$\exists f^{(n)}(a)$ $\quad T_n^{f,a}(x) = f(a) + f'(a)\cdot(x-a) + \dots + \frac{1}{n!}\cdot f^{(n)}(a)\cdot(x-a)^n$ $\quad$ 25-1

Věta $f^{(n)}(a) \in \mathbb{R}$ $\quad \lim\limits_{x\to a} \frac{f(x)-P(x)}{(x-a)^n} = 0 \iff P(x) = T_n^{f,a}(x).$

$\operatorname{st} P \le n$

Věta (Taylor) Nechť funkce f má vlastní $(n+1)$-ní derivaci v intervalu $[a,x]$ a nechť φ je spojitá funkce v $[a,x]$ a má vlastní nenulovou derivaci v (a,x). Pak existuje $\xi \in (a,x)$ tak, že

$$f(x) - T_n^{f,a}(x) = \frac{1}{n!}\cdot\frac{\varphi(x)-\varphi(a)}{\varphi'(\xi)}\cdot f^{(n+1)}(\xi)\cdot(x-\xi)^n.$$

Důsledek $\exists \xi_1 \in (a,x)$

$$R_n^{f,a}(x) = f(x) - T_n^{f,a}(x) = \frac{1}{(n+1)!}\cdot f^{(n+1)}(\xi_1)\cdot(x-a)^{n+1}.$$

Věta (Cauchy) 4.9 $\quad f,g$ spojité na $[a,b]$ $\exists f'$ na (a,b) $\exists g' \neq 0$ vlastní na (a,b)

Pak $\exists \xi \in (a,b)$ tak, že $\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$

25 - 2

Dk: Definujme pro $t \in [a, x]$:

$$F(t) = f(x) - \left[f(t) + \frac{f'(t)}{1!}\cdot(x-t) + \frac{f''(t)}{2!}\cdot(x-t)^2 + \dots + \frac{f^{(n)}(t)}{n!}\cdot(x-t)^n \right].$$

Platí
- F je spojitá na $[a, x]$ ($\exists f^{(n+1)}$ vlastní $\Rightarrow f^{(n)}$ spojitá)
- $F(x) = 0$
- $F(a) = f(x) - T_n^{f,a}(x)$
- $\exists F'$ na (a, x)

- φ spojitá na $[a, x]$
- φ' vlastní a $\neq 0$ na (a, x)

Podle Cauchyovy věty o střední hodnotě V 4.9. $\exists \xi \in (a, x)$

$$\frac{0 - (f(x) - T_n^{f,a}(x))}{\varphi(x) - \varphi(a)} = \frac{F(x) - F(a)}{\varphi(x) - \varphi(a)} = \frac{F'(\xi)}{\varphi'(\xi)}.$$

Nyní

$$F'(\xi) = 0 - \Big[\cancel{f'(t)} + \cancel{f'(t)(-1)} + \cancel{f''(t)(x-t)} + \cancel{f''(t)\frac{2(x-t)}{2!}}(-1) + f'''(t)\frac{(x-t)^2}{2!} + \cancel{\dots +} f^{(n)}(t)\cdot\cancel{\frac{n\cdot(x-t)^{n-1}}{n!}}\cdot(-1) + f^{(n+1)}(t)\cdot\frac{(x-t)^n}{n!} \Big]_{t=\xi}$$

$$= -f^{(n+1)}(\xi)\cdot\frac{(x-\xi)^n}{n!}$$

Tedy

$$f(x) - T_n^{f,a}(x) = -F'(\xi)\cdot\frac{\varphi(x) - \varphi(a)}{\varphi'(\xi)} = f^{(n+1)}(\xi)\cdot\frac{(x-\xi)^n}{n!}\cdot\frac{\varphi(x) - \varphi(a)}{\varphi'(\xi)}$$

$\square$

$$R_n^{f,a}(x) = f(x) - T_n^{f,a}(x) = \frac{1}{(n+1)!} \cdot f^{(n+1)}(\xi) \cdot (x-a)^{n+1} \quad \xi \in (a,x)$$ 25 - 3

Taylorovy řady elementárních funkcí

e^x $\quad T_n^{e^x,0}$ $\quad (e^x)' = e^x$ $\quad (e^x)'' = e^x$ $\quad e^0 = 1$

$$T_n^{e^x,0} = 1 + 1\cdot(x-0) + \dots + \frac{1}{n!}\cdot 1\cdot(x-0)^n = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!}$$

x pevné $\quad e^x - T_n^{e^x,0}(x) = \frac{1}{(n+1)!}\cdot e^\xi \cdot x^{n+1}$ $\quad \exists \xi \in [0,x]$

(resp $\exists \xi \in [x,0]$ pro $x<0$)

$$\left|\frac{1}{(n+1)!}\cdot e^\xi \cdot x^{n+1}\right| \leq \frac{1}{(n+1)!}\cdot e^{|x|}\cdot |x|^{n+1} \xrightarrow{n\to\infty} 0$$

$$\lim_{n\to\infty} e^x - T_n^{e^x,0}(x) = 0 \implies e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$\sin x$, $\quad T_n^{\sin x,0}$ $\quad f = \sin x$ $\quad f' = \cos x$ $\quad f'' = -\sin x$, $\quad f''' = -\cos x$

$\sin 0 = 0$, $\cos 0 = 1$

$$T_n^{\sin x,0} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

x pevné $\quad \left|\sin x - T_n^{\sin x,0}(x)\right| = \left|\frac{1}{(n+1)!}\cdot f^{(n+1)}(\xi)\cdot x^{n+1}\right| \leq \frac{|x|^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$$

analogicky $\quad T_n^{\cos x,0} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

$\log(1+x) = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n}{n} \qquad x \in (-1,1)$

$(\log(1+x))' = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n \overset{\text{"}\int\text{"}}{\Longrightarrow} \log(1+x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1}$

$(\arctan x)' = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n \overset{\text{"}\int\text{"}}{\Longrightarrow} \arctan x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$

$x \in (-1,1)$

$\boxed{\text{Definice}}$ Nechť f, g jsou funkce a $a \in \mathbb{R}^*$. Řekneme, že funkce f je v bodě a malé o od g, jestliže platí $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$. Píšeme $f(x) = o(g(x))$ pro $x \to a$.

Např. příklady: a) $\lim_{x \to 0} \frac{\sin x - (x - \frac{x^3}{3!})}{x^3} = 0$ Podle V 4.18.

$\sin x = T_3^{\sin x, 0}$

Tedy $\sin x = x - \frac{x^3}{3!} + o(x^3)$. Analogicky $e^x = 1 + x + \frac{x^2}{2!} + o(x^2)$.

b) platí (i) $x^k \cdot o(x^l) = o(x^{k+l})$ (ii) $o(x^k) \cdot o(x^l) = o(x^k \cdot x^l)$

$k, l \in \mathbb{N}$ (iii) $o(x^k + o(x^l)) = o(x^k)$ pro $k < l$

(i) $\lim_{x \to 0} \frac{x^k \cdot o(x^l)}{x^{k+l}} = \lim_{x \to 0} \frac{o(x^l)}{x^l} = 0$ ✓

(ii) $\lim_{x \to 0} \frac{o(x^k) \cdot o(x^l)}{x^{k+l}} = \lim \frac{o(x^k)}{x^k} \cdot \lim \frac{o(x^l)}{x^l} = 0 \cdot 0$

25-5

Příklady: a) $\lim_{x\to 0} \frac{x-\sin x}{x^3} =$ Víme $\sin x = x - \frac{x^3}{3!} + o(x^3)$

$= \lim_{x\to 0} \frac{x-\left(x-\frac{x^3}{3!}+o(x^3)\right)}{x^3} \overset{?}{=} \lim_{x\to 0} \frac{\frac{x^3}{3!}}{x^3} - \lim_{x\to 0} \frac{o(x^3)}{x^3} = \frac{1}{6} - 0$

b) Nalezněte $a, b \in \mathbb{R}$ tak, aby existovala vlastní $\lim_{x\to 0} \frac{\sin x \cdot e^x - ax - bx^2}{x^3}$. Limitu spočtěte.

(iv) $o(x^k) + o(x^k) = o(x^k)$

$\sin x = x - \frac{x^3}{3!} + o(x^3)$, $e^x = 1 + x + \frac{x^2}{2} + o(x^2)$

$\sin x \cdot e^x = \left(x - \frac{x^3}{6} + o(x^3)\right) \cdot \left(1 + x + \frac{x^2}{2} + o(x^2)\right) =$

$= x + x^2 + \frac{x^3}{2} + o(x^3) - \frac{x^3}{6} + o(x^3) + o(x^3) = x + x^2 + \frac{x^3}{3} + o(x^3)$

$= \lim_{x\to 0} \frac{x + x^2 + \frac{x^3}{3} + o(x^3) - ax - bx^2}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{3} + o(x^3)}{x^3} = \frac{1}{3}$

Položíme $a = 1$, $b = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \quad + \frac{x^n}{n!} + \dots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2} - i\cdot\frac{x^3}{3!} + \frac{x^4}{4!} \dots = \cos x + i\cdot\sin x$$

$$\sin x = x - \frac{x^3}{3!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\boxed{e^{ix} = \cos x + i\sin x}$$

e^x je $2\pi i$ periodická

$$(\cos x + i\cdot\sin x)^n = (e^{ix})^n = e^{ixn} = \cos(xn) + i\cdot\sin(xn)$$

$$\boxed{\cancel{e^{a+b}}\ e^a\cdot e^b = e^{a+b} \quad a,b\in\mathbb{C}}$$

$\cos(x+y) = \dots$ $\qquad$ $\sin(x+y) = \dots$

$$\cos(x+y) + i\sin(x+y) = e^{i\cdot(x+y)} = e^{ix}\cdot e^{iy} =$$

$$= (\cos x + i\cdot\sin x)\cdot(\cos y + i\cdot\sin y) = \cos x\cdot\cos y - \sin x\cdot\sin y$$

$$+ i\cdot\sin x\cos y + i\cdot\cos x\cdot\sin y$$

c) Spočtěte $T_3^{f,0}$ $\qquad f(x) = e^{\sin x}$

$T_3^{f,0}$ $\quad f(x) = e^{\sin x}$ $\quad f(0) = 0$ $\quad$ 25 – 7

(A) $f' = e^{\sin x} \cdot \cos x \Big|_0 = \underline{1}$ $\quad f'' = e^{\sin x} \cdot \cos x \cdot \cos x + e^{\sin x} \cdot (-\sin x)\Big|_0 = 1$

$f''' = e^{\sin x} \cdot \cos^3 x + \ldots \underset{0}{\sin x} + e^{\sin x} \cdot \underset{0}{\cos x \cdot (-\sin x)} + e^{\sin x}(-\cos x)\Big|_0 = 0$

$T_3^{f,0} = 1 + \underline{x} + \underline{\frac{x^2}{2}} + \underline{0}$

(B) $e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + o(y^3)$ $\qquad y = \sin x$

$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3!} + \underbrace{o(\sin^3 x)}_{= o(x^3)} =$

$\sin x = x - \frac{x^3}{3!} + o(x^3)$

$\sin x = x + o(x^2)$

$= 1 + x - \frac{x^3}{3!} + o(x^3) + \frac{(x + o(x^2))^2}{2} + \frac{(x + o(x^2))^3}{3!} + o(x^3) =$

$= \underline{1 + x} - \cancel{\frac{x^3}{3!}} + \underline{\frac{x^2}{2}} + o(x^3) + \cancel{\frac{x^3}{3!}} + o(x^3)$