$0 \le a_{n+1} \le a_n$. Pak $\sum_{n=1}^{\infty} a_n$ K $\iff$ $\sum_{n=1}^{\infty} 2^n \cdot a_{2^n}$ K

Dk: Pro $k \in \mathbb{N}$

$$s_k = \sum_{j=1}^{k} a_j \quad \text{a} \quad A_k = \sum_{j=0}^{k} 2^j \cdot a_{2^j}$$

"$\Leftarrow$" Označme $A = \sum_{j=0}^{\infty} 2^j \cdot a_{2^j}$, pak $A \in \mathbb{R}$.

Nechť $m \in \mathbb{N}$ a nalezneme $k \in \mathbb{N}$, $m < 2^k$. Pak $A_{k-1} \le A$ a

$$s_m \le a_1 + \underbrace{(a_2 + a_3)}_{\le 2 \cdot a_2} + \underbrace{(a_4 + a_5 + a_6 + a_7)}_{\le 4 \cdot a_4} + \dots + \underbrace{(a_{2^{k-1}} + \dots + a_{2^k - 1})}_{\le 2^{k-1} \cdot a_{2^{k-1}}}$$

$$\le \sum_{j=0}^{k-1} 2^j \cdot a_{2^j} = A_{k-1} \le A.$$

Tedy s_m je shora omezená (a rostoucí) $\Rightarrow \exists \lim_{m \to \infty} s_m \in \mathbb{R}$.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ K

"$\Rightarrow$" označme $B = \sum_{n=1}^{\infty} a_n \in \mathbb{R}$. Zvolme $k \in \mathbb{N}$ a nalezneme $m \in \mathbb{N}$, aby $2^k \le m$. Pak $s_m \le B$ a platí

$$B \ge s_m \ge a_1 + \underbrace{a_2}_{\ge 1 \cdot a_2} + \underbrace{(a_3 + a_4)}_{\ge 2 \cdot a_4} + \underbrace{(a_5 + a_6 + a_7 + a_8)}_{\ge 4 \cdot a_8} + \dots + \underbrace{(a_{2^{k-1}+1} + \dots + a_{2^k})}_{\ge 2^{k-1} \cdot a_{2^k}}$$

$$\ge a_1 + \sum_{j=1}^{k} 2^{j-1} \cdot a_{2^j} = a_1 + \frac{1}{2} \sum_{j=1}^{k} 2^j \cdot a_{2^j} \ge \frac{1}{2} A_k \Rightarrow A_k \le 2 \cdot B$$

A_k shora omezená $\Rightarrow \exists \lim_{k \to \infty} A_k \in \mathbb{R}$

$\Rightarrow \sum_{n=1}^{\infty} 2^n \cdot a_{2^n}$ K $\square$

Ozn: $\sum_{n=1}^{\infty} \frac{1}{n}$ D $\quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$\sum_{n=1}^{\infty} \frac{1}{n^5} \in \mathbb{Q}$? $\in \mathbb{R} \setminus \mathbb{Q}$?