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Planar Segment Processes with Reference Mark Distributions, Modeling and Estimation

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Abstract The paper deals with planar segment processes given by a density with respect to the Poisson process. Parametric models involve reference distributions of directions and/or lengths of segments. These distributions generally do not coincide with the corresponding observed distributions. Statistical methods are presented which first estimate scalar parameters by known approaches and then the reference distribution is estimated non-parametrically. Besides a general theory we offer two models, first a Gibbs type segment process with reference directional distribution and secondly an inhomogeneous process with reference length distribution. The estimation is demonstrated in simulation studies where the variability of estimators is presented graphically.

Keywords Conditional intensity · Segment process · Semiparametric estimation

Mathematics Subject Classification (2010) MSC 60D05 · MSC 60G55

1 Introduction

The present research addresses an important problem in the statistics of spatial marked point processes given by a density with respect to the Poisson process. Observing a realization of spatial data which shall be fitted to such a model we first estimate the parameters by a

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method of point estimation. However, among the quantities to be estimated there may appear also the reference mark distribution which need not coincide with the observed mark distribution of the process. Both the scalar parameters and the reference mark distribution are needed e.g. when we try to simulate the model. This distribution can be also parametrized by a subjective choice of model, cf. Beneš et al. (2017). In the present paper the main aim is to estimate the reference mark distribution non-parametrically, i.e. in total to use a semiparametric approach instead of a fully parametric one.

An early paper (Baddeley and Turner 2000) mentions parameter estimation of a marked point process by means of the maximum pseudolikelihood method but the authors do not identify our problem. Much more attention is paid to it in Møller and Helisova (2010) where the marks form radii of circles centered at the points of a point process given by a density with respect to the Poisson process. The random set corresponding to the union of circles in a compact window is investigated. Since an exact method is not available the authors use an approximation what means that estimation of the distribution of radii is done by methods for a Boolean model. Then an MCMC maximum likelihood method (Møller and Waagepetersen 2004) is used for the estimation of parameters of the point process. A recent paper by Dereudre et al. (2014) deals with the same model as Møller and Helisova (2010), their goal is to use the Takacs-Fiksel estimator instead of the computationally demanding maximum likelihood method.

In our work we present a solution of the problem for another random set. We deal with the planar segment process (Chiu et al. 2013; Pawlas 2014) having a density of exponential form with respect to a Poisson process. We consider first a model with reference directional distribution and in the end a model with reference length distribution of segments. The difference in comparison to marked point processes presented in the literature is that when the directional distribution is present we are not completely on Euclidean spaces. Our main tool is the derived relation between the reference and observed mark distribution for the segment process. The basic asymptotic properties of the Takacs-Fiksel method of estimation are known, see Coeurjolly et al. (2012). In the present paper we pay attention to the computing of semiparametric estimators using simulated data and quantifying the variability for small sample size.

First some background from spatial point processes having a density is presented. Then in Section 3 a general formula for the mark distributions is derived. In Section 4 we present a Gibbs type segment process with reference directional distribution and fixed lengths of segments. Interactions enter the model by means of intersections, cf. Večeřa and Beneš (2016) and Večeřa (2016). An approximation from Baddeley and Nair (2012) is used to avoid the problem with unknown moments. The Takacs-Fiksel estimator for this model is developed in Section 5 and tested on data simulated by MCMC algorithm from Geyer and Møller (1994). The estimator of the reference directional distribution is evaluated numerically. In Section 6 we address an old problem of the existence of a stationary process with given conditional intensity from the previous Section. Section 7 presents a model with reference length distribution of segments while the reference directional distribution is uniform. This model is an inhomogeneous Poisson process with a condition on segments to lie entirely in a circular window. The maximum likelihood method of estimation is available, see Section 8, we use the isotropy of the process to simplify the computation.

2 Spatial Point Process Given by a Density

Consider a bounded Borel set $B \subset \mathbb{R}^d$ with Lebesgue measure $|B| > 0$ and a measurable space $(\mathbf{N}, \mathcal{N})$ of integer-valued finite measures on B. $\mathcal{N}$ is the smallest σ-algebra which

makes the mappings $\mathbf{x} \mapsto \mathbf{x}(A)$ measurable for all Borel sets $A \subset B$. A random element having a.s. values in $(\mathbf{N}, \mathcal{N})$ is called a finite point process. Let a Poisson point process η on B have finite intensity measure λ with no atoms and distribution P_η on $\mathcal{N}$. We consider a finite point process X on B given by a density p w.r.t. P_η, i. e. with distribution P_X

$$\mathrm{d}P_X(\mathbf{x}) = p(\mathbf{x})\,\mathrm{d}P_\eta(\mathbf{x}), \ \mathbf{x} \in \mathbf{N}, \tag{1}$$

where $p : \mathbf{N} \to \mathbb{R}_+$ is measurable satisfying

$$\int_{\mathbf{N}} p(\mathbf{x})\,\mathrm{d}P_\eta(\mathbf{x}) = 1.$$

As described in Baddeley ([2007]), p. 61, integer-valued finite measures can be represented in this context by n-tuples of points corresponding to their support (n is variable). We have

$$\mathbb{P}(X \in A) = \tag{2}$$

$$= e^{-\lambda(B)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_B \dots \int_B \mathbf{1}[(x_1, \dots, x_n) \in A] p(x_1, \dots, x_n)\lambda(\mathrm{d}x_1) \dots \lambda(\mathrm{d}x_n),$$

$A \in \mathcal{N}$. For an hereditary density p the distribution of the process is alternatively determined by the conditional intensity

$$\lambda^*(\mathbf{x}, u) = \frac{p(\mathbf{x} \cup u)}{p(\mathbf{x})}, \ \mathbf{x} \in \mathbb{N}, \ u \notin \mathbf{x}.$$

An important tool is the Georgii-Nguyen-Zessin (GNZ) formula

$$\mathbb{E}\left[\sum_{u \in X} q(u, X \setminus u) \right] = \int_B \mathbb{E}[\lambda^*(X, u)q(u, X)]\lambda(\mathrm{d}u), \tag{3}$$

valid for any measurable test function q on $B \times \mathbf{N}$.

3 Segment Process with Reference Mark Distribution

In the paper we study random segment processes in the plane $\mathbb{R}^2$. A segment is a closed set which will be parametrized by its centre $y = (y_1, y_2) \in \mathbb{R}^2$, length $r > 0$ and direction $\varphi \in [0, \pi)$. There is a bijection ι between the parametric space $\{u = (y, r, \varphi)\}$ and the system of segments as a subsystem of the space of closed sets in $\mathbb{R}^2$. Throughout the paper we use exclusively the parametric representation of segments, omitting the bijection ι in some expressions, e.g. $z \in u$ means a point z of the segment u, etc.

A segment process can be considered as a marked point process with two marks corresponding to the length and direction of a segment. Let $B \subset \mathbb{R}^2$ be bounded measurable,

$$Y = B \times S, \quad S = (0, e_1] \times [0, \pi), \tag{4}$$

where $e_1 > 0$ is an upper bound for the segment length, $[0, \pi)$ is the manifold of axial directions. Further $(\mathbf{N}, \mathcal{N})$ is a measurable space of integer-valued finite measures on Y. Let the Poisson process η on Y have intensity measure λ,

$$\lambda(\mathrm{d}u) = \lambda(\mathrm{d}(y, r, \varphi)) = \frac{1}{e_1\pi}\mathrm{d}y\mathrm{d}r\mathrm{d}\varphi, \tag{5}$$

where on the right hand side we have a multiple of Lebesgue measure on Y. Let the segment process X have an hereditary density p with respect to η

$$p(\mathbf{x}) = c\tau^{n(\mathbf{x})} \prod_{u \in \mathbf{x}} g(r, \varphi) \exp(bD(\mathbf{x}) + aE(\mathbf{x})), \ \mathbf{x} \in \mathbf{N}, \tag{6}$$

where $c > 0$ is a normalizing constant, $b \in \mathbb{R}$, $a \leq 0$, and $\tau > 0$ are parameters, g is the density of reference length-direction distribution, $n(\mathbf{x})$ is number of segments in the configuration $\mathbf{x}$. Further

$$D(\mathbf{x}) = \sum_{u \in \mathbf{x}} d(u), \quad E(\mathbf{x}) = \sum_{\{u,v\} \subset \mathbf{x}}^{\neq} \Phi(u, v) \tag{7}$$

(sum over pairs of different segments from $\mathbf{x}$) for some nonnegative functions d, Φ on Y, Y^2, respectively, Φ is called the pair potencial, E energy function in the theory of Gibbs processes. The conditional intensity is

$$\lambda^*(\mathbf{x}, u) = \tau g(r, \varphi) \exp \left(bd(u) + a \sum_{v \in \mathbf{x} \setminus \{u\}} \Phi(u, v) \right).$$

Let ρ be the intensity function of the process X, i.e. for $A \subset Y$

$$\mathbb{E}X(A) = \int_A \rho(u)\lambda(\mathrm{d}u).$$

Using $q(u, \mathbf{x}) = \mathbf{1}_{[u \in A]}$ in the GNZ formula we obtain

$$\rho(u) = \mathbb{E}\lambda^*(X, u). \tag{8}$$

For $A \subset B$ measurable and the point process ν of segment centres, denote

$$\kappa(A) = \mathbb{E} \sum_{y \in \nu} 1_A(y).$$

For $D \subset S$ let

$$C(A \times D) = \mathbb{E} \sum_{(y,\xi) \in X} 1_A(y)1_D(\xi).$$

The measure ω_D, $\omega_D(A) = C(A \times D)$, is absolutely continuous with respect to κ and the Radon-Nikodym density $y \mapsto P^y(D)$, such that

$$\omega_D(A) = \int_A P^y(D)\kappa(\mathrm{d}y), \tag{9}$$

is the Palm mark distribution of X at y. Let $f_X^{(y)}$ be the density of the Palm mark distribution of length and direction of a typical segment of the process X at the location $y \in B$.

Proposition 1 *For each $y \in B$ we have*

$$f_X^{(y)}(\xi) = \frac{\rho(y, \xi)}{\int_S \rho(y, \xi)\mathrm{d}\xi}. \tag{10}$$

Proof For Borel sets $A \subset B$, $D \subset S$ it holds using the Campbell theorem

$$C(A \times D) = \int_Y 1_A(y)1_D(\xi)\rho(y, \xi)\mathrm{d}\xi\mathrm{d}y.$$

Specially for $D = S$ we have

$$\kappa(A) = \int_A \int_S \rho(y, \xi) \mathrm{d}\xi \, \mathrm{d}y.$$

Then from (9) we have

$$\int_A P^y(D) \int_S \rho(y, \xi) \mathrm{d}\xi \, \mathrm{d}y = \int_A \int_D \rho(y, \xi) \mathrm{d}\xi \, \mathrm{d}y$$

for all Borel sets $A \subset B$, finally

$$P^y(D) = \frac{\int_D \rho(y, \xi) \mathrm{d}\xi}{\int_S \rho(y, \xi) \mathrm{d}\xi}$$

and (10) follows. □

4 Segment Process with Reference Directional Distribution

In this Section we assume that the segment length $r > 0$ is fixed and for a bounded Borel set $B \subset \mathbb{R}^2$ we deal with the

$$Y = B \times [0, \pi). \tag{11}$$

A segment $u = (y, \varphi) \in Y$ has center y and axial orientation φ. Consider a measurable space $(\mathbf{N}, \mathcal{N})$ of integer-valued finite measures on Y, alternatively of the supports of these measures.

We deal with the unit Poisson segment process η with the intensity measure

$$\lambda(\mathrm{d}u) = \mathrm{d}y \frac{1}{\pi} \mathrm{d}\varphi$$

on Y. Let the segment process X have a density p with respect to η, we consider

$$p(\mathbf{x}) = c \exp(a \, N(\mathbf{x})) \tau^{n(\mathbf{x})} \prod_{u_i \in \mathbf{x}} g(\varphi_i), \ \mathbf{x} \in \mathbf{N}, \tag{12}$$

where $N(\mathbf{x})$ is the total number of intersections between segments, g a reference probability density on $[0, \pi)$, φ_i direction of i−th segment u_i, $a \leq 0$, $\tau > 0$ are parameters, c a normalizing constant. That means in the general model (6) we have $b = 0$, g depends only on φ and the pair potential is

$$\Phi(u, v) = \mathbf{1}_{[u \uparrow v]}, \tag{13}$$

i.e. indicator of the event that segment u hits segment v. This model belongs to the more general class of facet processes (Večeřa and Beneš 2017). The conditional intensity is

$$\lambda^*(\mathbf{x}, u) = \tau g(\varphi) \exp(a N_{\mathbf{x}}(u)), \tag{14}$$

where $N_{\mathbf{x}}(u)$ is the number of segments of $\mathbf{x} \setminus \{u\}$ hit by the segment u.

Example Let the direction density g be that of von Mises distribution on $[0, \pi)$ with parameters $\kappa \geq 0$, $\mu \in \mathbb{R}$,

$$g(\phi) = h(\kappa) \exp(\kappa \cos(2(\phi - \mu))), \ \phi \in [0, \pi], \quad h(\kappa) = \frac{1}{\pi I_0(\kappa)},$$

$I_0(\kappa)$ is the modified Bessel function of the first kind and order 0. Then

$$p(\mathbf{x}) = c(\theta) \exp(\langle \theta, G(\mathbf{x}) \rangle), \quad \mathbf{x} \in \mathbf{N}, \tag{15}$$

where

$$\theta = (a, \log(\tau h(\kappa)), \kappa),$$

$$G(\mathbf{x}) = (N(\mathbf{x}), n(\mathbf{x}), \sum_{x_i \in \mathbf{x}} \cos(2(\varphi_i - \mu))),$$

segments $x_i = (u_i, \varphi_i)$ have centres $u_i \in B$ and directions $\varphi_i \in [0, \pi)$. The normalizing constant $c(\theta)$ is defined as

$$c(\theta) = \frac{1}{\int \exp\{\langle \theta, G(\mathbf{x}) \rangle\} \mathrm{d}P_\eta(\mathbf{x})}.$$

Also

$$\{\theta \in \mathbb{R}^3 : \int \exp\{\langle \theta, G(\mathbf{x}) \rangle\} \mathrm{d}P_\eta(\mathbf{x}) < \infty\}$$

is the largest set of θ such that the density (15) is well defined.

In the present paper we want to relax the unimodality assumption of the reference directional distribution and deal with a general density g. Let $f_X^{(y)}$ be the density of the Palm mark distribution P^y of direction of a typical segment of the process X at the location $y \in B$. For $u = (y, \varphi)$ we get from (14) that

$$\rho(u) = \tau g(\varphi) \mathbb{E}_X \exp(a N_\mathbf{x}(u)) \tag{16}$$

is the intensity function of X, cf. (8). The expectation $\mathbb{E}_X$ with respect to the distribution of X is not analytically tractable, therefore (Baddeley and Nair 2012) suggest an approximation

$$\mathbb{E}_X e^{a N_\mathbf{x}(u)} \approx \mathbb{E}_{\eta(\rho)} e^{a N_\mathbf{x}(u)}, \tag{17}$$

where $\eta(\rho)$ is a Poisson process with intensity function ρ.

Proposition 2 *For $u = (y, \varphi) \in Y$ we have*

$$\mathbb{E}_{\eta(\rho)} e^{a N_\mathbf{x}(u)} = \exp\left((e^a - 1) \int_{M_u} \rho(x) \mathrm{d}x \right),$$

where $M_u = \{x \in Y; \ x \uparrow u\}$.

Proof Using (2) we have

$$\mathbb{E}_{\eta(\rho)} e^{a N_\mathbf{x}(u)} =$$

$$= \exp\left(-\int_Y \rho(x) \mathrm{d}x \right) \sum_{n=0}^{\infty} \frac{1}{n!} \int_Y \ldots \int_Y \exp(a N_{(x_1, \ldots, x_n)}(u)) \rho(x_1) \mathrm{d}x_1 \ldots \rho(x_n) \mathrm{d}x_n$$

$$= e^{-\int_Y \rho(x) \mathrm{d}x} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_Y \exp(a 1_{[u \uparrow x_i]}) \rho(x_i) \mathrm{d}x_i = \exp\left(\int_Y (e^{a 1_{[u \uparrow x]}} - 1) \rho(x) \mathrm{d}x \right) =$$

$$= \exp\left((e^a - 1) \int_{M_u} \rho(x) \mathrm{d}x \right).$$

$\square$

From Proposition 1 we have

$$\rho(y, \beta) = C_y f_X^{(y)}(\beta), \ \beta \in [0, \pi),$$

for normalizing constants $C_y, \ y \in B$. Assuming that there exists a stationary segment process in $\mathbb{R}^2$ with given conditional intensity (this time the conditional intensity cannot be defined by means of densities, but from the energy function) corresponding to (14), we have that $C = C_y, \ f_X^{(y)} = f_X$ do not depend on y in the extension of B onto the whole $\mathbb{R}^2$. Under this assumption, discussed in Section 6, we estimate (β is the direction of segment x)

$$\int_{M_u} f_X(\beta) \mathrm{d}x \approx \int_{\mathbb{R}^2 \times [0, \pi)} \mathbf{1}_{[x \uparrow u]} f_X(\beta) \mathrm{d}x = r^2 \int_0^{\pi} |\sin(\beta - \varphi)| f_X(\beta) \mathrm{d}\beta.$$

Using (16) we can then express the desired density g approximately as

$$g(\varphi) \approx \frac{C f_X(\varphi)}{\tau \exp((e^a - 1)C r^2 J(\varphi))}, \tag{18}$$

where

$$J(\varphi) = \int_0^{\pi} |\sin(\varphi - \beta)| f_X(\beta) \mathrm{d}\beta.$$

5 Semiparametric Estimation Using Takacs-Fiksel Approach

In this section we suggest a method of estimation of parameters C, a, τ and the density g from the previous section using the Takacs-Fiksel method. From formula (3) we obtain innovation equations

$$\delta_{\mathbf{x}}(q) = \sum_{u \in X} q(u, X \setminus u) - \int_Y \lambda^*(X, u) q(u, X) \lambda(\mathrm{d}u) = 0 \tag{19}$$

and solve them for various test functions q. We take λ^* from (14) where we insert approximation (18) for unknown g. First the density f_X is estimated using a kernel estimator for directional data (Mardia and Jupp 1999). Then put

$$\beta(a, r, C, \varphi) = \exp((e^a - 1)r^2 C J(\varphi)),$$

and we estimate C, a from the system of Takacs-Fiksel equations:

$$\sum_{u \in \mathbf{x}} N_{\mathbf{x}}(u) = \frac{\pi |B| C}{J} \sum_{i=1}^{J} \frac{f_X(\varphi_i) N_{\mathbf{x}}(u_i) e^{a N_{\mathbf{x}}(u_i)}}{\beta(a, r, C, \varphi_i)},$$

$$n(\mathbf{x}) = \frac{\pi |B| C}{J} \sum_{i=1}^{J} \frac{f_X(\varphi_i) e^{a N_{\mathbf{x}}(u_i)}}{\beta(a, r, C, \varphi_i)}.$$

Here in the innovations equations we take score functions $q(u, \mathbf{x}) = N_{\mathbf{x}}(u)$ and $q = 1$, respectively, the integrals in the second term of (19) are evaluated by Monte Carlo method using J independent simulations of segments uniformly distributed in Y. Then we plug the estimators of C, a in a formula obtained by integrating (18):

$$\tau = \frac{\pi C}{J} \sum_{i=1}^{J} \frac{f_X(\varphi_i)}{\beta(a, r, C, \varphi_i)}$$

and finally estimate g from (18).

Table 1 Empirical mean, standard deviation (sd) and coefficient of variation (CV) of Takacs-Fiksel estimates of scalar parameters in the model having density (12) with reference directional distribution. It is based on 100 simulations, the two cases correspond to $a = -0.5$, $a = -3$

	True	Mean	sd	CV
a	-0.5	-0.496	0.071	0.14
τ	1000	1011	154.7	0.15
a	-3	-3.03	0.356	0.12
τ	1000	976	141.0	0.14

A numerical study is based on twice 100 simulated realizations of segment process with parameters $\kappa = 1, \mu = 0, \tau = 1000, r = 0.12$ on $[0, 1]^2 \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. The two cases I, II investigated are $a = -0.5$, $a = -3$, respectively. The results are in Table 1, Figs. 1 and 2.

In Table 1 we observe a small difference between the true and mean values for both estimates of a and τ. The coefficient of variation

$$CV = \frac{sd}{|\text{mean}|}$$

is also comparable despite the fact that the model II involves more interactions (inhibition of intersections) than the model I. In Fig. 1 we can observe how the kernel estimator of the observed (Palm mark) directional distribution differs from the true reference directional distribution (von Mises). The results in Fig. 2 suggest that the estimate of the reference density is slightly better (smaller bias and variability) for the case I than for the case II. We conclude that the approximation (17) works well in the Takacs-Fiksel method here.
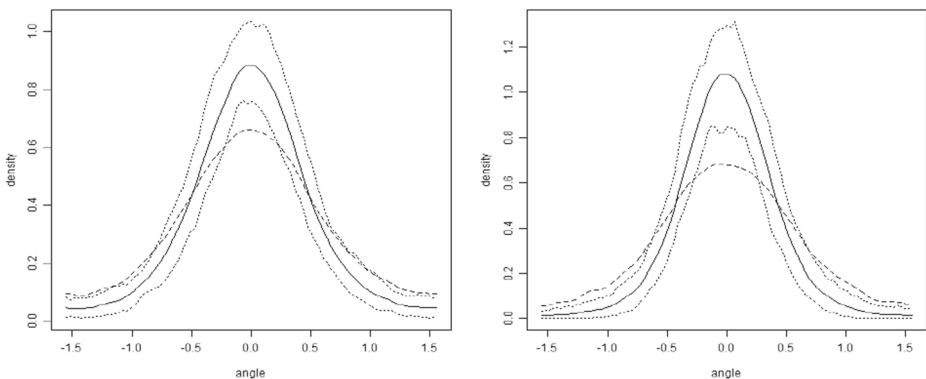

Fig. 1 Kernel estimation of the observed directional density based on 100 simulations of the segment process X, $a = -0.5$ (left), $a = -3$ (right). The average kernel estimator of the observed directional density (full line) compared to the true reference density (dashed line) of von Mises distribution with parameters $\mu = 0$, $\kappa = 1$. The envelopes (dotted lines) correspond to empirical 90% confidence interval for the kernel estimator, pointwise in 100 points on horizontal axis
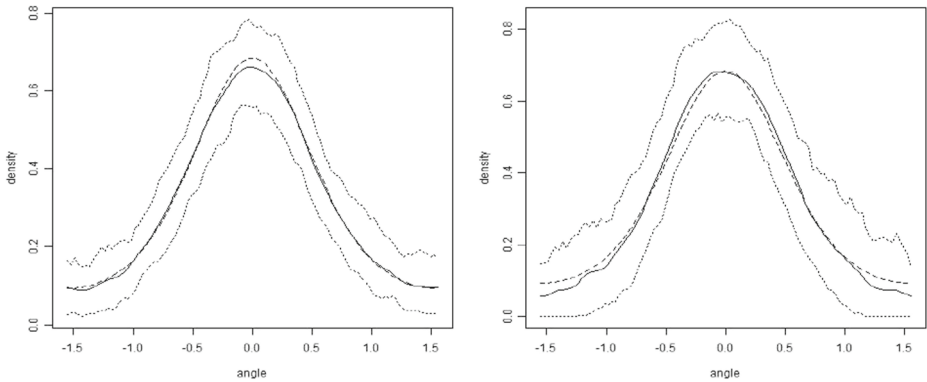
Fig. 2 Semiparametric estimation based on 100 simulations of the segment process X, $a = -0.5$ (left), $a = -3$ (right). The average estimator of the reference density (full line) compared to the true reference density (dashed line) of von Mises distribution with parameters $\nu = 0$, $\kappa = 1$. The envelopes (dotted lines) correspond to empirical 90% confidence interval for the estimated reference density, pointwise in 100 points on horizontal axis

6 Existence of a Stationary Process with Given Conditional Intensity

In order to be able to use the approximation (18) correctly we need a sufficient condition for the existence of a stationary Gibbs segment process in $\mathbb{R}^2$ with prescribed conditional intensity. Various conditions are present in the literature starting with Ruelle (1970). We shall use a recent work of Dereudre (2017) who deals with the concept of an hereditary energy function, invariant with respect to shifts, satisfying the finite range property. While his results are formulated for point processes in $\mathbb{R}^d$, what we need here from Dereudre (2017) is valid also for particle processes in the sense of Schneider and Weil (2008). This straightforward extension of a part of Dereudre (2017) from Gibbs point process to Gibbs particle process is presented in Novotná and Beneš (2017), specially also for segment processes in the plane. In this section we keep the parametric representation $u = (y, \varphi)$ of segments with fixed length r from Section 4. We extend the space of segments to

$$Y' = \mathbb{R}^2 \times [0, \pi),$$

$\mathbf{N}'$ will be the system of integer-valued locally finite measures on Y'. We consider the energy function E in (7) with pair potential Φ in (13) on Y'^2. The energy function is hereditary if for each $\mathbf{x} \in \mathbf{N}'$, $y \in supp\, \mathbf{x}$ we have

$$E(\mathbf{x}) < \infty \implies E(\mathbf{x} \setminus \{y\}) < \infty,$$

where $supp$ is the support of a measure. In the following for $B \subset \mathbb{R}^2$ bounded Borel set we write

$$B' = B \times [0, \pi) \subset Y'.$$

Definition 1 A measurable function $f : \mathbf{N}' \to \mathbb{R}$ is local (on B) if there exists $B \subset \mathbb{R}^2$ bounded Borel such that for all $\mathbf{x} \in \mathbf{N}'$ it holds

$$f(\mathbf{x}) = f(\mathbf{x}_{B'}),$$

where $\mathbf{x}_{B'}$ is the restriction of $\mathbf{x}$ onto B'.

Denote $\oplus$ Minkowski addition of sets, A^c is the complement of a set A and $b(0, R)$ is a circle in $\mathbb{R}^2$ centered in zero with radius R.

Definition 2 An energy function E has finite range $R > 0$ if for all $B \subset \mathbb{R}^2$ bounded Borel sets the local energy

$$E_B(\mathbf{x}) = E(\mathbf{x}) - E(\mathbf{x}_{B'^c})$$

is a local function on $B \oplus b(0, R)$.

Proposition 3 *The energy function E from (7) with pair potential (13) on Y'^2 has finite range property with $R = r$.*

Proof Let $B \subset \mathbb{R}^2$ be bounded Borel set, $\mathbf{x} \in \mathbf{N}$. We have to show that

$$E_B(\mathbf{x}) = E_B(\mathbf{x}_{(B \oplus b(0,r))'}),$$

i.e.

$$E(\mathbf{x}_{B'}) + \sum_{u \in B', v \in B'^c} \Phi(u, v) = E(\mathbf{x}_{(B \oplus b(0,r))'}) - E(\mathbf{x}_{(B \oplus b(0,r))' \cap B'^c}).$$

This is true since to both sides exactly intersections of such pairs of segments contribute, which have either both centres in B or one centre in B and the other in $(B \oplus b(0, r)) \cap B^c$. $\qquad\square$

Corollary 1 *There exists a stationary segment process in $\mathbb{R}^2$ with conditional intensity (14).*

Proof The energy function E from (7) with pair potential (13) is nonnegative, invariant with respect to shifts, hereditary and has finite range. Analogously to Dereudre (2017) and Novotná and Beneš (2017) the existence of a stationary segment process is guaranteed. $\qquad\square$

The uniqueness issue is more complex, see Dereudre (2017) for Gibbs point process in $\mathbb{R}^d$. Here the uniqueness is not investigated, we deal with an existing stationary process in order to use the approximation (18).

7 The Segment Process with Reference Length Distribution

Consider a circle $B \subset \mathbb{R}^2$ centered in the origin with diameter $e_1 > 0$. Let $L_o = (0, e_1]$ be the interval of segment lengths. Then

$$Y = B \times L_o \times [0, \pi) \tag{20}$$

is the space of segments $u = (y, r, \varphi) \in Y$ which have centre y, length r and axial direction φ.

We consider the Poisson segment process η with the intensity measure

$$\lambda(\mathrm{d}u) = \mathrm{d}y \frac{1}{e_1} \mathrm{d}r \frac{1}{\pi} \mathrm{d}\varphi$$

on Y. Let the segment process X have a density p w.r.t. η :

$$p(\mathbf{x}) = c 1_{[\mathbf{x} \subset B]} \exp(b \, D(\mathbf{x})) \tau^{n(\mathbf{x})} \prod_{u_i \in \mathbf{x}} f_1(r_i), \tag{21}$$

$b \in \mathbb{R}$, $\tau > 0$, c is the normalizing constant, r_i is the length of i−th segment u_i, f_1 is a reference probability density on L_o and

$$D(\mathbf{x}) = \sum_{u \in \mathbf{x}} d(u), \quad d(u) = \max_{z \in u} \frac{||z||}{e_1}.$$

That means in the general model (6) we have $a = 0$, $g = f_1$ depends only on the length variable r, marginal reference directional distribution is uniform. Moreover a factor is added in (21) which forces the segments to lie completely in the window.

For $u \subset B$ the quantity $d(u) \in (0, \frac{1}{2}]$ is the normalized distance of the most distant point of u from the centre of B. For positive, negative values of parameter b more, less distant segments from the origin prevail, respectively, cf. Fig. 3.

The corresponding conditional intensity is

$$\lambda^*(\mathbf{x}, u) = 1_{[\mathbf{x} \cup \{u\} \subset B]} \tau f_1(r) \exp(b d(u)).$$

For the intensity function it holds

$$\begin{aligned} \rho(u) &= \mathbb{E}\lambda^*(X, u) = \tau f_1(r) \exp(b d(u)), \ u \subset B, \\ \rho(u) &= 0, \qquad\qquad\qquad\qquad\qquad\quad u \cap (\mathbb{R}^2 \setminus B) \neq \emptyset. \end{aligned}$$

There are no interactions among the segments in the model (21) with statistics $D(\mathbf{x})$, X is in fact an inhomogeneous Poisson process with unknown reference density f_1 and a condition $X \subset B$. The process X is isotropic since both the reference Poisson process and the density p are invariant with respect to rotations around the origin. We denote Ay rotation of $y \in \mathbb{R}^2$, where A is an orthonormal matrix 2×2, analougously Au is rotation of the segment u and finally we use the same symbol $A\varphi$ for the corresponding direction of the segment u after rotation Au. Then also the intensity function is invariant with respect to rotations, i.e.

$$\rho(u) = \rho(Au)$$
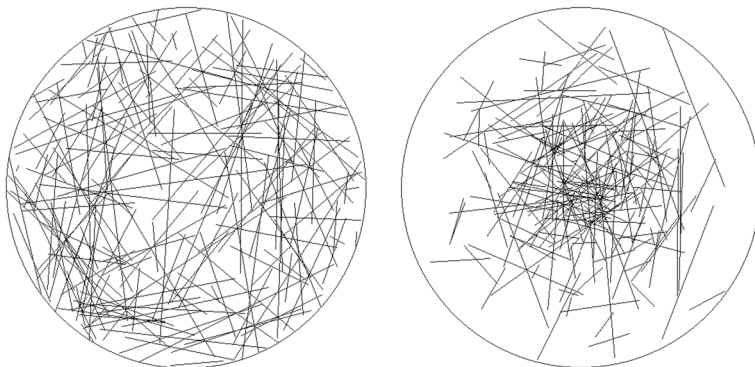
for all $u \in Y$ and all rotations A.

Fig. 3 Simulated realizations of the process with $e_1 = 1$ and the density (21) having parameters $b = 10$, $\tau = 3$ (left) and $b = -10$, $\tau = 4000$ (right), f_1 corresponds to beta distribution with parameters $\alpha = 2$, $\beta = 4$

Let $f_X^{(y)}$ be the bivariate (Palm) density of the distribution of length and direction of the segment centered at y, from Proposition 1 we have for $y \in B$ a normalizing constant $C_y > 0$ such that

$$f_1(r) = \frac{C_y f_X^{(y)}(r, \varphi)}{e^{bd(u)}}. \tag{22}$$

From the isotropy of X it holds

$$f_X^{(y)}(r, \varphi) = f_X^{(Ay)}(r, A\varphi) \tag{23}$$

and therefore

$$C_y = C_{Ay}$$

for all $y \in B$ and all rotations A.

8 Semiparametric Estimation Using the Maximum Likelihood Method

The aim is to estimate parameters b, τ and density f_1 from simulated data. We are using again a semiparametric approach so that f_1 is not parametrized. Because of inhomogeneity of X the solution has to be discretized, but we make use of isotropy. In the parametric part, since the process is Poisson we use maximum likelihood method for parameter estimation. For an observed realization $\mathbf{x}$ the likelihood is defined as

$$\mathcal{L} = \prod_{u \in \mathbf{x}} \rho(\mathbf{x} \setminus u, u) \exp\left(-\int_{v \subset B} \rho(\mathbf{x}, v)\lambda(\mathrm{d}v)\right).$$

The logarithmic likelihood

$$\log \mathcal{L} = \log(\tau)n(\mathbf{x}) + bD(\mathbf{x}) + \sum_{u \in \mathbf{x}} \log f_1(r) - \tau \int_{u \subset B} f_1(r)e^{bd(u)}\lambda(\mathrm{d}u)$$

has to be maximized with respect to τ, b. We have

$$\frac{\partial \log \mathcal{L}}{\partial \tau} = \frac{n(\mathbf{x})}{\tau} - \int_{u \subset B} f_1(r)e^{bd(u)}\lambda(\mathrm{d}u),$$

$$\frac{\partial \log \mathcal{L}}{\partial b} = D(\mathbf{x}) - \tau \int_{u \subset B} d(u)f_1(r)e^{bd(u)}\lambda(\mathrm{d}u).$$

Using (22) we obtain equations

$$n(\mathbf{x}) = \tau \int_{u \subset B} C_y f_X^{(y)}(r, \varphi)\lambda(\mathrm{d}u), \tag{24}$$

$$D(\mathbf{x}) = \tau \int_{u \subset B} C_y d(u) f_X^{(y)}(r, \varphi)\lambda(\mathrm{d}u).$$

In the estimation procedure we proceed in several steps:

(i) Consider discrete levels of $||y||$ like $0 < w_1 < \cdots < w_k < e_1/2$ and put $y_j = (0, w_j)^T \in B$. For $\Delta_f = \frac{e_1}{2k}$ let

$$w_j = (j-1)\Delta_f + \frac{\Delta_f}{2}, \; j = 1, \ldots, k,$$

$$Y_j = B_j \times L_o \times [0, \pi), \quad B_j = \{y \in B, \, (j-1)\Delta_f < ||y|| \le j\Delta_f\}, \; j = 1, \ldots, k.$$

The kernel estimator of the bivariate densities $f_X^{(y_j)}$, $j = 1, \ldots, k$, is evaluated from the observed data in each class, i.e. from the sample of segments $u_j^{(i)} = (y_j^{(i)}, r_j^{(i)}, \varphi_j^{(i)})$, $i = 1, \ldots, m_j$ centered in B_j, $j = 1, \ldots, k$. Because of (23) these segments are first rotated by such $A_j^{(i)}$ that

$$A_j^{(i)} y_j^{(i)} = q y_j$$

for some $q > 0$. Then we apply kernel estimation to each sample

$$A_j^{(i)} u_j^{(i)}, i = 1, \ldots, m_j,$$

to estimate the length-direction density $f_X^{(y_j)}$, $j = 1, \ldots, k$. Since the length component has values on a compact L_o, we use here a system of beta kernels (Chen 1999). The angular component is a circular variable, cf. Mardia and Jupp (1999), jointly we use a bivariate product kernel.

(ii) Since the length of the longest possible segment centered in j−th class is

$$l_j = 2 \sqrt{\left(\frac{e_1}{2}\right)^2 - (\Delta_f(j-1))^2},$$

we use the step $\Delta_c^{(j)} = l_j/m$ for numerical integration of (22) with $y = y_j$ and a fixed φ using Simpson rule, to express unknown constants $C_{y_j} = C_j$, $j = 1, \ldots, k$, as functions of b.
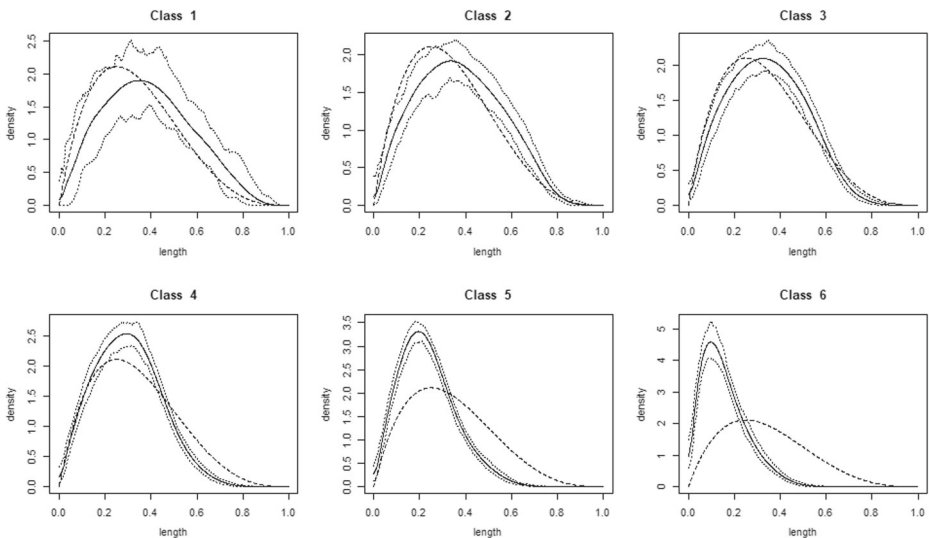

Fig. 4 Kernel estimation of the observed length density based on 60 simulations of the inhomogeneous segment process X, $b = 3, \tau = 900$. In each of six classes the average kernel estimator of the observed length density (full line) is compared to the true reference density (dashed line) of beta distribution with parameters $\alpha = 2$, $\beta = 4$. The envelopes (dotted lines) correspond to empirical 90% confidence interval for the kernel estimator, pointwise in 100 points on horizontal axis

Table 2 Empirical mean, standard deviation (sd) and coefficient of variation (CV) of the maximum likelihood estimates of scalar parameters in the model (21) with reference length density. Based on a sample of 60 simulated realizations of the segment process with parameters $b = 3$, $\tau = 900$, $\alpha = 2$, $\beta = 4$, $e_1 = 1$

	True	Mean	sd	CV
b	3	3.051	0.481	0.16
τ	900	948.9	186.2	0.19

(iii)   Integrals in the equations (24) are evaluated by Monte Carlo simulation of M segments in Y uniformly randomly. Let n of them lie completely in B, denoted $\bar{u}_j^{(i)} = (\bar{y}_j^{(i)}, \bar{r}_j^{(i)}, \bar{\varphi}_j^{(i)})$, $i = 1, \ldots, n_j$ where n_j of them are centered in B_j, $\sum_{j=1}^k n_j = n$. Then

$$\frac{D(\mathbf{x})}{n(\mathbf{x})} = \frac{\sum_{j=1}^k C_j \sum_{i=1}^{n_j} d(\bar{u}_j^{(i)}) f_X^{(\bar{y}_j^{(i)})}(\bar{r}_j^{(i)}, \bar{\varphi}_j^{(i)})}{\sum_{j=1}^k C_j \sum_{i=1}^{n_j} f_X^{(\bar{y}_j^{(i)})}(\bar{r}_j^{(i)}, \bar{\varphi}_j^{(i)})} \tag{25}$$

is an equation with a single variable b which is solved numerically.

(iv)   Having estimated b we obtain C_j from step (ii) and then τ from any of the equations

$$\tau = \frac{4M \, n(\mathbf{x})}{\pi^2 e_1^3 \sum_{j=1}^k C_j \sum_{i=1}^{n_j} f_X^{(\bar{y}_j^{(i)})}(\bar{r}_j^{(i)}, \bar{\varphi}_j^{(i)})},$$

$$\tau = \frac{4M \, D(\mathbf{x})}{\pi^2 e_1^3 \sum_{j=1}^k C_j \sum_{i=1}^{n_j} d(\bar{u}_j^{(i)}) f_X^{(\bar{y}_j^{(i)})}(\bar{r}_j^{(i)}, \bar{\varphi}_j^{(i)})}$$

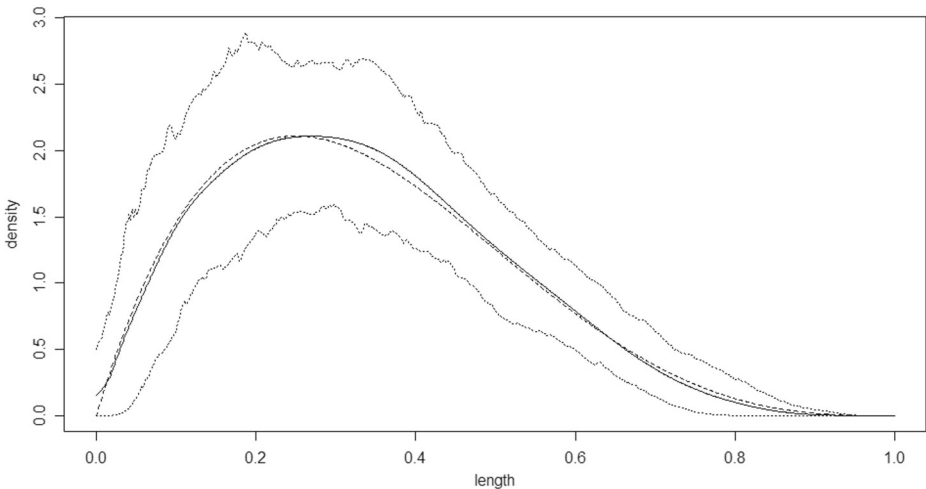which are Monte Carlo analogues of the equations in (24).

Fig. 5 Semiparametric estimation of reference length density based on 60 simulations, respectively, of the segment process X, $b = 3$, $\tau = 900$. The average (from first four classes) estimator of the reference density (full line) compared to the true reference density (dashed line) of beta distribution with parameters $\alpha = 2$, $\beta = 4$. The envelopes (dotted lines) correspond to empirical 90% confidence interval for the estimated reference density, pointwise in 100 points on horizontal axis

(v) Finally the estimator of the reference length density f_1 is obtained by plugging these estimators in (22) and renormalizing. We choose $\varphi = 0$, from the levels of y those close to the boundary are mostly variable and therefore not used. We can average results of several levels of y.

A numerical study is based on a sample of 60 simulated independent realizations of the segment process with parameters $b = 3, \tau = 900, \alpha = 2, \beta = 4, e_1 = 1$. In step (i) of the estimation procedure $k = 6$ classes were considered. In Fig. 4 there are kernel estimates of the observed length density in all six classes separately. We can observe the difference between the estimated observed length distribution and the true reference beta distribution, which increases along classes towards the boundary of the window. The results of estimation of parameters b, τ (step (iii), (iv)) are in Table 2. We observe still a reasonably small difference between the true and the mean value of estimates of b and τ, but it is larger than in the Gibbs model with directional distribution in Section 5, also CV is slightly larger. This can be justified by the fact that the present model is inhomogeneous and the estimation procedure is more complex. The semiparametric estimator of the reference length density f_1 (step (v)) is in Fig. 5, here we do not consider two outer classes where the inhomogeneity is the highest (since there are little admissible directions of longer segments close to the border). We observe a small bias of the estimator of the reference density, but the variability is larger than in the first model in Section 5 from the same reasons as we argued to Table 2.

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