X elementární integrand

$X_t = \xi_0 1_{\{0\}} + \sum_{i=1}^{N} \xi_i 1_{(t_i, t_{i+1}]}$ $\quad \xi_i$ je $\mathcal{F}_{t_i}$ měřitelná n. vel. s nejvýše kon. mnoha hodnotami

$0 = t_0 = t_1 < t_2 < \dots < t_{N+1} = T$

$\mathcal{E}(\mathcal{F}_t)$

L^p-integrátor

Z $\quad$ Z je zprava spojitý a $\mathcal{F}_t$-adaptovaný

$$\sup\left\{ \left(E\left(\int X\,dZ\right)^p\right)^{1/p},\ X \in \mathcal{E}(\mathcal{F}_t),\ |X| \le 1 \right\} \le K < \infty$$

$$|Z|_{I_p} = \sup\left\{ \left(E\left(\int X\,dZ\right)^p\right)^{1/p},\ \text{---}\ \right\} \qquad X \longmapsto \int X\,dZ$$

lineární na $\mathcal{E}(\mathcal{F}_t)$

Tvrdenie 3: Z $\mathcal{F}_t$-adaptovaný, sprava spojitý $|Z|_v(\omega) < \infty \ \forall \omega$

Pokiaľ $E|Z|_v^p < \infty$, tak Z je L^p-integrátor.

Dôkaz:

$$\left|\int X\, dZ\right| = \left|\xi_0 Z_0 + \sum_{i=1}^{N} \xi_i (Z_{t_{i+1}} - Z_{t_i})\right| \le |\xi_0||Z_0| + \sum_{i=1}^{N} |\xi_i||Z_{t_{i+1}} - Z_{t_i}|$$

a pokiaľ $|X| \le 1$

tak $|\xi_i| \le 1$

$$\le |Z_0| + \sum_{i=1}^{N} |Z_{t_{i+1}} - Z_{t_i}| \le |Z|_v$$

$$\Rightarrow \forall\, |X| \le 1 \quad E\left|\int X\, dZ\right|^p \le E|Z|_v^p < \infty.$$

$\blacksquare$

Tvrzení 4: (stoch. integrál charakterizuje martingal)

Buď M spojitý (zprava spojitý), $\mathcal{F}_t$-adaptovaný, $E|M_t| < \infty$ $\forall t \in [0,T]$

Pak M je $\mathcal{F}_t$-martingal $\boxed{\text{právě když}}$

$$E \int X \, dM = 0 \quad \forall X \in \mathcal{E}(\mathcal{F}_t), \; X_0 = 0$$

Důkaz: Je-li M martingal

$$\int X \, dM = 0 \cdot M_0 + \sum_{i=1}^{N} \xi_i (M_{t_{i+1}} - M_{t_i})$$

$$E \int X \, dM = \sum_{i=1}^{N} E\left[\xi_i (M_{t_{i+1}} - M_{t_i})\right] = \sum_{i=1}^{N} E\left[E\left(\xi_i (M_{t_{i+1}} - M_{t_i}) \middle| \mathcal{F}_{t_i}\right)\right] = 0$$

$$E\left[\xi_i \underbrace{E\left(M_{t_{i+1}} - M_{t_i} \middle| \mathcal{F}_{t_i}\right)}_{= 0 \text{ s.j.}}\right] = 0$$

$E\int X\,dM = 0$ volme $X_F = 1_F 1_{(s,t]}$ $F \in \mathcal{F}_s$

ukážeme, že M musí splňovat $E[M_t | \mathcal{F}_s] = M_s$ s.j.

$$0 = E[1_F(M_t - M_s)] \Rightarrow \int_F M_t\,dP = \int_F M_s\,dP \quad \forall F \in \mathcal{F}_s$$

$$\int_\Omega 1_F (M_t - M_s)\,dP = \int_F M_t - M_s\,dP = 0$$

$$\underbrace{\int_F E[M_t|\mathcal{F}_s]\,dP}_{\mathcal{F}_s\text{-měř.}} \overset{\forall F \in \mathcal{F}_s}{=} \int_F M_t\,dP = \int_F \underbrace{M_s}_{\text{je } \mathcal{F}_s\text{-měř.}}\,dP$$

Q.E.D.

Pokud $E\int X\,dM \geq 0 \quad \forall X \in \mathcal{E}(\mathcal{F}_t) \quad X_0 = 0$

$\Leftrightarrow$ M je submartingal

Tvrzení 5: Zprava spojitý $\mathcal{F}_t$-adaptovaný L_2-martingal je L_2-integrátor.

(ověříme $\sup\{E(\int X\,dM)^2,\ X \in \mathcal{E}(\mathcal{F}_t),\ |X| \leq 1\} \leq K < \infty$)

Důkaz: Buď $X \in \mathcal{E}(\mathcal{F}_t)$

$$E\left(\int X\,dM\right)^2 = E\left(\xi_0 M_0 + \sum_{i=1}^{N} \xi_i (M_{t_{i+1}} - M_{t_i})\right)^2 =$$

$$= E\Big(\xi_0^2 M_0^2 + \sum_{i=1}^{N} \xi_i^2 \left(M_{t_{i+1}} - M_{t_i}\right)^2 + 2\sum_{i=1}^{N} \xi_0 \xi_i M_0 \left(M_{t_{i+1}} - M_{t_i}\right)$$

$$+ \sum_{i \neq j}\sum \xi_i \xi_j \left(M_{t_{i+1}} - M_{t_i}\right)\left(M_{t_{j+1}} - M_{t_j}\right) \Big)$$

$$E\left[\xi_i^2 \left(M_{t_{i+1}} - M_{t_i}\right)^2\right] = E\left[E\left(\xi_i^2 \left(M_{t_{i+1}} - M_{t_i}\right)^2 \,\middle|\, \mathcal{F}_{t_i}\right)\right] =$$

$$= E\left[\xi_i^2 \underbrace{E\left(M_{t_{i+1}}^2 - 2M_{t_{i+1}} M_{t_i} + M_{t_i}^2 \,\middle|\, \mathcal{F}_{t_i}\right)}\right] = E\left[\xi_i^2 \left(M_{t_{i+1}}^2 - M_{t_i}^2\right)\right]$$

$$\overset{s.j.}{=} E\left(M_{t_{i+1}}^2 \,\middle|\, \mathcal{F}_{t_i}\right) - 2M_{t_i} \underbrace{E\left(M_{t_{i+1}} \,\middle|\, \mathcal{F}_{t_i}\right)}_{= M_{t_i}\ s.j.} + M_{t_i}^2 = E\left(M_{t_{i+1}}^2 \,\middle|\, \mathcal{F}_{t_i}\right) - M_{t_i}^2 = E\left(M_{t_{i+1}}^2 - M_{t_i}^2 \,\middle|\, \mathcal{F}_{t_i}\right)$$

$i<j$ $\quad E\left[\xi_i \xi_j (M_{t_{i+1}} - M_{t_i})(M_{t_{j+1}} - M_{t_j})\right] = E\left[E\left(\underbrace{\xi_i \xi_j (M_{t_{i+1}} - M_{t_i})}_{\mathcal{F}_{t_j}\text{-mért.}}(M_{t_{j+1}} - M_{t_j}) \middle| \mathcal{F}_{t_j}\right)\right]$

$$E\left[\xi_i \xi_j (M_{t_{i+1}} - M_{t_i}) \underbrace{E\left(M_{t_{j+1}} - M_{t_j} \middle| \mathcal{F}_{t_j}\right)}_{=0}\right] = 0$$

$$E\left(\int X\,dM\right)^2 = E\left(\xi_0^2 M_0^2 + \sum_{i=1}^{N} \xi_i^2 (M_{t_{i+1}}^2 - M_{t_i}^2)\right) \le E\left(M_0^2 + \underbrace{\sum_{i=1}^{N} (M_{t_{i+1}}^2 - M_{t_i}^2)}_{M_{t_{N+1}}^2 - M_{t_1}^2}\right) =$$

$\xi_0^2 \le 1 \qquad \xi_i^2 \le 1$

$M_{t_{N+1}}^2 = M_T^2 \qquad M_{t_1}^2 = M_0^2$

$$\boxed{= E M_T^2 < \infty}$$