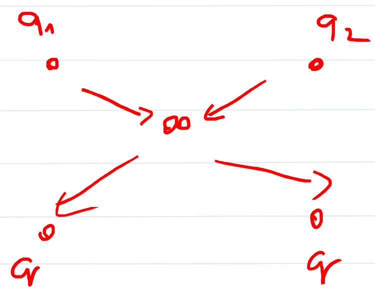
1.3 Dvě kuličky nesoucí náboj q1 a q2 se ve vzdálenosti r přitahují silou o velikosti F1. Po doteku se v téže vzdálenosti odpuzují silou F2. Určete náboje q1, q2. (pro určitost uvažujte q1>0)

$\rightarrow q_2 < 0$

$$|F_1| = F_1 = -k\frac{q_1 q_2}{r^2} \qquad k = \frac{1}{4\pi\varepsilon_0}$$



$$q = \frac{q_1+q_2}{2}$$

Velikost $|F_2| = F_2 = k\frac{q^2}{r^2} \Rightarrow |q| = \sqrt{\frac{F_2 r^2}{k}}$

1) $|q_1| > |q_2| \rightarrow q > 0$

$$q = \sqrt{\frac{r^2 F_2}{k}}$$

$$q_2 = 2q - q_1$$

$\Downarrow$

$$\frac{r^2 F_1}{k} = -q_1(2q - q_1)$$

$$0 = q_1^2 - 2q_1\sqrt{\frac{r^2F_2}{k}} - \frac{r^2F_1}{k}$$

řešení bereme $q_1 > 0$

$$q_1 = \sqrt{\frac{r^2}{k}}\left(\sqrt{F_2} + \sqrt{F_1+F_2}\right)$$

$$q_2 = 2q - q_1 = \sqrt{\frac{r^2}{k}}\left(\sqrt{F_2} - \sqrt{F_1+F_2}\right)$$

2) $|q_1| < |q_2| \rightarrow q < 0$

$$q = -\sqrt{\frac{r^2F_2}{k}}$$

$$q_2 = 2q - q_1 = -\left(q_1 + 2\sqrt{\frac{r^2F_2}{k}}\right) = -2|q| - q_1$$

$$0 = q_1^2 + 2q_1\sqrt{\frac{r^2F_2}{k}} - \frac{r^2F_1}{k}$$

bereme $q_1 > 0$

$$q_1 = \sqrt{\frac{r^2}{k}}\left(\sqrt{F_1+F_2} - \sqrt{F_2}\right)$$

$$q_2 = -\sqrt{\frac{r^2}{k}}\left(\sqrt{F_1+F_2} + \sqrt{F_2}\right)$$

Řešení

$$q_1 = \sqrt{\frac{r^2}{k}}\left(\sqrt{F_1+F_2} \pm \sqrt{F_2}\right), \quad q_2 = -\sqrt{\frac{r^2}{k}}\left(\sqrt{F_1+F_2} \mp \sqrt{F_2}\right)$$

VEKTOROVÁ ALGEBRA

A,B,C – VEKTORY $\quad A = (a_1, a_2, a_3)$ atd.

f, g – skaláry

– Sumační notace

$A.B = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i$

$A\times B = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$

$A.B = B.A \qquad A\times B = -B\times A$

$f(A.B) = (fA).B = A.(fB) \qquad f(A\times B) = (fA)\times B = A\times(fB)$

$A(B\cdot C) \neq (A\cdot B)C \qquad A\times(B\times C) \neq (A\times B)\times C$

$A\cdot(B+C) = A\cdot B + A\cdot C \qquad A\times(B+C) = A\times B + A\times C$

$A.B = 0 \qquad A\perp B$

$A\times B = 0 \qquad A\parallel B$

$A.A = |A|^2 \qquad A\times A = 0$

$A\times(B\times C) = B(A.C) - C(A.B)$ $\quad$ BÁC – CÁB

$A.(B\times C) = C.(A\times B) = B.(C\times A)$

$(A\times B).(C\times D) = (A.C)(B.D) - (B.C)(A.D)$

$(A\times B)^2 = |A|^2|B|^2 - (A.B)^2$

VEKTOROVÝ SOUČIN POMOCÍ LEVI-CIVITOVA TENZORU

$\varepsilon_{ijk} =$
- 1 $\quad$ i,j,k – sudé permutace čísel 1,2,3 (123 – 312 – 231)
- 0 $\quad$ alespoň dva indexy stejné
- −1 $\quad$ ijk – liché permutace 321 – 132 – 213

$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji}$

$(A\times B)_i = \varepsilon_{ijk} A_j B_k$ – i-tá složka součinu

$A = \vec{A}$

$B = \vec{B}$

$C = \vec{C}$

Derivace polí – gradient

$f(\vec{r}) \qquad \vec{r} = (x, y, z) \qquad d\vec{r} = (dx, dy, dz)$

$$f(\vec{r} + d\vec{r}) = \underbrace{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}_{} + f(\vec{r})$$

posun o $d\vec{r}$

$$df(\vec{r}) = f(\vec{r} + d\vec{r}) - f(\vec{r}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot d\vec{r}$$

vektor
‖
gradient f
grad f

zavedeme operátor

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

nabla

$$\text{grad} f = \nabla f$$

$$df(\vec{r}) = \nabla f \cdot d\vec{r}$$

Derivace ve směru $\vec{v}$ $\qquad \dfrac{d f(\vec{r})}{d\vec{v}} = \nabla f \cdot \vec{v}$

jedn. vektor

Další s ∇ $A_x(\vec{r})$ $A_y(\vec{r})$ $A_z(\vec{r})$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

divergence $\nabla \cdot \vec{A} = \mathrm{div}\,\vec{A}$

$\nabla \cdot \vec{A} \neq \vec{A} \cdot \nabla$

skalární pole — operator

$\vec{A}\cdot\nabla = A_x \frac{\partial}{\partial x} + \dots$

$\nabla f \neq f \nabla$

a $\nabla \times \vec{A}$

vektor

$$(\nabla \times \vec{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$(\quad)_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$(\quad)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

oper

$\nabla \times \vec{A} \neq \vec{A} \times \nabla$

vekt. pole

rotace $\mathrm{rot}\,\vec{A} = \nabla \times \vec{A}$

$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ....

OPERÁTORY - 1

Hamiltonův operátor $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$

A, B, C - VEKTOROVÁ POLE

f, g - skalární funkce

$\nabla . A = \text{div } A$

$\nabla \times A = \text{rot } A$

$\nabla f = \text{grad } f$

Pozor! $\nabla f \neq f \nabla \Rightarrow$ OPERÁTOR $\rightarrow \left(f\frac{\partial}{\partial x_1}, f\frac{\partial}{\partial x_2}, f\frac{\partial}{\partial x_3}\right)$

VEKTOR $\rightarrow \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$

$\nabla . A \neq A . \nabla \longrightarrow a_i \frac{\partial}{\partial x_i} = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3}$ = operátor

$\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = \frac{\partial a_i}{\partial x_i}$ = skalár

---

TENZOR IDENTITY $\overleftrightarrow{I}$

$\overleftrightarrow{I} . A = A$

$\nabla \vec{r} = \overleftrightarrow{I}$

$A . \text{grad } \vec{r} = A . \nabla \vec{r}$

k-tá složka

$A . \nabla \vec{r}\,\big|_k = (A . \nabla)\, \vec{r}\,\big|_k = \left(a_j \frac{\partial}{\partial x_j}\right) x_k =$

$= a_j \frac{\partial x_k}{\partial x_j} = a_j \delta_{kj} = a_k$

$\Downarrow$

$\underline{\underline{A . \nabla \vec{r} = A}}$

$\text{grad } \vec{r} = \overleftrightarrow{I}$

$\text{div } \vec{r} = n$ - dimenze

$\text{rot } \vec{r} = \emptyset$

OPERÁTORY 2

2. derivace

$\nabla \times \nabla = \emptyset$ $\qquad \nabla . \nabla = \nabla^2 = \Delta$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$\text{div grad } f = \nabla . (\nabla f) = (\nabla . \nabla) f = \Delta f$

$\text{rot grad } f = \nabla \times (\nabla f) = (\nabla \times \nabla) f = \emptyset$

$\text{div rot } A = \nabla . (\nabla \times A) = A . (\nabla \times \nabla) = \emptyset$

$\text{div grad } A = (\nabla . \nabla) A = \Delta A$ - vekt. pole

$\text{rot rot } A = \nabla \times (\nabla \times A) = \nabla(\nabla . A) - (\nabla . \nabla) A = \nabla(\nabla . A) - \Delta A$

$\Rightarrow \Delta A = \text{grad div } A - \text{rot rot } A$

IDENTITY

konstanta z hlediska derivací

$\text{grad } (fg) = \nabla (fg) = \nabla (f_c g) + \nabla (f g_c) = f \nabla g + g \nabla f$

$\text{grad } (f(g(r))) = \frac{df}{dg} \text{ grad } g$

Pozor! $(\nabla f) \times (\nabla g) \neq \emptyset$ jen pro $f = g$.

$\text{grad } (A . B) = \nabla (A . B) = \nabla (A_c . B) + \nabla (A . B_c)$

ale lze ukázat že

$A \times (\nabla \times B) = \nabla (A_c . B) - (A . \nabla) B \Rightarrow \nabla (A_c . B) = A \times (\nabla \times B) + (A . \nabla) B$

a podobně

$\nabla (B_c . A) = B \times (\nabla \times A) + (B . \nabla) A$

Po dosazení pak

$\text{grad } (A . B) = (A . \nabla) B + (B . \nabla) A + A \times \text{rot } B + B \times \text{rot } A$

$A \times \text{rot } A = A \times (\nabla \times A) = \nabla (A_c . A) - (A . \nabla) A$

$\nabla (A . A) = \nabla (A_c . A) + \nabla (A . A_c) \Rightarrow \nabla (A_c . A) = \frac{1}{2} \nabla (A . A)$

Po dosazení

$A \times \text{rot } A = \frac{1}{2} \text{ grad } (A . A) - (A . \nabla) A$

OPERÁTORY 3

$\text{div } (fA) = \nabla . (fA) = \nabla . (f_c A) + \nabla . (f A_c) =$

$= f \nabla . A + \nabla f . A_c = f \text{ div } A + A . \text{grad } f$

$\text{div } (A \times B) = \nabla . (A \times B) = \nabla . (A_c \times B) + \nabla . (A \times B_c) =$

$= -A . (\nabla \times B) + B . (\nabla \times A) = B . \text{rot } A - A . \text{rot } B$

$\text{rot } (fA) = \nabla \times (fA) = \nabla \times (f_c A) + \nabla \times (f A_c) =$

$= f (\nabla \times A) + \nabla f \times A_c = f (\nabla \times A) - A \times \nabla f =$

$= f \text{ rot } A - A \times \text{grad } f$

$\text{rot } (A \times B) = \nabla \times (A \times B) = \nabla \times (A_c \times B) + \nabla \times (A \times B_c)$

$\overset{\text{BAC-CAB}}{=} A (\nabla . B) - (A . \nabla) B + (B . \nabla) A - B (\nabla . A)$

$= (B . \nabla) A - (A . \nabla) B + A \text{ div } B - B \text{ div } A$

$\vec{F} = (F_x, F_y, F_z)$

2. Určete divergenci a rotaci následujících vektorových $F_x(x,y,z)$, $F_y(x,y,z)$, $F_z(x,y,z)$ polí: a) $\vec{F} \equiv (\underbrace{x+y}_{F_x = x+y},\ \underbrace{-x+y}_{F_y=-x+y},\ \underbrace{-2z}_{F_z=-2z})$; b) $\vec{F} \equiv (2y,\ 2x+3z,\ 3y)$; c) $\vec{F} \equiv (x^2-z^2,\ 2,\ 2xz)$ ($F_z = 2xz$). Je-li rot $\vec{F}$ = 0, najděte skalární pole takové, aby $\vec{F}$ = grad φ.

$\mathrm{div}\,\vec{F} =$

a) $\mathrm{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 1 + 1 - 2 = 0$

b) $\mathrm{div}\,\vec{F} = 2 + 0 + 0 = 2$

c) $\mathrm{div}\,\vec{F} = 2x + 0 + 2x = 4x$

1, 2, 3 x, y, z

$\vec{A} \times \vec{B} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$

$\nabla \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$

a) $\mathrm{rot}\,\vec{F} = (0-0,\ 0-0,\ -1-1) = (0, 0, -2)$

b) $\mathrm{rot}\,\vec{F} = (3-3,\ 0-0,\ 2-2) = (0,0,0)$

c) $\mathrm{rot}\,\vec{F} = (0,\ -2z-2z,\ 0) = (0, -4z, 0)$

$\vec{F} = \nabla f$ $\quad$ $\mathrm{rot}\,\vec{F} = 0$ $\quad$ prob) $\vec{F} = (2y,\ 2x+3z,\ 3y)$

$\frac{\partial f}{\partial x} = 2y \Rightarrow 2xy + C_x$

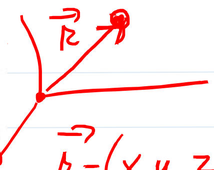
$\frac{\partial f}{\partial y} = 2x + 3z \Rightarrow 2xy + 3yz + C_y$

$\frac{\partial f}{\partial z} = 3y \Rightarrow 3yz + C_z$

$\underline{f = 2xy + 3yz + C}$

Určete gradient následujících skalárních polí ($\vec{r}$ je radiusvektor): a) r ✓, b) r^2, DÚ c) r^3, d) $\frac{1}{r}$ ✓ e) $\frac{1}{r^2}$, DÚ f) $\frac{1}{r^3}$, g) $\vec{c}\cdot\vec{r}$, h) $\frac{\vec{c}\cdot\vec{r}}{r}$, DÚ i) $\frac{\vec{c}\cdot\vec{r}}{r^2}$, j) $\frac{\vec{c}\cdot\vec{r}}{r^3}$. ✓

[$\frac{\vec{r}}{r}$, $2\vec{r}$, $3r\vec{r}$, $\frac{-\vec{r}}{r^3}$, $\frac{-2\vec{r}}{r^4}$, $\frac{-3\vec{r}}{r^5}$, $\vec{c}$, $\frac{r^2\vec{c}-(\vec{c}\cdot\vec{r})\vec{r}}{r^3}$, $\frac{r^2\vec{c}-2(\vec{c}\cdot\vec{r})\vec{r}}{r^4}$, $\frac{r^2\vec{c}-3(\vec{c}\cdot\vec{r})\vec{r}}{r^5}$; zde i v následujících příkladech řešení s podmínkou $r \neq 0$, pokud výrazy při $r \to 0$ neomezeně rostou.]



$\vec{c} = (c_x, c_y, c_z)$ konstantní vektor

$\vec{r} = (x, y, z)$ ✓

$r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$

$1/r = (x^2+y^2+z^2)^{-1/2}$

$r^{-m} = (x^2+y^2+z^2)^{-m/2}$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{\ }} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-x}{(x^2+y^2+z^2)^{3/2}} = -\frac{x}{r^3}$$

$$\frac{\partial}{\partial x} r^{-m} = \frac{-mx}{(x^2+y^2+z^2)^{\frac{m}{2}+1}} = \frac{-mx}{r^{m+2}} = -mx\, r^{-(m+2)}$$

$$\nabla r = \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{\vec{r}}{r}$$

$$\nabla \frac{1}{r} = \left(-\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3}\right) = -\frac{\vec{r}}{r^3}$$ – pole bodového náboje

$$\nabla \frac{1}{r^m} = \nabla r^{-m} = \left(-\frac{mx}{r^{m+2}}, -\frac{my}{r^{m+2}}, -\frac{mz}{r^{m+2}}\right) = -\frac{m\vec{r}}{r^{m+2}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\vec{r} = (x_1, x_2, x_3)$

$$\nabla(\vec{c}\cdot\vec{r}) = \nabla(c_x x + c_y y + c_z z) = (c_x, c_y, c_z) = \vec{c}$$

$$\nabla(\vec{r}\cdot\vec{c}) = \underbrace{\nabla\vec{r}}_{\overleftrightarrow{I}}\cdot\vec{c} = \overleftrightarrow{I}\cdot\vec{c} = \vec{c}$$

$$\overleftrightarrow{I} = I_{ij} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

$\delta_{ij} = 0$ pro $i \neq j$

$\delta_{ii} = 1$ $i = j$

$$\overleftrightarrow{I}\cdot\vec{c}\,\big|_i = \delta_{ij} c_j = c_i$$

$$\nabla\left(\frac{\vec{c}\cdot\vec{r}}{r^3}\right) = \nabla\left(\frac{c_x x + c_y y + c_z z}{(x^2+y^2+z^2)^{3/2}}\right) =$$

$$\boxed{\begin{aligned} f &= \vec{c}\cdot\vec{r} \\ g &= r^{-3} \end{aligned}} = (x^2+y^2+z^2)^{-3/2}$$

$$= \left(\frac{c_x}{r^3} - 3\frac{x}{r^5}(\vec{c}\cdot\vec{r}),\ \frac{c_y}{r^3} - 3\frac{y}{r^5}(\vec{c}\cdot\vec{r}),\ \frac{c_z}{r^3} - 3\frac{z}{r^5}(\vec{c}\cdot\vec{r})\right)$$

$$= \frac{\vec{c}}{r^3} - 3\frac{(\vec{c}\cdot\vec{r})}{r^5}\vec{r}$$

$$\nabla\left(\frac{\vec{c}\cdot\vec{r}}{r^3}\right) = (\vec{c}\cdot\vec{r})\underbrace{\nabla r^{-3}}_{-\frac{3\vec{r}}{r^5}} + r^{-3}\overbrace{\nabla(\vec{c}\cdot\vec{r})}^{\vec{c}} =$$

$$= \vec{c}\, r^{-3} - 3\frac{\vec{r}}{r^5}(\vec{c}\cdot\vec{r}) = \frac{\vec{c}}{r^3} - 3\frac{(\vec{c}\cdot\vec{r})}{r^5}$$

$$\nabla(fg) =$$

$$\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x},\ f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y},\ f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right) =$$

$$= f\underbrace{\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)}_{\nabla g} + g\underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}_{\nabla f} = f\nabla g + g\nabla f$$

$$\frac{\partial}{\partial x}(fg) = f\frac{\partial}{\partial x}g + g\frac{\partial f}{\partial x} = (\vec{c}\cdot\vec{r})\left(-\frac{3}{2}\right)\frac{2x}{(x^2+y^2+z^2)^{5/2}} +$$

$$+ r^{-3}c_x = (\vec{c}\cdot\vec{r})\frac{-3x}{r^5} + \frac{c_x}{r^3}$$

Určete divergenci a rotaci následujících vektorových polí:

a) $\vec{r}$, b) $\dfrac{\vec{r}}{r}$, c) $\dfrac{\vec{r}}{r^2}$, d) $\dfrac{\vec{r}}{r^3}$, e) $\dfrac{\vec{c}}{r}$. – $\dfrac{\vec{r}}{r^n}$ DÚ

[3, $\vec{0}$; $\dfrac{2}{r}$, $\vec{0}$; $\dfrac{1}{r^2}$, $\vec{0}$; 0, $\vec{0}$; $-\dfrac{\vec{c}\cdot\vec{r}}{r^3}$, $\dfrac{\vec{c}\times\vec{r}}{r^3}$. Všimněme si zejména příkladu d), který odpovídá Coulombově poli bodového náboje, umístěného v počátku souřadnic. Toto pole má všude kromě počátku div $\vec{F} = 0$ v souladu s Poissonovou rovnicí. Naproti tomu lze dokázat i opačně, že jediné pole, které vyhoví této podmínce je právě pole Coulombovo. Rozložíme-li totiž obecné pole $\vec{F}$ do mocninné řady se zápornými exponenty (pole neomezeně roste při $r \neq 0$) a najdeme divergenci obecného členu této řady:

$$\operatorname{div} \frac{\vec{r}}{r^\alpha} = \frac{\operatorname{div} \vec{r}}{r^\alpha} + \vec{r} \cdot \operatorname{grad} \frac{1}{r^\alpha} = \frac{3-\alpha}{r^\alpha} = 0$$

, zjistíme, že jediný nenulový člen této řady odpovídá $\alpha = 3$, tedy právě Coulombově poli.]

1.1.9. Poměr velikostí dvoubodových nábojů opačných znamének je n, vzdálenost obou nábojů je d. Dokažte, že povrch nulového potenciálu je kulová plocha. Vypočítejte poloměr R této plochy a vzdálenost jejího středu od jednoho z nábojů.