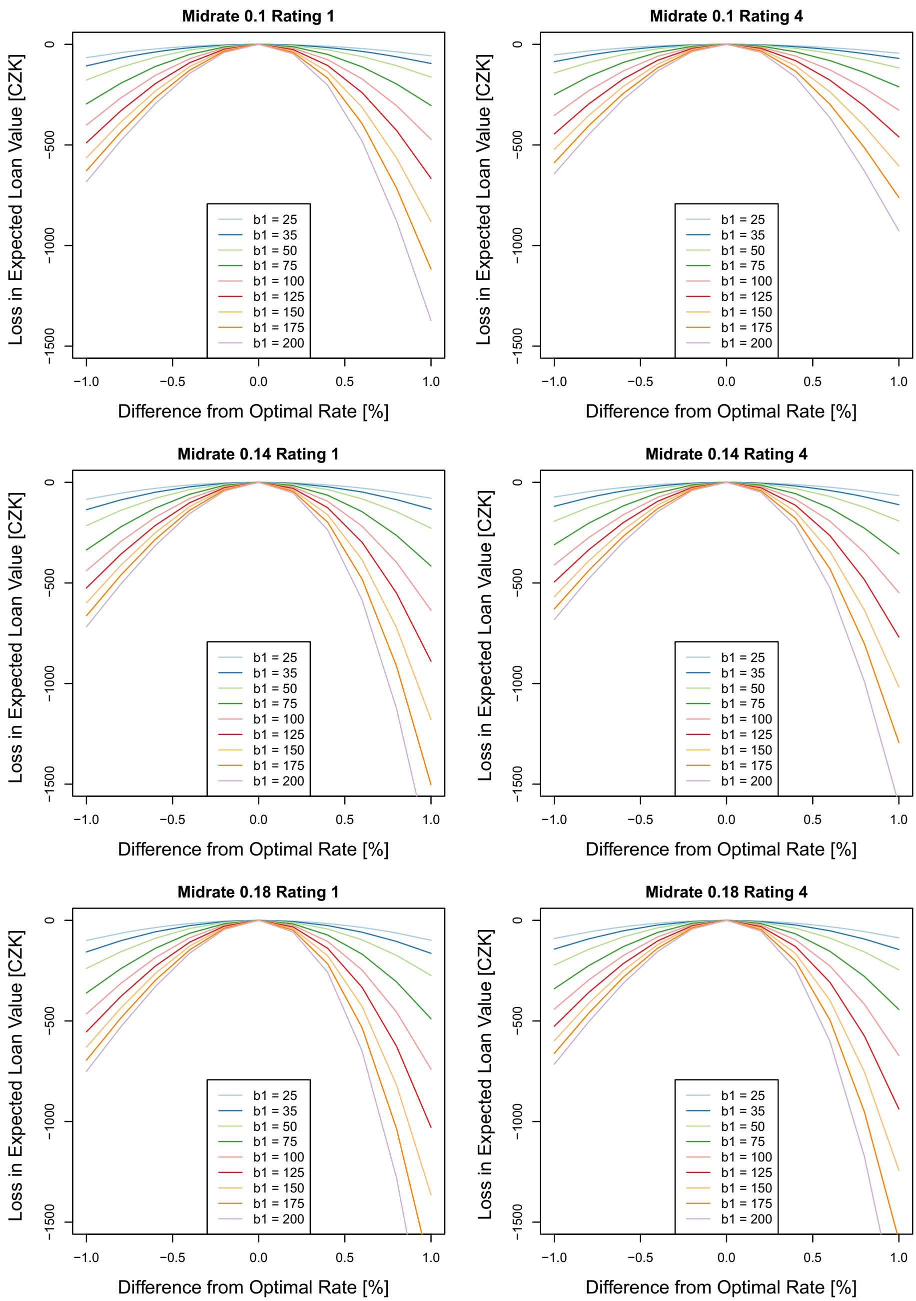

Fig. 6 Comparison of losses to the company caused by not offering optimal interest rate to the customer

Table 4 Differences in objective function for ±1% change in offered rate against the optimal value for midrate 14%

Rating →	b_1 ↓	Absolute (CZK)				Relative (%)			
		1	2	3	4	1	2	3	4
−1%	25	84	82	78	73	1.4	1.5	1.6	1.7
	35	136	132	127	119	2.2	2.3	2.5	2.7
	50	215	210	203	193	3.2	3.4	3.7	4.1
	75	336	331	322	310	4.5	4.8	5.3	5.9
	100	439	433	424	410	5.5	5.9	6.4	7.3
	125	525	519	510	495	6.2	6.6	7.3	8.3
	150	599	594	583	568	6.8	7.3	8	9.1
	175	663	657	646	629	7.3	7.8	8.6	9.7
	200	718	711	699	682	7.7	8.3	9	10.2
+1%	25	79	76	72	66	1.4	1.4	1.4	1.5
	35	133	128	121	111	2.2	2.3	2.4	2.5
	50	228	220	208	192	3.4	3.6	3.8	4.1
	75	416	403	383	356	5.6	5.9	6.3	6.8
	100	636	617	589	548	7.9	8.4	8.9	9.7
	125	889	864	824	769	10.5	11.1	11.8	12.9
	150	1179	1142	1092	1017	13.4	14	15	16.3
	175	1503	1457	1388	1293	16.6	17.3	18.4	20
	200	1866	1808	1721	1600	20.1	21	22.3	24

These properties are essential as they capture the decision-dependent uncertainty which is present in the optimization model. We focused on calculation of losses caused by not offering optimal interest rate. We saw that especially for the more interest rate-sensitive customers, the losses can be very high and any kind of simplification of the joint optimization model can lead to wrong decisions. That implies that it is important to consider the decision-dependent nature of the model in its entirety and ignoring it, even only in parts, can lead to a significant reduction of profits for the company.

4 Conclusion

In this article, we have considered an asset–liability management problem of a consumer loan, where, due to the possibility of the customer not accepting the loan and, upon acceptance, prepaying or defaulting on the loan, we formulate the problem as a non-linear multistage stochastic program with endogenous source of uncertainty. There, two groups of decisions appear: first, the initial fixed rate decision and second, the decisions associated with the asset–liability management policy. The fixed rate decision on the loan affects the future (uncertain) loan evolution and hence its value. The presented optimization problem allows the determination of such interest rate and optimal loan management taking all the contingencies into account through a set of conditional, decision-dependent scenario probabilities. We focused on the formulation of the problem in the theoretical part, where all the features of

Table 5 Differences in objective function for ±1% change in offered rate against the optimal value for midrate 16%

Rating →	b_1 ↓	Absolute (CZK)				Relative (%)			
		1	2	3	4	1	2	3	4
−1%	25	93	90	87	82	1.3	1.3	1.4	1.4
	35	147	144	139	132	1.9	2	2.1	2.2
	50	228	224	218	209	2.7	2.8	3	3.3
	75	349	345	337	327	3.6	3.9	4.1	4.6
	100	452	447	440	427	4.4	4.6	5	5.5
	125	540	535	526	512	5	5.3	5.7	6.3
	150	615	609	599	583	5.5	5.8	6.2	6.9
	175	679	673	662	645	5.9	6.2	6.7	7.4
	200	734	728	716	699	6.2	6.6	7.1	7.8
+1%	25	89	87	82	77	1.2	1.3	1.3	1.4
	35	149	145	138	129	1.9	2	2.1	2.2
	50	253	246	236	221	3	3.1	3.2	3.5
	75	456	444	428	404	4.8	5	5.3	5.6
	100	693	677	652	618	6.7	7	7.4	8
	125	967	944	911	863	8.9	9.3	9.9	10.6
	150	1281	1251	1205	1144	11.4	11.9	12.6	13.5
	175	1637	1599	1541	1459	14.2	14.8	15.6	16.7
	200	2040	1992	1916	1811	17.3	18	19	20.2

this problem, especially the decision-dependent randomness and its implementation into the program, were described.

The practical part was then devoted to the solutions of the program. First, we have investigated the performance of a single optimal solution in the stochastic program's scenarios. The optimal strategy was to borrow only for the shortest time period as these loans are, in general, the cheapest. Such a strategy also allows great flexibility, which is beneficial, for example, when the loan is prepaid. As a drawback, it increases *interest rate risk*. The exposure to interest rate risk could possibly be addressed by the implementation of various risk constraints (e.g. a chance constraint (Telser 1955), a Value-at-Risk constraint (Risk Metrics 1995), a conditional Value-at-Risk constraint (Rockafellar and Uryasev 2000, 2002) a second-order stochastic dominance constraint (Hadar and Russell 1969; Dentcheva and Ruszczynski 2003)) or application of robustness or contamination approaches (Dupačová and Kopa 2012, 2014). This however falls out of the scope of this paper and applying it would make our problem even more computationally demanding, possibly intractable.

Second, we have implemented and solved the stochastic program under several parameter settings, capturing different customer's properties to determine what the implications of such a model would be on decisions made by the company. There, we noticed different actions on customers with different strengths in decision-dependent uncertainty. This was the case for both the changes in interest rate sensitivity and the changes in rating. This demonstrates that decision-dependent randomness needs to be considered in this problem and that the model which takes it into account produces strong results.

Table 6 Differences in objective function for ±1% change in offered rate against the optimal value for midrate 18%

Rating →	b_1 ↓	Absolute (CZK)				Relative (%)			
		1	2	3	4	1	2	3	4
−1%	25	100	98	95	90	1.1	1.2	1.2	1.3
	35	157	154	149	143	1.6	1.7	1.8	1.9
	50	239	236	230	222	2.2	2.4	2.5	2.7
	75	361	357	351	340	3.1	3.2	3.4	3.7
	100	465	461	453	441	3.7	3.9	4.1	4.4
	125	554	549	539	526	4.2	4.4	4.7	5
	150	630	624	614	598	4.6	4.8	5.1	5.5
	175	695	690	678	661	5	5.2	5.5	6
	200	751	745	733	715	5.3	5.5	5.8	6.3
+1%	25	99	96	92	87	1.1	1.1	1.2	1.2
	35	164	159	153	145	1.7	1.8	1.8	1.9
	50	274	268	259	246	2.6	2.7	2.8	3
	75	489	479	464	443	4.1	4.3	4.5	4.8
	100	740	725	704	672	5.8	6.1	6.4	6.8
	125	1029	1011	982	938	7.8	8.1	8.5	9
	150	1364	1340	1298	1243	10	10.4	10.8	11.5
	175	1748	1713	1664	1589	12.5	12.9	13.5	14.3
	200	2186	2143	2073	1979	15.3	15.8	16.5	17.4

Finally, we were interested in the incurred losses which are caused by the company not offering the optimal interest rate. We found that such costs depend greatly on interest rate sensitivity. Moreover, we saw that it is more costly to offer a higher rate than a lower rate compared to the optimal rate . This is due to two reasons—a lower probability of accepting the loan hurts more than lower interest rate revenues and the default rate increases with the higher interest rate. Both effects are a consequence of decision-dependent uncertainty. This, again, underlines the need to capture the dependence between the offered interest rate and default probability correctly and not neglect the relationship.

5 Appendix

In the figures and tables below, we present the results of the sensitivity analysis for other midrates as discussed in Sect. 3.2. First, we show optimal solutions of the program for each rating and interest rate sensitivity of the customer. Then, we present the tables with losses associated with ±1% difference from the optimal interest rate, the objective value contour plot for the program with a customer of rating 2 and a fixed initial interest rate decision (Figs. 7, 8, 9, 10, 11, 12, 13, 14). Objective function surfaces can be seen in the interactive mode available at https://plot.ly/~rusy/.

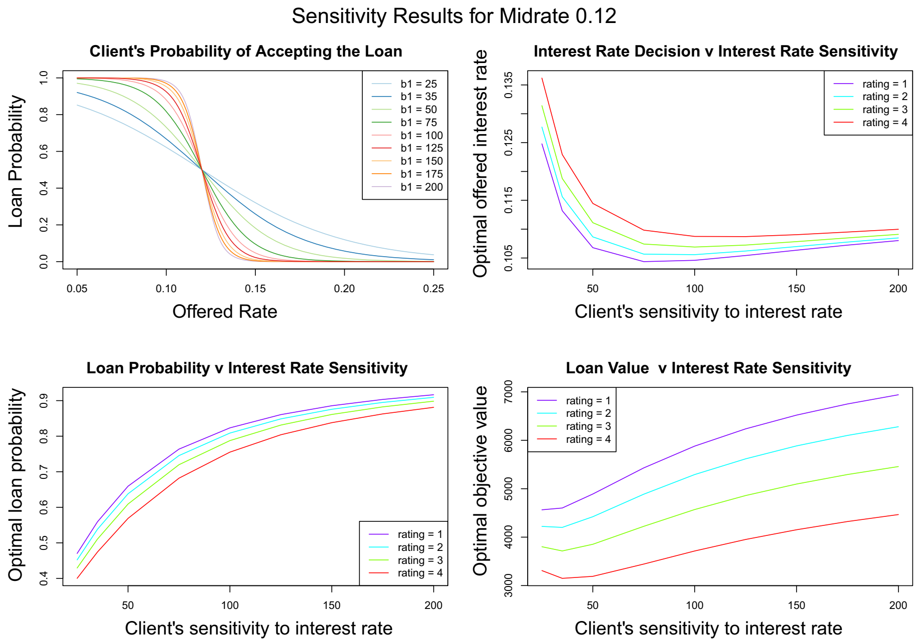

Fig. 7 Sensitivity analysis results for midrate 0.12

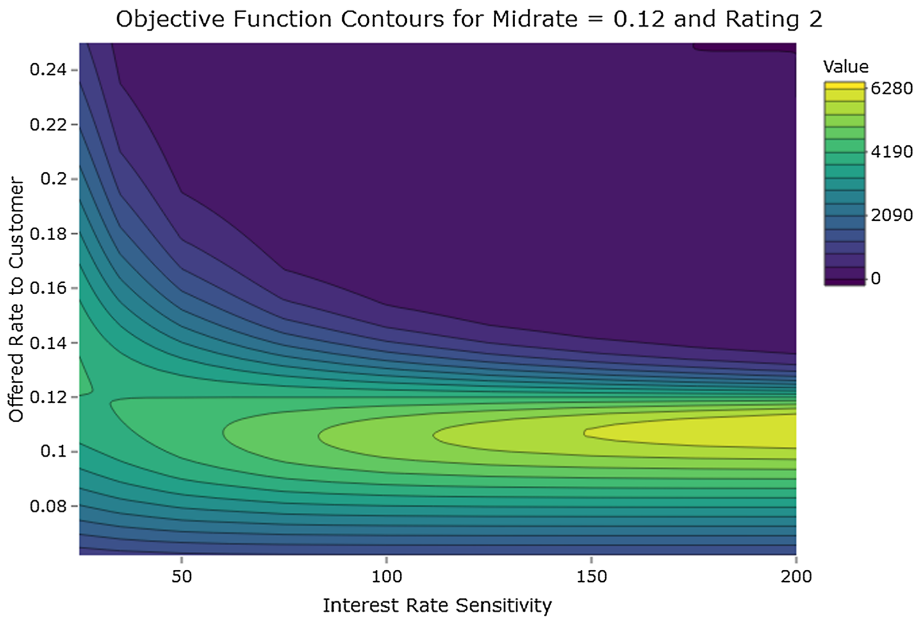

Fig. 8 Contour plot constructed from objective values of the program when fixing offered interest rate r for different values of interest rate sensitivity b_1

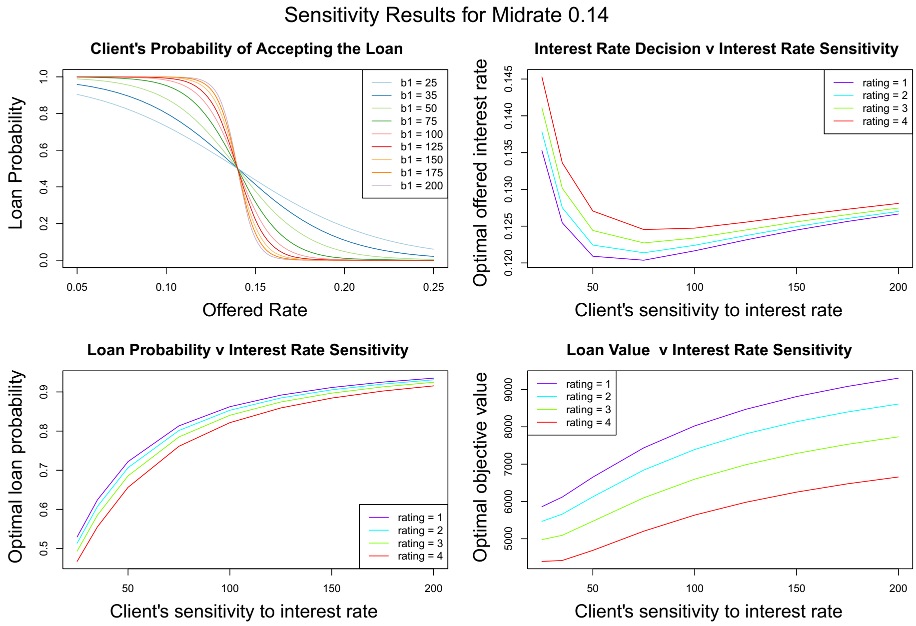

Fig. 9 Sensitivity analysis results for midrate 0.14

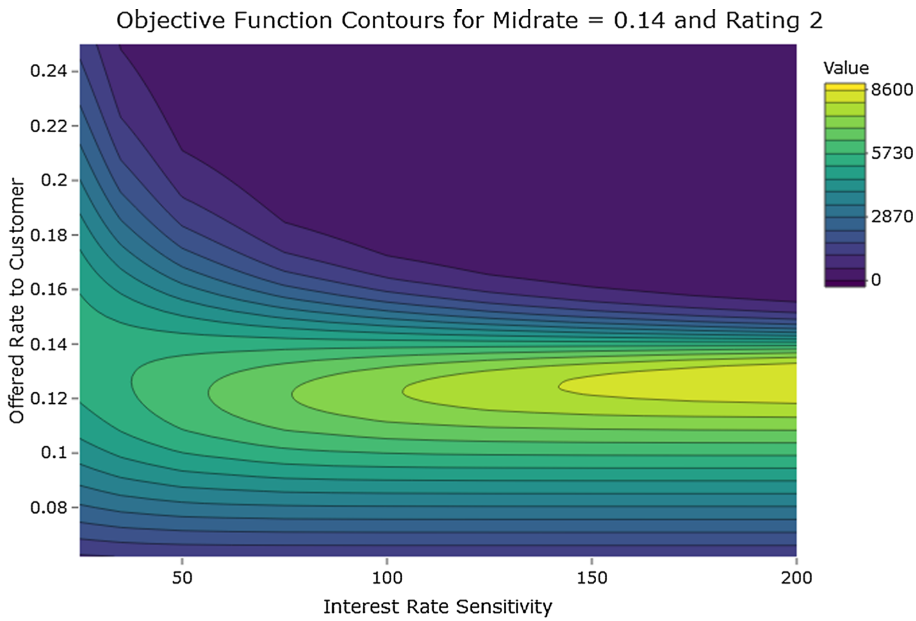

Fig. 10 Contour plot constructed from objective values of the program when fixing offered interest rate r for different values of interest rate sensitivity b_1

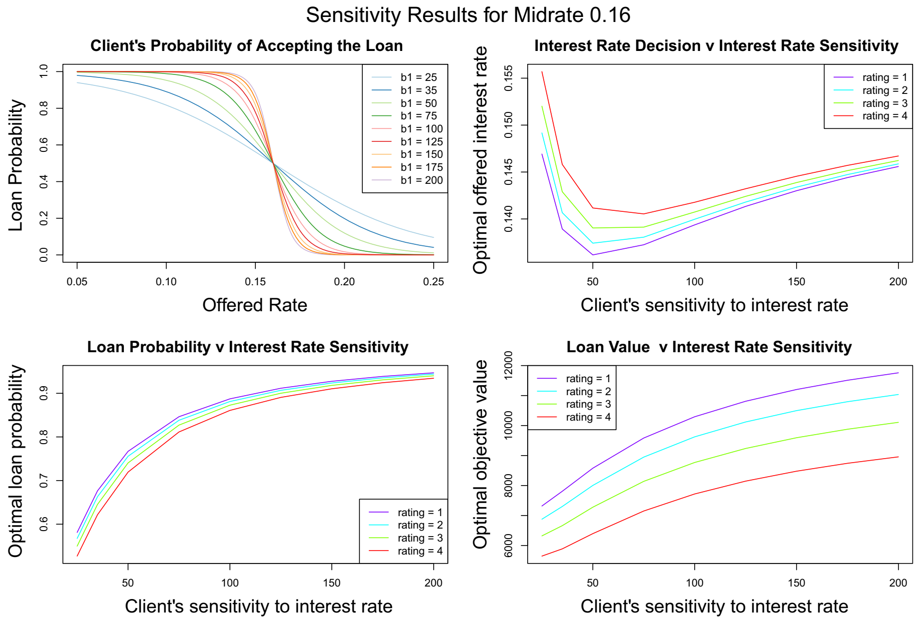

Fig. 11 Sensitivity analysis results for midrate 0.16

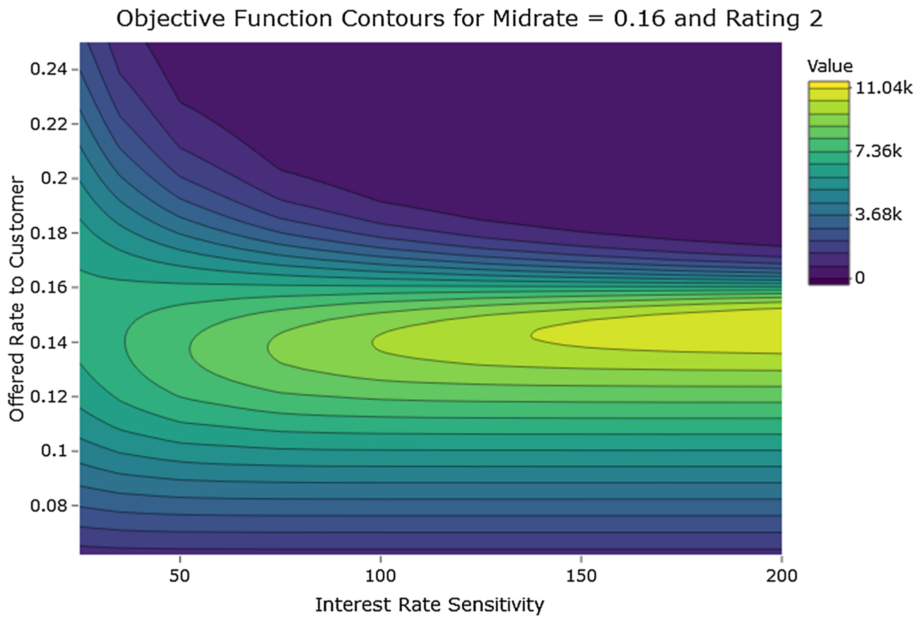

Fig. 12 Contour plot constructed from objective values of the program when fixing offered interest rate r for different values of interest rate sensitivity b_1

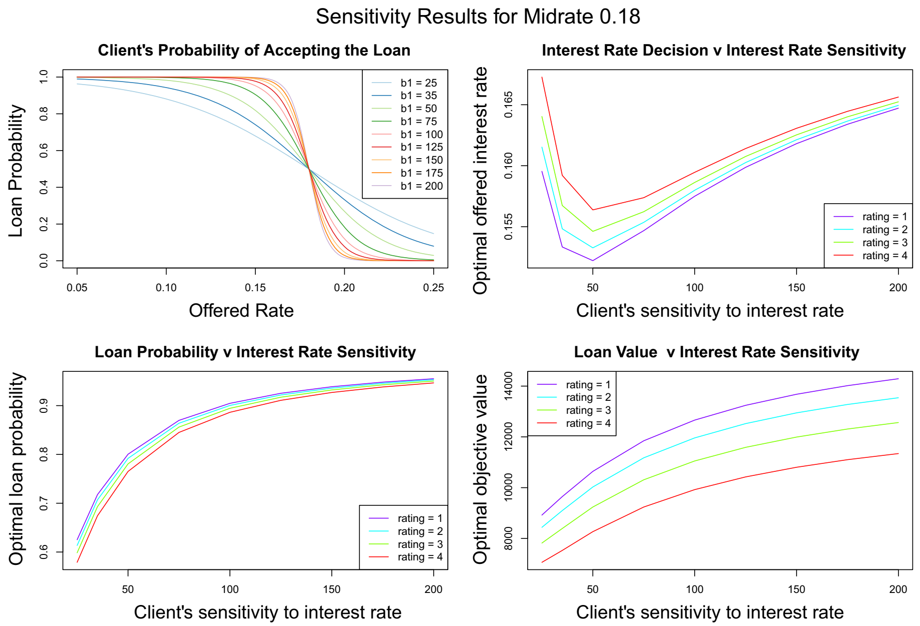

Fig. 13 Sensitivity analysis results for midrate 0.18

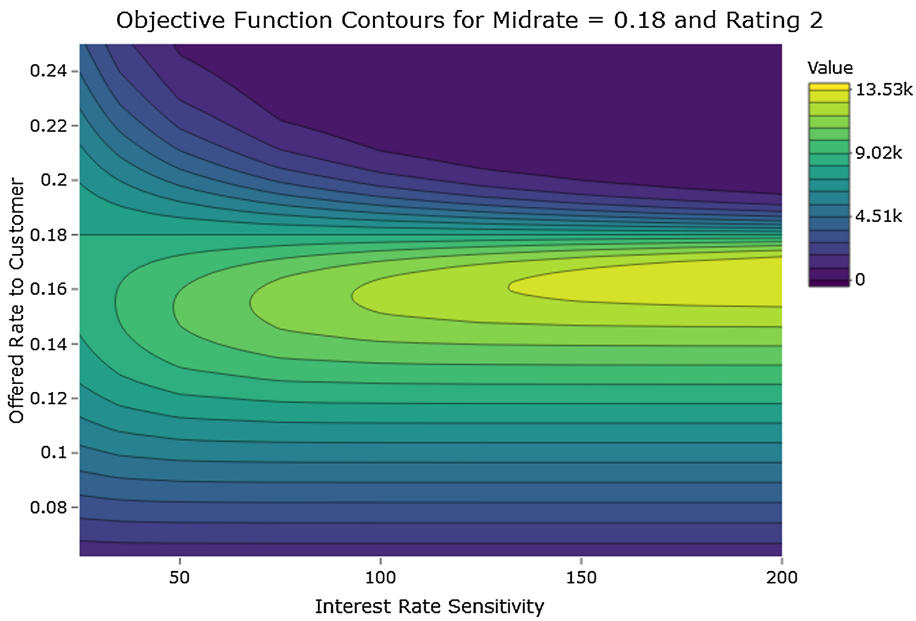

Fig. 14 Contour plot constructed from objective values of the program when fixing offered interest rate r for different values of interest rate sensitivity b_1

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