6. 1.

a) ? $\cos(0,1)$

$f(x) = \cos(x) \quad \rightsquigarrow \quad T_n^{f,a}(x)$

$\longrightarrow a = 0 \; ; \; x = 0{,}1 \; ; \; x - a = 0{,}1$

$$|f(x) - T_n^{f,a}(x)| \underset{\text{Lagrange}}{=} \left| \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-a)^{n+1} \right|$$

$$\leq \frac{1}{(n+1)!} (0{,}1)^{n+1}$$

Řešme: $\frac{1}{(n+1)!}(0{,}1)^{n+1} < 10^{-4}$ $\longrightarrow$ platí pro $n = 3$

? platí nerovnost pro 2 ?

$$\Rightarrow \cos(0{,}1) \doteq T_3^{\cos,0}(0{,}1) = \left(1 - \frac{x^2}{2}\right)\Big|_{x=0{,}1} = 1 - \frac{1}{200}$$

$$= \frac{199}{200}$$

Závěr: $\left|\cos(0{,}1) - \frac{199}{200}\right| < 10^{-4}$

b) $\lg(1{,}01)$

$$f(x) = \lg(1+x) \;;\; a = 0 \;;\; x = 0{,}01$$

$$\left|\lg(1+x) - T_n^{f,0}(x)\right| \le \frac{1}{(n+1)!}\left|f^{(n+1)}(\xi)\,\underbrace{(x-a)^{n+1}}_{(0{,}01)^{n+1}}\right|$$

$$\xi \in (a, x)$$

Hypo: Stačí $n = 1$

$$f'(x) = \frac{1}{1+x} \;;\; f''(x) = -\frac{1}{(1+x)^2}$$

Pro $x > 0$: $|f'(x)| < 1$, $|f''(x)| < 1$

Pro $n = 1$: $\cancel{|\lg(x) - T_1^{f,0}(}$

$$\left|\lg(1{,}01) - \underbrace{T_1^{f,0}(0{,}01)}_{=\frac{0{,}01}{1} = 0{,}01}\right| \le \frac{1}{2}(0{,}01)^2 = \frac{1}{2}\cdot 10^{-4}$$

1. $e^x = 1 + x + \frac{x^2}{2} + o(x^2)$ $\qquad$ $\underline{x \to 0}$

$\sin x = x - \frac{x^3}{6} + o(x^3)$

$e^x \sin x = x + x^2 + \frac{x^3}{2} \cancel{+} - \frac{x^3}{6}$

$+ x\, o(x^2) - \frac{x^4}{6} - \frac{x^5}{12} - \frac{x^3}{6} \cdot o(x^2) + o(x^3)\left(1 + x + \frac{x^2}{2} + o(x^2)\right)$

$= x + x^2 + \frac{x^3}{3} + o(x^3)$, pour $x \to 0$.

$$\lim_{x \to 0} \frac{e^x \sin x - x(1+x)}{x^3} = \lim_{x \to 0} \frac{\frac{x^3}{3} + o(x^3)}{x^3} = \frac{1}{3}$$

6. $(1+y)^p = \sum_{k=0}^{N} \binom{p}{k} y^k + o(y^N)$, $y \to 0$.

$\quad \hookrightarrow \frac{p(p-1) \quad (p-k+1)}{k(k-1) \dots 1}$; $p > 0$

$\sqrt{1 - 2x + x^3} = \left(1 + (-2x + x^3)\right)^{1/2} \Rightarrow p = \frac{1}{2}$

$= 1 + p(-2x + x^3) + \frac{p(p-1)}{2}(-2x + x^3)^2 + \frac{p(p-1)(p-2)}{3!}(-2x + x^3)^3$

$+ \underbrace{o\left((-2x + x^3)^3\right)}_{o(x^3)}$

$= 1 + p(-2x + x^3) + \frac{p(p-1)}{2}(4x^2) + \frac{p(p-1)(p-2)}{6}(-8x^3) + o(x^3)$

$= 1 - x - \frac{1}{2}x^2 + x^3\left(\frac{1}{2} - \frac{8}{6}\left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right)\right) + o(x^3)$

6.

$$\sqrt[3]{1+(-3x+x^2)} = 1 + \frac{1}{3}(-3x+x^2) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}(-3x+x^2)^2 +$$

$$+ \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}(-3x+x^2)^3 + o(x^3)$$

$$= 1 + \frac{1}{3}(-3x+x^2) - \frac{1}{9}(9x^2-6x^3) + \frac{10}{6}(-x^3) + o(x^3)$$

$$= 1 - x - \frac{2}{3}x^2 + x^3\left(+\frac{6}{9} - \frac{10}{6}\right) + o(x^3)$$

$$\lim_{x\to 0} \frac{\sqrt{1-2x+x^3} - \sqrt[3]{1-3x+x^2} - \frac{x^2}{6}}{x^3} =$$

$$\lim_{x\to 0} \frac{\cancel{1}-\cancel{x}-\frac{x^2}{2} + o(x^3) - \left(\cancel{1}-\cancel{x}-\frac{2}{3}x^2 - x^3\right) - \frac{x^2}{6}}{x^3} =$$

$$= \lim_{x\to 0} \frac{x^3 + o(x^3)}{x^3} = 1$$