INTRODUCTORY BANKING

Seminar 1B – Financial Mathematics

Magda Pečená
Institute of Economic Studies, Faculty of Social Sciences,
Charles University in Prague, Czech Republic

18 and 25 February 2021

Financial Mathematics

NPV concept

Net present value (NPV) **is the present value of all future cash flows of a project or financial instrument (incl. the intitial investment (inicial cash outflow)).** Because the time-value of money dictates that money is worth more now than it is in the future, the value of a project is not simply the sum of all future cash flows. Those future cash flows must be discounted because the money earned in the future is worth less today.
In order to calculate NPV, we must discount each future cash flow in order to get the present value of each cash flow, and then we sum those present values associated with each time period.

NPV takes into account initial investment (i.e. minus *Cash flow in t0)*
PV – just the future cash inflows

$$PV = \frac{Cash\ flow\ in\ t1}{(1+r)} + \frac{Cash\ flow\ in\ t2}{(1+r)^2} + \frac{Cash\ flow\ in\ t3}{(1+r)^3} + \cdots + \frac{Cash\ flow\ in\ tn}{(1+r)^n}$$

or

$$PV = \sum_{t=1}^{n} \frac{Cash\ flow\ in\ t}{(1+r)^t}$$

NPV concept – Example - Project

Suppose you as the investor are looking at investing in a project (new oven) for your company that would increase the efficiency of producing bread (≈ increase the production). This new piece of oven costs $100,000 and has a four year life expectancy. Your analysts are projecting that the new oven will produce additional cash flows of $28,000 in Year 1, $36,000 in Year 2, $38,000 in Year 3 and $36,000 in Year 4. The rate of return of an alternative project is 6%. What is the net present value of your potential investment? Would you purchase the oven ?

Initial investment

				Discount factor		Present value of cash flows
	t0	-	100 000	1,00000		- 100 000,00

Additional cash flows

	t1	28 000	0,94340		26 415,09
	t2	36 000	0,89000		32 039,87
	t3	38 000	0,83962		31 905,53
	t4	36 000	0,79209		28 515,37

Discount rate **6%**

Present value of the project	**118 875,87**
Net present value of the project	**18 875,87**

Bond valuation

The valuation of a straight (vanilla) bond that pays constant annual coupons regularly and the principal at the end of maturity.

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} + \frac{M}{(1+r)^T}$$

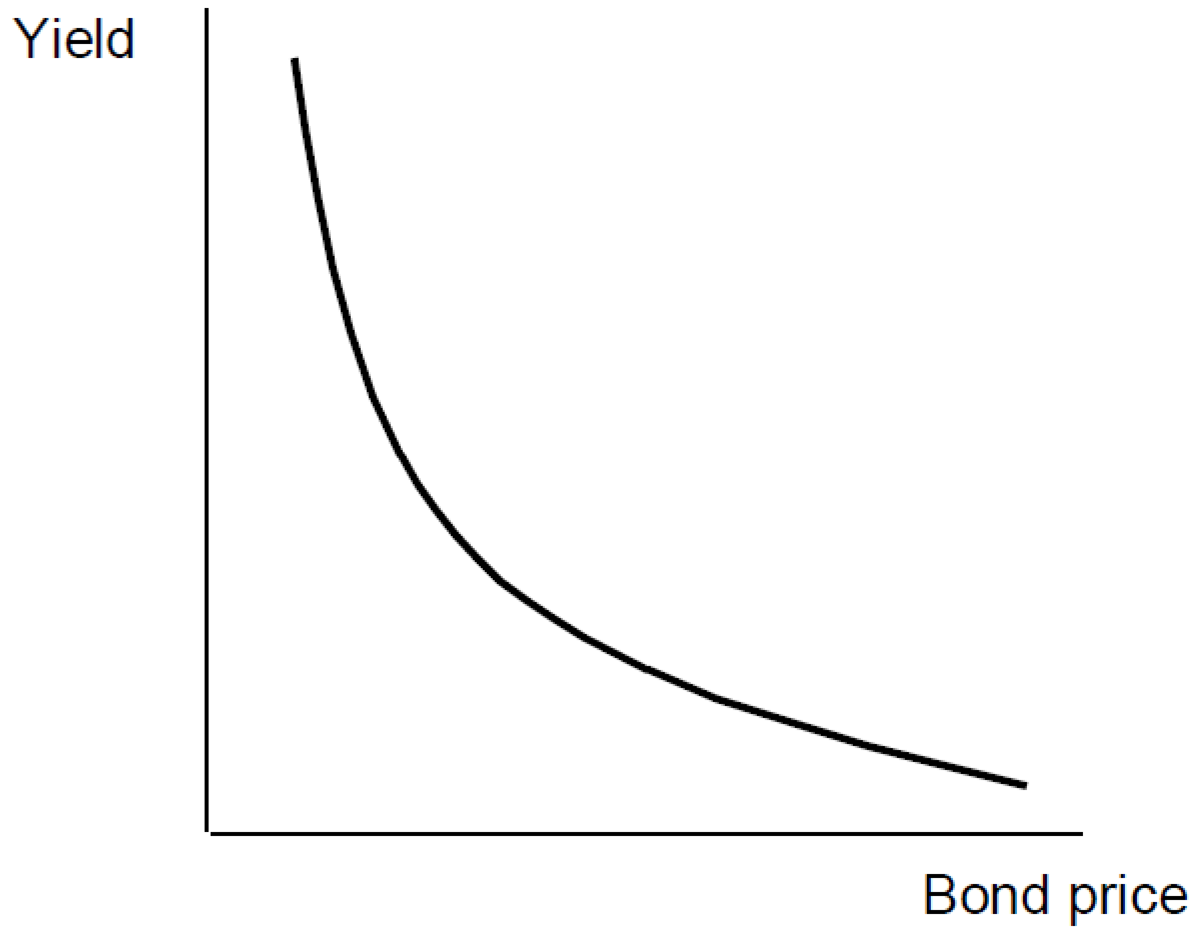
$$P_0 = \sum_{t=1}^{T} \frac{\frac{C_t}{m}}{\left(1+\frac{r}{m}\right)^{m\times t}} + \frac{M}{\left(1+\frac{r}{m}\right)^{m\times T}}$$

P_0 – market value of the bond
r – required rate of return
C_t – coupon at time t
M – face value of the bond
T – maturity

$$P_0 = M \times \left[\frac{c}{r} - \frac{c-r}{r\times(1+r)^n}\right]$$

P_0 – market value of the bond
M – face value of the bond
r – required rate of return in an interest period
c – coupon rate in an interest period (in %)
T – number of interest periods until the bond's maturity

The inverse (non-linear) relationship between bond prices and yield valuation



NPV concept – Example - Bond

Today, a bond was issued by ABC company rated A- with the following parameters: a CZK 1,000 principal, 5-year maturity and 5,5% coupon paid semiannually. Assume that a discount rate of 5,4% reflects the riskiness of an A- borrower. Calculate the expected issuig price of the bond.

discount rate (expected yield)	*5,40%*
discount rate/2	*2,70%*

Period		***Coupon payment (5,5%/2)***	***Principal payment***	***Discount factor***	***PV of the cash flow***
1	30.06.2021	27,5		0,974	26,777
2	31.12.2021	27,5		0,948	26,073
3	30.06.2022	27,5		0,923	25,388
4	31.12.2022	27,5		0,899	24,720
5	30.06.2023	27,5		0,875	24,070
6	31.12.2023	27,5		0,852	23,437
7	30.06.2024	27,5		0,830	22,821
8	31.12.2024	27,5		0,808	22,221
9	30.06.2025	27,5		0,787	21,637
10	31.12.2025	27,5	1000	0,766	787,186

Expected issuing price	**1004,331**

*Here period = m*t, where m = 2 and t starts at 0.5*

1. What is an anuity ? What is a perpetuity? Give an example of such a product.

Annuity

- An annuity is an instrument that generates the same sum of money for a certain period, the number of instalments is known and finite.
- Examples of an annuity are a mortgage loans or money paid each month to a retiree.

$$PV(I_0) = C \times \left[\frac{1}{r} - \frac{1}{r \times (1+r)^T} \right]$$

PV	– PV of instalments
C	– instalment
r	– interest rate
T	– maturity

…if instalments are paid at the end of the period

(if it is paid twice a year: r/2 instead of r and 2t instead of t)

Perpetuity

A perpetuity (or perpetual annuity) is an annuity that is payable for a period of time without any fixed end, i.e. its principal is not to be paid. (e.g. a consol bond, stock)

$$PV\ (I) = \frac{C}{(1+r)} + \frac{PV}{(1+r)}$$

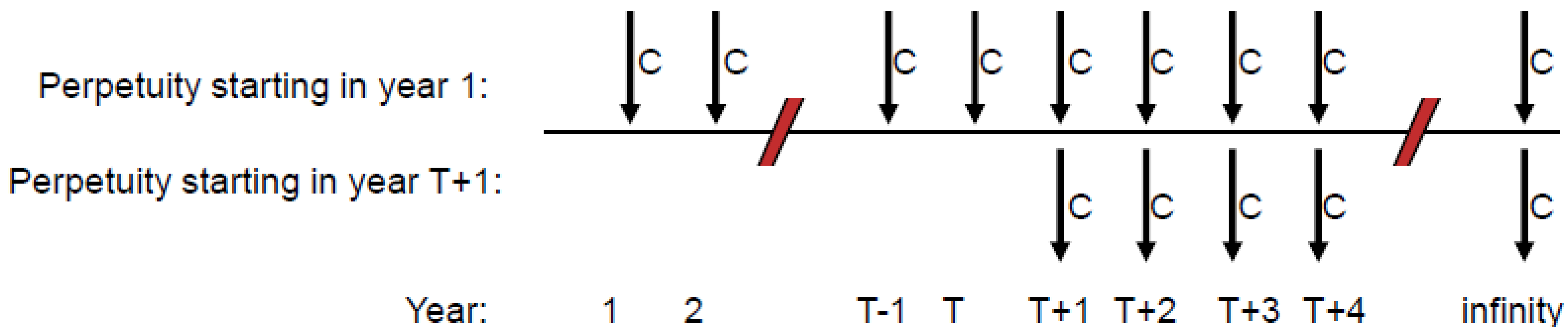
$$PV\ (I) = \frac{C}{r}$$

PV (I) – present value of instalments
C – coupon
r – interest rate

Perpetuity

Annuity (present value of annuity payments) is the difference between two perpetuities in time.

You start getting C in the first year but since year T+1, you have to give the money back.



Hence the present value of a perpetuity starting in year 1 and ending in year T is:

$$PV = \frac{C}{r} - \frac{C}{(1+r)^T r} = C\left(\frac{1}{r} - \frac{1}{(1+r)^T r}\right)$$

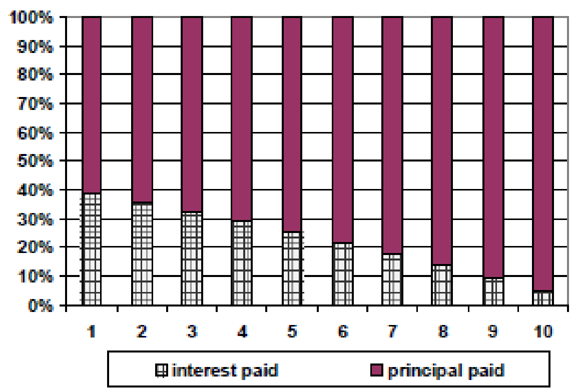
Task 1 (Annuity)

Calculate an instalment for a CZK 1 million loan with 10-year maturity, interest rate at 5% and yearly instalments.

Principal	1 000 000
Interest rate	5%
Maturity	10
Instalments per year	1
Annuity	?

$$C = \frac{PV(I_0)}{\left[\frac{1}{r} - \frac{1}{r \times (1+r)^T}\right]} = \frac{1,000,000}{\left[\frac{1}{5\%} - \frac{1}{5\% \times (1+5\%)^{10}}\right]} = 129,504.57$$

Year	Instalment	Interest paid	Principal paid	Principal left	Discount factor	PV of instalments
1	129,505	50,000	79,505	920,495	0.9524	123,338
2	129,505	46,025	83,480	837,016	0.9070	117,464
3	129,505	41,851	87,654	749,362	0.8638	111,871
4	129,505	37,468	92,036	657,325	0.8227	106,544
5	129,505	32,866	96,638	560,687	0.7835	101,470
6	129,505	28,034	101,470	459,217	0.7462	96,638
7	129,505	22,961	106,544	352,673	0.7107	92,036
8	129,505	17,634	111,871	240,802	0.6768	87,654
9	129,505	12,040	117,464	123,338	0.6446	83,480
10	129,505	6,167	123,338	0	0.6139	79,505
	1,295,046	**295,046**	**1,000,000**			**1,000,000**



Task 2 (mortgage)

You want to buy a flat worth CZK 2,000,000. A bank will provide you with a mortgage of up to 70% of the flat's purchase price. Calculate an instalment for this mortgage provided its 15-year maturity, an interest rate of 6% and monthly instalments.

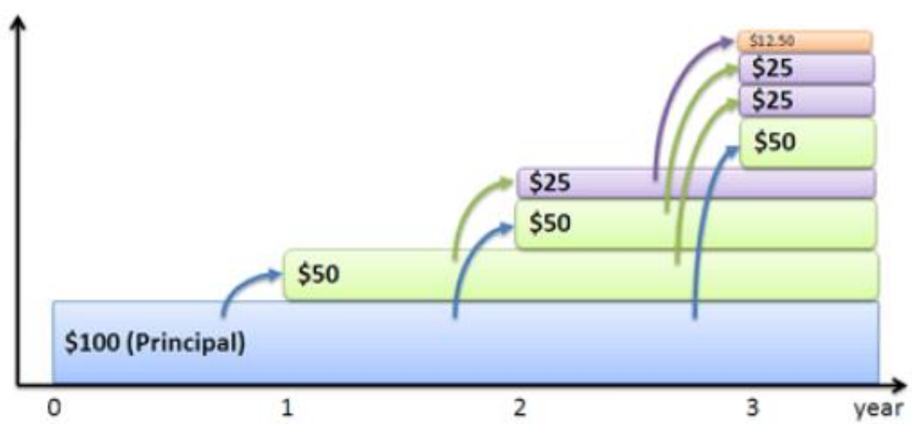
$$C = \frac{PV(I_0)}{\left[\frac{1}{r/m} - \frac{1}{r/m \times (1+r/m)^{m \times T}}\right]} = \frac{1400\,000}{\left[\frac{1}{6\%/12} - \frac{1}{6\%/12 \times (1+6\%/12)^{12 \times 15}}\right]} = 11{,}814.00$$

Types of interest

1. Simple interest — interest is calculated only on the principal, and no interest on interest occurs.

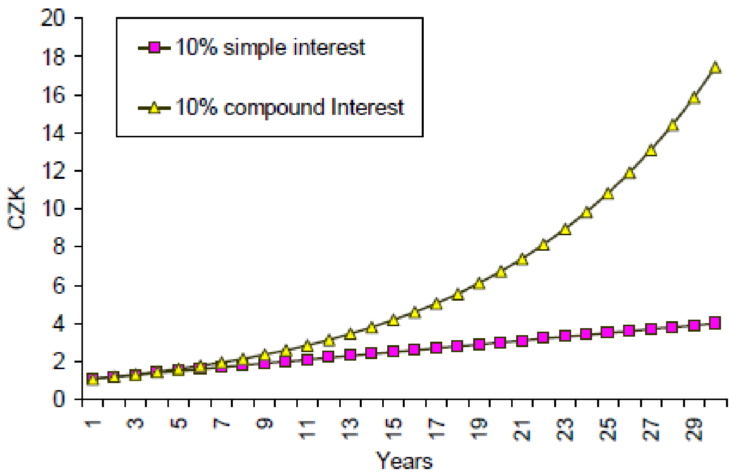
2. Compound interest (interest upon interest) — interest is payable not only on the principal but also on sums of interest as they accumulate.

Figure: Compound interest



Task 3 (Types of interest)

Show the difference between compound and simple interest based on a deposit yielding 10% and a maturity of 1, 5, 10, 30, 50 and 100 years.



	Simple interest			*Compound interest*			
Year	**Initial Amount**	**Interest**	**Final Amount**	**Initial Amount**	**Interest**	**Final Amount**	**Difference**
1	1	0.10	1.10	1	0.10	1.10	**0.00**
5	1.40	0.10	1.50	1.46	0.15	1.61	**0.11**
10	1.90	0.10	2.00	2.36	0.24	2.59	**0.59**
30	3.90	0.10	4.00	15.86	1.59	17.45	**13.45**
50	5.90	0.10	6.00	106.72	10.67	117.39	**111.39**
100	10.90	0.10	11.00	12,527.83	1,252.78	13,780.61	**13,769.61**

Frequency of interests

Show that the following formula holds for continuous interest:

$FV(I_0) = PV(I_0) \times e^{r \times t}$

1) $$FV = PV \times \left(1 + \frac{r}{m}\right)^{m \times T} = PV \times \left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{rT} = PV \times \left[\left(1 + \frac{1}{n}\right)^{n}\right]^{rT}$$ where $n = m / r$

2) We know that $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n} = e$

3) Hence $$FV = PV \times e^{rt}$$

Effective Interest Rate, Example

Considering a variety of interest frequencies, it would be difficult to compare these interest rates. Therefore a new variable has been introduced: effective interest rate, sometimes denoted as annual percentage rate or APR, corresponds to an annual nominal interest rate r_N compounded m-times a year.

$$r_{ef} = \left(1 + \frac{r_N}{m}\right)^m - 1$$

You are looking at different banks to find the best investment choice for deposit. You have CZK 1,000,000 and (for sake of simplicity) a one year horizon. What bank would you recommend:

Bank	Rate (p.a.)	Compounding	APR (Effective interest rate)
Unicreditbank	2,35%	1	2,350%
CSOB	2,33%	4	2,350%
Moneta Money Bank	2,35%	2	2,364%
Bank Creditas	2,34%	12	2,365%

Long-term bonds

A bond is a debt instrument with a maturity of over 1 year. We can distinguish many types of bonds, for example according to:

a) **Coupon:** Zero-coupon vs. coupon bonds

b) **Coupon-rate variability:** Floating-rate vs. fixed-rate bond

c) **Issuer:** Public sector vs. financial institutions vs. corporates

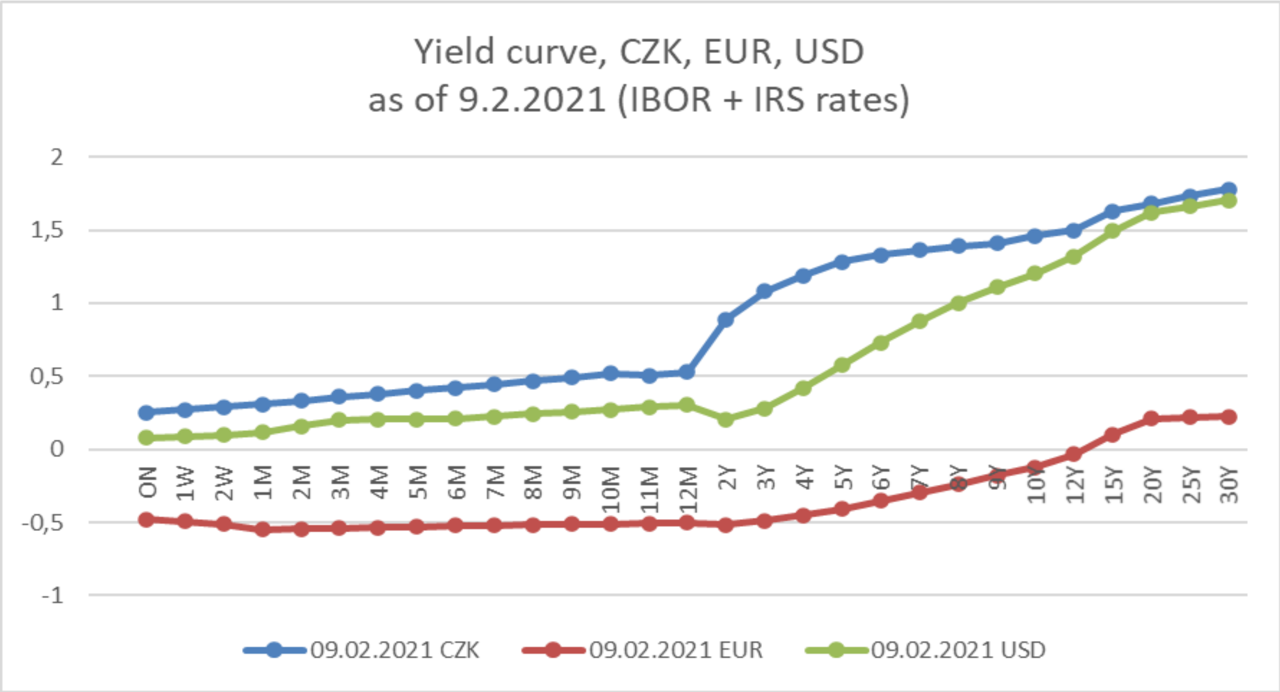
d) **Embedded options:** Callable vs. putable vs. convertible bonds

e) **Maturity:** Short-term vs. medium-term vs. long-term bonds.

2. What is a yield curve ? What is it used for ? Give examples.

Yield curve

The yield curve shows the relationship between maturity and yields (yield curve of risk free or risky assets)



Spot rate vs. Forward rate

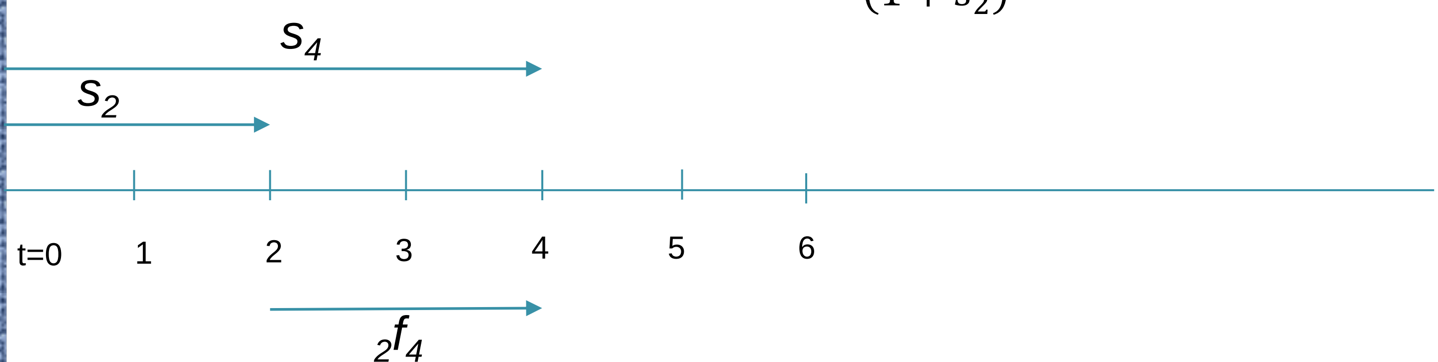
Yield curve – graphical representation of spot rates (yields) of bonds/instruments with different maturities (in p.a. terms).

It can be calculated based on spot rate on the further future date and a closer future date and the number of years until the further future date and closer future date.

Example **(all rates in p.a. terms)**

2-year interest rate ${}_2f_4$ related to the future period which starts at the time 2 and ends at time 4.

$$(1 + {}_2f_4)^2 = \frac{(1 + s_4)^4}{(1 + s_2)^2}$$

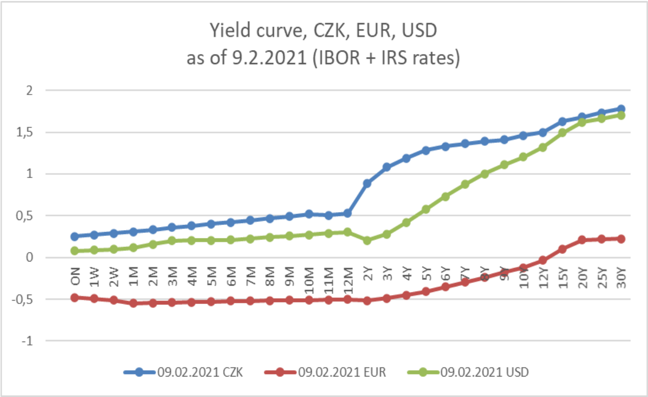


Financial Mathematics
Spot rate vs. Forward rate

IBOR+IRS	09.02.2021		
Tenor	CZK	EUR	USD
ON	0,25	-0,48	0,08
1W	0,27	-0,49	0,09
2W	0,29	-0,51	0,10
1M	0,31	-0,55	0,12
2M	0,33	-0,55	0,16
3M	0,36	-0,54	0,20
4M	0,38	-0,53	0,20
5M	0,40	-0,53	0,21
6M	0,42	-0,52	0,21
7M	0,44	-0,52	0,22
8M	0,47	-0,52	0,24
9M	0,49	-0,51	0,26
10M	0,52	-0,51	0,27
11M	0,50	-0,51	0,29
12M	0,53	-0,50	0,31
2Y	0,89	-0,52	0,20
3Y	1,08	-0,49	0,28
4Y	1,19	-0,45	0,42
5Y	1,28	-0,41	0,58
6Y	1,33	-0,35	0,73
7Y	1,36	-0,30	0,87
8Y	1,39	-0,24	1,00
9Y	1,41	-0,18	1,11
10Y	1,46	-0,12	1,20
12Y	1,50	-0,03	1,32
15Y	1,63	0,10	1,49
20Y	1,68	0,21	1,62
25Y	1,73	0,22	1,66
30Y	1,78	0,22	1,70

Calculate CZK, EUR and USD forward rate $_2f_4$

from the yield curve:



Estimates ?

CZK

EUR

USD

Yield to maturity

The yield to maturity (YTM) is an average return paid to an investor if he or she holds a bond until its maturity. It is hard to compute without software, so "hand" computing is possible either by iterations (a trial and error method) or approximation (e.g. the Hawawini-Vory's approximate yield to maturity, or AYTM)

If the bond currently trading at 101,334 has exactly two years to maturity and pays 4 % p.a. coupon, the yield to maturity is closest to:

a) 3.3 %

b) 4.2 %

c) 2.9 %

d) 4.1 %

(by trial and error…but is there a yield you can exclude immediately, why?)

See the PV of a bond formula on slide 4

Accrued interest (1/4)

Accrued interest is a part of the coupon that compensates the Buyer (or the Seller) for the non-obtaining of the accrual part of the coupon.

$$P_D = P_C \pm AI$$

P_D – dirty price
P_C – clear (market) price of the bond
AI – accrued interest

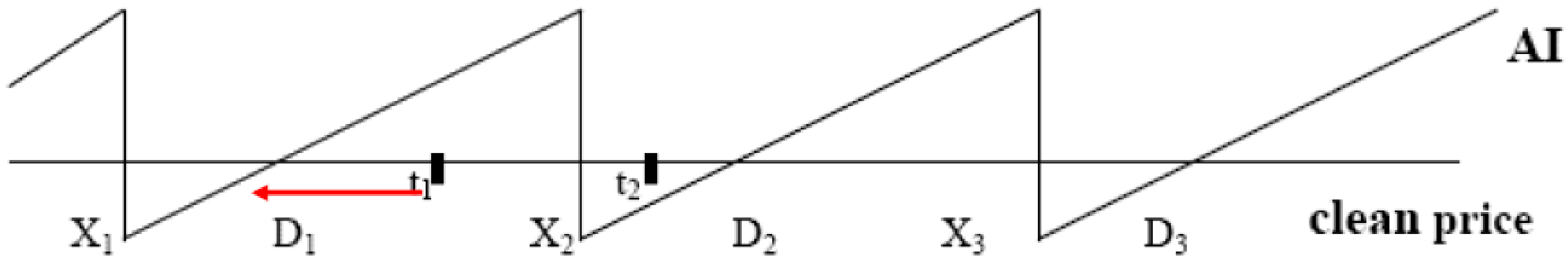
Accrued interest (2/4)

The equation above shows that a bond's dirty price is equal to a bond's price adjusted by AI, which can be both positive and negative based on the date of a bond's sale.

When calculating AI, we should know the ex-dividend day, which is decisive for a coupon payoff for an investor. Whoever owns the bond on that day will receive a coupon.

However, the coupon is to be paid on a dividend day, which usually follows 3–4 days after the ex-dividend day. Two different dates of the sale of the bond are shown in the following figures.

Accrued interest (3/4)

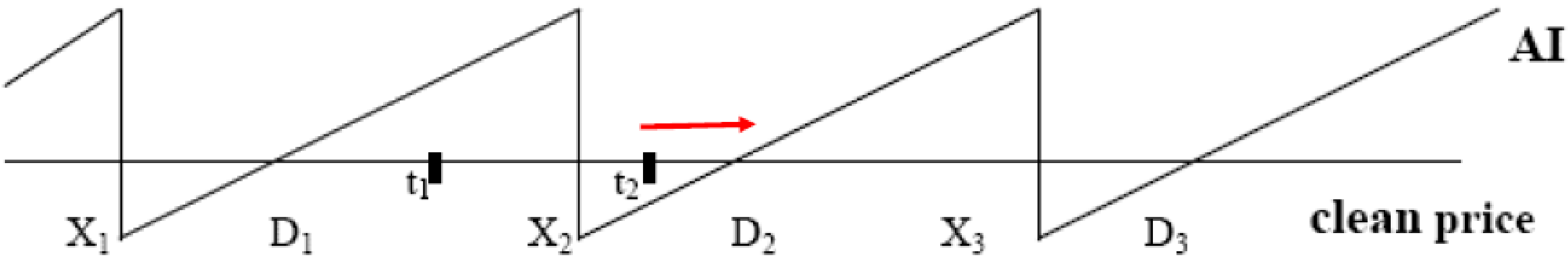


If the deal is done at time t_1, the buyer is to be compensated for holding a bond in period (D_1, t_1)…buyer pays more, i.e. $P_D = P_C + AI$

$$AI = \frac{t_1 - D_1}{360} \times C$$

AI	– accrued interest
t_1	– the date of a bond's sale
D_1	– a dividend day
X_1	– an ex-dividend day
C	– annual coupon

Accrued interest (4/4)



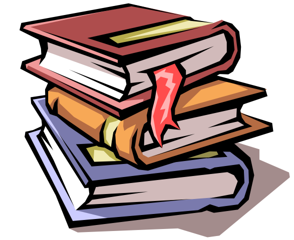
If the deal is done at time t_2, the buyer is to be compensated for holding a bond in period (t_2, D_2)…buyer pays less, i.e. $P_D = P_C - AI$

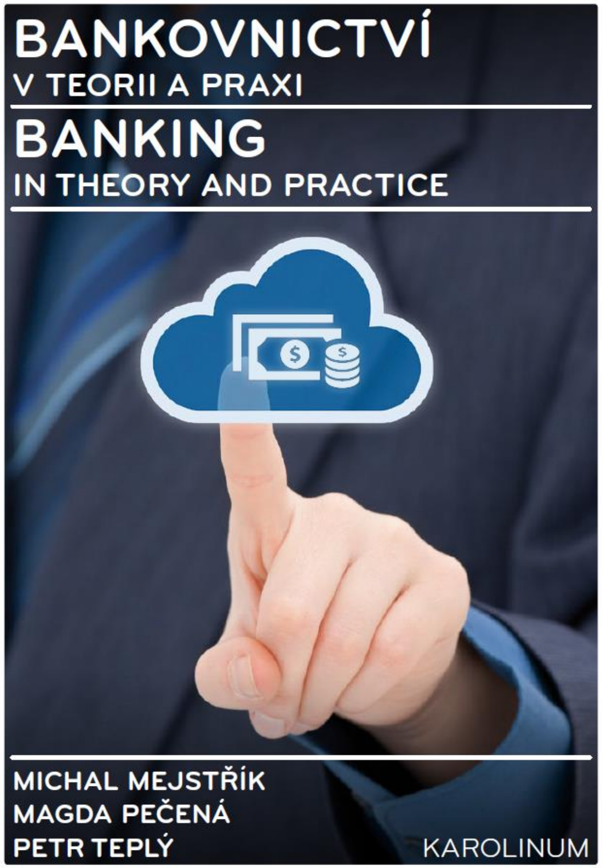
$$AI = \frac{D_2 - t_2}{360} \times C$$

AI	– accrued interest
t_2	– the date of a bond's sale
D_2	– a dividend day
X_2	– an ex-dividend day
C	– annual coupon

Financial Mathematics

Sources





Brealey, R. A., Myers, S. C.: Principles of Corporate Finance, 8th edition, McGraw-Hill, 2006

Capinski, M., Zastawiak M.: Mathematics for Finance, Springer, 2004

Cipra, T.: Matematika cenných papírů, HZ Praha, 2000

Damodaran, A.: Investment Valuation: Tools and Techniques for Determining the Value of Any Asset, John Wiley & Sons, 2002

Dvořák, P., Radová, J., Málek, J.: Finanční matematika pro každého, 6. vydání, Grada, 2008

Jílek, J.: Kapitálový a derivátový trh, Bankovní institut, 1997

Kislingerová, E. et al.: Manažerské finance, 1. vydání, C. H. Beck, 2004

Reilly, F., Brown, K.: Investment Analysis and Portfolio Management, 6th ed, Dryden, New York, 2000

Radová, J. et al.: Finanční matematika pro každého – příklady, 1. vydání, Grada, 2008

Sinkey, J. F.: Commercial Bank Management in the Financial Services Industry, 6th edition, Prentice Hall, 2002

www.csob.cz

www.damodaran.com

Thank you for your attention.