## Extra loops

In this section it will be proved that Moufang loops with squares in the nucleus coincide with loops fulfilling the identity $xy{\cdot}xz = x(yz{\cdot}x)$. Such loops are called extra loops. The section concludes by a construction of extra loops that encompasses the loop of octonions, which is probably the most well known Moufang loop.

Other results of this section include a proof that all nuclei are associative subloops (i.e., groups).

**From autotopisms to nuclear elements.** Let $Q$ be a loop and let $\alpha, \beta, \gamma \in \mathrm{Sym}(Q)$.

(1) $(\alpha, \mathrm{id}_Q, \gamma) \in \mathrm{Atp}(Q) \Rightarrow \exists\, a \in N_\lambda(Q)$ such that $\alpha = \gamma = L_a$;
(2) $(\mathrm{id}_Q, \beta, \gamma) \in \mathrm{Atp}(Q) \Rightarrow \exists\, a \in N_\rho(Q)$ such that $\beta = \gamma = R_a$; and
(3) $(\alpha, \beta, \mathrm{id}_Q) \in \mathrm{Atp}(Q) \Rightarrow \exists\, a, b \in N_\mu(Q)$ such that $ab = 1$, $\alpha = R_a = R_b^{-1}$ and $\beta = L_a^{-1} = L_b$.

*Proof.* If $(\alpha, \mathrm{id}_Q, \gamma) \in \mathrm{Atp}(Q)$, then $\alpha(x)y = \gamma(xy)$ for all $x, y \in Q$. Setting $y = 1$ yields $\alpha = \gamma$, setting $x = 1$ provides $ay = \gamma(y)$, where $a = \alpha(1)$.

Suppose that $(\alpha, \beta, \mathrm{id}_Q) \in \mathrm{Atp}(Q)$. Then $\alpha(x)\beta(y) = xy$ for all $x, y \in Q$. Put $a = \alpha(1)$ and $b = \beta(1)$. Substitutions $x = 1$ and $y = 1$ give $\beta = L_a^{-1}$, where $a = \alpha(1)$, and $\alpha = R_b^{-1}$, where $b = \beta(1)$. Thus $x/b \cdot a\backslash y = xy$ for all $x, y \in Q$. Putting $x = b$ provides $L_b = L_a^{-1} = \beta$, and $y = a$ yields $R_b^{-1} = R_a = \alpha$. Therefore $L_aL_b = R_aR_b = \mathrm{id}_Q$ and thus $1 = \mathrm{id}_Q(1) = L_aL_b(1) = ab = R_aR_b(1) = ba$. □

**LIP and RIP elements.** Let $Q$ be a loop. An element $a \in Q$ is said to be a *LIP element* if there exists $b \in Q$ such that $L_a^{-1} = L_b$. Arguments used in case of LIP loops may be applied without a change to show that if $L_a^{-1} = L_b$, then $b = 1/a = a\backslash 1$. Hence $b$ may be denoted by $a^{-1}$. If $x \in Q$, then $a^{-1}(ax) = x = a(a^{-1}x)$.

*RIP elements* are defined symmetrically. An element that is both RIP and LIP is called an *IP element*.

**Nuclei and inverse properties.** If $(L_a, \mathrm{id}_Q, L_a) \in \mathrm{Atp}(Q)$, $Q$ a loop, then $(L_a^{-1}, \mathrm{id}_Q, L_a^{-1}) \in \mathrm{Atp}(Q)$. Therefore for each $a \in N_\lambda(Q)$ there exists $b \in N_\lambda(Q)$ such that $L_a^{-1} = L_b$. This shows that *elements of the left nucleus satisfy the LIP, and that $N_\lambda(Q)$ is closed under inverses.* Similarly *elements of the right nucleus satisfy the RIP, and $N_\rho(Q)$ is closed under inverses.*

If $c \in N_\mu(Q)$, then $(R_c^{-1}, L_c, \mathrm{id}_Q) \in \mathrm{Atp}(Q)$. By the statement above there exist $a, b \in N_\mu(Q)$ such that $R_c^{-1} = R_a = R_b^{-1}$ and $L_c = L_a^{-1} = L_b$. Hence $c = b$. This means that *each element of a middle nucleus is an IP element, and $N_\mu(Q)$ is closed under inverses.*

**Nuclei are groups.** *Let $Q$ be a loop. Then each of sets $N_\lambda(Q)$, $N_\mu(Q)$ and $N_\rho(Q)$ is an associative subloop of $Q$ (i.e., a group).*

*Proof.* Suppose that $a, b \in N_\lambda(Q)$. Then

$$(L_a, \mathrm{id}_Q, L_a)(L_b, \mathrm{id}_Q, L_b) = (L_aL_b, \mathrm{id}_Q, L_aL_b) \in \mathrm{Atp}(Q).$$

Therefore there exists $c \in N_\lambda(Q)$ such that $L_c = L_aL_b$. Since $c = L_c(1) = L_aL_b(1) = ab$, we have $ab \in N_\lambda(Q)$ for all $a, b \in N_\lambda(Q)$. If $a, b, c \in N_\lambda(Q)$, then $a \cdot bc = ab \cdot c$. This proves that $N_\lambda(Q)$ is a subsemigroup of $Q$ in which every element possesses an inverse. That makes $N_\lambda(Q)$ a group.

The case of $N_\rho(Q)$ can be obtained by mirroring. To prove that $N_\mu(Q)$ is a semigroup closed under inverses start from

$$(R_a^{-1}, L_a, \mathrm{id}_Q)(R_b^{-1}, L_b, \mathrm{id}_Q) = (R_{ab}^{-1}, L_{ab}, \mathrm{id}_Q), \text{ for all } a, b \in N_\mu(Q).$$

□

**The nucleus.** If $Q$ is a loop, then $N(Q) = N_\lambda(Q) \cap N_\rho(Q) \cap N_\mu(Q)$ is called the *nucleus* of $Q$. In general all three nuclei may by pairwise distinct. Since each of them is a subloop of $Q$, the nucleus always is an associative subloop of $Q$. In some cases, like in Moufang loops, all three nuclei coincide and are equal to $N(Q)$.

**Inverted Moufang indentities.** Let $Q$ be a Moufang loop. Then

$$(xy \cdot z)x^{-1} = x(y \cdot zx^{-1}) \text{ and } x^{-1}(y \cdot zx) = (x^{-1}y \cdot z)x.$$

*Proof.* This is essentially only one identity since $x = (x^{-1})^{-1}$. The identity can be also expressed as $xy{\cdot}z = x(y{\cdot}zx^{-1})x$. The right hand is equal to $xy{\cdot}(zx^{-1}{\cdot}x) = xy{\cdot}z$, by (Mm). □

The argument might be reversed. Since both

$$(xy \cdot z)(x\backslash 1) = x(y \cdot z(x\backslash 1)) \text{ and } (xy \cdot z)(1/x) = x(y \cdot z(1/x))$$

yield the IP property, as may be verified readily, each of them is an equivalent formulation of the Moufang identity.

**An identity induced by squares in nucleus.** Let $Q$ be a Moufang loop such that $x^2 \in N(Q)$ for every $x \in Q$. Then $Q$ satisfies the identity

$$(xy \cdot z)x = x(y \cdot zx). \tag{mE}$$

*Proof.* $(xy \cdot z)x = (xy \cdot z)(x^{-1} \cdot x^2) = ((xy \cdot z)x^{-1})x^2 = (x(y \cdot zx^{-1}))x^2 = x(y \cdot (zx^{-1})x^2) = x(y \cdot zx)$. □

**Equivalence of the extra identities.** The identity (mE) is equivalent to each these two identities:

$$xy \cdot xz = x(yx \cdot z) \text{ and} \tag{lE}$$

$$zx \cdot yx = (z \cdot xy)x. \tag{rE}$$

*Proof.* Let us first verify that each of the three identities yields a flexible IP loop. The flexibility may be obtained by setting $z = 1$. Further on, only (mE) and (lE) will be considered since (rE) is a mirror image of (lE).

In the case of $(xy \cdot z)x = x(y \cdot zx)$ set $z = 1/x$ to get the RIP, and $y = x\backslash 1$ to obtain the LIP. For $xy \cdot xz = x(yx \cdot z)$ set $z = x\backslash 1$ to get the RIP. To obtain the LIP consider first the equality

$$xy \cdot (x \cdot yz) = x(yx \cdot yz) = x \cdot y(xy \cdot z).$$

The RIP implies the existence of two sided inverses. Setting $z = (xy)^{-1}$ gives $xy \cdot (x \cdot y(xy)^{-1}) = xy$. Hence $x \cdot y(xy)^{-1} = 1$. Since $x^{-1}$ is the two sided inverse, $y(xy)^{-1} = x^{-1}$. Applying the RIP yields $y = x^{-1}(xy)$.

Writing $(xy \cdot z)x = x(y \cdot zx)$ as $(xy \cdot z/x)x = x \cdot yz$ shows that $Q$ satisfies (mE) if and only if

$$\forall x \in Q \ (L_x, R_x^{-1}, R_x^{-1}L_x) \in \mathrm{Atp}(Q).$$

Expressing $xy \cdot xz = x(yx \cdot z)$ as $(x(y/x) \cdot xz = x \cdot yz$ yields the formulation

$$\forall x \in Q \ (L_xR_x^{-1}, L_x, L_x) \in \mathrm{Atp}(Q).$$

Since we are dealing with IP loops, a switching of coordinates and the identity $IR_x^{-1}I = L_x$ provide

$$(L_xR_x^{-1}, L_x, L_x) \in \mathrm{Atp}(Q) \ \Leftrightarrow \ (L_x, R_x^{-1}, L_xR_x^{-1}) \in \mathrm{Atp}(Q).$$

The rest follows from the flexibility. □

**Extra loops are Moufang loops with squares in the nucleus.** Identities (mE), (rE) and (lE) are known as the *extra* identities. A loop satisfying an extra identity is said to be an *extra loop*. A loop $Q$ is extra if and only if $Q$ is a Moufang loop such that $x^2 \in N(Q)$ for each $x \in Q$.

*Proof.* As shown above, Moufang loops with nuclear squares fulfil (mE). To prove the converse consider an extra loop $Q$. Both $(L_xR_x^{-1}, L_x, L_x)$ and $(L_x, R_x^{-1}, R_x^{-1}L_x)$ are autotopisms for each $x \in Q$. Hence

$$(R_x^{-1}L_x, L_x, L_x)(L_x^{-1}, R_x, L_x^{-1}R_x) = (R_x^{-1}, L_xR_x, R_x)$$

is an autotopism of $Q$ for each $x \in Q$ too. This means that $Q$ is a Moufang loop since these autotopisms describe the identity (rM).

By the (mM) identity, $(L_x, R_x, L_xR_x) \in \mathrm{Atp}(Q)$ for every $x \in Q$. Therefore

$$(L_x, R_x, L_xR_x)(L_x^{-1}, R_x, L_x^{-1}R_x) = (\mathrm{id}_Q, R_x^2, L_xR_xL_x^{-1}R_x)$$

is an autotopism for each $x \in Q$. Hence $x^2 = R_x^2(1) \in N_\rho(Q) = N(Q)$, for each $x \in Q$. □

**The centre.** For a loop $Q$ put

$$Z(Q) = \{a \in N(Q);\ ax = xa \text{ for every } x \in Q.\}$$

This is the *centre* of $Q$. An element $a \in N(Q)$ thus belongs to the centre if and only if $L_a = R_a$.

Central elements are IP elements since $Z(Q) \subseteq N_\mu(Q)$. If $a, b \in Z(Q)$, then $L_{ab} = L_{ba} = L_bL_a = R_bR_a = R_{ab}$ and $L_{a^{-1}} = L_a^{-1} = R_a^{-1} = R_{a^{-1}}$. That makes $Z(Q)$ a subgroup of $N(Q)$.

A subloop $Z$ of $Q$ is said to be *central* if $Z \leq Z(Q)$.

Consider a central subloop $Z \leq Q$. If $x, y \in Q$ and $a, b \in Z$, then $xa = yb$ implies $y = xc = cx$, where $c = ab^{-1} = b^{-1}a$. This shows that $Q$ may be partitioned into cosets $xZ = Zx$. We have $xZ \cdot yZ = xyZ$ for all $x, y \in Q$, and this defines the structure of a factor loop $Q/Z$. (Later we shall pay attention to conditions under which a factor loop $Q/S$ may be defined if $S \leq Q$ is not necessary central.)

**Involutory Moufang loops are groups.** A loop $Q$ is said to be *involutory* if $x^2 = 1$ for all $x \in Q$. As is well known, involutory groups are commutative, and thus coincide with the class of elementary abelian 2-groups. Let us observe that the same is true for Moufang loops.

They are commutative since if $x, y \in Q$, then $xy \cdot yx = xy^2x = x^2 = 1$, and that implies $yx = (xy)^{-1} = xy$. Hence $y = x^2y = x(xy) = xyx$ for all $x, y \in Q$. They are associative since $zx \cdot y = y \cdot zx = y \cdot zx^{-1} = x(y \cdot zx^{-1})x = xy \cdot z = z \cdot xy$.

**A journey to octonions.** A 4-element vector space may be represented by a triangle. The vertices correspond to nonzero vectors. The sum of two distinct vertices is the third vertex.

An 8-element vector space may be represented by a Fano plane. The vertices correspond to nonzero vectors. The sum of two distinct vertices is the vertex that completes the line passing through the two vertices.

Suppose that $v_0$, $v_1$ and $v_2$ are pairwise distinct nonzero elements of a 8-element vector space $V$ such that $v_2 \neq v_0 + v_1$. Define a sequence of vectors $v_i$, $i \geq 0$, by setting $v_{i+3} = v_i + v_{i+1}$. Thus

$$v_3 = v_0 + v_1,\ v_4 = v_1 + v_2,\ v_5 = v_2 + v_3 = v_0 + v_1 + v_2, v_6 = v_3 + v_4 = v_0 + v_2,$$
$$v_7 = v_4 + v_5 = v_0,\ v_8 = v_5 + v_6 = v_1 \text{ and } v_9 = v_6 + v_7.$$

Hence $v_i$, $0 \leq i \leq 6$, are all nonzero vectors of $V$, the indices $i$ may be computed modulo 7, and $\{v_i, v_{i+1}, v_{i+3}\}$, $0 \leq i \leq 6$, are all lines of the Fano plane that is induced by $V$.

We have obtained a representation of Fano plane upon an oriented cycle of 7 elements. Let it be called a *circular representation of Fano plane*.

Let us get oriented. An oriented triangle may be thought as a representation of the quaternion group $Q_8$. If the triangle is oriented as $(a_0\ a_1\ a_2)$, then there exists a unique quaternion group upon $\{a_i, -a_i;\ 0 \leq i \leq 2\} \cup \{-1, 1\}$ such that $a_i^2 = -1$ and $a_i a_{i+1} = a_{i-1}$, $i \in \mathbb{Z}_3$.

Suppose now that each line of a Fano plane obtains one of two possible orientations. This yields seven oriented 3-cycles each of which is further on interpreted as a quaternion group. Elements $-1$ and 1 are considered to be common for all of the seven quaternion groups. Denote by $U$ their union. Any two elements $x, y \in U$ occur in one of these groups, and so their product is well defined. This makes $U$ a loop, and this loop is diassociative. The set $Z = \{-1, 1\}$ is a central subgroup, and $U/Z \cong (V, +)$.

The question is whether there exists an orientation that makes $U$ a Moufang loop (and thus also an extra loop). In fact, there exist several such orientations. However, loops produced by these orientations are mutually isomorphic.

The orientation that is standardly used to produce a Moufang loop is based upon the circular representation of Fano plane. This results in setting $U = \{e_i, -e_i;\ 0 \leq i \leq 6\} \cup \{-1, 1\}$ where $-1$ is a central element equal to each $e_i^2$, and $e_i e_{i+1} = e_{i+3}$, $0 \leq i \leq 6$. This loop is known as the loop of *octonions*. More precisely $U$ is the loop of octonion units, similarly as $\{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$ is the group of quaternion units. (Quaternions $\mathbb{H}$ are a division ring upon $\mathbb{R}^4$ and octonions $\mathbb{O}$ are an algebra upon $\mathbb{R}^8$.)

In fact $U$ is up to isomorphism the only Moufang loop $Q$ of order 16 for which there exists a central subloop $Z = \{1, z\}$ such that $x^2 = z$ for each $x \in Q \setminus Z$.

We shall now show that if such a loop $Q$ exists, then it has to be isomorphic to $U$. However, the very existence of $Q$ will be verified later.

*Proof.* First note that there exists an 8-element vector space $V$ such that $(V, +) \cong Q/Z$. This is because $Q/Z$ is an involuntory Moufang loop. If $x, y \in Q \setminus Z$ and $y \notin \{x, xz\}$, then $\langle x, y\rangle/Z$ is isomorphic to Klein group, and $\langle x, y\rangle$ is a group isomorphic to the group of quaternions $Q_8$. This follows from the diassociativity of $Q$ (a direct proof is also possible). Hence $xy = yxz$ and $xyx = y$.

Denote the nonzero vectors of $V$ by $v_i$, $0 \leq i \leq 6$, so that each $\{v_i, v_{i+1}, v_{i+3}\}$ is a line of the corresponding Fano plane. For $i \in \{0, 1, 2\}$ choose any $e_i$ such that $e_i Z = v_i$. Set $e_3 = e_0 e_1$, $e_4 = e_1 e_2$, $e_5 = e_2 e_3$ and $e_6 = e_3 e_4$. Then $e_i Z = v_i$ for every $i \in \{0, \ldots, 6\}$. The choice and definitions of $e_i$ establish an orientation $(v_i, v_{i+1}, v_{i+3})$ of a line for $i \in \{0, 1, 2, 3\}$. It remains to observe that the Moufang law forces out this orientation for the remaining values of $i$ as well. Now, $e_4 e_5 = e_1 e_2 \cdot e_2 e_3 = e_2 e_1 z \cdot z e_3 e_2 = e_2 e_1 \cdot e_3 e_2 = e_2 \cdot e_1 e_3 \cdot e_2 = e_2 e_0 e_2 = e_0$. Similarly, $e_5 e_6 = e_2 e_3 \cdot e_3 e_4 = e_3 \cdot e_2 e_4 \cdot e_3 = e_3 e_1 e_3 = e_1$. Finally, $e_6 e_0 = e_3 e_4 \cdot e_0 = (e_0 e_1 \cdot e_4) e_0 = z(e_1 e_0 \cdot e_4) e_0 = z(e_1 \cdot e_0 e_4 e_0) = z e_1 e_4 = e_4 e_1 = e_2$. □

**A construction using quadratic forms.** Let $V$ be a vector space over a field $F$. A mapping $g\colon V \to F$ is said to be a *quadratic form* if $h\colon (x, y) \mapsto g(x + y) - g(x) - g(y)$ is a bilinear form $V \times V \to F$ and $g(\lambda x) = \lambda^2 g(x)$ for all $x \in V$ and $\lambda \in F$. If $\mathrm{char}(F) = 2$, then the bilinear form is alternating (which means that $h(x, x) = 0$ for every $x \in V$).

Recall that if $h\colon V \times V \to F$ is alternating and bilinear, then $h(x, y) = -h(y, x)$, for all $x, y \in F$ (no assumption on $\mathrm{char}(F)$ is being made here). Recall also that

a multilinear mapping $f\colon V^n \to F$ is said to be *alternating* if $f(x_1, \ldots, x_n) = 0$ whenever $x_i = x_j$, where $1 \le i < j \le n$. If $\sigma \in S_n$ and $f$ is alternating, then $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (-1)^{\mathrm{sgn}(\sigma)} f(x_1, \ldots, x_n)$. Alternating multilinear mappings in characteristic two thus are symmetric.

**Theorem.** *Let $V$ be a vector space over a field $F$,* $\mathrm{char}(F) = 2$*, and let $q\colon V \times V \to F$ be such that for each $v \in V$ the mapping $x \mapsto q(x, v)$ is a quadratic form, while the mapping $x \mapsto q(v, x)$ is a linear form. Put $Q = V \times F$ and define a binary operation upon $Q$ by*

$$(u, a)(v, b) = (u + v, q(u, v) + a + b)$$

*and assume that $q(u + v, u) = q(u, u) + q(v, u)$ for all $u, v \in V$. Then $(Q, \cdot)$ is a Moufang loop. Furthermore, the mapping $A\colon V \times V \times V \to F$ defined by $A(u, v, w) = q(u + v, w) + q(u, w) + q(v, w)$ is an alternating trilinear mapping, and a triple of elements $((u, a), (v, b), (w, c)) \in Q^3$ is associative if and only if $A(u, v, w) = 0$.*

*Proof.* The multilinearity of $A$ follows directly from the assumptions on $q$. Clearly, $A(u, v, w) = A(v, u, w)$ and $A(u, u, v) = 0$, for any $u, v, w \in V$. To verify that $A$ is an alternating trilinear form it thus remains to show that $A(u, v, u) = 0$. This follows from $q(u+v, u) + q(u, u) + q(v, u) = q(u, u) + q(v, u) + q(u, u) + q(v, u) = 0$.

The neutral element is $(0, 0)$ since $q(u, 0) = q(0, u)$ for all $u \in V$. The operation $\cdot$ thus yields a loop. Each element $(0, a)$ is central and $(u, a) = (u, 0)(0, a)$. A triple $((u, a), (v, b), (w, c))$ is thus associative if and only if the triple $((u, 0), (v, 0), (w, 0))$ is associative. Now, $((u, 0) \cdot (v + w, q(v, w)) = (u + v + w, q(u, v + w) + q(v, w))$ is equal to $(u + v, q(u, v)) \cdot (w, 0) = (u + v + w, q(u + v, w) + q(u, v))$ if and only if $A(u, v, w) = q(u, w) + q(v, w) + q(u + v, w)$ is equal to 0 since $q(u, v + w) = q(u, v) + q(u, w)$.

To verify that $(Q, \cdot)$ is a Moufang loop it suffices to show that

$$(u, 0)(v, 0) \cdot (w, 0)(u, 0) = (u, 0)((v, 0)(w, 0) \cdot (u, 0)).$$

Note that $(v, 0)(w, 0) \cdot (u, 0) = (v+w, q(v, w))(u, 0) = (v+w+u, q(v+w, u)+q(v, w))$. The left hand side of the Moufang identity is

$$(u + v, q(u, v)) \cdot (w + u, q(w, u)) = (v + w, q(u, v) + q(w, u) + q(u + v, w + u)),$$

while the right hand side is equal to

$$(u, 0) \cdot (u+v+w, q(v+w, u)+q(v, w)) = (v+w, q(u, u+v+w)+q(v+w, u)+q(v, w)).$$

The question thus is whether

$$q(u, v) + q(u + v, u) + q(w, u) + q(u + v, w) = q(u, v) + q(u, u) + q(v, u) + q(w, u) + q(u + v, w)$$

is equal to

$$q(u, v) + q(u, u) + q(u, w) + q(v + w, u) + q(v, w).$$

That really holds since $q(v, u) + q(w, u) + q(v + w, u) = A(v, w, u)$ is equal to $A(u, v, w) = q(u + v, w) + q(u, w) + q(v, w)$. □

**Parameters for quadratic forms.** Let $F$ be a field of characteristic 2, let $V$ be a vector space over $F$, and let $b_1, \ldots, b_n$ be a basis of $V$. A quadratic form $g\colon V \to F$ is fully determined by values of $g$ at $b_i$ and $b_i + b_j$, $1 \le i < j \le n$. This fact follows from the formula

$$g\left(\sum \lambda_i b_i\right) = \sum_i \lambda_i^2 g(b_i) + \sum_{i<j} \lambda_i \lambda_j (g(b_i) + g(b_j) + g(b_i + b_j))$$

that may be easily proved. Whenever $g(b_i)$, $g(b_j)$ and $g(b_i + b_j)$ are given, then the formula defines a quadratic form.

Suppose now that $n = 3$ and set

$$q\left(\sum \lambda_i b_i, \sum \nu_i b_i\right) = \lambda_1^2(\nu_1+\nu_2+\nu_3) + \lambda_2^2(\nu_2+\nu_3) + \lambda_3^2\nu_3 + \lambda_1\lambda_2\nu_3 + \lambda_1\lambda_3\nu_2 + \lambda_2\lambda_3\nu_1.$$

Fixing any value of the second coordinate thus yields a quadratic form, while fixing any value for the first coordinate provides a linear form. Set $A(u,v,w) = q(u,w) + q(v,w) + q(u+v,w)$. This is a trilinear form. Suppose that $u = \sum \lambda_i b_i$, $v = \sum \rho_i b_i$ and $w = \sum \nu_i b_i$. Since $\lambda_i^2 + \rho_i^2 = (\lambda+\rho_i)^2$ the first part of the formula defining $q$ contributes nothing to $A(u,v,w)$. Let now $\{i,j,k\} = \{1,2,3\}$. Then $\lambda_i\lambda_j\nu_k + \rho_i\rho_j\nu_k + (\lambda_i+\rho_i)(\lambda_j+\rho_j)\nu_k = \lambda_i\rho_j\nu_k + \lambda_j\rho_i\nu_k$. Since $A(u,v,w)$ is obtained by summing over all $i,j,k$, there has to be $A(u,v,w) = \det(u,v,w)$ (i.e., a determinant of the matrix in which the columns are formed by coefficients of $u$, $v$ and $w$, respectively). Thus $A(u,v,u) = 0$, and that shows that $q$ as defined can be used to build a Moufang loop on $V \times F$. The operation of the loop is $(u,a)(v,b) = (u+v, q(u,v)+a+b)$.

Use now the same formula for $F = \{0,1\}$. To see that the construction yields a Moufang loop in which $(u,a)(u,a) = (0,1)$ whenever $u \neq 0$ we have to show that if at least one of $\lambda_i \in F$ is nonzero and $u = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$, then $q(u,u) = 1$.

It is easy to verify that

$$q(u,u) = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_1 + \lambda_2 + \lambda_3.$$

This yields 1 if $\lambda_1 = \lambda_2 = \lambda_3 = 1$. If $\lambda_3 = 0$, then the formula to consider is $\lambda_1\lambda_2 + \lambda_1 + \lambda_2$. That is equal to 0 if and only if $\lambda_1 = \lambda_2 = 0$. We can thus conclude by stating:

**The loop of octonion units.** There exists a Moufang loop $Q$ upon elements $\pm 1, \pm e_0, \ldots, \pm e_6$ such that $(-1)(\pm e_i) = \mp e_i$, $e_i^2 = -1$, $-1$ is a central element, $e_i e_j = -e_j e_i$ if $0 \leq i \leq 6$, and

$$e_i e_{i+1} = e_{i+3},\ e_i e_{i+2} = e_{i-1},\ e_i e_{i+3} = -e_{i+1}$$

for each $i \in \{0,\ldots,6\}$, with the indices computed modulo 7.

Let $V$ be a vector space over $F$ with nonzero vectors $v_0, \ldots, v_6$ such that $v_i + v_{i+1} = v_{i+3}$ for each $i \in \{0,\ldots,6\}$. Denote by $\pi$ the mapping $\pm e_i \mapsto v_i$, $\pm 1 \mapsto 0$. Then $\pi$ is a homomorphism $(Q,\cdot) \to (V,+)$. A triple $(x,y,z) \in Q^3$ is associative if and only if $\det(\pi(x),\pi(y),\pi(z)) = 0$ since the trilinear mapping $A$ coincides with the determinant. The associativity is thus equivalent to linear dependence.