Omega 102 (2021) 102338

Contents lists available at ScienceDirect

Omega

journal homepage: www.elsevier.com/locate/omega

Risk-aversion in data envelopment analysis models with diversification☆

Lukáš Adam [a,b], Martin Branda [c,*]

[a] Faculty of Electrical Engineering, Czech Technical University in Prague, Prague, Czech Republic
[b] Department of Computer Science and Engineering, Southern University of Science and Technology, Shenzhen, China
[c] Charles University, Faculty of Mathematics and Physics, Department of Probability and Mathematical Statistics, Sokolovská 83, Prague 186 75, Czech Republic

ARTICLE INFO

Article history:
Received 10 February 2020
Accepted 8 September 2020
Available online 15 September 2020

MSC:
90C15
90B36
90C10

Keywords:
Risk aversion
DEA
Diversification
Spectral risk measures
Second order stochastic dominance
Shadow risk spectrum

ABSTRACT

We deal with data envelopment analysis models with diversification which can identify investment opportunities efficient with respect to several inputs and outputs represented by risk and return measures. Moreover, they enable to project the inefficient investment opportunity to the efficient frontier and suggest how to revise its structure. However, the current DEA models does not take into account the individual risk aversion of a particular investor. We will introduce several approaches based on the spectral risk measures which deal with this drawback. These approaches are then compared in the empirical study. Note that all considered models as well as risk aversion are consistent with the second order stochastic dominance.

1. Introduction

This paper is focused on DEA models suitable for accessing efficiency of investment opportunities available on financial markets. Traditional DEA models were applied in finance by papers [1–4]. However, these models ignored the diversification between considered investment opportunities which appears when risk measures are considered as the inputs. Therefore diversification-consistent DEA (DC-DEA) models were introduced and investigated, cf [5–7]. We postpone basic overview of the main contributions to this class of DEA models to Section 1.1.

In this paper, our primary interest is the link between DC-DEA models and risk aversion. Risk aversion refers to a person's behavior with respect to random payouts, where a risk averse investor seeks to reduce the degree of uncertainty. Pratt [8] and Arrow [9] provided the basis for the assessment of risk aversion within the theory of utility functions, see also [10] for generalizations. Fulga [11] proposed a new risk measure called Expected Shortfall with Loss Aversion (ESLA) which is calculated with the downside part of the portfolio return distribution and captures the investors preferences in the mean-risk models. We turn our attention to so called spectral risk measures, cf [12]. These risk measure use a risk spectrum to weights the quantiles of the random returns, thus they enable to take into account risk aversion of an investor. Papers [13–17] investigated deeply the relations with the utility theory and risk aversion, see Section 1.1 for deeper review.

To our best knowledge, research on the risk aversion within the DEA literature is limited to a few articles which deal with the moment criteria, i.e. expected value, variance, and skewness of the random returns. Briec et al. [18] realized that duality offers information about the investors risk aversion via the shadow prices associated with their mean-variance efficiency measure. Their approach enabled to estimate the weights of the mean and variance criteria such that the particular efficient portfolio is rendered as optimal for the investor. Therefore, the weighted criteria were labeled as a shadow (indirect) mean-variance utility function. The procedure for identifying the utility function of the investor has been generalized by Briec et al. [19] for the mean-variance-skewness criteria and by Briec and Kerstens [20] for the multi-period mean-variance efficiency measure. We will propose a similar approach to identify the shadow risk spectrum leading to a

☆ Area: Data-Driven Decision Making and Analytics; Sustainable Operations. This manuscript was processed by Associate Editor Sungmook Lim.

* Corresponding author.
E-mail address: branda@karlin.mff.cuni.cz (M. Branda).

https://doi.org/10.1016/j.omega.2020.102338
0305-0483/

shadow spectral risk measure which renders the DC-DEA efficient portfolio as optimal which is according to our best knowledge a new result. Moreover, we will show that our approach is fully consistent with the second order stochastic dominance (SSD) efficiency which is well established tool in finance, cf [21–23]. Every rational risk-averse investor should invest into a SSD efficient investment opportunity and avoid to the inefficient ones. The DC-DEA models with particular inputs (CVaRs) and output (expectation) have been shown to be equivalent with the SSD efficiency tests by Branda and Kopa [7,24]. However, the SSD efficient frontier contains infinitely many portfolios. Under mild conditions, by minimizing a spectral risk measure with investor's risk spectrum, we can obtain an ideal investment opportunity which is SSD efficient. However, such ideal opportunity need not to be directly available on the market or its construction can be too expensive, e.g., because of transaction costs. Therefore, we suggest four ways how to measure distance between the ideal choice and the available investment opportunities. First comparison is based on the portfolio weights which corresponds to a possible revision by selling/buying necessary assets. The second approach is based on the mean–risk characterization for each opportunity, i.e. the distance compares the values of these measures for two portfolios. The third one – called inverse – uses a reconstruction of the investor's (shadow) risk spectrum which is possible for any efficient investment opportunity as we will show later in this paper. Then, the reconstructed risk spectrum is compared with the investor's one. Finally, we extend the DC-DEA score to measure the relative improvement of inputs and outputs which is necessary to reach the ideal investment opportunity (instead of nearest one). Note that all these distances can be also used as measures of super-efficiency for efficient investment opportunities, because they provide alternative measures for ranking of the efficient ones with respect to the preferences of an investor.

In the numerical study, we apply our approaches to industry representative portfolios from the US market which are listed in the Kenneth French library. First, we derive their rankings with respect the DC-DEA scores which are consistent with the SSD efficiency. Then, we select two basic parametric spectra and compute the ideal portfolios for several choices of the parameters which correspond to various risk aversions. These portfolios are then compared with the initial representative portfolios and their DC-DEA projections to the efficient frontier. Finally, we will investigate robustness and stability of the proposed measures using two extreme scenarios.

The paper is organized as follows. Section 2 reviews the DEA models with diversification based on directional distance measure and their relation to SSD efficiency. Moreover, it introduces the spectral risk measures and risk spectra. In Section 3, we introduce the reconstruction of the shadow risk spectrum and propose several ways how to measure the distance to an ideal investment opportunity. Section 4 contains a numerical study where the introduced approaches are applied to representative portfolios from the US market. The formal and axiomatic definitions are postponed to the Appendix A, technical proofs to the Appendix B.

1.1. Literature review

This part provides an overview of the contributions to the diversification-consistent DEA models and spectral risk measures. We believe that readers familiar with these topics can skip this short subsection.

First two moments of the random returns were considered as the input and output by Briec et al. [18], and the third moment (skewness) was added by papers [19,25] and further elaborated by Kerstens et al. [26]. Briec and Kerstens [20] extended the mean-variance models to dynamic settings. The first backtesting analysis was proposed by Brandouy et al. [27]. A general class of diversification-consistent models was proposed by Lamb and Tee [5] who considered several risk and return measures as the inputs and outputs. Branda [6] used general deviation measures as the inputs of the models with diversification and investigated the strength of the models. In paper [24], DEA models equivalent to the second order stochastic dominance tests were provided. DEA models based on a directional distance measure were proposed by Branda [7] who considered coherent risk measures as the inputs and return measures as the outputs. It was even shown, that under particular choice of the inputs and outputs, the obtained models are equivalent to the stochastic dominance efficiency tests. Moreover, their project inefficient investment opportunities to the efficient frontier, when the projection can serve as a suggestion for investors how to revise their (inefficient) portfolios. Traditional DEA models were used to approximate the efficient frontier and to assess performance of portfolios by Liu et al. [28]. The author of [29] discussed models with Value at Risk inputs which does not belong to any above mentioned classes but still is a popular risk measure. Recently, Branda and Kopa [30] proposed the DEA models which are consistent with the higher order stochastic dominance tests. Tarnaud and Leleu [31] revised the set of axioms which define the financial technology set to allow the generalization to a multi-moment framework. Moreover, they investigated the influence on the measures of technical efficiency on a sample set of US stocks. Choi and Min [32] designed new performance measure based on diversification-consistent DEA models and confirmed empirical efficiency of well-diversified portfolios. The DEA models with diversification were extended to dynamic (multiperiod) settings by Lin et al. [33]. Their approach decomposes the overall efficiency of mutual funds in the whole investment period into efficiencies at individual intervals taking into account dynamic dependence among the investment periods. Zhou et al. [34] proposed a DEA frontier improvement approach under the mean-variance framework which provides investor with a rebalancing strategy as well as an improved DEA frontier which approximates the portfolio efficient frontier better than the traditional DEA model does. They applied their approach to evaluate mutual fund performance. Essid et al. [35] combined the DEA game cross-efficiency approach and the maverick index to introduce a tool for portfolio selection and verified that the resulting portfolios are well-diversified using assets listed in Paris Stock Exchange. Recently, Lin and Li [36] proposed two super-efficiency DEA models with diversification for discriminating the performance of efficient mutual funds and applied the models to evaluate the performance of mutual funds in the American market.

The spectral risk measures were proposed by Accerbi [12] as a special class of coherent risk measures. Since then, these measures were studied deeply by many papers. Relations of these measures with respect to the Ross and Arrow-Pratt-risk aversion were investigated by Brandtner and Kürsten [13]. Wächter and Mazzoni [16] clarified the relation between decisions of a risk-averse decision makers, based on expected utility theory on the one hand, and spectral risk measures on the other. Brandtner and Kürsten [14] revisit this procedure with respect to the axiomatic foundation of the underlying decision rules. Brandtner [15] fixed some issues which were opened by paper [17] with respect to the exponential and power spectral risk measures and how they reflect the risk-aversion of an investor.

2. Diversification consistent DEA models

The main goal of portfolio optimization methods is to design a portfolio with high return and low risk. We assume that the possible portfolio consists of n assets; each represented by its random return $R_1, \dots, R_n$ with known distribution. If the whole budget is invested into the risky assets and the short sales are not allowed, all possible returns can be represented by

$$\mathcal{X} = \left\{ \sum_{i=1}^{n} x_i R_i : \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0 \right\}, \tag{1}$$

Note that other choices of the set are also possible, see, e.g., [6]. We assume that all portfolio random returns contained in the set of investment opportunities $\mathcal{X}$ have finite expected value which is sufficient for the use of the CVaR risk measures later. We review several DEA models with diversification in which we take into account risk aversion later.

2.1. Diversification-consistent DEA model based on directional distance measure

The diversification-consistent DEA models are based on a set of inputs and outputs. In [7], the models where coherent risk measures are used as the inputs were proposed. Coherent risk measures introduced by Artzner et al. [37] are functionals denoted by $\mathcal{R}$ that satisfy translation equivariance (if a risky position is increased by a constant, then its risk decreases by the same constant); positive homogenity (risk of a position multiplied by a positive constant is increased proportionally); subadditivity (risk of a sum of two risky positions is lower or equal than the sum of risk of individual positions); and finally monotonicity (if one risky position has almost surely higher random return than another, than its risk is lower or equal). Formal definition can be found in Appendix A, Definition A.1. Coherent risk measures are always convex which is important to reach globally optimal solutions.

The return functionals are used as the outputs to quantify the return (right) tail of the distribution. Any functional $\mathcal{E}$ is called a return measure if there exists a coherent risk measure $\mathcal{R}$ such that

$$\mathcal{E}(X) = -\mathcal{R}(X).$$

For the diversification-consistent DEA we consider J return measures $\mathcal{E}_j$ as the outputs and K coherent risk measures $\mathcal{R}_k$ as the inputs. This corresponds to risk-shaping which enables to better compare the distribution of various investment opportunities. Since both coherent risk and return measures can take positive as well as negative values, the DEA models proposed by paper [7] were based on the directional distance measures. For a benchmark portfolio $X_0 \in \mathcal{X}$, we define the directions as

$$e_j(X_0) = \max_{X \in \mathcal{X}} \mathcal{E}_j(X) - \mathcal{E}_j(X_0), \tag{2}$$

$$d_k(X_0) = \mathcal{R}_k(X_0) - \min_{X \in \mathcal{X}} \mathcal{R}_k(X). \tag{3}$$

These directions denote the maximal possible improvements over the return and risk measures over the benchmark portfolio X_0.

Although Branda [7] proposed several DEA models with different properties and relations to Pareto–Koopmans efficiency, we will focus on the strongest one which is related to the stochastic dominance efficiency test. This relation will later enable us to consider the risk aversion of a particular investor. This DEA model measures necessary relative improvements to reach the efficient frontier with respect to each input and output separately:

$$\begin{aligned}
&\mathit{minimize}_{\theta_k, \varphi_j, x_i} \frac{1 - \frac{1}{K}\sum_{k=1}^{K} \theta_k}{1 + \frac{1}{J}\sum_{j=1}^{J} \varphi_j} \\
&\text{s.t. } \mathcal{E}_j\left(\sum_{i=1}^{n} R_i x_i\right) \geq \mathcal{E}_j(X_0) + \varphi_j \cdot e_j(X_0),\ \ j = 1, \ldots, J, \qquad (4)\\
&\mathcal{R}_k\left(\sum_{i=1}^{n} R_i x_i\right) \leq \mathcal{R}_k(X_0) - \theta_k \cdot d_k(X_0),\ \ k = 1, \ldots, K, \\
&\sum_{i=1}^{n} x_i = 1,\ x_i \geq 0,\ \varphi_j \geq 0,\ \theta_k \geq 0.
\end{aligned}$$

Due to (3), φ_j denotes the fraction of the improvement of the optimal portfolio $\sum_{i=1}^{j} R_i x_i$ over the maximal possible improvement in return $\mathcal{E}_j$. Similarly, θ_k denotes the fraction of the improvement of the optimal portfolio $\sum_{i=1}^{j} R_i x_i$ over the maximal possible improvement in risk $\mathcal{R}_j$. The objective function then balances the mean improvement in risks in the numerator and the mean improvements in returns in the denominator. A formal explanation can be found in Appendix A, Remark A.1.

We say that X_0 is efficient if the optimal value of (4) equals one, otherwise it is inefficient. The former means $\varphi_j = \theta_k = 0$ and no improvement over the benchmark portfolio X_0 in all measures is possible. Even though (4) is a non-convex problem, it admits a convex reformulation as shown in [7]. Because the optimal solution of (4) is always efficient, see [7], the investors, who identify their benchmark portfolios as inefficient, can use the optimal solution to revise their benchmarks.

2.2. CVaR and second order stochastic dominance efficiency

The most popular coherent risk measure is the Conditional Value at Risk (CVaR). For a confidence level $\alpha \in (0, 1)$, it is roughly defined as the "expected value of $(1-\alpha)$*100% worst losses". It takes the tail of the return distribution into account. If the distribution of X is continuous, CVaR can be expressed as the conditional mean of losses over the value at risk (quantile)

$$\mathrm{CVaR}_\alpha(X) = \mathbb{E}[-X \mid -X \geq \mathrm{VaR}_\alpha(X)].$$

Note that we need to use $-X$ to switch from returns to losses and that the value of risk is already computed from losses. A formal definition was proposed by Rockafellar and Uryasev [38], see also Definition A.2 in the Appendix A.

As shown by Branda [7], it is reasonable to choose in (4) CVaRs at different levels as risk measures and expected return as the single return measure. We propose the following problem which is a special case of (4):

$$\begin{aligned}
&\mathit{minimize}_{\theta_k, \varphi, x_i} \frac{1 - \frac{1}{S-1}\sum_{k=1}^{S-1} \theta_k}{1 + \varphi} \\
&\mathbb{E}\left(\sum_{i=1}^{n} R_i x_i\right) \geq \mathbb{E}(X_0) + \varphi \cdot e(X_0), \qquad (5)\\
&\mathrm{CVaR}_{k/S}\left(\sum_{i=1}^{n} R_i x_i\right) \leq \mathrm{CVaR}_{k/S}(X_0) - \theta_k \cdot d_k(X_0),\ \ k = 1, \ldots, S-1, \\
&\sum_{i=1}^{n} x_i = 1,\ x_i \geq 0,\ \varphi \geq 0,\ \theta_k \geq 0.
\end{aligned}$$

with the directions

$$\begin{aligned}
e(X_0) &= \max_{X \in \mathcal{X}} \mathbb{E}[X] - \mathbb{E}[X_0], \\
d_k(X_0) &= \mathrm{CVaR}_{k/S}(X_0) - \min_{X \in \mathcal{X}} \mathrm{CVaR}_{k/S}(X),\ \ k = 1, \ldots, S-1. \qquad (6)
\end{aligned}$$

As we will see soon, a portfolio is DC-DEA efficient with respect to model (5) if and only if it is non-dominated with respect to the second-order stochastic dominance (SSD). SSD efficiency is related to the utility theory, see, e.g. Lévy [23]. We say that the investment opportunity dominates another opportunity with respect to the second-order stochastic dominance (SSD) if and only if its expected utility is higher or equal for all (nondecreasing) concave utility functions with strict inequality for at least one utility function. Then an investment opportunity is SSD efficient if there is no other investment opportunity which dominates it. All rational risk-averse investor should prefer SSD efficient opportunities.

Formal definition can be found in Definition Appendix A.3 in the Appendix A.

DEA models related to SSD were proposed also by Kuosmanen [39], Lozano and Gutiérrez [40]. We repeat the important theorem derived in paper [7] which proposed the equivalence between the DC-DEA efficiency and the SSD efficiency.

Theorem 2.1. *An investment opportunity $X_0 \in \mathcal{X}$ is DEA efficient according to model (5) (its optimal value equals to one) if and only if it is SSD efficient.*

Using the ideas of Branda [7], we can show that (5) is equivalent to the following reformulation

$$\begin{aligned}
&\text{minimize}_{\tilde{\theta}_k,\tilde{\varphi},t,\tilde{x}_i,\xi_k,u_{sk}} t - \frac{1}{S-1}\sum_{k=1}^{S-1}\tilde{\theta}_k \\
&t + \tilde{\varphi} = 1, \qquad (7)\\
&\frac{1}{S}\sum_{s=1}^{S}\sum_{i=1}^{n}\tilde{x}_i r_{is} \geq t\cdot\mathbb{E}[X_0] + \tilde{\varphi}\cdot e(X_0),\\
&\xi_k + \frac{1}{S-k}\sum_{s=1}^{S}u_{sk} \leq t\cdot \mathrm{CVaR}_{k/S}(X_0) - \tilde{\theta}_k\cdot d_k(X_0),\ k=1,\dots,S-1,\\
&u_{sk} \geq -\sum_{i=1}^{n}\tilde{x}_i r_{is} - \xi_k,\ s=1,\dots,S, k=1,\dots,S-1,\\
&\sum_{i=1}^{n}\tilde{x}_i = t,\ \tilde{x}_i \geq 0,\ \tilde{\varphi}\geq 0,\ \tilde{\theta}_k \geq 0,\ t\geq 0,\ u_{sk}\geq 0.
\end{aligned}$$

Since CVaR can be rewritten via linear constraints, problem (7) can be understood as a linear program, i.e. large instances with many assets (large n) and many scenarios (large S) can be solved to optimality.

2.3. Spectral risk measures and risk aversion

The spectral risk measures (SRM) were proposed by Accerbi [12] as a special class of coherent risk measures. They are defined as the weighted quantiles of the random returns. Even though they are usually considered for a continuous distribution, we restrict the discussion to our case of discrete distribution. For a general formulation, see Definition A.4 in the Appendix A. So called risk spectra serve as the weights. We say that a sequence $\{\phi_s\}_{s=1}^S$ is an admissible empirical risk spectrum if it is:

1. positive: $\phi_s \geq 0$,
2. non-increasing: $\phi_s \geq \phi_{s'}$ if $s < s'$,
3. normalized: $\sum_{s=1}^S \phi_s = 1$.

Investors can identify their risk aversion by choosing a risk spectrum ϕ and then obtain the admissible empirical risk spectrum by setting

$$\phi_s = \frac{\phi(s/S)}{\sum_{s=1}^S \phi(s/S)}. \qquad (8)$$

Let $X_{[1]} \leq X_{[2]} \leq \dots \leq X_{[S]}$ be the sorted equiprobable realizations of X. Then the empirical spectral risk measure is defined as

$$M_\phi^S(X) = -\sum_{s=1}^S \phi_s X_{[s]}. \qquad (9)$$

Since $X_{[1]}$ is the smallest return (-largest loss) and since ϕ_s is non-increasing, minimizing the empirical risk measure assigns largest weights to worst scenarios. Moreover, CVaR is a special case of the empirical spectral risk measure, see formula (A.2) in Appendix A.

As we discussed above, any rational investor wants to minimize the empirical spectral risk measure M_ϕ^S. Since expression (9) requires to sort the scenarios, it is not useful for optimization problem. Therefore, we derive its alternative expression using CVaRs in the next lemma. Its proof is presented in Appendix B.

Lemma 2.1. *Let the distribution of X be finite discrete with equiprobable realizations. Consider an admissible empirical risk spectrum $\{\phi_s\}_{s=1}^S$. Then the spectral risk measure (9) can be expressed as*

$$M_\phi^S(X) = \sum_{s=1}^S \mu_s \mathrm{CVaR}_{1-s/S}^S(X) = \mu_S\mathbb{E}(-X) + \sum_{s=1}^{S-1}\mu_s \mathrm{CVaR}_{1-s/S}^S(X), \qquad (10)$$

where the weights can be obtained from the empirical risk spectrum as

$$\mu_s = s(\phi_s - \phi_{s+1}),\ s=1,\dots,S, \qquad (11)$$

setting $\phi_{S+1} \equiv 0$.

3. DEA With diversification and risk aversion

In this section, we will focus on the ideal investor investment opportunities and their distances to other opportunities which include the initial ones as well as those obtained as optimal solutions of the DC-DEA models, i.e. projections to the efficient frontier. We will also propose an inverse approach which leads to a shadow risk spectrum and shadow SRM.

The previous section summarized two ways of obtaining an efficient portfolio. The DC-DEA approach considers a benchmark portfolio X_0 and solves the DC-DEA problem (5) to obtain an efficient portfolio. This may be understood as projecting (improving) the benchmark portfolio on the efficient frontier. This portfolio is SSD efficient due to Lemma 2.1. In the empirical spectrum risk approach, the investors identify their risk spectrum ϕ corresponding to their risk aversion and they find their ideal investment opportunity by minimizing the spectral risk measure

$$\text{minimize}_{X\in\mathcal{X}} \qquad M_\phi^S(X). \qquad (12)$$

They can also derive the weights $\mu_1,\dots,\mu_S$ and solve the problem

$$\text{minimize}_{X\in\mathcal{X}} \qquad \mu_S\mathbb{E}(-X) + \sum_{s=1}^{S-1}\mu_s\mathrm{CVaR}_{1-s/S}^S(X). \qquad (13)$$

Problems (12) and (13) are equivalent due to Lemma 2.1. Moreover, the empirical risk spectrum ϕ and the auxiliary variables are related by

$$\begin{aligned}
\mu_s &= s(\phi_s - \phi_{s+1}), \qquad s=1,\dots,S,\\
\phi_s &= \sum_{t=s}^S \frac{\mu_t}{t}, \qquad s=1,\dots,S. \qquad (14)
\end{aligned}$$

We summarize these relations in Fig. 1. The left two blocks correspond to the DC-DEA approach while the right four blocks to the empirical spectral risk measure approach. So far, we did not show any connection between these two approaches. We will provide it later in Theorem 3.2 which states that every DC-DEA efficient portfolio corresponds to some empirical risk spectrum called the shadow risk spectrum.

We know that problems (12) and (13) are equivalent. The latter is preferred because it posses a linear programming reformulation, i.e. it can be solved for large number of assets and scenarios. An important property of the optimal solution of (12) or (13) is the SSD efficiency, i.e. the optimal portfolio can be a reasonable choice for a risk-averse investor.

Lemma 3.1. *If the empirical risk spectrum $\{\phi_s\}_{s=1}^S$ is decreasing, then the optimal portfolio obtained by solving (12) is SSD efficient. If is merely non-decreasing and if the solution of (12) is unique, then it is SSD efficient.*

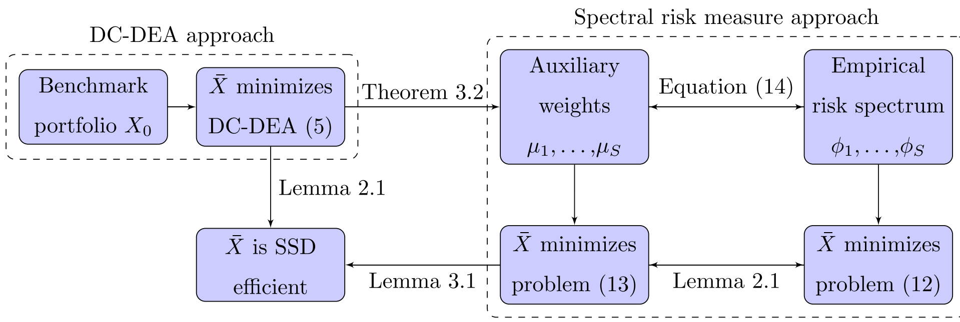

Fig. 1. Relation of the various concepts from the manuscript.

3.1. Inverse approach and shadow risk spectrum

In this part, we will propose an inverse approach which enables us to derive the shadow risk spectrum from the DC-DEA model (5). This risk spectrum can be then compared with the individual spectrum of a particular investor. We start with the following theorem which is related to the general DC-DEA model (4), i.e. it is applicable even if we move outside of the SSD efficiency which we are mainly interested in. Its proof can be found in the Appendix B.

Theorem 3.1. *Let all directions (3) be positive. Let $\bar{x}$ be the optimal weights of (4) for the benchmark portfolio X_0 and denote $\bar{X} = \sum_{i=1}^n R_i\bar{x}_i$ the corresponding random return of the optimal (efficient) portfolio. Defining weights*

$$\lambda_j = \frac{1}{e_j(X_0)}\sum_{k=1}^K \frac{\mathcal{R}_k(\bar{X}) - \mathcal{R}_k(X_0) + d_k(X_0)}{d_k(X_0)},$$

$$\mu_k = \frac{1}{d_k(X_0)}\sum_{j=1}^J \frac{\mathcal{E}_j(\bar{X}) - \mathcal{E}_j(X_0) + e_j(X_0)}{e_j(X_0)}, \tag{15}$$

then $\bar{x}$ is an optimal solution to the following convex problem

$$\begin{aligned} &\text{minimize} \ -\sum_{j=1}^J \lambda_j \mathcal{E}_j\left(\sum_{i=1}^n R_i x_i\right) + \sum_{k=1}^K \mu_k \mathcal{R}_k\left(\sum_{i=1}^n R_i x_i\right), \\ &\text{s.t.} \ \sum_{i=1}^n x_i = 1, \quad x_i \ge 0. \end{aligned} \tag{16}$$

The assumption on that all directions (3) are positive is reasonable. In the opposite case, X_0 is already an efficient portfolio.

Now we are ready to formulate the result which shows how to reconstruct the shadow risk spectrum in the DC-DEA model (5) which is consistent with the second-order stochastic dominance efficiency. The proof is postponed again to the Appendix B.

Theorem 3.2. *Let all directions (6) be positive. Let $\bar{x}$ be the optimal weigths of (5) for the benchmark portfolio X_0 and denote $\bar{X} = \sum_{i=1}^n R_i\bar{x}_i$ the corresponding random return of the optimal (efficient) portfolio. Defining weights*

$$\mu_S = \frac{1}{e(X_0)}\sum_{s=1}^{S-1} \frac{\text{CVaR}^S_{s/S}(\bar{X}) - \text{CVaR}^S_{s/S}(X_0) + d_s(X_0)}{d_s(X_0)},$$

$$\mu_s = \frac{1}{d_{S-s}(X_0)} \frac{\mathbb{E}(\bar{X}) - \mathbb{E}(X_0) + e(X_0)}{e(X_0)}, \qquad s = 1, \dots, S-1, \tag{17}$$

then $\bar{x}$ is an optimal solution to the following convex problem

$$\begin{aligned} &\text{minimize} \ \mu_S \mathbb{E}\left(-\sum_{i=1}^n R_i x_i\right) + \sum_{s=1}^{S-1} \mu_s \text{CVaR}^S_{1-s/S}\left(\sum_{i=1}^n R_i x_i\right), \\ &\text{s.t.} \ \sum_{i=1}^n x_i = 1, \quad x_i \ge 0. \end{aligned} \tag{18}$$

We realize now that problem (18) is a special case of (13). Therefore, it corresponds to minimization of some shadow spectral risk measure for which the efficient portfolio is rendered as optimal. The shadow risk spectrum can be computed in a simple way: for $\mu_1 \dots, \mu_S$ from (17) compute the shadow risk spectrum $\phi_1, \dots, \phi_S$ via (14). Remind that the term shadow is inspired by Briec et al. [18] who applied similar approach to their mean-variance DEA model and obtained weights leading to a shadow mean-variance utility.

3.2. Distances to the ideal investment opportunity

To summarize, any investor with risk spectrum ϕ should use the empirical spectral risk measure to get his/her ideal portfolio:

$$M_\phi(X) = -\sum_{s=1}^S \phi_s X_{[s]} = \sum_{s=1}^S \mu_s \text{CVaR}_{1-s/S}(X)$$

under the choice of the weights (11). However, such portfolio does not need to be available to the investor for several reasons. It may be too diversified, i.e. composed from too many original assets, or the investor is limited to one fund only. Therefore we will introduce several distances to help investors choose the most suitable investment opportunity.

First, we define the distance in the portfolio weights space. Denote by $\hat{X}$ the ideal investment opportunity and by $\hat{x}_i$ the corresponding portfolio weights. Then the l_1-distance between X and $\hat{X}$ can be defined as

$$\text{dist}^x(X, \hat{X}) = \sum_{i=1}^n |x_i - \hat{x}_i|.$$

Its interpretation is simple. It measures the necessary revision of assets, i.e. if $x_i - \hat{x}_i$ is negative, asset i must be bought, in opposite case it must be sold. If transaction costs are associated with a change in the portfolio structure, this distance will help the investor to estimate their amount.

We can also measure the distance in the input-output space represented by risk $\mathcal{R}_k$ and return $\mathcal{E}_j$ measures by

$$\text{dist}^{\mathcal{E},\mathcal{R}}(X, \hat{X}) = \sum_{j=1}^J |\mathcal{E}_j(X) - \mathcal{E}_j(\hat{X})| + \sum_{k=1}^K |\mathcal{R}_k(X) - \mathcal{R}_k(\hat{X})|,$$

Using the axiom of translation equivariance (R1), this distance can be interpreted as a sum of differences in the deterministic (constant) positions between X and $\hat{X}$. At the same time, the investor can assess the difference in the risk–return profile between X and $\hat{X}$.

The next distance is measured in the risk spectrum space ϕ. As we have shown in Section 3.1, we can apply the inverse approach to get the risk spectrum to each projection of the inefficient investment opportunity. Then, the l_1-distance which compares the

Table 1
Distances between DC-DEA optimal and ideal portfolios in weight x dimension.

			Exponential spectrum						Power spectrum					
	DEA		$k=0.4$		$k=1.4$		$k=10$		$\gamma=0.3$		$\gamma=0.65$		$\gamma=1$	
	Rnk	Score	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist
Agric	43	0.04	34	2.00	33	0.41	7	0.76	4	0.72	18	0.68	29	2.00
Food	18	0.13	34	2.00	40	0.59	2	0.72	11	0.75	40	0.81	29	2.00
Soda	13	0.19	34	2.00	2	0.22	11	0.78	22	0.78	2	0.59	29	2.00
Beer	12	0.19	34	2.00	36	0.53	1	0.70	2	0.70	37	0.75	29	2.00
Smoke	32	0.08	34	2.00	20	0.35	25	0.84	24	0.78	24	0.69	29	2.00
Toys	41	0.05	34	2.00	21	0.36	17	0.80	15	0.76	22	0.69	29	2.00
Fun	1	1.00	2	0.69	48	2.00	48	2.00	48	2.00	48	2.00	1	0.00
Books	37	0.06	34	2.00	18	0.35	9	0.78	12	0.75	9	0.66	29	2.00
Hshld	28	0.10	34	2.00	39	0.56	3	0.72	5	0.73	39	0.79	29	2.00
Clths	6	0.36	5	1.74	44	1.09	43	1.17	44	1.08	44	1.09	4	1.74
Hlth	29	0.10	34	2.00	10	0.30	32	0.87	34	0.82	15	0.68	29	2.00
MedEq	7	0.28	14	2.00	37	0.53	41	1.08	42	0.89	42	0.85	29	2.00
Drugs	16	0.15	34	2.00	22	0.36	28	0.84	24	0.78	29	0.70	29	2.00
Chems	20	0.13	14	2.00	3	0.23	15	0.79	25	0.79	3	0.61	29	2.00
Rubbr	15	0.15	14	2.00	30	0.39	38	0.96	21	0.77	11	0.66	29	2.00
Txtls	11	0.20	3	1.54	45	1.46	45	1.48	45	1.45	45	1.46	3	1.57
BldMt	33	0.08	34	2.00	9	0.29	30	0.86	35	0.82	14	0.67	29	2.00
Cnstr	40	0.05	34	2.00	13	0.33	19	0.81	19	0.77	13	0.66	29	2.00
Steel	46	0.03	34	2.00	29	0.38	11	0.78	8	0.74	19	0.68	29	2.00
FabPr	38	0.06	14	2.00	7	0.28	29	0.84	33	0.81	8	0.65	29	2.00
Mach	27	0.10	14	2.00	6	0.27	28	0.84	31	0.81	7	0.65	29	2.00
ElcEq	39	0.06	34	2.00	17	0.35	20	0.81	17	0.77	17	0.68	29	2.00
Autos	30	0.10	14	2.00	1	0.19	5	0.74	6	0.73	1	0.50	29	2.00
Aero	8	0.28	7	1.92	42	0.65	42	1.13	39	0.85	41	0.82	7	1.92
Ships	3	0.53	1	0.53	48	2.00	48	2.00	48	2.00	48	2.00	2	1.22
Guns	24	0.11	34	2.00	16	0.35	35	0.91	38	0.85	34	0.72	29	2.00
Gold	47	0.02	34	2.00	27	0.37	8	0.77	7	0.74	12	0.66	29	2.00
Mines	45	0.03	34	2.00	24	0.36	12	0.78	14	0.75	11	0.66	29	2.00
Coal	48	0.02	34	2.00	31	0.40	13	0.79	10	0.74	28	0.70	29	2.00
Oil	44	0.04	34	2.00	34	0.43	4	0.74	1	0.69	17	0.68	29	2.00
Util	26	0.11	34	2.00	38	0.55	6	0.74	3	0.71	38	0.78	29	2.00
Telcm	22	0.12	34	2.00	25	0.36	16	0.80	13	0.75	20	0.68	29	2.00
PerSv	42	0.04	34	2.00	32	0.41	15	0.79	9	0.74	32	0.71	29	2.00
BusSv	4	0.50	8	1.96	43	0.74	44	1.24	43	0.96	43	0.93	8	1.96
Comps	18	0.13	34	2.00	4	0.24	18	0.80	27	0.79	4	0.61	29	2.00
Chips	10	0.23	9	1.98	35	0.49	40	1.00	30	0.81	27	0.70	9	1.98
LabEq	2	0.79	4	1.64	46	1.47	46	1.51	46	1.47	46	1.47	5	1.75
Paper	25	0.11	34	2.00	12	0.33	31	0.86	32	0.81	26	0.70	29	2.00
Boxes	31	0.09	34	2.00	15	0.34	25	0.84	26	0.79	21	0.68	29	2.00
Trans	24	0.11	34	2.00	11	0.32	33	0.88	36	0.83	25	0.69	29	2.00
Whlsl	19	0.13	34	2.00	20	0.35	21	0.82	21	0.77	23	0.69	29	2.00
Rtail	9	0.26	14	2.00	5	0.27	25	0.83	29	0.81	6	0.65	29	2.00
Meals	5	0.41	14	2.00	24	0.36	36	0.92	41	0.87	35	0.73	29	2.00
Banks	36	0.07	34	2.00	14	0.34	34	0.90	37	0.84	33	0.71	29	2.00
Insur	14	0.16	14	2.00	26	0.37	37	0.93	40	0.85	36	0.74	29	2.00
RlEst	22	0.12	6	1.91	41	0.63	39	0.98	16	0.77	32	0.71	6	1.91
Fin	34	0.08	34	2.00	8	0.29	23	0.83	28	0.80	5	0.65	29	2.00
Other	35	0.07	34	2.00	28	0.37	22	0.82	18	0.77	30	0.71	29	2.00

obtained risk spectrum $\hat{\phi}$ and the investor's one ϕ is

$$\operatorname{dist}^{\phi}(X, \hat{X}) = \sum_{s=1}^{S} |\phi_s - \hat{\phi}_s|.$$

This distance can help capture the possible difference in the risk aversion expressed by the risk spectra.

We can consider the ideal investment opportunity $\hat{X}$ as the projection in general DC-DEA model (4), i.e. we can fix the weights x to the optimal weights $\hat{x}$ from the spectral risk measure minimization problem (12). If all directions e_j, d_k are positive, then we obtain

$$\frac{\mathcal{E}_j(\hat{X}) - \mathcal{E}_j(X_0)}{e_j(X_0)} = \hat{\varphi}_j, \;\; j = 1, \ldots, J,$$
$$\frac{\mathcal{R}_k(X_0) - \mathcal{R}_k(\hat{X})}{d_k(X_0)} = \hat{\theta}_k, \;\; k = 1, \ldots, K, \tag{19}$$

These values can be inserted into the fractional objective function of DC-DEA model (4) leading to the directional distance measure

$$\operatorname{dist}^{dd}(X, \hat{X}) = \frac{1 - \frac{1}{K}\sum_{k=1}^{K} \hat{\theta}_k}{1 + \frac{1}{J}\sum_{j=1}^{J} \hat{\varphi}_j} \tag{20}$$

It quantifies the ratio of relative necessary improvement needed to reach the efficient ideal portfolio in the input and output space, thus it is fully comparable with the optimal value of the DC-DEA model. However, some of the values of $\hat{\varphi}_j, \hat{\theta}_k$ can be also negative. This appears when the risk of the ideal investment opportunity is lower than the risk of benchmark or the return is higher, which is both forbidden by the original DC-DEA model (4). We can avoid the negative values by setting

$$\frac{\max\{\mathcal{E}_j(\hat{X}) - \mathcal{E}_j(X_0), 0\}}{e_j(X_0)} = \hat{\varphi}_j, \;\; j = 1, \ldots, J, \tag{21}$$

Table 2
Distances between DC-DEA optimal and ideal portfolios in input-output dimension.

	DEA		Exponential spectrum						Power spectrum					
			$k = 0.4$		$k = 1.4$		$k = 10$		$\gamma = 0.3$		$\gamma = 0.65$		$\gamma = 1$	
	Rnk	Score	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist
Agric	43	0.04	43	3.04	32	0.09	6	0.16	23	0.36	24	0.12	43	3.41
Food	18	0.13	48	3.06	37	0.12	5	0.16	37	0.37	34	0.14	48	3.43
Soda	13	0.19	25	2.99	3	0.04	23	0.19	2	0.35	1	0.10	25	3.36
Beer	12	0.19	47	3.06	34	0.11	1	0.15	30	0.36	30	0.13	48	3.43
Smoke	32	0.08	29	3.01	15	0.06	20	0.18	23	0.36	9	0.11	29	3.38
Toys	41	0.05	35	3.02	22	0.07	14	0.17	23	0.36	16	0.11	35	3.40
Fun	1	1.00	2	0.40	48	3.34	48	3.42	48	3.27	48	3.37	1	0.00
Books	37	0.06	37	3.02	22	0.07	12	0.17	12	0.36	9	0.11	37	3.40
Hshld	28	0.10	47	3.06	36	0.11	3	0.16	35	0.37	32	0.13	46	3.43
Clths	6	0.36	5	2.38	44	0.60	44	0.72	44	0.68	44	0.63	5	2.75
Hlth	29	0.10	23	2.99	8	0.04	26	0.20	10	0.36	9	0.11	23	3.36
MedEq	7	0.28	10	2.78	39	0.22	39	0.38	39	0.43	39	0.28	10	3.16
Drugs	16	0.15	30	3.01	18	0.06	20	0.18	30	0.36	16	0.11	30	3.39
Chems	20	0.13	13	2.91	27	0.07	36	0.25	6	0.36	36	0.15	13	3.29
Rubbr	15	0.15	11	2.80	38	0.19	38	0.35	38	0.40	38	0.25	11	3.18
Txtls	11	0.20	4	2.05	45	0.92	45	1.02	45	0.94	45	0.95	4	2.43
BldMt	33	0.08	19	2.98	5	0.04	29	0.20	9	0.36	16	0.11	19	3.35
Cnstr	40	0.05	32	3.01	16	0.06	16	0.17	14	0.36	4	0.11	33	3.39
Steel	46	0.03	39	3.03	29	0.08	9	0.17	23	0.36	21	0.12	39	3.40
FabPr	38	0.06	18	2.97	2	0.04	31	0.21	6	0.35	19	0.11	18	3.35
Mach	27	0.10	16	2.95	5	0.04	33	0.22	6	0.35	28	0.12	16	3.32
ElcEq	39	0.06	33	3.01	19	0.07	17	0.18	17	0.36	9	0.11	33	3.39
Autos	30	0.10	12	2.90	27	0.08	35	0.25	1	0.34	35	0.14	12	3.28
Aero	8	0.28	7	2.62	42	0.36	42	0.51	42	0.52	42	0.42	7	3.00
Ships	3	0.53	1	0.13	47	3.09	47	3.16	47	3.01	47	3.11	2	0.36
Guns	24	0.11	21	2.98	9	0.05	30	0.20	23	0.36	24	0.12	21	3.35
Gold	47	0.02	41	3.03	28	0.08	7	0.16	14	0.36	13	0.11	41	3.40
Mines	45	0.03	38	3.03	24	0.07	9	0.17	14	0.36	9	0.11	38	3.40
Coal	48	0.02	41	3.03	30	0.08	9	0.17	32	0.36	24	0.12	41	3.40
Oil	44	0.04	44	3.04	33	0.09	2	0.16	23	0.36	27	0.12	44	3.42
Util	26	0.11	45	3.05	35	0.11	4	0.16	35	0.37	31	0.13	45	3.43
Telcm	22	0.12	36	3.02	24	0.07	13	0.17	23	0.36	16	0.11	36	3.40
PerSv	42	0.04	41	3.03	31	0.08	12	0.17	33	0.37	27	0.12	41	3.41
BusSv	4	0.50	6	2.57	43	0.42	43	0.56	43	0.56	43	0.47	6	2.95
Comps	18	0.13	22	2.98	1	0.04	27	0.20	3	0.35	3	0.11	22	3.36
Chips	10	0.23	9	2.71	40	0.27	40	0.42	40	0.45	40	0.33	9	3.09
LabEq	2	0.79	3	1.85	46	1.13	46	1.24	46	1.14	46	1.16	3	2.23
Paper	25	0.11	27	3.00	12	0.05	24	0.19	17	0.36	9	0.11	27	3.37
Boxes	31	0.09	28	3.01	14	0.06	21	0.18	17	0.36	5	0.11	28	3.38
Trans	24	0.11	20	2.98	7	0.04	29	0.20	12	0.36	21	0.12	20	3.35
Whlsl	19	0.13	31	3.01	18	0.06	18	0.18	28	0.36	13	0.11	31	3.39
Rtail	9	0.26	15	2.93	13	0.06	34	0.24	9	0.36	33	0.14	15	3.30
Meals	5	0.41	14	2.92	20	0.07	37	0.25	35	0.37	37	0.15	14	3.30
Banks	36	0.07	25	2.99	11	0.05	26	0.20	23	0.36	19	0.11	25	3.36
Insur	14	0.16	17	2.96	7	0.04	32	0.22	28	0.36	29	0.13	17	3.34
RlEst	22	0.12	8	2.66	41	0.32	41	0.46	41	0.47	41	0.37	8	3.03
Fin	34	0.08	26	3.00	10	0.05	22	0.19	6	0.36	2	0.11	26	3.37
Other	35	0.07	34	3.02	24	0.07	15	0.17	32	0.37	22	0.12	34	3.39

$$\frac{\max\{\mathcal{R}_k(X_0)-\mathcal{R}_k(\hat{X}),0\}}{d_k(X_0)}=\hat{\theta}_k,\ \ k=1,\dots,K, \tag{22}$$

which are nonzero if there is a possible improvement, i.e. the ideal investment opportunity has lower risk or higher return. This can be interpreted as the investor "pays" for increasing the return and decreasing the risk, whereas decreasing the return and increasing the risk does not cost anything.

4. Numerical study

In this section, we apply and compare all approaches introduced above and discuss recommendations for investors on how to take risk aversion into account. We use the industry representative portfolios of US stock market which are listed in the Kenneth French online library. Each portfolio represents a branch of the industry, so alone it is not well diversified as was confirmed by paper [7]. We employed Matlab 2019a to perform the computations.

To make the explanation clearer, we summarize the terminology used below:

- Representative portfolio – original portfolio consisting of only one US industry branch.
- DC-DEA optimal portfolio – optimal solution of the DC-DEA model (5), resp. (7) composed from several representative portfolios and, at the same time, the projection of one representative portfolio to the DC-DEA (=SSD) efficient frontier.
- Ideal portfolio – optimal solution of the spectral risk measure minimization problem (12), resp. (13) for a particular risk spectrum. It also lies on the SSD efficient frontier.

We use the following parametric risk spectra:

1. Exponential risk spectrum:

$$\phi_k(p)=\frac{k\cdot e^{-k\cdot p}}{1-e^{-k}},\ \ k>0, \tag{23}$$

Table 3
Distances between DC-DEA optimal and ideal portfolios in risk-spectrum ϕ dimension.

	DEA		Exponential spectrum						Power spectrum					
			$k=0.4$		$k=1.4$		$k=10$		$\gamma=0.3$		$\gamma=0.65$		$\gamma=1$	
	Rnk	Score	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist
Agric	43	0.04	43	0.38	39	0.14	6	0.98	3	0.62	38	0.27	43	0.48
Food	18	0.13	48	0.49	45	0.25	1	0.90	6	0.62	46	0.36	48	0.58
Soda	13	0.19	26	0.32	14	0.08	21	1.04	15	0.64	17	0.24	26	0.42
Beer	12	0.19	45	0.45	42	0.21	2	0.91	1	0.61	43	0.32	45	0.55
Smoke	32	0.08	29	0.32	21	0.10	19	1.03	21	0.65	26	0.26	29	0.42
Toys	41	0.05	35	0.35	27	0.12	14	1.01	13	0.63	31	0.26	35	0.45
Fun	1	1.00	2	0.10	46	0.34	48	1.34	46	0.84	42	0.31	1	0.00
Books	37	0.06	36	0.35	25	0.11	12	1.00	7	0.62	24	0.25	36	0.45
Hshld	28	0.10	47	0.46	44	0.22	3	0.92	8	0.62	44	0.34	47	0.56
Clths	6	0.36	1	0.08	41	0.21	45	1.18	45	0.73	10	0.21	4	0.22
Hlth	29	0.10	22	0.29	9	0.07	26	1.06	26	0.65	15	0.24	22	0.39
MedEq	7	0.28	8	0.19	37	0.13	43	1.16	42	0.71	15	0.24	8	0.29
Drugs	16	0.15	32	0.34	25	0.11	22	1.04	27	0.65	39	0.27	32	0.44
Chems	20	0.13	14	0.25	2	0.05	35	1.08	29	0.66	9	0.21	14	0.36
Rubbr	15	0.15	10	0.22	15	0.08	38	1.12	39	0.68	11	0.21	10	0.33
Txtls	11	0.20	7	0.16	48	0.63	46	1.28	48	0.88	48	0.58	3	0.09
BldMt	33	0.08	17	0.26	5	0.06	33	1.08	33	0.66	12	0.23	17	0.37
Cnstr	40	0.05	30	0.33	20	0.10	16	1.02	12	0.63	20	0.24	30	0.43
Steel	46	0.03	40	0.36	33	0.12	10	1.00	9	0.63	32	0.26	40	0.46
FabPr	38	0.06	12	0.24	5	0.06	34	1.08	29	0.66	8	0.21	12	0.35
Mach	27	0.10	11	0.24	5	0.06	37	1.09	31	0.66	7	0.21	11	0.35
ElcEq	39	0.06	32	0.34	23	0.11	17	1.02	16	0.64	29	0.26	32	0.44
Autos	30	0.10	13	0.25	1	0.05	31	1.08	22	0.65	3	0.19	13	0.36
Aero	8	0.28	3	0.15	37	0.13	44	1.16	43	0.71	6	0.21	5	0.27
Ships	3	0.53	16	0.26	28	0.12	40	1.13	44	0.73	34	0.27	16	0.37
Guns	24	0.11	23	0.29	12	0.08	29	1.07	36	0.67	29	0.26	23	0.40
Gold	47	0.02	39	0.36	30	0.12	7	0.99	4	0.62	25	0.25	39	0.46
Mines	45	0.03	34	0.35	22	0.11	8	1.00	6	0.62	21	0.24	34	0.45
Coal	48	0.02	41	0.36	35	0.13	10	1.00	11	0.63	33	0.27	41	0.46
Oil	44	0.04	44	0.40	40	0.16	5	0.96	2	0.62	41	0.28	44	0.50
Util	26	0.11	46	0.46	43	0.22	4	0.93	10	0.63	45	0.34	46	0.55
Telcm	22	0.12	39	0.36	33	0.12	13	1.01	14	0.64	35	0.27	39	0.46
PerSv	42	0.04	42	0.37	38	0.14	11	1.00	17	0.64	40	0.28	42	0.47
BusSv	4	0.50	6	0.16	35	0.13	42	1.15	41	0.70	4	0.20	7	0.28
Comps	18	0.13	15	0.26	3	0.06	29	1.07	24	0.65	6	0.20	15	0.36
Chips	10	0.23	4	0.15	31	0.12	41	1.15	40	0.70	2	0.19	6	0.27
LabEq	2	0.79	6	0.16	47	0.50	47	1.32	47	0.87	47	0.45	2	0.03
Paper	25	0.11	27	0.32	19	0.10	24	1.05	25	0.65	30	0.26	28	0.42
Boxes	31	0.09	28	0.32	18	0.09	21	1.04	20	0.65	23	0.25	28	0.42
Trans	24	0.11	19	0.28	8	0.07	31	1.08	34	0.66	16	0.24	19	0.38
Whlsl	19	0.13	32	0.34	26	0.11	19	1.03	24	0.65	36	0.27	32	0.44
Rtail	9	0.26	21	0.29	7	0.07	27	1.07	31	0.66	18	0.24	22	0.39
Meals	5	0.41	20	0.28	14	0.08	32	1.08	35	0.67	22	0.25	20	0.39
Banks	36	0.07	25	0.31	17	0.09	25	1.06	31	0.66	27	0.26	25	0.41
Insur	14	0.16	18	0.27	11	0.08	37	1.09	37	0.68	20	0.24	18	0.37
RlEst	22	0.12	9	0.19	17	0.09	39	1.12	38	0.68	1	0.18	9	0.30
Fin	34	0.08	24	0.30	11	0.08	23	1.04	19	0.64	13	0.23	24	0.41
Other	35	0.07	37	0.35	30	0.12	15	1.01	19	0.64	38	0.27	37	0.45

2. Power risk spectrum:

$$\phi_\gamma(p) = \gamma \cdot p^{\gamma-1}, \ \ \gamma \in (0, 1]. \tag{24}$$

We consider several choices of the parameters for the exponential spectrum, $k \in \{0.4, 1.4, 10\}$ and the power spectrum, $\gamma \in \{0.3, 0.65, 1.0\}$ which reflect different risk aversions of investors. For the exponential spectrum, the higher values of parameters are related to higher aversion to risk, whereas for the power spectrum the opposite is true. Below we will simplify the notion and identify the risk spectrum by its parameter, for example $k = 1.4$ means exponential spectrum with this parameter value. We believe that this cannot cause any misunderstanding.

For reader's convenience we summarize the steps:

1. For all representative portfolios:
   1.1. Apply the DC-DEA model consistent with SSD (5) and obtain the DC-DEA optimal portfolio.
   1.2. Reconstruct the shadow risk spectrum using (17).
2. For the risk spectra (23), (24) and their parameters $k \in \{0.4, 1.4, 10\}$, $\gamma \in \{0.3, 0.65, 1.0\}$:
   2.1. Derive the empirical risk spectrum and compute auxiliary weights using (11).
   2.2. Find the ideal investment opportunity by solving problem (12), resp. (13).
3. Measure the distances between the ideal investment opportunities and the investigated ones.
4. Rank the available investment opportunities according to the distances.

Tables 1–10 present the numerical results. Since our main goal is the comparison with the DC-DEA, the first columns always contain the DEA scores and its ranking. First, we compare the ideal portfolios for several choices of risk spectra and the optimal portfolios of DC-DEA model which corresponds to projecting the representative portfolios onto the efficient frontier, cf. Tables 1–3. Then, we investigate the rankings based on the distances between representative and ideal portfolios, cf. Tables 5–7. Tables 4, 8 provide correlations

Table 4
Rank correlations between DC-DEA optimal and ideal portfolios.

	Exponential spectrum			Power spectrum		
	$k=0.4$	$k=1.4$	$k=10$	$\gamma=0.3$	$\gamma=0.65$	$\gamma=1$
x	0.692	−0.350	−0.621	−0.648	−0.507	0.597
input-output	0.696	−0.354	−0.710	−0.508	−0.621	0.695
ϕ	0.630	−0.136	−0.678	−0.721	0.108	0.629

Table 5
Distances between representative and ideal portfolios in weight x dimension.

			Exponential spectrum						Power spectrum					
	DEA		$k=0.4$		$k=1.4$		$k=10$		$\gamma=0.3$		$\gamma=0.65$		$\gamma=1$	
	Rnk	Score	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist
Agric	43	0.04	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Food	18	0.13	26	2.00	28	2.00	6	1.89	29	2.00	28	2.00	25	2.00
Soda	13	0.19	26	2.00	3	1.59	2	1.55	1	1.50	2	1.45	25	2.00
Beer	12	0.19	26	2.00	5	1.89	7	1.90	29	2.00	28	2.00	25	2.00
Smoke	32	0.08	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Toys	41	0.05	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Fun	1	1.00	1	0.69	28	2.00	29	2.00	29	2.00	28	2.00	1	0.00
Books	37	0.06	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Hshld	28	0.10	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Clths	6	0.36	26	2.00	7	1.98	9	2.00	8	1.97	6	1.98	25	2.00
Hlth	29	0.10	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
MedEq	7	0.28	26	2.00	4	1.83	29	2.00	6	1.89	5	1.88	25	2.00
Drugs	16	0.15	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Chems	20	0.13	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Rubbr	15	0.15	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Txtls	11	0.20	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
BldMt	33	0.08	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Cnstr	40	0.05	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Steel	46	0.03	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
FabPr	38	0.06	26	2.00	28	2.00	29	2.00	9	1.99	28	2.00	25	2.00
Mach	27	0.10	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
ElcEq	39	0.06	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Autos	30	0.10	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Aero	8	0.28	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Ships	3	0.53	2	1.31	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Guns	24	0.11	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Gold	47	0.02	26	2.00	6	1.96	5	1.80	3	1.66	4	1.88	25	2.00
Mines	45	0.03	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Coal	48	0.02	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Oil	44	0.04	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Util	26	0.11	26	2.00	28	2.00	4	1.75	5	1.84	28	2.00	25	2.00
Telcm	22	0.12	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
PerSv	42	0.04	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
BusSv	4	0.50	26	2.00	2	1.44	1	1.44	4	1.68	1	1.28	25	2.00
Comps	18	0.13	26	2.00	28	2.00	29	2.00	7	1.91	28	2.00	25	2.00
Chips	10	0.23	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
LabEq	2	0.79	3	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Paper	25	0.11	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Boxes	31	0.09	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Trans	24	0.11	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Whlsl	19	0.13	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Rtail	9	0.26	26	2.00	28	2.00	8	1.97	29	2.00	28	2.00	25	2.00
Meals	5	0.41	26	2.00	1	1.32	3	1.70	2	1.56	3	1.53	25	2.00
Banks	36	0.07	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Insur	14	0.16	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
RlEst	22	0.12	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Fin	34	0.08	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00
Other	35	0.07	26	2.00	28	2.00	29	2.00	29	2.00	28	2.00	25	2.00

between rankings bases on DC-DEA and various distances. Finally, the robustness of the criteria with respect to two extreme scenarios is evaluated in Tables 9, 10.

Concerning the results of the DC-DEA model, see Table 1, only one representative portfolio (Fun) was identified as efficient, because it has the highest mean return. This is consistent with our previous findings, because the representative portfolios alone are not well diversified. Therefore, interdisciplinary portfolios composed from several representative portfolios are necessary to reach the DC-DEA as well as SSD efficient frontiers.

In principle, the distance in the portfolio weight space prefers those interdisciplinary portfolios which are similar to the structure of the ideal portfolios, see Table 1. Therefore, the ranking can be very different as the structure of the ideal portfolios changes. In particular, the DC-DEA efficient representative portfolio Fun is ranked as second for the parameter value $k=0.4$ and it is the worse for the remaining parameter values for the exponential

Table 6
Distances between representative and ideal portfolios in input-output dimension.

			Exponential spectrum						Power spectrum					
	DEA		$k = 0.4$		$k = 1.4$		$k = 10$		$\gamma = 0.3$		$\gamma = 0.65$		$\gamma = 1$	
	Rnk	Score	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist
Agric	43	0.04	22	1.08	40	3.37	39	3.42	39	3.26	40	3.39	12	0.90
Food	18	0.13	44	2.35	3	0.73	3	0.78	3	0.62	3	0.75	44	2.71
Soda	13	0.19	37	2.03	7	0.99	7	1.04	7	0.89	7	1.01	39	2.40
Beer	12	0.19	45	2.52	2	0.51	2	0.56	1	0.40	2	0.53	45	2.89
Smoke	32	0.08	17	0.97	23	2.14	23	2.19	23	2.04	23	2.16	21	1.31
Toys	41	0.05	9	0.84	33	3.17	33	3.22	33	3.07	33	3.19	9	0.73
Fun	1	1.00	2	0.40	38	3.34	40	3.42	40	3.27	38	3.37	1	0.00
Books	37	0.06	19	0.98	29	2.74	29	2.79	29	2.63	29	2.76	16	1.05
Hshld	28	0.10	39	2.09	8	1.04	8	1.09	8	0.93	8	1.06	40	2.45
Clths	6	0.36	27	1.30	15	1.70	15	1.77	15	1.61	15	1.72	29	1.66
Hlth	29	0.10	14	0.94	24	2.15	24	2.20	24	2.04	24	2.17	20	1.28
MedEq	7	0.28	33	1.77	10	1.20	12	1.29	12	1.18	11	1.23	35	2.14
Drugs	16	0.15	36	1.92	9	1.10	9	1.15	9	0.99	9	1.12	38	2.28
Chems	20	0.13	12	0.87	25	2.25	25	2.31	25	2.15	25	2.27	19	1.19
Rubbr	15	0.15	5	0.65	26	2.40	26	2.46	26	2.30	26	2.42	14	0.99
Txtls	11	0.20	32	1.71	44	4.64	44	4.69	44	4.54	44	4.66	24	1.34
BldMt	33	0.08	3	0.48	36	3.31	36	3.36	36	3.21	36	3.33	2	0.31
Cnstr	40	0.05	18	0.98	37	3.34	37	3.39	37	3.23	37	3.36	11	0.75
Steel	46	0.03	38	2.08	45	5.00	45	5.05	45	4.89	45	5.02	31	1.71
FabPr	38	0.06	30	1.47	43	4.39	43	4.45	43	4.29	43	4.41	17	1.10
Mach	27	0.10	4	0.64	30	2.83	30	2.89	30	2.73	30	2.85	7	0.68
ElcEq	39	0.06	7	0.75	31	3.01	31	3.07	31	2.91	31	3.04	10	0.74
Autos	30	0.10	10	0.85	32	3.09	32	3.15	32	2.99	32	3.11	8	0.73
Aero	8	0.28	25	1.22	17	1.75	18	1.83	18	1.68	18	1.78	27	1.59
Ships	3	0.53	13	0.90	41	3.86	41	3.93	41	3.77	41	3.88	4	0.53
Guns	24	0.11	20	1.00	20	2.05	20	2.10	20	1.95	20	2.07	25	1.35
Gold	47	0.02	47	3.98	47	6.90	47	6.95	47	6.80	47	6.92	47	3.61
Mines	45	0.03	43	2.32	46	5.25	46	5.30	46	5.14	46	5.27	33	1.95
Coal	48	0.02	48	6.89	48	9.81	48	9.86	48	9.71	48	9.83	48	6.52
Oil	44	0.04	23	1.14	34	3.23	34	3.28	34	3.12	34	3.25	15	1.02
Util	26	0.11	42	2.18	6	0.94	6	0.99	6	0.83	6	0.96	42	2.54
Telcm	22	0.12	35	1.84	11	1.20	10	1.25	10	1.10	10	1.22	37	2.21
PerSv	42	0.04	6	0.70	39	3.36	38	3.41	38	3.25	39	3.38	5	0.57
BusSv	4	0.50	41	2.17	4	0.80	4	0.89	4	0.77	4	0.82	43	2.55
Comps	18	0.13	24	1.21	19	1.85	19	1.90	19	1.75	19	1.87	26	1.55
Chips	10	0.23	26	1.28	16	1.70	16	1.77	16	1.62	16	1.72	28	1.65
LabEq	2	0.79	31	1.59	13	1.38	13	1.49	13	1.37	13	1.42	34	1.97
Paper	25	0.11	28	1.31	18	1.75	17	1.81	17	1.65	17	1.77	30	1.67
Boxes	31	0.09	21	1.01	22	2.14	22	2.19	22	2.03	22	2.16	23	1.33
Trans	24	0.11	15	0.97	21	2.08	21	2.13	21	1.97	21	2.10	22	1.33
Whlsl	19	0.13	34	1.82	12	1.22	11	1.27	11	1.11	12	1.24	36	2.18
Rtail	9	0.26	40	2.13	5	0.85	5	0.91	5	0.77	5	0.87	41	2.51
Meals	5	0.41	46	2.58	1	0.38	1	0.48	2	0.44	1	0.41	46	2.96
Banks	36	0.07	1	0.40	35	3.24	35	3.29	35	3.13	35	3.26	3	0.50
Insur	14	0.16	29	1.35	14	1.65	14	1.71	14	1.55	14	1.67	32	1.72
RlEst	22	0.12	17	0.97	42	3.90	42	3.95	42	3.79	42	3.92	6	0.60
Fin	34	0.08	11	0.85	28	2.56	28	2.61	28	2.45	28	2.58	13	0.99
Other	35	0.07	8	0.84	27	2.51	27	2.57	27	2.41	27	2.54	18	1.11

spectrum, i.e. for the investors with high risk aversion. We can observe similar results for the power spectrum where the parameter $\gamma = 1$ leads directly to the (degenerate) ideal portfolio Fun. On the other hand, the worst representative portfolio according to DC-DEA model Coal is not so bad for any parameter of risk spectra, e.g. it is ranked as tenth for $\gamma = 0.3$.

Looking into Table 2, the distances in the input-output (mean-CVaRs) space, the rankings of Fun and Coal are similar. This can be explained as a similar structure of the interdisciplinary portfolios leads to similar mean-risk profile.

As representative portfolio Fun is efficient, we cannot reconstruct its risk-spectrum using Theorem 3.2. However, if we realize that Fun has the highest mean return, we can use a constant risk spectrum equal to one, which corresponds to maximizing the expected value. Therefore Fun is the best for $\gamma = 1$, the second best for $k = 0.4$, but it is bad for investors with higher risk aversion. If we focus on the DC-DEA second best representative portfolio LabEq, we can see that for some investors ($k = 0.4,\ \gamma = 1$) its projection to the efficient frontier can really correspond to the risk aversion preferences. Even projection of coal need not to be far from ideal portfolios of some investors ($k = 10,\ \gamma = 0.3$).

Table 4 contains rank correlations between DC-DEA scores and the distances. It shows that the original DC-DEA ranking corresponds rather to low risk aversion, i.e. there are significantly positive correlations with $k = 0.4,\ \gamma = 1$. On the other hand, the other parameters suitable for investors with higher risk aversion provide different (opposite) ranking which is confirmed by significantly negative rank correlations.

Now, we can focus on the distances between representative and ideal portfolios. Table 5 confirms that if the representative portfolio is contained in the ideal one, than it is closer. For instance, Fun is contained in ideal portfolios for $k = 0.4,\ \gamma = 1$. On the other hand, LabEq (ranked as second by the DC-DEA model) is not contained in any ideal portfolio, so the distance is the worst possible, i.e. it is equal to two which means which represents a complete revision of the portfolio. A slightly different situation appears in the

Table 7
Distances between representative and ideal portfolios according to directional distance.

	DEA		Exponential spectrum						Power spectrum					
			$k=0.4$		$k=1.4$		$k=10$		$\gamma=0.3$		$\gamma=0.65$		$\gamma=1$	
	Rnk	Score	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist	Rnk	Dist
Agric	43	0.04	41	0.33	43	0.05	43	0.07	43	0.11	43	0.06	41	0.38
Food	18	0.13	20	0.46	10	0.18	13	0.22	11	0.42	12	0.20	19	0.46
Soda	13	0.19	12	0.47	7	0.20	9	0.27	8	0.46	8	0.23	13	0.46
Beer	12	0.19	13	0.47	6	0.24	6	0.30	4	0.56	6	0.26	13	0.46
Smoke	32	0.08	25	0.45	30	0.09	29	0.14	28	0.23	29	0.10	25	0.45
Toys	41	0.05	37	0.36	41	0.06	41	0.08	41	0.14	41	0.06	38	0.40
Fun	1	1.00	1	0.85	14	0.15	19	0.19	24	0.28	15	0.17	1	1.00
Books	37	0.06	34	0.38	37	0.07	38	0.10	37	0.16	37	0.08	34	0.41
Hshld	28	0.10	24	0.45	20	0.13	25	0.16	22	0.30	23	0.14	25	0.45
Clths	6	0.36	10	0.47	12	0.17	14	0.22	16	0.34	13	0.19	11	0.47
Hlth	29	0.10	23	0.45	26	0.10	23	0.17	23	0.28	26	0.12	22	0.45
MedEq	7	0.28	4	0.48	4	0.28	3	0.37	5	0.55	3	0.34	5	0.47
Drugs	16	0.15	11	0.47	13	0.16	8	0.27	7	0.47	12	0.20	10	0.47
Chems	20	0.13	26	0.44	21	0.13	24	0.17	25	0.27	22	0.15	27	0.45
Rubbr	15	0.15	15	0.46	19	0.13	21	0.19	20	0.31	17	0.16	17	0.46
Txtls	11	0.20	43	0.29	35	0.07	36	0.11	36	0.17	35	0.09	43	0.33
BldMt	33	0.08	33	0.38	32	0.08	31	0.13	32	0.20	31	0.10	32	0.43
Cnstr	40	0.05	40	0.34	41	0.06	40	0.08	41	0.14	41	0.06	39	0.38
Steel	46	0.03	45	0.24	46	0.03	46	0.05	46	0.08	46	0.04	45	0.28
FabPr	38	0.06	44	0.28	39	0.06	37	0.10	38	0.16	38	0.07	44	0.32
Mach	27	0.10	31	0.40	27	0.10	27	0.14	29	0.23	27	0.12	28	0.44
ElcEq	39	0.06	35	0.37	39	0.06	39	0.09	39	0.15	39	0.07	33	0.41
Autos	30	0.10	36	0.36	29	0.09	33	0.12	34	0.19	30	0.10	35	0.40
Aero	8	0.28	6	0.48	8	0.20	8	0.27	10	0.42	7	0.24	6	0.47
Ships	3	0.53	29	0.42	24	0.12	26	0.16	27	0.23	25	0.14	36	0.40
Guns	24	0.11	16	0.46	22	0.12	15	0.21	14	0.35	20	0.15	16	0.46
Gold	47	0.02	47	0.18	47	0.03	47	0.03	47	0.06	47	0.03	47	0.20
Mines	45	0.03	46	0.23	45	0.04	45	0.05	45	0.08	45	0.04	46	0.27
Coal	48	0.02	48	0.13	48	0.02	48	0.02	48	0.04	48	0.02	48	0.15
Oil	44	0.04	40	0.34	44	0.05	44	0.06	44	0.10	44	0.05	40	0.38
Util	26	0.11	22	0.45	17	0.14	22	0.18	18	0.33	19	0.15	23	0.45
Telcm	22	0.12	17	0.46	18	0.14	16	0.20	15	0.35	18	0.16	16	0.46
PerSv	42	0.04	38	0.35	42	0.05	42	0.07	42	0.12	42	0.06	38	0.40
BusSv	4	0.50	3	0.49	2	0.31	2	0.39	3	0.59	2	0.37	3	0.48
Comps	18	0.13	21	0.45	16	0.14	21	0.19	21	0.30	16	0.16	20	0.46
Chips	10	0.23	7	0.48	9	0.18	11	0.24	13	0.36	9	0.21	7	0.47
LabEq	2	0.79	2	0.50	3	0.29	5	0.35	6	0.52	4	0.33	2	0.48
Paper	25	0.11	20	0.46	25	0.12	17	0.19	17	0.33	24	0.14	21	0.46
Boxes	31	0.09	27	0.44	28	0.09	28	0.14	26	0.24	28	0.11	27	0.45
Trans	24	0.11	18	0.46	23	0.12	18	0.19	19	0.32	22	0.14	18	0.46
Whlsl	19	0.13	14	0.46	15	0.15	12	0.23	12	0.41	14	0.17	14	0.46
Rtail	9	0.26	8	0.48	5	0.27	4	0.36	2	0.61	5	0.32	8	0.47
Meals	5	0.41	5	0.48	1	0.43	1	0.52	1	0.73	1	0.51	4	0.47
Banks	36	0.07	32	0.40	36	0.07	34	0.12	33	0.19	36	0.08	31	0.43
Insur	14	0.16	9	0.47	11	0.17	10	0.26	9	0.43	10	0.21	9	0.47
RlEst	22	0.12	42	0.32	34	0.08	35	0.11	35	0.17	34	0.09	42	0.37
Fin	34	0.08	30	0.40	32	0.08	30	0.13	30	0.21	32	0.10	30	0.43
Other	35	0.07	28	0.42	33	0.08	32	0.12	32	0.20	33	0.09	30	0.43

input-output dimension. Fun is ranked in a similar way, however, the mean-CVaRs profile of LabEq is similar to that of ideal portfolios for $k=0.4$, $\gamma=1$, see Table 6. We can see that Coal is a bad choice for any investor regardless of the risk profile. Finally, Table 7 shows ranking for the directional distance. Maybe surprisingly, Fun is ranked well only for $k=0.4$ and $\gamma=1$, but not for the other choices of risk spectra. On the other hand, LabEq is very close to the ideal portfolios for all investors. Maybe a surprising “winner” is Meals (ranked 5 by DC-DEA) which is very close to the ideal portfolio for all considered risk-spectra. We can also mention Coal, which remains the worst for all risk spectra.

In comparison with Table 4, interpretation of Table 8 containing the rank correlations is not so straightforward. The correlations do not depend only on the parameters, but also on the selected distance. We stress that we compare SSD efficient portfolios with inefficient ones. Ranking according to the distance in the portfolio weight dimension x is weakly positively correlated with DC-DEA ranking. On the other hand, the ranking based on the input-output (mean-CVaRs) distance is similar as in Table 4, i.e. there are strong positive correlations for $k=0.4$, $\gamma=1$, and strong negative correlations otherwise. If we consider the distance based on the risk-spectra, we can see strong positive correlations with the DC-DEA ranking for all considered risk spectra, which is a bit surprising. Our suggestion is therefore to select the distance according to the purpose, i.e. when the goal is to reduce the revision, then measure the distance in the weight dimension, whereas when we would like to compare the mean-risk (mean-CVaRs) profile, then the input-output distance is more suitable.

We accessed robustness of considered criteria by applying a simple approach, adding extreme scenarios to existing return realizations and looking at their impact. The extreme scenarios are created from the average of the observed returns by subtracting, resp. by adding three times the standard deviation, i.e. the first scenario represents a crisis, whereas the second one is optimistic. Tables 9, 10 contain correlations between original rankings and rankings after including extreme scenarios with respect to all considered criteria. We can observe that in most cases the correlations are close to one, i.e. the extreme scenarios have (almost) no

Table 8
Rank correlations between DC-DEA optimal and ideal portfolios.

	Exponential spectrum			Power spectrum		
	$k=0.4$	$k=1.4$	$k=10$	$\gamma=0.3$	$\gamma=0.65$	$\gamma=1$
x	0.419	0.344	0.310	0.226	0.298	0.247
input-output	−0.151	0.625	0.611	0.611	0.622	−0.250
directional distance	0.847	0.902	0.871	0.842	0.901	0.823

Table 9
Rank correlations between criteria based on unperturbed data and data with added scenario *mean-3std* (first three rows correspond to distances between DC-DEA optimal and ideal portfolios, the following three rows refer to distances between representative and ideal portfolios).

	DEA	Exponential spectrum			Power spectrum		
		$k=0.4$	$k=1.4$	$k=10$	$\gamma=0.3$	$\gamma=0.65$	$\gamma=1$
x	0.989	0.844	0.822	0.870	0.858	0.535	0.704
input-output	0.989	0.989	0.854	0.989	0.301	0.366	0.989
ϕ	0.989	0.921	0.977	0.991	0.964	0.835	0.990
x	0.989	0.999	0.863	0.892	0.460	0.824	1.000
input-output	0.989	0.531	0.997	0.998	0.997	0.997	0.994
directional distance	0.989	0.976	0.996	0.997	0.992	0.987	0.998

Table 10
Rank correlations between criteria based on unperturbed data and data with added scenario *mean+3std* (first three rows correspond to distances between DC-DEA optimal and ideal portfolios, the following three rows refer to distances between representative and ideal portfolios).

	DEA	Exponential spectrum			Power spectrum		
		$k=0.4$	$k=1.4$	$k=10$	$\gamma=0.3$	$\gamma=0.65$	$\gamma=1$
x	0.991	0.825	0.868	0.913	0.763	0.976	0.915
input-output	0.991	0.994	0.931	0.993	0.912	0.899	0.994
ϕ	0.991	0.895	0.947	0.991	0.995	0.980	0.878
x	0.991	0.825	0.945	0.953	0.959	1.000	1.000
input-output	0.991	1.000	1.000	1.000	1.000	1.000	1.000
directional distance	0.991	1.000	0.999	0.999	1.000	0.999	1.000

influence and the criteria are stable/robust. Some instability can be observed for the input-output distance (for $\gamma \in \{0.3, 0.65\}$) in Table 9 where it can be explained by the fact that the risk measures are highly influenced by the crisis scenario.

5. Conclusions

In this paper, we have focused on risk aversion in data envelopment analysis models with diversification which are consistent with the second order stochastic dominance. Our approach relies on using spectral risk measures which can take into account the risk aversion of a particular investor by the choice of the risk spectrum.

We have proposed a way to detect the risk aversion which is implicitly contained in the DC-DEA model. This approach allowed us to reconstruct the shadow risk spectrum. By minimizing the corresponding (shadow) spectral risk measure, we get a DC-DEA and at the same time an SSD efficient portfolio. This portfolio can be then compared with other efficient portfolios. We have proposed several ways how to measure the distance to the ideal investment opportunity which represent the best choice for the investor taking into account individual risk aversion. In general, the distances can be classified to four cases according to the dimension where they measure the distance: portfolio weight, input-outputs, risk spectra, and directional.

Numerical study showed that the DC-DEA model implicitly prefers the expected value criterion over the risk, which corresponds to lower risk aversion. We can also conclude that the ranking provided by the DEA scores can be very different from the ranking based on the distances from the ideal investment opportunity, which was supported by the negative rank correlations obtained for several choices of the risk spectra, mainly for higher risk aversion. Thus, the use of an inappropriate criterion leads to meaningless results. Our suggestion is that if the investor risk aversion is known, than it must be taken into account using one of the proposed approaches. On the other hand, if we do not have any estimate of the risk aversion, a projection to the efficient frontier using the DC-DEA model can be sufficient.

We have also accessed robustness of the proposed criteria with respect to two extreme scenarios. We can conclude that most of them are stable. We have observed instability only for the input-output space distance which is based on risk measures which are highly dependent on the return realizations.

All considered approaches require performing two steps to sort the investment opportunities. Future research will be focused on finding a single step procedure. Extension to multiperiod models is also of interest.

Acknowledgements

This work was supported by National Natural Science Foundation of China (Grant No. 61850410534), the Grant Agency of the Czech Republic (19-28231X) and the OP RDE (OP VVV) funded project Research Center for Informatics, reg. No: CZ02.1.01/0.0./0.0./16 019/0000765.

Appendix A. Formal and axiomatic definitions

Coherent risk measures were introduced by Artzner et al. [37]. Let $\mathcal{L}_p(\Omega)$ denote the Lebesgue space of functionals defined on some Ω with finite p-th moment.

Definition A.1. We say that $\mathcal{R} : \mathcal{L}_p(\Omega) \to \mathbb{R}$, for some $p \in \{1, \dots, \infty\}$, is a coherent risk measure if it satisfies

(R1) translation equivariance: $\mathcal{R}(X+c)=\mathcal{R}(X)-c$ for all $X\in\mathcal{L}_p(\Omega)$ and constants $c\in\mathbb{R}$,
(R2) positive homogenity: $\mathcal{R}(0)=0$, and $\mathcal{R}(\lambda X)=\lambda\mathcal{R}(X)$ for all $X\in\mathcal{L}_p(\Omega)$ and all $\lambda\geq 0$,
(R3) subadditivity: $\mathcal{R}(X_1+X_2)\leq\mathcal{R}(X_1)+\mathcal{R}(X_2)$ for all $X_1,X_2\in\mathcal{L}_p(\Omega)$,
(R4) monotonicity: $\mathcal{R}(X_1)\leq\mathcal{R}(X_2)$ when $X_1\geq X_2$, $X_1,X_2\in\mathcal{L}_p(\Omega)$.

Note that the axioms (R2) and (R3) imply convexity.

A formal definition of Conditional Value at Risk (CVaR) for general distribution of X can be found in paper by Rockafellar and Uryasev [38].

Definition A.2. For $X\in\mathcal{L}_1(\Omega)$ and $\alpha\in[0,1)$, Conditional Value at Risk (CVaR) is defined as the mean of losses in the α-tail distribution with the distribution function:

$$F_\alpha(\eta)=\begin{cases}\frac{F(\eta)-\alpha}{1-\alpha}, & \text{if } \eta\geq\mathrm{VaR}_\alpha(X),\\ 0, & \text{otherwise},\end{cases}$$

where $F(\eta)=P(-X\leq\eta)$ and $\mathrm{VaR}_\alpha(X)=\min_\eta\{\eta \text{ s.t. } F(\eta)\geq\alpha\}$.

Rockafellar and Uryasev [38] also showed that CVaR_α can be computed using the minimization formula:

$$\mathrm{CVaR}_\alpha(X)=\min_{\xi\in\mathbb{R}}\xi+\frac{1}{1-\alpha}\mathbb{E}[\max\{(-X-\xi),0\}], \tag{A.1}$$

where $\mathrm{VaR}_\alpha(X)$ is the optimal solution. This formula is often used within the optimization problems because it does not require explicit evaluation of VaR which can be highly demanding. If the distribution of random returns X is finite discrete with equiprobable realizations, denoting $X_{[1]}\leq\cdots\leq X_{[S]}$ their sorted version starting with the smallest return, CVaR follows the simple formula

$$\mathrm{CVaR}_{k/S}(X)=-\mathbb{E}[X\mid X\leq X_{[S-k]}]=-\frac{1}{S-k}\sum_{s=1}^{S-k}X_{[s]}, \tag{A.2}$$

for $k=1,\dots,S-1$.

We propose formal definitions of the second order stochastic dominance relation and efficiency based on twice cumulative distribution functions. However, this approach is equivalent to the definitions based on concave utility functions, see Lévy [23].

Definition A.3. Let $F_X(t)=P(X\leq t)$ denote the cdf of $X\in\mathcal{X}\subseteq\mathcal{L}_1(\Omega)$, the twice cumulative probability distribution function is defined by

$$F_X^{(2)}(t)=\int_{-\infty}^{t}F_X(\eta)\,\mathrm{d}\eta.$$

Then the investment opportunity X dominates $\tilde{X}$ with respect to the second-order stochastic dominance (SSD), $\tilde{X}\preceq_{SSD}X$, if and only if

$$F_X^{(2)}(t)\leq F_{\tilde{X}}^{(2)}(t),\ \ \forall t\in\mathbb{R}, \tag{A.3}$$

The relation is strict, i.e. $\tilde{X}\prec_{SSD}X$, if the inequality is strict for at least one $t\in\mathbb{R}$. We say that an investment opportunity $X\in\mathcal{X}$ is SSD efficient if there is no other $\tilde{X}\in\mathcal{X}$ for which $X\prec_{SSD}\tilde{X}$.

Acerbi [12] defined the class of spectral risk measures which are based on admissible risk spectra.

Definition A.4. An element $\phi\in\mathcal{L}_1([0,1])$ is called an admissible risk spectrum, cf., if it is

(A1) positive: for all $I\subseteq[0,1]$ holds

$$\int_I\phi(p)dp\geq 0,$$

(A2) non-increasing: for all $q\in(0,1)$ and $\varepsilon>0$ such that $[q-\varepsilon,q+\varepsilon]\subset[0,1]$, holds

$$\int_{q-\varepsilon}^{q}\phi(p)dp\geq\int_{q}^{q+\varepsilon}\phi(p)dp,$$

(A3) normalized:

$$\|\phi\|=\int_0^1\phi(p)dp=1.$$

Then, the spectral risk measure (SRM) is defined as the weighted quantiles of the random returns

$$M_\phi(X)=-\int_0^1F_X^{-1}(p)\,\phi(p)\,dp \tag{A.4}$$

where we consider the quantile function

$$F_X^{-1}(p)=\min\{x:\ F_X(x)\geq p\},\ \ p\in[0,1]. \tag{A.5}$$

As an important special case, we can obtain CVaR for $\alpha\in[0,1)$

$$\mathrm{CVaR}_\alpha(X)=\frac{-1}{1-\alpha}\int_0^{1-\alpha}F_X^{-1}(p)\,dp$$

for the risk spectrum

$$\phi(p)=\frac{1}{1-\alpha}\mathbb{I}\{0\leq p\leq 1-\alpha\}.$$

Note that Acerbi [12] considered so called expected shortfall instead of CVaR, which can lead to some misunderstanding. However, if we consider finite discrete distribution of X with equiprobable realizations, there is a simple relation between these two measures:

$$\mathrm{CVaR}^S_{1-s/S}(X)=\mathrm{ES}^S_{s/S}(X)=\frac{-1}{s}\sum_{t=1}^{s}X_{[t]},\ \ s=1,\dots,S. \tag{A.6}$$

Remark A.1. We can interpret DC-DEA model (4). Due to (3), the first constraint implies

$$0\leq\varphi_j\leq\frac{\mathcal{E}_j\left(\sum_{i=1}^n R_ix_i\right)-\mathcal{E}_j(X_0)}{e_j(X_0)}\leq\frac{\mathcal{E}_j\left(\sum_{i=1}^n R_ix_i\right)-\mathcal{E}_j(X_0)}{\max_{X\in\mathcal{X}}\mathcal{E}_j(X)-\mathcal{E}_j(X_0)}\leq 1.$$

It can be showed that the inequalities turn into equalities to obtain

$$\varphi_j=\frac{\mathcal{E}_j\left(\sum_{i=1}^n R_ix_i\right)-\mathcal{E}_j(X_0)}{\max_{X\in\mathcal{X}}\mathcal{E}_j(X)-\mathcal{E}_j(X_0)},$$

and thus φ_j denotes the fraction of the improvement of the optimal portfolio $\sum_{i=1}^j R_ix_i$ over the maximal possible improvement in return $\mathcal{E}_j$. Similarly, θ_k denotes the fraction of the improvement of the optimal portfolio $\sum_{i=1}^j R_ix_i$ over the maximal possible improvement in risk $\mathcal{R}_j$. The objective function then balances the mean improvement in risks in the numerator and the mean improvements in returns in the denominator.

Appendix B. Auxiliary results and proofs

Theorem B.1. *Consider a closed convex set X, Lipschitz continuous functions $f_1,\dots,f_J$ and $g_1,\dots,g_K$, any scalars a_j, c_k and positive scalars b_k and d_k which further satisfy*

$$d_k\geq c_k-\min_{x\in X}g_k(x). \tag{B.1}$$

Consider further the optimization problem

$$\begin{aligned}&\text{minimize}_{x\in X,\theta,\varphi}\ \frac{1-\frac{1}{K}\sum_{k=1}^K\theta_k}{1+\frac{1}{J}\sum_{j=1}^J\varphi_j}\\ &\text{s.t. } f_j(x)\geq a_j+\varphi_jb_j,\\ &\qquad g_k(x)\leq c_k-\theta_kd_k,\\ &\qquad \theta_k\geq 0,\ \ \varphi_j\geq 0\end{aligned} \tag{B.2}$$

and denote its optimal solution $(\bar{x}, \bar{\theta}, \bar{\varphi})$. *Define*

$$\lambda_j = \frac{1}{b_j}\sum_{k=1}^{K}\frac{g_k(\bar{x}) - c_k + d_k}{d_k},$$
$$\mu_k = \frac{1}{d_k}\sum_{j=1}^{J}\frac{f_j(\bar{x}) - a_j + b_j}{b_j}. \tag{B.3}$$

Then these constants are positive. Moreover, if $f_j(\bar{x}) > a_j$ *and* $g_k(\bar{x}) < c_k$, *then* $\bar{x}$ *is stationary point of*

$$\text{minimize}_{x\in X}\ \sum_{j=1}^{J}\lambda_j(-f_j(x)) + \sum_{k=1}^{K}\mu_k g_k(x). \tag{B.4}$$

Moreover, if f_j *are concave and* h_k *are convex, then (B.4) is a convex problem and* $\bar{x}$ *is a global minimum of (B.4).*

Proof. The structure of the objective of (B.2) implies that we need to maximize both φ_j and θ_k. Since $f_j(\bar{x}) > a_j$ and $g_k(\bar{x}) < c_k$, for the optimal solution we have $\bar{\varphi}_j > 0$ and $\bar{\theta}_k > 0$. Thus, $(\bar{x}, \bar{\theta}, \bar{\varphi})$ is a local minimum of

$$\begin{aligned} &\text{minimize}_{x\in X,\theta,\varphi}\ \frac{1 - \frac{1}{K}\sum_{k=1}^{K}\theta_k}{1 + \frac{1}{J}\sum_{j=1}^{J}\varphi_j} \\ &\text{s.t.}\ f_j(x) \ge a_j + \varphi_j b_j, \\ &\qquad g_k(x) \le c_k - \theta_k d_k. \end{aligned} \tag{B.5}$$

For any θ_k we due to (B.1) obtain

$$\theta_k \le \frac{c_k - g_k(x)}{d_k} \le 1,$$

which implies that the numerator in the objective of (B.5) is non-negative around $(\bar{x}, \bar{\theta}, \bar{\varphi})$. This further implies that locally around this point we have

$$\varphi_j = \frac{f_j(x) - a_j}{b_j},$$
$$\theta_k = \frac{c_k - g_k(x)}{d_k}.$$

Plugging this into (B.5) yields that $\bar{x}$ is a local minimum of

$$\begin{aligned} \text{minimize}_{x\in X}\ \frac{1 - \frac{1}{K}\sum_{k=1}^{K}\frac{c_k - g_k(x)}{d_k}}{1 + \frac{1}{J}\sum_{j=1}^{J}\frac{f_j(x) - a_j}{b_j}} &= \frac{\frac{1}{K}\sum_{k=1}^{K}\frac{g_k(x) - c_k + d_k}{d_k}}{\frac{1}{J}\sum_{j=1}^{J}\frac{f_j(x) - a_j + b_j}{b_j}} \\ &\approx \frac{\sum_{k=1}^{K}\frac{g_k(x) - c_k + d_k}{d_k}}{\sum_{j=1}^{J}\frac{f_j(x) - a_j + b_j}{b_j}}, \end{aligned} \tag{B.6}$$

where in the last step we mean that multiplying the objective by the positive scalar $\frac{J}{K}$ does not change the optimal solution. Denote the objective function of (B.6) by h. Due to [41, Theorem 8.15] we obtain $0 \in \partial h(\bar{x}) + N_X(\bar{x})$. Here, ∂h is the (Clarke) subdifferential of h and N_X is the (Clarke) normal cone to X which is a shortened form to write the Lagrange multipliers. From the subdifferential product rule derived in Lemma Appendix B.1 we obtain

$$0 \in \frac{\left(\sum_{k=1}^{K}\frac{1}{d_k}\partial g_k(\bar{x})\right)\left(\sum_{j=1}^{J}\frac{f_j(\bar{x}) - a_j + b_j}{b_j}\right) - \left(\sum_{k=1}^{K}\frac{g_k(\bar{x}) - c_k + d_k}{d_k}\right)\left(\sum_{j=1}^{J}\frac{1}{b_j}\partial f_j(\bar{x})\right)}{\left(\sum_{j=1}^{J}\frac{f_j(\bar{x}) - a_j + b_j}{b_j}\right)^2} + N_X(\bar{x}).$$

Since the normal cone $N_X(\bar{x})$ is a cone, we have $N_X(\bar{x}) = tN_X(\bar{x})$ for any positive scalars t. Thus, the previous relation amounts to

$$0 \in \left(\sum_{k=1}^{K}\frac{1}{d_k}\partial g_k(\bar{x})\right)\left(\sum_{j=1}^{J}\frac{f_j(\bar{x}) - a_j + b_j}{b_j}\right) - \left(\sum_{k=1}^{K}\frac{g_k(\bar{x}) - c_k + d_k}{d_k}\right)\left(\sum_{j=1}^{J}\frac{1}{b_j}\partial f_j(\bar{x})\right) + N_X(\bar{x}).$$

Due to [42, Theorem 10.20] we get

$$\begin{aligned} 0 \in &\left(\sum_{k=1}^{K}\frac{1}{d_k}\partial g_k(\bar{x})\right)\left(\sum_{j=1}^{J}\frac{f_j(\bar{x}) - a_j + b_j}{b_j}\right) + \\ &+\left(\sum_{k=1}^{K}\frac{g_k(\bar{x}) - c_k + d_k}{d_k}\right)\left(\sum_{j=1}^{J}\frac{1}{b_j}\partial(-f_j)(\bar{x})\right) + N_X(\bar{x}). \end{aligned}$$

Using the definition (B.3), we arrive at

$$0 \in \sum_{j=1}^{J}\lambda_j\partial(-f_j)(\bar{x}) + \sum_{K=1}^{K}\mu_k\partial g_k(\bar{x}) + N_X(\bar{x}).$$

Thus, $\bar{x}$ is a stationary point of problem (B.4). If all $-f_j$ and g_k are convex, their Clarke subdifferential coincides with their convex subdifferential. Moreover, from the theory of the convex programming we infer that $\bar{x}$ is an optimal solution of the convex program (B.4). □

Lemma B.1. *Consider Lipschitz functions* f *and* g *and a point* x *such that* $g(x) > 0$. *Then*

$$\partial\left(\frac{f(x)}{g(x)}\right) \subset \frac{\partial f(x)g(x) - f(x)\partial g(x)}{(g(x))^2},$$

where ∂ *stands for the Clarke subdifferential.*

Proof. From [42, Theorem 10.20] we obtain

$$\partial\left(\frac{1}{g(x)}\right) \subset -\frac{\partial g(x)}{(g(x))^2},$$

while from [42, Exercise 10.21] we have

$$\partial\big(f(x)h(x)\big) \subset \partial f(x)h(x) + f(x)\partial h(x).$$

Combining these two results with $h = \frac{1}{g}$ yields the lemma statement. □

Proof of Lemma 2.1. Due to (A.2) we have

$$\sum_{s=1}^{S}\mu_s \text{CVaR}^S_{1-s/S}(X) = \sum_{s=1}^{S}\mu_s\frac{-1}{s}\sum_{t=1}^{s}X_{[t]} = \sum_{t=1}^{S}\left(\sum_{s=t}^{S}\mu_s\frac{-1}{s}\right)X_{[t]}.$$

Since the right-hand side needs to equal to $M^S_\phi(X) = -\sum_{t=1}^{S}\phi_t X_{[t]}$, we solve the following system of equations

$$\phi_t = \sum_{s=t}^{S}\frac{\mu_s}{s}.$$

Setting $\phi_{S+1} \equiv 0$, we have

$$\mu_s = s(\phi_s - \phi_{s+1}),$$

which is precisely (11). Due to (A.2), we have

$$\text{CVaR}^S_0(X) = -\frac{1}{S}\sum_{s=1}^{S}X_{[s]} = -\frac{1}{S}\sum_{s=1}^{S}X_s = -\mathbb{E}(X),$$

which implies the second equality in (10). □

Proof of Lemma 3.1. It was shown by [43] that a portofolio is SSD efficient if there is no other portfolio which has lower or equal values of CVaRs on the set of levels $\{0, 1/S, \ldots, 1-1/S\}$ with one inequality strict. This correspond to the multicriteria efficiency which can be ensured by the aggregate function approach, see, e.g., [44]. □

Proof of Theorem 3.1. We apply Theorem Appendix B.1 with $f_j(x) = \mathcal{E}_j\left(\sum_{i=1}^n R_i x_i\right)$, $g_k(x) = \mathcal{R}_k\left(\sum_{i=1}^n R_i x_i\right)$, $a_j = \mathcal{E}_j(X_0)$, $b_j = e_j(X_0)$, $c_k = \mathcal{R}_k(X_0)$, $d_k = d_k(X_0)$ and $X = \{x| \ \sum_{i=1}^n x_i = 1, \ x_i \geq 0\}$. First observe that X is convex and compact and that

$$d_k = d_k(X_0) = \mathcal{R}_k(X_0) - \min_{X\in\mathcal{X}} \mathcal{R}_k(X) = c_k - \min_{x\in X} g_k(x).$$

Thus, condition (B.1) is satisfied. Since weights (B.3) amount to (15) and problem (B.4) to (16), we obtain the theorem statement. □

Proof of Theorem 3.2. It follows directly from Theorem 3.1 with $J = 1$, $K = S-1$, $\mathcal{E}_1(X) = \mathbb{E}(X)$ and $\mathcal{R}_s(X) = \text{CVaR}_{1-s/S}(X)$. □

References

[1] Murthi B, Choi YK, Desai P. Efficiency of mutual funds and portfolio performance measurement: a non-parametric approach. Eur J Oper Res 1997;98(2):408– 418. doi:10.1016/S0377-2217(96)00356-6.
[2] Basso A, Funari S. A data envelopment analysis approach to measure the mutual fund performance. Eur J Oper Res 2001;135(3):477– 492. doi:10.1016/S0377-2217(00)00311-8.
[3] Chen Z, Lin R. Mutual fund performance evaluation using data envelopment analysis with new risk measures. OR Spectrum 2006;28(3):375–98. doi:10.1007/s00291-005-0032-1.
[4] Basso A, Funari S. Constant and variable returns to scale dea models for socially responsible investment funds. Eur J Oper Res 2014;235(3):775– 783. doi:10.1016/j.ejor.2013.11.024.
[5] Lamb JD, Tee K-H. Data envelopment analysis models of investment funds. Eur J Oper Res 2012;216(3):687– 696. doi:10.1016/j.ejor.2011.08.019.
[6] Branda M. Diversification-consistent data envelopment analysis with general deviation measures. Eur J Oper Res 2013;226(3):626– 635. doi:10.1016/j.ejor.2012.11.007.
[7] Branda M. Diversification-consistent data envelopment analysis based on directional-distance measures. Omega (Westport) 2015;52:65– 76. doi:10.1016/j.omega.2014.11.004.
[8] Pratt JW. Risk aversion in the small and in the large. Econometrica 1964;32(1/2):122–36.
[9] Arrow K. Aspects of the theory of risk-bearing. Yrjö Jahnssonin Säätiö; 1965.
[10] Ross S. Some stronger measures of risk aversion in the small and the large with applications. Econometrica 1981;49(3):621–38.
[11] Fulga C. Portfolio optimization under loss aversion. Eur J Oper Res 2016;251(1):310– 322. doi:10.1016/j.ejor.2015.11.038.
[12] Acerbi C. Spectral measures of risk: a coherent representation of subjective risk aversion. Journal of Banking & Finance 2002;26(7):1505– 1518. doi:10.1016/S0378-4266(02)00281-9.
[13] Brandtner M, Kürsten W. Decision making with expected shortfall and spectral risk measures: the problem of comparative risk aversion. Journal of Banking & Finance 2015;58:268– 280. doi:10.1016/j.jbankfin.2015.03.012.
[14] Brandtner M, Kürsten W. Consistent modeling of risk averse behavior with spectral risk measures: wächter/mazzoni revisited. Eur J Oper Res 2017;259(1):394– 399. doi:10.1016/j.ejor.2016.12.027.
[15] Brandtner M. “Spectral risk measures: properties and limitations”: comment on dowd, cotter, and sorwar. Journal of Financial Services Research 2016;49(1):121–31. doi:10.1007/s10693-014-0204-8.
[16] Wächter HP, Mazzoni T. Consistent modeling of risk averse behavior with spectral risk measures. Eur J Oper Res 2013;229(2):487– 495. doi:10.1016/j.ejor.2013.03.001.
[17] Dowd K, Cotter J, Sorwar G. Spectral risk measures: properties and limitations. Journal of Financial Services Research 2008;34(1):61–75. doi:10.1007/s10693-008-0035-6.
[18] Briec W, Kerstens K, Lesourd JB. Single-period markowitz portfolio selection, performance gauging, and duality: a variation on the luenberger shortage function. J Optim Theory Appl 2004;120(1):1–27. doi:10.1023/B:JOTA.0000012730.36740.bb.
[19] Briec W, Kerstens K, Jokung O. Mean-variance-skewness portfolio performance gauging: a general shortage function and dual approach. Manage Sci 2007;53(1):135–49. doi:10.1287/mnsc.1060.0596.
[20] Briec W, Kerstens K. Multi-horizon markowitz portfolio performance appraisals: a general approach. Omega (Westport) 2009;37(1):50– 62. doi:10.1016/j.omega.2006.07.007.
[21] Hadar J, Russell WR. Rules for ordering uncertain prospects. American Economic Review 1969;59(1):25–34.
[22] Hanoch G, Levy H. The efficiency analysis of choices involving risk. Review of Economic Studies 1969;36(3):335–46.
[23] Lévy H. Stochastic dominance: investment decision making under uncertainty. Springer, 3rd ed; 2016.
[24] Branda M, Kopa M. On relations between dea-risk models and stochastic dominance efficiency tests. Central European Journal of Operations Research 2014;22(1):13–35. doi:10.1007/s10100-012-0283-2.
[25] Joro T, Na P. Portfolio performance evaluation in a mean variance skewness framework. Eur J Oper Res 2006;175(1):446– 461. https://doi.org/10.1016/j.ejor.2005.05.006.
[26] Kerstens K, Mounir A, de Woestyne IV. Geometric representation of the mean-variance-skewness portfolio frontier based upon the shortage function. Eur J Oper Res 2011;210(1):81– 94. doi:10.1016/j.ejor.2010.09.014.
[27] Brandouy O, Kerstens K, de Woestyne IV. Frontier-based vs. traditional mutual fund ratings: afirst backtesting analysis. Eur J Oper Res 2015;242(1):332– 342. doi:10.1016/j.ejor.2014.11.010.
[28] Liu W, Zhou Z, Liu D, Xiao H. Estimation of portfolio efficiency via dea. Omega (Westport) 2015;52:107– 118. doi:10.1016/j.omega.2014.11.006.
[29] Branda M. Mean-value at risk portfolio efficiency: approaches based on data envelopment analysis models with negative data and their empirical behaviour. 4OR 2016;14(1):77–99. doi:10.1007/s10288-015-0296-5.
[30] Branda M, Kopa M. DEA Models equivalent to general nth order stochastic dominance efficiency tests. Operations Research Letters 2016;44(2):285– 289. doi:10.1016/j.orl.2016.02.007.
[31] Tarnaud AC, Leleu H. Portfolio analysis with dea: prior to choosing a model. Omega (Westport) 2018;75:57– 76. doi:10.1016/j.omega.2017.02.003.
[32] Choi H-S, Min D. Efficiency of well-diversified portfolios: evidence from data envelopment analysis. Omega (Westport) 2017;73:104– 113. doi:10.1016/j.omega.2016.12.008.
[33] Lin R, Chen Z, Hu Q, Li Z. Dynamic network dea approach with diversification to multi-period performance evaluation of funds. OR Spectrum 2017;39(3):821–60. doi:10.1007/s00291-017-0475-1.
[34] Zhou Z, Xiao H, Jin Q, Liu W. DEA Frontier improvement and portfolio rebalancing: an application of china mutual funds on considering sustainability information disclosure. Eur J Oper Res 2018;269(1):111– 131. doi:10.1016/j.ejor.2017.07.010.
[35] Essid H, Ganouati J, Vigeant S. A mean-maverick game cross-efficiency approach to portfolio selection: an application to paris stock exchange. Expert Syst Appl 2018;113:161– 185. doi:10.1016/j.eswa.2018.06.040.
[36] Lin R, Li Z. Directional distance based diversification super-efficiency dea models for mutual funds. Omega, accepted 2019:102096. doi:10.1016/j.omega.2019.08.003.
[37] Artzner P, Delbaen F, Eber J-M, Heath D. Coherent measures of risk. Mathematical Finance 1999;9(3):203–28. doi:10.1111/1467-9965.00068.
[38] Rockafellar R, Uryasev S. Conditional value-at-risk for general loss distributions. Journal of Banking & Finance 2002;26(7):1443–71. doi:10.1016/S0378-4266(02)00271-6.
[39] Kuosmanen T. Performance measurement and best-practice benchmarking of mutual funds: combining stochastic dominance criteria with data envelopment analysis. Journal of Productivity Analysis 2007;28(1):71–86. doi:10.1007/s11123-007-0045-7.
[40] Lozano S, Gutiérrez E. Data envelopment analysis of mutual funds based on second-order stochastic dominance. Eur J Oper Res 2008;189(1):230– 244. doi:10.1016/j.ejor.2007.04.014.
[41] Rockafellar RT, Wets RJ-B. Variational analysis, 317. Springer Science & Business Media; 2009.
[42] Clarke F. Functional analysis, calculus of variations and optimal control, 264. Springer Science & Business Media; 2013.
[43] Kopa M, Chovanec P. A second-order stochastic dominance portfolio efficiency measure. Kybernetika 2008;44:243–58.
[44] Kaisa M. Nonlinear multiobjective optimization. International Series in Operations Research & Management Science, 12. Boston, USA: Kluwer Academic Publishers; 1999.