Cvičení nmma 101 – 5.1.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} := \lim_{N \to +\infty} \sum_{n=0}^{N} \frac{x^n}{n!} = \lim_{N \to +\infty} \left(\frac{x^0}{0!} + \frac{x^1}{1!} + \ldots + \frac{x^N}{N!} \right)$$

$$\Rightarrow \quad T_N^{\exp, 0}(x) = \sum_{n=0}^{N} \frac{x^n}{n!}$$

$$\Rightarrow \quad \exp(x) = \sum_{n=0}^{N} \frac{x^n}{n!} + o(x^N), \text{ pro } x \to 0.$$

tj. $\lim_{x \to 0} \left| \exp x - \sum_{n=0}^{N} \frac{x^n}{n!} \right| = 0$

Při: $$\lim_{x \to 0} \frac{\exp(x) - 1 - x}{x^2} = \lim_{x \to 0} \frac{\overbrace{1 + x + \frac{x^2}{2} + o(x^2)}^{\exp(x)} - 1 - x}{x^2}$$

$$= \lim_{x \to 0} \frac{1}{2} + \frac{o(x^2)}{x^2} = \frac{1}{2}$$

4) $$\lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \lim_{x \to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) - 1 + \frac{x^2}{2}}{x^4}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)$$

$$= \lim_{x \to 0} \left(\frac{1}{24} + \frac{o(x^5)}{x^5} \cdot x \right) = \frac{1}{24}$$

2. $f(x) = x \lg x$

$\overset{?}{\cdot}$ $T_3^{f,1}(x) = f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2} + f'''(1)\frac{(x-1)^3}{6}$

$f'(x) = \lg x + 1$; $f''(x) = \frac{1}{x}$; $f'''(x) = -\frac{1}{x^2}$

$f(1) = 0$; $f'(1) = 1$; $f''(1) = 1$; $f'''(1) = -1$

$\Rightarrow T_3^{f,1}(x) = x - 1 + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6}$

---

1. $f(x) = \operatorname{tg} x = \frac{\sin x}{\cos x}$

$$x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^6) : 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) = x + \frac{x^3}{3}$$

$$-\left(x - \frac{x^3}{2} + \frac{x^5}{24} + o(x^6)\right)$$

$$\frac{x^3}{3} - \frac{4x^5}{120} + o(x^6)$$

$$-\left(\frac{x^3}{3} - \frac{x^5}{6} + o(x^6)\right)$$

$$\frac{2}{15}x^5 + o(x^6)$$

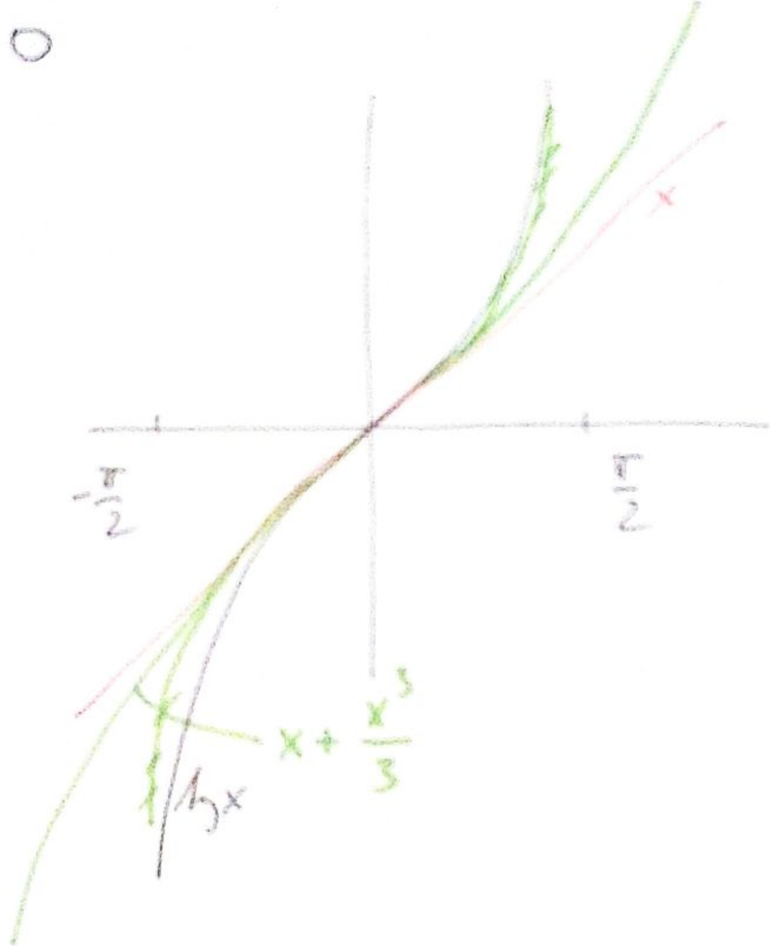
$\frac{16}{120} = \frac{4}{30} = \frac{2}{15}$

$$\operatorname{tg} x = x + \frac{x^3}{3} + \underbrace{\frac{\frac{2}{15}x^5 + o(x^6)}{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)}}_{\overset{?}{=} o(x^4) \text{ per } x \to 0.}$$

$$\lim_{x\to 0} \frac{1}{x^4}\,\frac{\frac{2}{15}x^5+o(x^6)}{1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5)} = \lim_{x\to 0} \underbrace{x}_{\to 0}\cdot\frac{\frac{2}{15}+\overbrace{\frac{o(x^6)}{x^6}}^{\to 0}\cdot x}{\underbrace{1-\frac{x^2}{2}+\frac{x^4}{24}+\underbrace{\frac{o(x^5)}{x^5}}_{\to 0}\cdot x^5}_{\to 1}}$$

$$\overset{AL}{=} 0$$

$$\Rightarrow \quad T_4^{tg,0}(x) = x + \frac{x^3}{3}$$



3) $f(x) = \cos(\sin x)$, $k=5$, $x_0 = 0$

$$\cos(y) = 1 - \frac{y^2}{2} + \frac{y^4}{24} + o(y^5)$$

$$\sin(x) = x - \frac{x^3}{6} + o(x^4)$$

$$f(x) = 1 - \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^4)\right)^2 + \frac{1}{24}\left(x - \frac{x^3}{6} + o(x^4)\right)^4 + o\left(\left(x - \frac{x^3}{6} + o(x^4)\right)^5\right) =$$

$$= 1 - \frac{x^2}{2} + x^4\left(-\frac{1}{6}\cdot 2\left(-\frac{1}{2}\right) + \frac{1}{24}\right) + o(x^5) + o\left(\left(x - \frac{x^3}{6} + o(x^4)\right.\right.$$

$$f(x) = 1 - \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^5), \qquad x \to 0$$

$$\lim_{x\to 0} \underbrace{\frac{o\left(\left(x - \frac{x^3}{6} + o(x^5)\right)^5\right)}{\left(x - \frac{x^3}{6} + o(x^5)\right)^5}}_{\Rightarrow 0} \cdot \frac{\left(x - \frac{x^3}{6} + o(x^5)\right)^5}{x^5} \overset{AL}{=}$$

$$0 \cdot \underbrace{\lim_{x\to 0}\left(1 - \frac{x^2}{6} + \frac{o(x^5)}{x^4}\cdot x^3\right)^5}_{=1} = 0.$$

---

5) $$\sin x - x = x - \frac{x^3}{6} + o(x^5) - x = -\frac{x^3}{6} + o(x^5)$$

$$\exp(x^3) = 1 + x^3 + o(x^3)$$

$$e^y = 1 + y + o(y)$$

$$\lim_{x\to 0} \frac{e^{x^3} - 1}{\sin x - x} = \lim_{x\to 0} \frac{1 + x^3 + o(x^3) - 1}{-\frac{x^3}{6} + o(x^5)} =$$

$$\lim_{x\to 0} \frac{x^3\left(1 + \frac{o(x^3)}{x^3}\right)}{x^3\left(-\frac{1}{6} + \underbrace{\frac{o(x^5)}{x^5}}_{\to 0}\cdot x\right)} = -6$$

6)

$$\lim_{n\to+\infty} n^4\left(\cos\frac{1}{n} - e^{-\frac{1}{2n^2}}\right) = -\frac{1}{12}$$

$x \sim \frac{1}{n}$

$$\lim_{x\to 0+} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4} =$$

+ Heine $\left\{\frac{1}{n}\right\}_{n=1}^{+\infty}$

$$\lim_{x\to 0+} \frac{\cancel{1} - \cancel{\frac{x^2}{2}} + \frac{x^4}{24} + o(x^4) - \left(\cancel{1} + \cancel{\left(-\frac{x^2}{2}\right)} + \frac{x^4}{4}\cdot\frac{1}{2} + o(x^4)\right)}{x^4}$$

$$= \lim_{x\to 0+} \frac{\cancel{x^4}}{\cancel{x^4}}\left(\frac{1}{24} + \underbrace{\frac{o(x^4)}{x^4}}_{\to 0} - \frac{1}{8} - \underbrace{\frac{o(x^4)}{x^4}}_{\to}\right) = -\frac{1}{12}$$