2. Show that a Markov chain $\{Z_1, \ldots, Z_n\}$ is a Markov random field with respect to the relation $i \sim j \Leftrightarrow |i-j| \leq 1$. Prove that the converse implication holds as follows: if $\{Z_1, \ldots, Z_n\}$ is a Markov random field with a probability density function satisfying $p(\boldsymbol{z}) > 0$ for all $\boldsymbol{z} = (z_1, \ldots, z_n)^T$ then it is a Markov chain.

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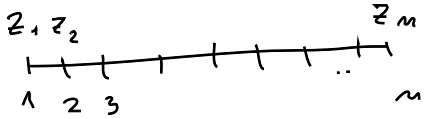

M. Chain: $p(z_i \mid z_{i-1}, \ldots, z_1) = p(z_i \mid z_{i-1})$

M.R.F.: $p(z_i \mid z_n, \ldots, z_{i+1}, z_{i-1}, \ldots, z_1) =$

$= p(z_i \mid z_{i+1}, z_{i-1})$

$L = \{1, \ldots, n\}$, $i \sim j \Leftrightarrow |i-j| = 1$

a) M.C. $\overset{?}{\Rightarrow}$ M.R.F.

$1 < i < n$: $\quad p(z_i \mid z_{-i}) = \dfrac{p(z_1, \ldots, z_n)}{p(z_{-i})} = (*)$

$p(z_{-i}) > 0$

$\hookrightarrow p(z_1, \ldots, z_n) = p(z_n \mid z_{n-1}, \ldots, z_1) \cdot p(z_{n-1} \mid z_{n-2}, \ldots, z_1) \cdot \ldots \cdot p(z_2 \mid z_1) \cdot p(z_1)$

$\overset{M.C.}{=} p(z_n \mid z_{n-1}) \cdot \ldots \cdot p(z_2 \mid z_1) \cdot p(z_1)$

$\hookrightarrow p(z_{-i}) = \int_S p(z_1, \ldots, z_n)\, \nu_i(dz_i)$ $\qquad [\nu_i = \nu]$

$$(*) = \frac{p(z_n \mid z_{n-1}) \cdot \ldots \cdot p(z_2 \mid z_1) \cdot p(z_1)}{\left(\int_S p(z_{i+1} \mid w_i)\, p(w_i \mid z_{i-1})\, \nu_i(dw_i)\right) p(z_n \mid z_{n-1}) \ldots p(z_{i+2} \mid z_{i+1}) \cdot p(z_{i-1} \mid z_{i-2}) \cdot \ldots \cdot p(z_1)}$$

$$= \frac{p(z_{i+1} \mid z_i)\, p(z_i \mid z_{i-1})}{\int_S p(z_{i+1} \mid w_i)\, p(w_i \mid z_{i-1})\, \nu_i(dw_i)} = (**)$$

$$p(z_i \mid z_{i+1}, z_{i-1}) = \frac{p(z_{i+1}, z_i, z_{i-1})}{\int_S p(z_{i+1}, w_i, z_{i-1})\, \nu_i(dw_i)} \overset{M.C.}{=}$$

$= p(z_{i+1} \mid z_i)\, p(z_i \mid z_{i-1}) \cdot p(z_{i-1})$

$$= \frac{p(z_{i+1} \mid z_i)\, p(z_i \mid z_{i-1})\, \cancel{p(z_{i-1})}}{\cancel{p(z_{i-1})} \int_S p(z_{i+1} \mid w_i)\, p(w_i \mid z_{i-1})\, \nu_i(dw_i)} = (**)$$

$\Rightarrow$ M.R.F. property holds with "$\sim$"

- for $i = 1$, $i = n$ similarly

b) assume $Z \sim$ M.R.F. on L with $i \sim j \Leftrightarrow |i-j| = 1$

We need to factorize joint densities: Hammersley, Clifford theorem

$$\pi(z) = \prod_{c \in \mathcal{C}} g_c(z_c) \overset{\text{for our ,,}\sim\text{``}}{=} g_\emptyset() \cdot \left(\prod_{i=1}^{n} g_i(z_i)\right)\left(\prod_{i=1}^{n-1} g_{i,i+1}(z_i, z_{i+1})\right)$$

$\hookrightarrow z_c = (z_i,\ i \in c)$